BACK DOWN TO EARTH

Wednesday, September 16, 2020 1:07 PM

DEF. THE <u>HOPF MAP</u> $S^3 \to S^2$ IS AS FOLLOWS

S^3 = SET OF UNIT VECTORS IN $\mathbb{C}^2$

$S^2 = \mathbb{C} \cup \{\infty\}$ = ONE POINT COMPACTIFICATION

$= \mathbb{CP}^1$ = SET OF COMPLEX LINES THRU 0 IN $\mathbb{C}^2$

EACH POINT IN S^3 DETERMINES SUCH A COMPLEX LINE.

$$S^2 \ni (z_1, z_2) \longmapsto \begin{cases} z_1/z_2 & \text{IF } z_2 \neq 0 \\ \infty & \text{IF } z_2 = 0 \end{cases}$$

$S^3 \xrightarrow{\eta} S^2$ STUDIED BY HEINZ HOPF 1930

IT IS KNOWN TO BE ESSENTIAL, I.E. NOT HOMOTOPIC TO A CONSTANT MAP.

THE PREIMAGE OF EACH POINT IN $S^2 = \mathbb{CP}^1$ IS THE SET OF UNIT VECTORS LYING THAT COMPLEX LINE, I.E. A CIRCLE IN $S^3 = \mathbb{R}^3 \cup \{\infty\}$

ANY TWO SUCH CIRCLES ARE LINKED. SEE NILES JOHNSON'S VIDEO.

A SIMILAR MAP $S^7 \xrightarrow{\nu} S^4$ CAN BE DEFINED USING QUATERNIONS INSTEAD OF COMPLEX #s, AND $S^{15} \xrightarrow{\sigma} S^8$ USING OCTONIONS.

THREE HOPF MAPS. USING REAL #s IS $S^1 \longrightarrow S^1$ DEGREE 2.

EACH MAP IS ESSENTIAL.

HIGHER HOMOTOPY GROUPS

DEF. $\pi_n(X, x_0)$ IS THE SET OF HOMOTOPY CLASSES OF MAPS

$$(I^n, \partial I^n) \longrightarrow (X, x_0)$$

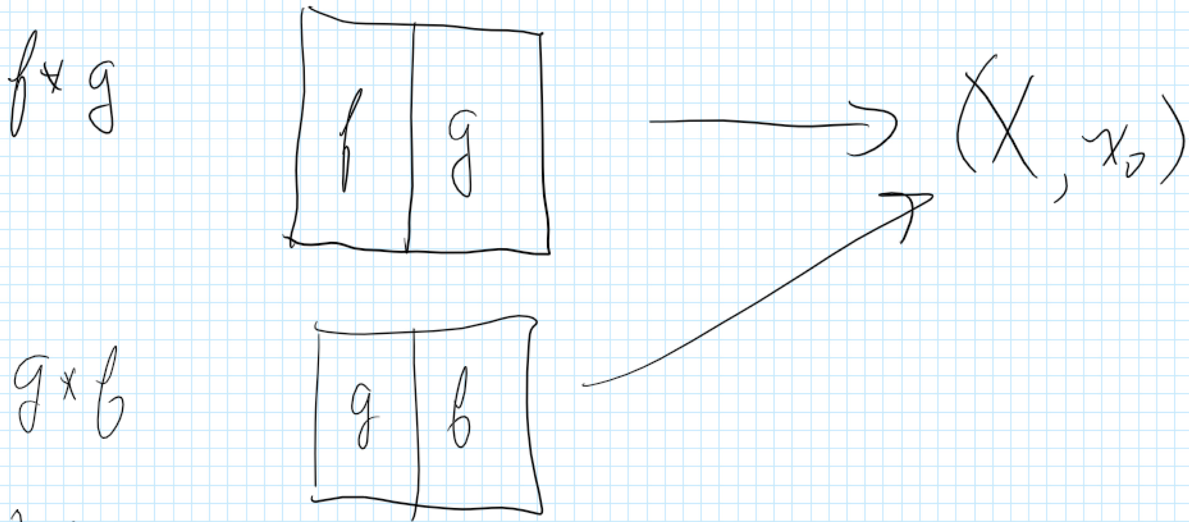
$[0,1]^n$

BOUNDARY $\approx S^{n-1}$

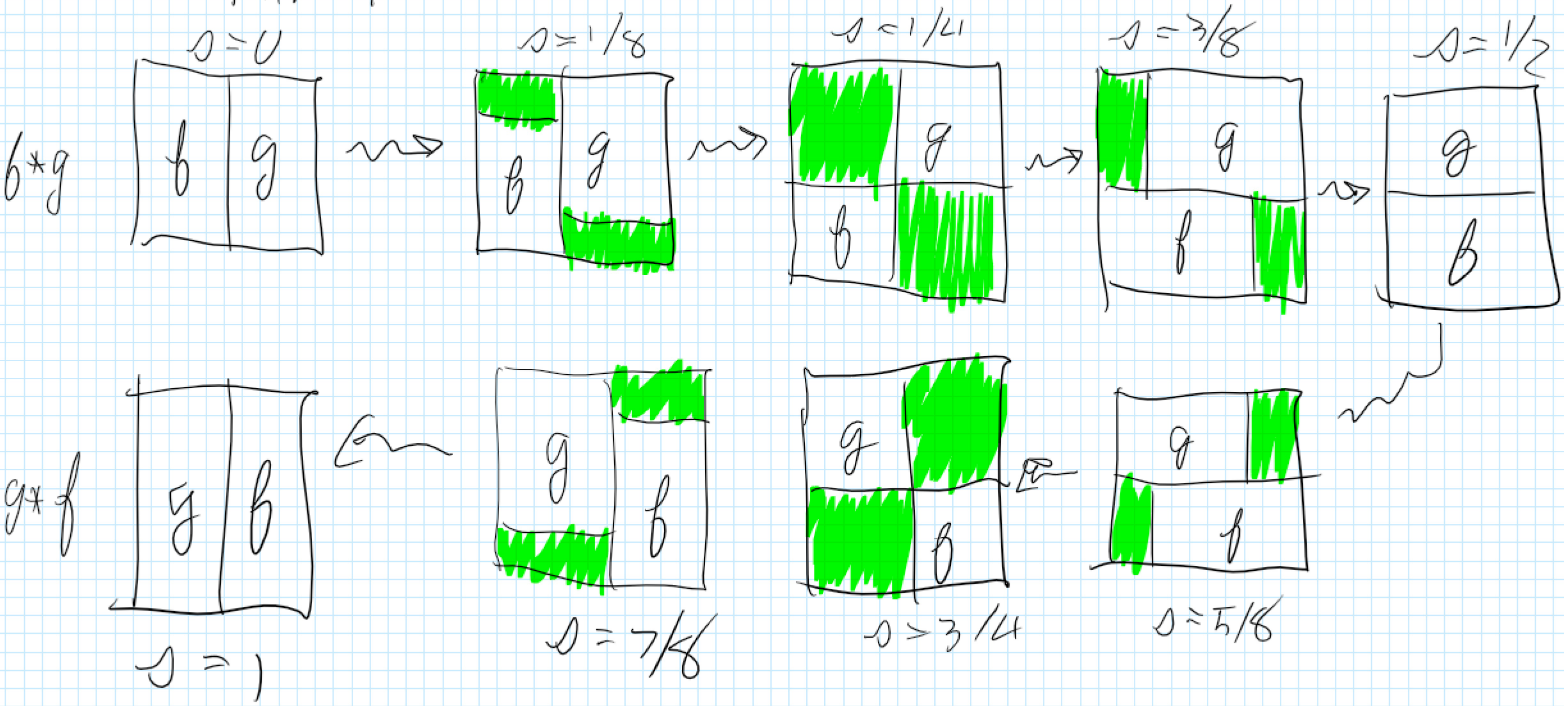
CAN DEFINE A GROUP STRUCTURE AS WE DID FOR $n=1$.

THEOREM $\pi_n(X, x_0)$ IS ABELIAN FOR $n > 1$.

PROOF FOR $n=2$. LET $f, g : (I^2, \partial I^2) \to (X, x_0)$



WILL SHOW A HOMOTOPY BETWEEN THEM



QED

HARD PROBLEM

COMPUTE $\pi_n(S^m, *)$ FOR ALL $m, n > 0$

KNOWN FACTS

$\cdots m, n \geq 0$

KNOWN FACTS

① FOR $m=1$ $\quad \pi_n(S^1,*) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & n>1 \end{cases}$

TO BE PROVED LATER

② FOR $n<m$, GROUP IS TRIVIAL

PROVED IN HATCHER

③ FOR $n=m$, IT IS $\mathbb{Z}$ GENERATED BY THE IDENTITY

④ $\pi_3(S^2) \approx \mathbb{Z}$ GENERATED BY HOPF MAP.

⑤ $\pi_n(S^m,*)$ IS FINITE ABELIAN EXCEPT

a) $n=m$ (SEE ABOVE)

b) m EVEN, $n=2m-1$

$\pi_{2l-1} S^{2l} \cong \mathbb{Z} \oplus$ FINITE GROUP

⑥ THERE A HOMOMORPHISM

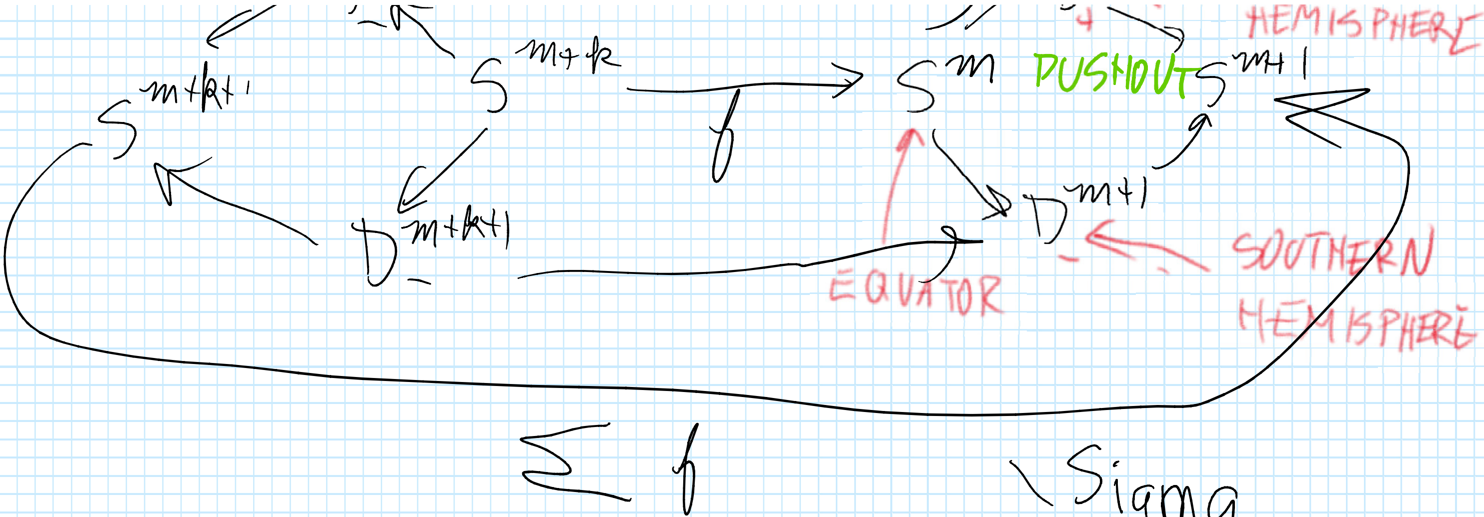
$k \geq 0 \qquad \pi_{m+k}(S^m,*) \longrightarrow \pi_{m+k+1}(S^{m+1},*)$

(THE SUSPENSION HOMOMORPHISM)

D^{m+k+1} $\quad D^{m+1}$ ← NORTHERN HEMISPHERE

$S^{m+k} \longrightarrow S^m$ PUSHOUTS S^{m+1}

S^{m+k+1}



\Sigma

THEOREM (HANS FREUDENTHAL 1935)

THE ABOVE IS AN ISOMORPHISM IF $k < m-1$ AND ONTO IF $k = m-1$

HENCE $\pi_{m+k}(S^m, *)$ DEPENDS ONLY ON k for $m > k+1$. CALL IT π_k, THE STABLE k-STEM.

KNOWN FOR $k \leq 90$.

k	0	1	2	3	4	5	6	7	8	...
π_k	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	

FOR $k > 0$ EACH GROUP IS FINITE ABELIAN. FOR $p > 2$, ITS p-COMPONENT IS KNOWN FOR $k \leq 2(p-1)p^3$

For $p = 2$ WE KNOW IT FOR $k \leq 90$.

A MAP
$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[r, "f"] \arrow[d] & (X, x_0) \\
(S^n, *) \arrow[ur, dashed, "\exists!"', "f^n"] &
\end{tikzcd}$$

CONSIDER THE SPACE OF ALL SUCH MAPS, $\Omega^n X$ N TH LOOP SPACE

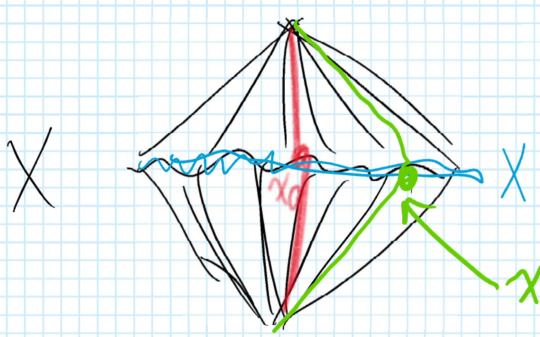
WITH COMPACT OPEN TOPOLOGY

Ω^n IS A FUNCTOR FROM THE CATEGORY OF POINTED SPACES TO ITSELF $X \longmapsto \Omega^n X$

ANOTHER SUCH FUNCTOR Σ^n, NTH SUSPENSION

$$\Sigma^n X = X \times S^n \big/ (X \times *) \cup (x_0 \times S^n)$$

FOR $n = 1$



SHRINK THE LINE TO A POINT

CONSIDER A MAP

$X \to \Sigma X \xrightarrow{f} Y$

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GREEN LINE $\longrightarrow$ CLOSED PATH

HENCE WE GET A MAP

$$X \xrightarrow{\hat{f}} \Omega Y$$

$$\mathrm{Map}(\Sigma X, Y) \approx \mathrm{Map}(X, \Omega Y)$$

Σ AND Ω ARE ADJOINT FUNCTORS.

TOWARD $\pi_1(S^1)$:

WILL USE THE MAP

$(\mathbb{R}^1, 0) \xrightarrow{p} (S^1, 1)$ = UNIT CIRCLE IN $\mathbb{C}$ WITH BASEPOINT 1

$$t \longmapsto e^{2\pi i t}$$

$$p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$$

DEFINE A HOM. $\mathbb{Z} \xrightarrow{\Phi} \pi_1(S^1, 1)$

FOR $n \in \mathbb{Z}$, CHOOSE w_n IN $\mathbb{R}$ FROM 0 TO n, e.g. $w_n(t) = nt$ FOR

0 TO n, e.g. $\omega_n(t) = nt$ FOR $0 \le t \le 1$

THEN $p\omega_n$ IS CLOSED IN S^1 AND REPRESENTS AN ELEMENT IN $\pi_1(S^1)$

WANT TO SHOW IT IS 1-1 AND ONTO

TWO CLAIMS

a) FOR ANY PATH $(I,0) \xrightarrow{f} (S^1, 1)$ STARTING AT THE BASEPOINT, THERE IS A UNIQUE PATH

$$(I,0) \xrightarrow{\tilde{f}} (\mathbb{R}, 0) \xrightarrow{p} (S^1, 1)$$

WITH $p\tilde{f} = f$

THIS MEANS Φ IS ONTO.

b) SIMILAR STATEMENT ABOUT HOMOTOPIES BETWEEN CLOSEDS IN S^1.

NEXT TIME.