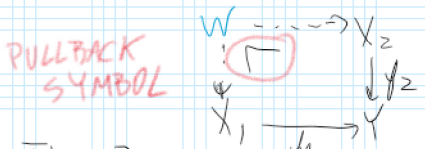


MORE CATEGORY THEORY

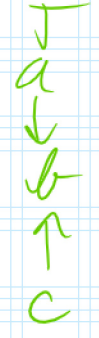
EXAMPLE

SUPPOSE WE HAVE MAPS OF SETS



(1)

$$\begin{aligned} X_1 &= F(a) \\ Y &= F(b) \\ X_2 &= F(c) \end{aligned}$$

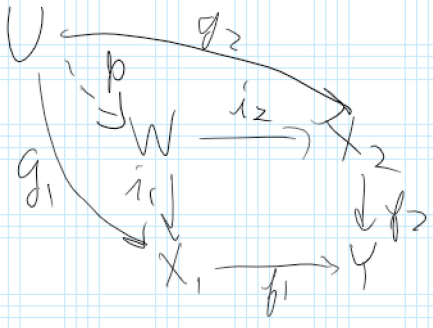


THEIR PULLBACK IS

$$W = \{ (x_1, x_2) \in (X_1 \times X_2) : f_1(x_1) = f_2(x_2) \in Y \}$$

IF Y IS A SINGLETON, $W = X_1 \times X_2$

PROP. (2)



A SET
SUPPOSE U HAS MAPS AS INDICATED WITH

$$f_2 \circ g_2 = f_1 \circ g_1$$

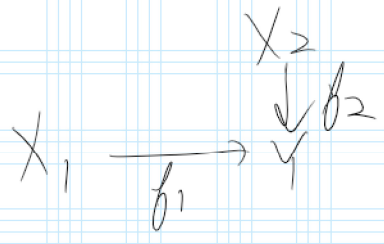
THEN $\exists!$ MAP $p: U \rightarrow W$ SUCH THAT $i_2 \circ p = g_2$ AND $i_1 \circ p = g_1$

PROOF: FOR $u \in U$, $p(u) = (g_1(u), g_2(u)) \in X_1 \times X_2$ AND $p(u) \in W$.

QED

THIS PROPERTY OF W IS CALLED A UNIVERSAL PROPERTY.

SUPPOSE

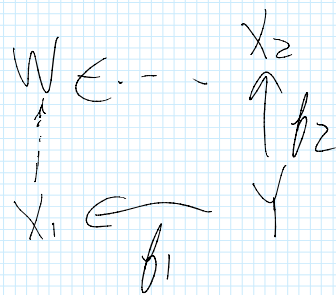


IS A DIAGRAM IN A CATEGORY \mathcal{C} .

WE COULD ASK FOR AN OBJECT W WITH THE SAME UNIVERSAL PROPERTY. IT MAY OR MAY NOT EXIST.

EXERCISE: DO THE ABOVE DISCUSSION
WITH ALL ARROWS REVERSED

EXAMPLE



W IS CALLED
A PUSHOUT

IF $Y = \emptyset$ EMPTY SET, THEN

$W = X_1 \sqcup X_2 =$ DISJOINT UNION
OF X_1 AND X_2 .

ANOTHER PERSPECTIVE

WE HAVE SEEN EXAMPLES OF

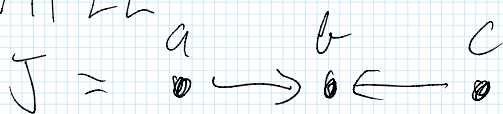
"NATURAL" CATEGORIES

ONE ALSO HAS

"SYNTHETIC" CATEGORIES

INVENTED TO STUDY THE ABOVE

EXAMPLE



J HAS THREE OBJECTS
AND TWO NON-IDENTITY
MORPHISMS

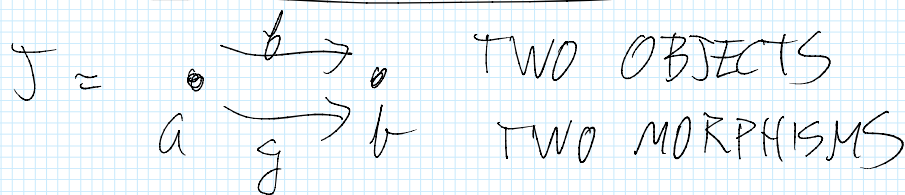
A FUNCTOR $J \xrightarrow{F} \mathcal{A} \text{ or } \mathcal{B}$ IS A

A FUNCTOR $J \xrightarrow{F} \text{Set}$ IS A
DIAGRAM OF THE FORM ①

DEF A CATEGORY IS SMALL IF ITS
COLLECTION OF OBJECTS IS A SET
RATHER THAN A PROPER CLASS.

DEF LET J BE A SMALL CATEGORY
AND \mathcal{C} AN ARBITRARY CATEGORY
AND $F: J \rightarrow \mathcal{C}$. THIS IS A "J-SHAPED"
DIAGRAM IN \mathcal{C} . ITS LIMIT IS AN
OBJECT W IN \mathcal{C} WITH A MORPHISM
TO $F(j)$ \forall OBJECT j IN J
SATISFYING A UNIVERSAL PROPERTY
AS IN PROP. 2. (IT MAY OR MAY NOT
EXIST). A COLIMIT OF F
IS AN OBJECT W' WITH
MORPHISMS $F(j) \rightarrow W'$ $\forall j \in \text{Ob}(J)$
SUCH THAT

EXAMPLE



$F: J \rightarrow \text{sets}$ is a diagram

$$\begin{array}{ccccccc}
 \beta_i = g_i & & W & \xrightarrow{i} & X & \xrightarrow{f} & Y & \xrightarrow{j} & W' & & j\beta = j'g \\
 & & & & & & \downarrow g & & & &
 \end{array}$$

ITS LIMIT $W = \{x \in X : f(x) = g(x)\}$

IT IS CALLED THE EQUALIZER OF f AND g .

ITS COLIMIT, THE COEQUALIZER

IS THE QUOTIENT OF Y OBTAINED BY EQUATING $f(x)$ and $g(x) \forall x \in X$.

PROP EVERY LIMIT IS AN EQUALIZER
 " COLIMIT " COEQUALIZER.

DEF \mathcal{C}^J (FOR J SMALL) IS

THE CATEGORY WHOSE OBJECTS ARE FUNCTORS $J \rightarrow \mathcal{C}$, I.E.

J -SHAPED DIAGRAMS WHERE MORPHISMS ARE COMM DIAGRAM SUCH AS

$$\begin{array}{ccccc}
 & & \beta_1 & & \beta_2 \\
 & & \searrow & & \swarrow \\
 X_1 & & & Y & & X_2 \\
 g_1 \downarrow & & & \downarrow g_0 & & \downarrow g_2 \\
 X'_1 & & \beta'_1 & & \beta'_2 & X'_2
 \end{array}$$

WE HAVE A DIAGONAL FUNCTOR

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$$

$X \longmapsto$ CONSTANT X -VALUED
DIAGRAM

RECALL ADJOINT FUNCTORS

$$X \in \mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \mathcal{D} \ni Y$$

$$\mathcal{C}(X, UY) \cong \mathcal{D}(FX, Y)$$

SET OF MORPHISMS
 $X \rightarrow UY$ IN \mathcal{C}

SET OF MORPHISMS
 $FX \rightarrow Y$ IN \mathcal{D}

F IS THE LEFT ADJOINT OF U
 U " RIGHT " F

EXAMPLE: THE LEFT & RIGHT ADJOINTS

OF $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$ ARE

$$\mathcal{C} \begin{array}{c} \xleftarrow{\text{LIMIT}} \\ \xleftarrow{\text{COLIMIT}} \end{array} \mathcal{C}^J \quad (\text{WHICH IS WHICH?})$$

EXAMPLE LET G BE A GROUP

$\mathcal{C} = \text{Top on Set}$

$\mathbb{B}G = \mathcal{J} = \text{SMALL CATEGORY WITH ONE OBJECT AND A MORPHISM } \forall \gamma \in G$

A FUNCTOR $\mathcal{J} \xrightarrow{F} \text{Top on Set}$

IS A SPACE OR SET X EQUIPPED WITH AN ACTION OF THE GROUP G

i.e. $\forall \gamma \in G$ WE HAVE A MAP $F(\gamma): X \rightarrow X$.

EXERCISE: SHOW THAT

① ITS LIMIT IS $X^G = \{x \in X : \gamma(x) = x \forall \gamma \in G\}$
= FIXED POINT SET

② ITS COLIMIT IS $X_G = \text{QUOTIENT OF } X$
ORBIT SPACE
DEFINED BY
 $\gamma(x) = x \quad \forall x \in X$
AND $\gamma \in G$

$$X^G \longrightarrow X \longrightarrow X_G$$

DEF A CATEGORY IS COMPLETE

(COCOMPLETE) IF ALL LIMITS

(COLIMITS) EXIST. IT IS

(LULU...)

BICOMplete IF BOTH EXIST

TM Set, Top, Inf, Ab IS
BICOMplete. CONVENIENT!

EXAMPLE (AMUSING). LET \mathcal{J} BE THE
EMPTY CATEGORY. (NO OBJECTS OR
MORPHISMS) AND LET \mathcal{C} BE COMPLETE.

$\mathcal{C}^{\mathcal{J}}$ HAS ONE OBJECT, THE EMPTY
DIAGRAM. WHAT IS ITS LIMIT?
IT IS AN OBJECT W WITH A CERTAIN
PROPERTY, IE A UNIQUE MAP
FROM EACH OBJECT U SATISFYING
(EMPTY CONDITION). W IS CALLED A
TERMINAL OBJECT IN \mathcal{C}

THE DUAL NOTION (FOR COCOMPLETE \mathcal{C})
IS AN INITIAL OBJECT W'
WITH A UNIQUE MORPHISM TO
EVERY OTHER OBJECT.

EVERY OTHER OBJECT.

IN Set THIS IS THE EMPTY SET
 Nup " TRIVIAL GROUP

WHEN THESE TWO OBJECTS ARE
THE SAME, WE SAY THE
BICOMplete CATEGORY \mathcal{C}
IS POINTED,

HOMEWORK DUE MONDAY
VIA GRADESCOPE.