THEOREM $\pi_1(S^1) \cong \mathbb{Z}$ = INTEGERS

Monday, September 21, 2020 7:59 AM

PROOF: WILL USE $p : \mathbb{R} \to S^1 \subset \mathbb{C}$

$t \mapsto e^{2\pi i t}$

$p^{-1}(1) \cong \mathbb{Z} \subset \mathbb{R}$. FOR ANY PATH W_n IN $\mathbb{R}$ FROM 0 TO n, $p(W_n)$ IS A CLOSED PATH IN S^1 REPRESENTING $\Phi(n) \in \pi_1(S^1)$. ANY TWO SUCH PATHS IN $\mathbb{R}$ ARE HOMOTOPIC, SO $\Phi : \mathbb{Z} \to \pi_1(S^1)$ IS WELL DEFINED

TWO CLAIMS ABOUT Φ THAT IMPLY IT IS AN ISOMORPHISM

a) FOR ANY PATH $(I,0) \xrightarrow{f} (S^1,1)$ STARTING AT $1 \in S^1$, $\exists!$ PATH $\hat{f} : (I,0) \to (\mathbb{R},0)$ WITH $p\hat{f} = f$

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & (\mathbb{R},0) \arrow[d, "p"] \\
(I,0) \arrow[r, "f"] \arrow[ur, dashed, "\exists!\,\hat{f}"] & (S^1,1)
\end{tikzcd}$$

$\hat{f}$ IS A <u>LIFTING</u> OF f

b)
$$\begin{tikzcd}
(I,0) \arrow[r, "\gamma"] \arrow[d, "i"] & (\mathbb{R},0) \arrow[d, "p"] \\
(I^2,I) \arrow[r, "h"] \arrow[ur, dashed, "\exists!\,\hat{h}"] & (S^1,1)
\end{tikzcd}$$

a) IMPLIES Φ IS ONTO. ANY CLOSED PATH IN $(S^1,1)$ LIFTS UNIQUELY

TO A PATH IN $\mathbb{R}$ STARTING AT 0 AND ENDING IN $p^{-1}(1) = \mathbb{Z}$.
HENCE EACH ELEMENT OF $\pi_1(S^1)$ IS IN THE IMAGE OF Φ

b) IMPLIES THAT Φ IS 1-1.
h COULD BE A HOMOTOPY BETWEEN TWO CLOSED PATHS IN S^1.
$\hat{h}$ WOULD BE A HOMOTOPY BETWEEN THEIR UNIQUE LIFTINGS

BOTH CLAIMS ARE SPECIALS OF

c)
$$\begin{tikzcd}
(X,x_0)\times\{0\} \arrow[r, "f"] \arrow[d, "i"] & (\mathbb{R},0) \arrow[d, "p"] \\
(X,x_0)\times(I,1) \arrow[r, "g"] \arrow[ur, dashed, "\exists! \hat{g}"] & (S^1,1)
\end{tikzcd}
\qquad
\begin{tikzcd}
(\hat{Y},\hat{y}_0) \arrow[d, "p"] \\
(Y,y_0)
\end{tikzcd}$$

COVERINGS

a) AND b) ARE THE CASES $X = pt$ AND $X = I$.

<u>DEFINITION</u> A <u>COVERING</u> IS A MAP $p:(\hat{Y},\hat{y}_0) \longrightarrow (Y,y_0)$ SUCH THAT EACH $y \in Y$ HAS A NEIGHBORHOOD U WHERE $p^{-1}(U) \approx U \times D$ FOR D DISCRETE. SUCH NEIGHBORHOOD ARE EVENLY COVERED

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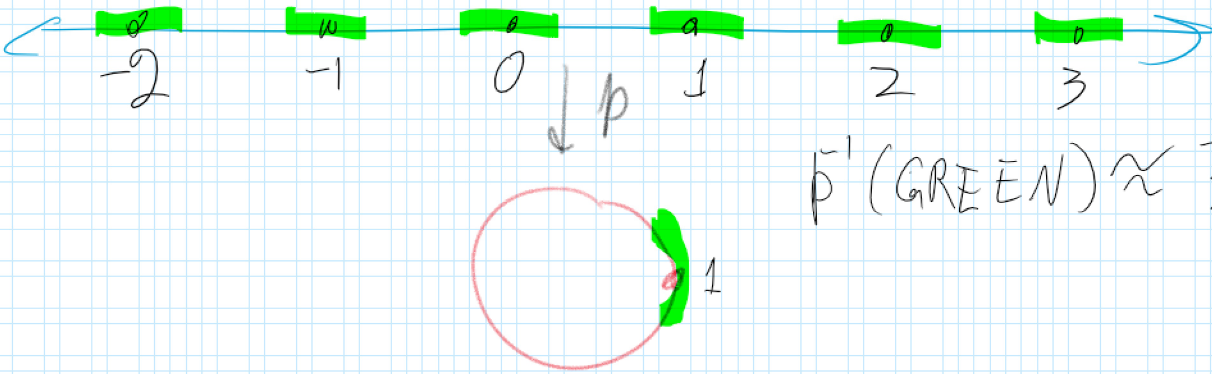
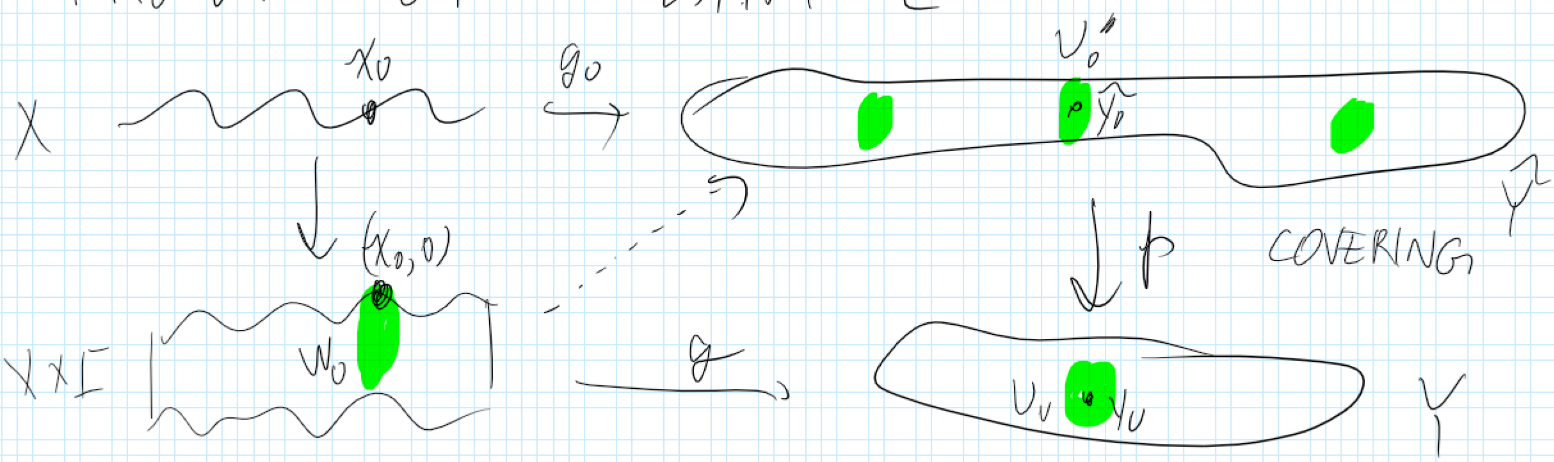


p^{-1}(GREEN) $\approx \mathbb{Z} \times$ GREEN

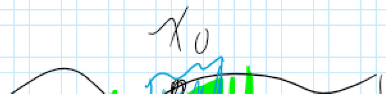
THE ARGUMENT SHOWS THAT THE MAP $\pi_1(\tilde{Y}, \tilde{y}_0) \longrightarrow \pi_1(Y, y_0)$ IS 1-1.

PROOF OF CLAIM C

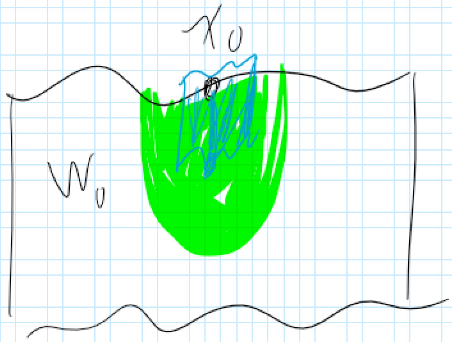


CHOOSE AN EVENLY COVERED NBD U_0 OF $y_0 \in Y$. LET $W_0 = g^{-1} U_0$. LET $\tilde{g}$ SEND W_0 TO $U_0' \subset \tilde{Y}$.

ZOOM IN ON $W_0 \subset X \times I$



$W \supset N \times [0, \epsilon]$ FOR A



$W_0 \supset N_0 \times [0, t_1]$ FOR A NEIGHBORHOOD N_0 OF x_0 AND SOME $t_1 > 0$.

LET $g(x_0, t_1) =: y_1 \in Y$ NEAR y_0

REPEAT AND GET A $t_2 > t_1$ WITH $y_2 = g(x_0, t_2)$. REPEAT THIS PROCESS UNTIL $t_n = 1$. WE HAVE EXTENDED $\tilde{g}$ TO $N_1 \times I$ FOR $x_0 \in N_1 \subset N_0$ AND N_1 A NEIGHBORHOOD OF x_0.

CAN CONTINUE AND EXTEND $\tilde{g}$ TO ALL OF $X \times I$. DETAILS ARE IN THE BOOK.

QED

COROLLARY: THE IDENTITY ON S^1, $\Phi(1)$ IS NOT HOMOTOPIC TO A CONSTANT MAP. $\Phi(0)$.

3 CONSEQUENCES

1. BROUWER FIXED POINT THEOREM
2. FUNDAMENTAL THEOREM OF ALGEBRA
3. HAM SANDWICH THEOREM

BROUWER FIXED POINT THEOREM (~1910)

$D^2 :=$ 2 DIMENSION DISK

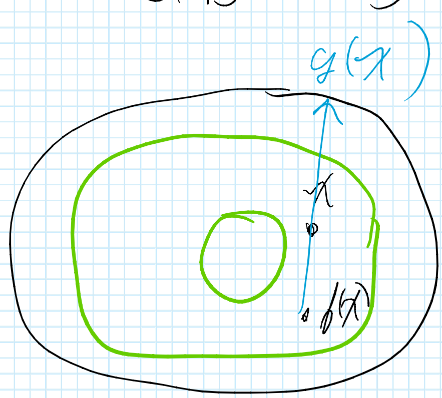
$$= \{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1 \}$$

LET $f : D^2 \to D^2$ BE CONTINUOUS

THEN $\exists x \in D^2$ WITH $f(x) = x$.

PROOF: ASSUME THERE IS NO SUCH POINT. DEFINE A MAP

$D^2 \xrightarrow{g} \partial D^2 \approx S^1$ AS FOLLOWS



THIS g IS CONTINUOUS AND

$g(x) = x$ FOR $x \in \partial D^2$.

THIS MAP GIVES US A HOMOTOPY

THIS MAP GIVES US A HOMOTOPY BETWEEN 1_{S^1} AND THE CONSTANT $g(0)$-VALUED MAP

THIS IS EXCLUDED BY THE COROLLARY. CONTRADICTION.

QED

KNOWN TO GENERALIZE TO MAPS $D^n \circlearrowleft$.

FUNDAMENTAL THEOREM OF ALGEBRA (GAUSS ~ 1800). LET $p(z) \in \mathbb{C}[z]$ NONCONSTANT A POLYNOMIAL. THEN $\exists z_0 \in \mathbb{C}$ WITH $p(z_0) = 0$. Hence

$p(z) = (z - z_0)\, q(z)$, ETC.

TOPOLOGICAL PROOF: ASSUME $p(0) \neq 0$ AND $p(z)$ IS MONIC, i.e. $p(z) = z^n +$ LOWER TERMS

SUPPOSE $p(z) \neq 0$ FOR $z \in \mathbb{C}$.

$$\mathbb{C} \xrightarrow{p} \mathbb{C} - \{0\} \simeq S^1 \xrightarrow{h} S^1$$

$$u \longmapsto \frac{u}{|u|}$$

FOR EACH $r > 0$ WE CAN RESTRICT

hp TO THE CIRCLE OF RADIUS ABOUT 0. THESE MUST ALL BE HOMOTOPIC, I.E. HAVE THE SAME DEGREE.

IDEA. FOR SMALL m

$p(me^{i\theta})$ IS CLOSE TO $p(0) = c_0 \neq 0$

THIS MAP HAS DEGREE 0.

FOR LARGE m,

$$p(me^{i\theta}) = (me^{i\theta})^n + \underbrace{\text{SMALLER TERMS}}_{\text{NEGLIGIBLE FOR } m \gg 0.}$$

THIS GIVES US A DEGREE n MAP $S^1 \longrightarrow S^1$

CONTRADICTION QED

BORSOK-ULAM THEOREM

LET $S^2 \longrightarrow \mathbb{R}^2$.

UNIT VECTORS $S^2 \subset \mathbb{R}^3$

$\exists\, x \in S^2$ SUCH $f(x) = f(-x)$.

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(KNOWN TO GENERALIZE TO

$S^n \longrightarrow \mathbb{R}^n$

PROOF: ASSUME NO SUCH x EXISTS

DEFINE $g : S^2 \longrightarrow S^1$

$$x \longmapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

$$g(-x) = -g(x)$$

TO BE CONTINUED.