MORE CATEGORY THEORY

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EXAMPLE

SUPPOSE WE HAVE MAPS OF SETS

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

PULLBACK SYMBOL

①

$X_1 = F(a)$
$Y = F(b)$
$X_2 = F(c)$

$$J \quad \begin{matrix} a \\ \downarrow \\ b \\ \uparrow \\ c \end{matrix}$$

THEIR PULLBACK IS

$$W = \{ (x_1, x_2) \in (X_1 \times X_2) : f_1(x_1) = f_2(x_2) \in Y \}$$

IF Y IS A SINGLETON, $W = X_1 \times X_2$

PROP. ②

$$\begin{tikzcd} U \arrow[dr, dashed, "p"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\ & W \arrow[r, "i_2"] \arrow[d, "i_1"] & X_2 \arrow[d, "f_2"] \\ & X_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

SUPPOSE A SET U HAS MAPS AS INDICATED WITH

$f_2 g_2 = f_1 g_1$

THEN $\exists!$ MAP $p : U \to W$ SUCH THAT $i_2 p = g_2$ AND $i_1 p = g_1$

PROOF: FOR $u \in U$, $p(u) = (g_1(u), g_2(u)) \in X_1 \times X_2$ AND $p(u) \in W$.

QED

THIS PROPERTY OF W IS CALLED A UNIVERSAL PROPERTY.

SUPPOSE

$$\begin{tikzcd} & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

IS A DIAGRAM IN A CATEGORY $\mathcal{C}$.

WE COULD ASK FOR AN OBJECT W WITH THE SAME UNIVERSAL PROPERTY.

IT MAY OR MAY NOT EXIST.

EXERCISE : DO THE ABOVE DISCUSSION WITH ALL ARROWS REVERSED

EXAMPLE

$$\begin{tikzcd} W & X_2 \arrow[l, dashed] \\ X_1 \arrow[u, dashed] & Y \arrow[l, "f_1"] \arrow[u, "f_2"'] \end{tikzcd}$$

W IS CALLED A <u>PUSHOUT</u>

IF $Y = \emptyset$ EMPTY SET, THEN

$W = X_1 \amalg X_2$ = DISJOINT UNION OF X_1 AND X_2.

ANOTHER PESPECTIVE

WE HAVE SEEN EXAMPLES OF "NATURAL" CATEGORIES

ONE ALSO HAS "SYNTHETIC" CATEGORIES INVENTED TO STUDY THE ABOVE

EXAMPLE

$$J = \overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longleftarrow \overset{c}{\bullet}$$

J HAS THREE OBJECTS AND TWO NON-IDENTITY MORPHISMS

A FUNCTOR $J \xrightarrow{F} \mathcal{S}et$ IS A

A FUNCTOR $J \xrightarrow{F} \mathcal{S}et$ IS A DIAGRAM OF THE FORM ①

DEF A CATEGORY IS SMALL IF ITS COLLECTION OF OBJECTS IS A SET RATHER THAN A PROPER CLASS.

DEF LET J BE A SMALL CATEGORY AND $\mathcal{C}$ AN ARBITRARY CATEGORY AND $F: J \to \mathcal{C}$. THIS IS A "J-SHAPED" DIAGRAM IN $\mathcal{C}$. ITS LIMIT IS AN OBJECT W IN $\mathcal{C}$ WITH A MORPHISM TO $F(j)$ $\forall$ OBJECT j IN J SATISFYING A UNIVERSAL PROPERTY AS IN PROP. 2. (IT MAY OR MAY NOT EXIST). A COLIMIT OF F IS AN OBJECT W' WITH MORPHISMS $F(j) \to W'$ $\forall j \in Ob(J)$ SUCH THAT ......

EXAMPLE $J = \begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$ TWO OBJECTS TWO MORPHISMS

$F : J \longrightarrow \mathcal{Sets}$ IS A DIAGRAM

$fi = gi$ $\quad W \xrightarrow{i} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{j} W'$ $\quad jf = jg$

ITS LIMIT $W = \{x \in X : f(x) = g(x)\}$

IT IS CALLED THE <u>EQUALIZER</u> OF f AND g.

ITS COLIMIT, THE <u>COEQUALIZER</u> IS THE QUOTIENT OF Y OBTAINED BY EQUATING $f(x)$ and $g(x)$ $\forall x \in X$.

<u>PROP</u> EVERY LIMIT IS AN EQUALIZER
" COLIMIT " COEQUALIZER.

DEF $\mathcal{C}^J$ (FOR J SMALL) IS THE CATEGORY WHOSE OBJECTS ARE FUNCTORS $J \to \mathcal{C}$, I.E. J-SHAPED DIAGRAMS WHERE MORPHISMS ARE COMM DIAGRAM SUCH AS

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] & Y \arrow[d, "g_0"] & X_2 \arrow[l, "f_2"'] \arrow[d, "g_2"] \\
X_1' \arrow[r, "f_1'"'] & Y' & X_2' \arrow[l, "f_2'"]
\end{tikzcd}$$

WE HAVE A DIAGONAL FUNCTOR

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^{J}$$

$X \longmapsto$ CONSTANT X-VALUED DIAGRAM

RECALL ADJOINT FUNCTORS

$$X \in \mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D} \ni Y \qquad (\perp)$$

$$\mathcal{C}(X, UY) \cong \mathcal{D}(FX, Y)$$

$\mathcal{C}(X, UY)$ = SET OF MORPHISMS $X \to UY$ IN $\mathcal{C}$

$\mathcal{D}(FX, Y)$ = SET OF MORPHISMS $FX \to Y$ IN $\mathcal{D}$

F IS THE LEFT ADJOINT OF U

U " RIGHT " F

EXAMPLE: THE LEFT & RIGHT ADJOINTS OF $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$ ARE

$$\mathcal{C} \underset{\text{COLIMIT}}{\overset{\text{LIMIT}}{\leftleftarrows}} \mathcal{C}^J$$

(WHICH IS WHICH?)

EXAMPLE LET G BE A GROUP

$\mathcal{C}$ = Top on Set

$BG = J$ = SMALL CATEGORY WITH ONE OBJECT AND A MORPHISM $\forall\ \gamma \in G$

A FUNCTOR $J \xrightarrow{F}$ Top on Set IS A SPACE on SET X EQUIPPED WITH AN ACTION OF THE GROUP G i.e. $\gamma \in G$ WE HAVE A MAP $F(\gamma): X \to X$.

EXERCISE: SHOW THAT

1. ITS LIMIT IS $X^G = \{x \in X : \gamma(x) = x\ \forall \gamma \in G\}$ = FIXED POINT SET

2. ITS COLIMIT IS X_G = QUOTIENT OF X DEFINED BY $\gamma(x) = x\ \forall x \in X$ AND $\gamma \in G$

ORBIT SPACE

$$X^G \longrightarrow X \longrightarrow X_G$$

<u>DEF</u> A CATEGORY IS <u>COMPLETE</u> (<u>COCOMPLETE</u>) IF ALL LIMITS (COLIMITS) EXIST. IT IS

(COLIMITS) EXIST. IT IS
BICOMPLETE IF BOTH EXIST

THM $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{G}rp$, $\mathcal{A}b$ IS BICOMPLETE. CONVENIENT!

EXAMPLE (AMUSING). LET J BE THE EMPTY CATEGORY (NO OBJECTS OR MORPHISMS) AND LET $\mathcal{C}$ BE COMPLETE. $\mathcal{C}^J$ HAS ONE OBJECT, THE EMPTY DIAGRAM. WHAT IS ITS LIMIT? IT IS AN OBJECT W WITH A CERTAIN PROPERTY, IE A UNIQUE MAP FROM EACH OBJECT U SATISFYING (EMPTY CONDITION). W IS CALLED A TERMINAL OBJECT IN $\mathcal{C}$

THE DUAL NOTION (FOR COCOMPLETE $\mathcal{C}$) IS AN INITIAL OBJECT W' WITH A UNIQUE MORPHISM TO EVERY OTHER OBJECT.

EVERY OTHER OBJECT.

IN $\mathcal{Set}$ THIS IS THE EMPTY SET

$\mathcal{Grp}$ " TRIVIAL GROUP

WHEN THESE TWO OBJECTS ARE THE SAME, WE SAY THE BICOMPLETE CATEGORY $\mathcal{C}$ IS POINTED.

HOMEWORK DUE MONDAY VIA GRADESCOPE.