Van Kampen Theorem

Monday, September 28, 2020 1:31 PM

LET $X = A \cup B$ WITH $A \cap B, A, B$ and X ALL PATH CONNECTED, AND $x_0 \in A \cap B$. WE HAVE DIAGRAM OF GROUPS

PUSHOUT SYMBOL

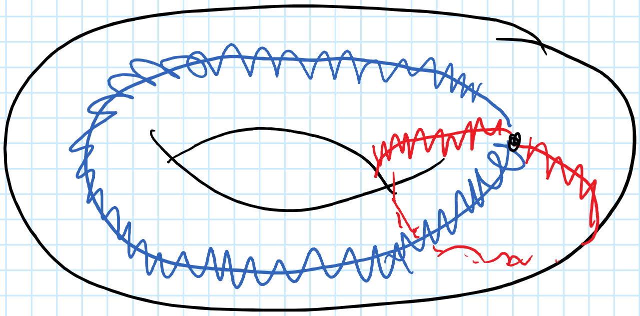
$$
\begin{tikzcd}
\pi_1(A \cap B) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \pi_1(A) \arrow[d, "f"] \\
\pi_1(B) \arrow[r, "g"'] & \text{PUSHOUT} \cong \pi_1(X) \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}
$$

4 EXAMPLE X = TORUS

A = NEIGHBORHOOD OF $S^1 \vee S^1$

$B = X - (S^1 \vee S^1)$

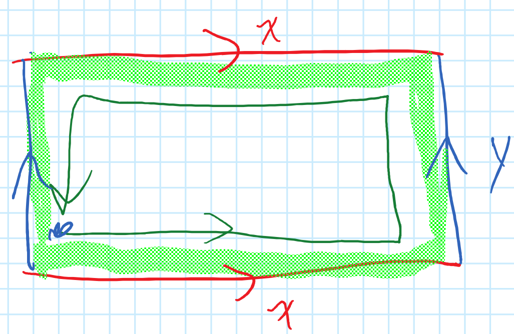
$A \cap B \simeq S^1 \times I$



CUTTING X ALONG THE TWO CIRCLES GIVES A RECTANGLE

$\pi_1(A \cap B) \simeq \mathbb{Z}$

$\pi_1 A \simeq F_2$ = FREE GROUP ON TWO GENERATORS



$\pi_1 B = 0$

B = INTERIOR OF RECTANGLE

A = NEIGHBORHOOD OF BOUNDARY GREEN

$A \cap B$ = GREEN CIRCLE

$$\begin{tikzcd} & F_2 \ni x, y \arrow[dr, "f"] & \\ \gamma \in \mathbb{Z} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & ? \\ & 0 \arrow[ur, "g"'] & \end{tikzcd}$$

GREEN CIRCLE γ GENERATES $\pi_1(A \cap B) \cong \mathbb{Z}$

RED + BLUE CIRCLES GENERATE $\pi_1(A) \cong F_2$

$\alpha(\gamma) = x y x^{-1} y^{-1} =: [x, y]$ = COMMUTATOR OF x AND y

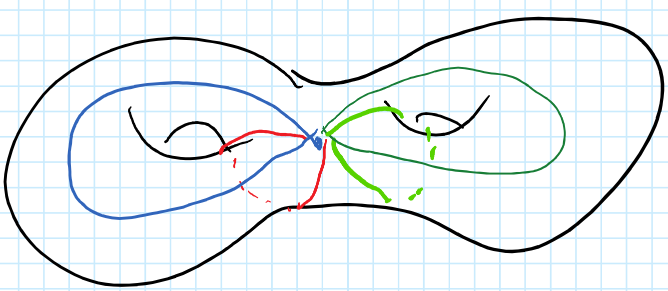
$= e$ IN $\pi_1(X)$

CONCLUSION

$$\pi_1(X) = [x, y : x y x^{-1} y^{-1}] \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$x y x^{-1} y^{-1} = e$$
$$x y x^{-1} = y$$
$$x y = y x$$

X = SURFACE OF GENUS 2

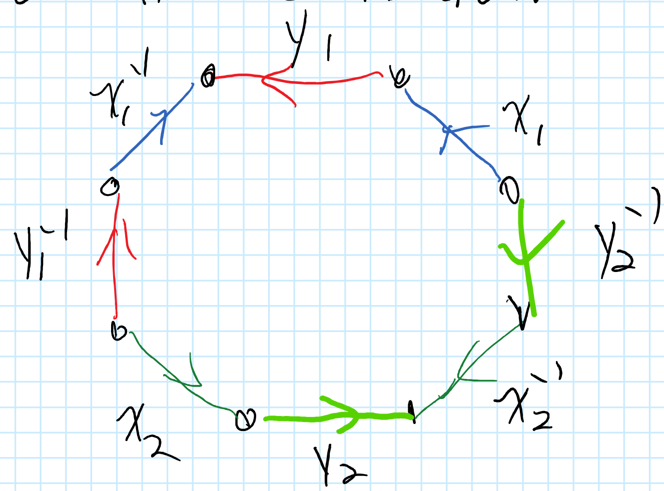


A = NBD OF $S^1 \vee S^1 \vee S^1 \vee S^1$

B = COMPLIMENT OF 4 CIRCLES $\simeq pt$

$A \cap B \simeq S^1$

CUTTING X ALONG 4 CIRCLES YIELDS AN OCTAGON



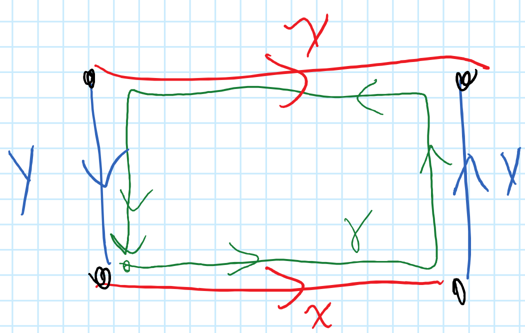
$$\pi_1(X) = [x_1, y_1, x_2, y_2 : x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1}]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad [x_1, y_1][x_2, y_2]$$

X = KLEIN BOTTLE

$\alpha(\gamma) = xyx^{-1}y$



$$\pi_1 X = [x, y : xyx^{-1}y]$$ NON ABELIAN

PROOF OF VAN KAMPEN THEOREM

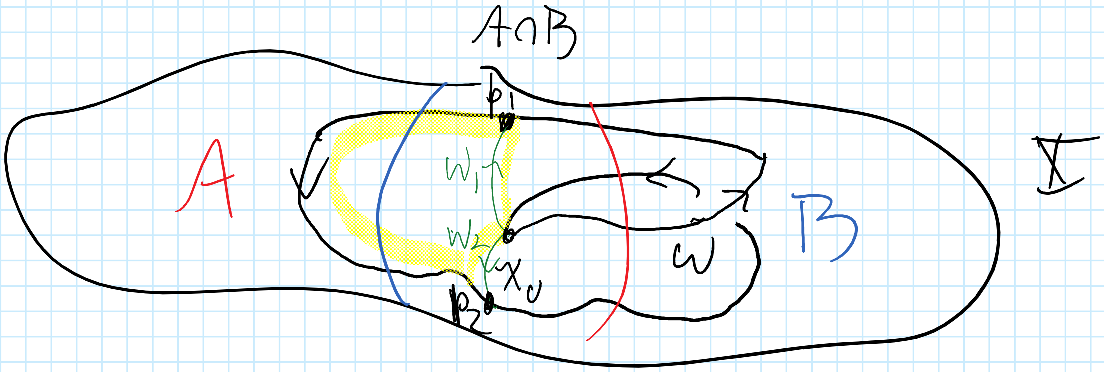
P = PUSHOUT GROUP

\begin{tikzcd}
 & \pi_1(A) \arrow[d, "\beta"] \arrow[dr] & \\
\pi_1(A\cap B) \arrow[ur, "\alpha"] \arrow[dr] & P \arrow[r, dashed, "\Phi"] & \pi_1(X) \\
 & \ &
\end{tikzcd}

$$\pi_1(A\cap B) \xrightarrow{\beta} \pi_1(B) \xrightarrow{g} \pi_1(X) \qquad P \overset{\alpha}{\dashrightarrow} \pi_1(X)$$

WANT TO SHOW Φ IS 1-1 AND ONTO

TO SHOW IT IS ONTO, LET w BE A CLOSED PATH IN X



CAN CHOOSE A FINITE # OF POINTS $p_1, \dots, p_{n-1} \in A\cap B$. LET $p_0 = x_0 = p_n$ SUCH THAT FOR EACH, THE PORTION OF w GOING FROM p_{i-1} to p_i LIES EITHER IN A OR IN B

FOR $1 \le i \le n-1$, LET w_i BE A PATH IN $A\cap B$ FROM x_0 TO p_i

NOTATION: LET $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$ SO THAT OUR SUBPATHS ARE

$w([s_{i-1}, s_i])$ $\qquad w(s_i) = p_i$

CONSIDER THE CLOSED PATHS

$\theta_i := w_{i-1} * w([s_{i-1}, s_i]) * w_i^{-1}$ FOR $1 \le i \le n$.

EACH LIE IN A OR B.

THE ELEMENT

$\theta_1 * \theta_2 * \cdots * \theta_n$ IS IN $\Phi(P)$

$$= \underbrace{\cancel{w_0} * w[0, s_1] * w_1^{-1}}_{\theta_1} * \underbrace{w_1 * w[s_1, s_2] * w_2^{-1}}_{\theta_2} \cdots w_{n-1} w[s_{n-1}, 1] \cdot \cancel{w_n^{-1}}$$

$$\simeq w[0, s_1] * w[s_1, s_2] \cdots * w[s_{n-1}, 1] = w$$

SO $w \in \operatorname{Im} \Phi$. Φ IS ONTO.

TO SHOW Φ IS 1-1, CONSIDER A HOMOTOPY BETWEEN CLOSED PATHS IN X, I.E. A MAP $h : I^2 \to X$ WITH CERTAIN CONDITIONS

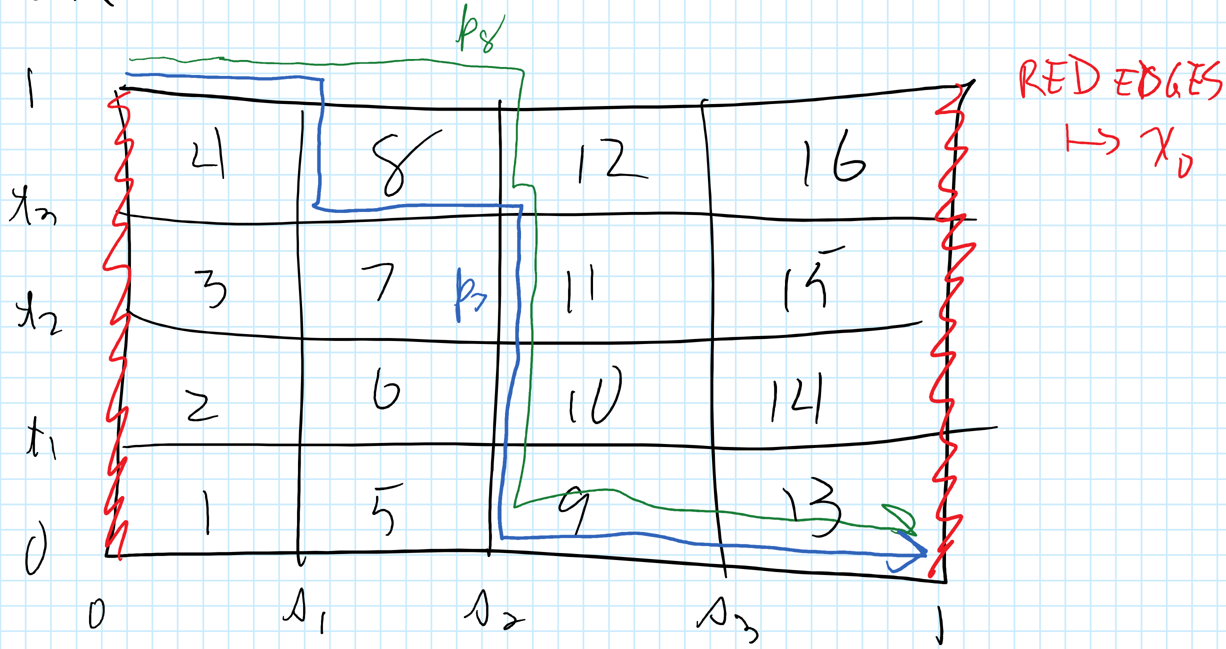
WE CAN FIND #

$0 = s_0 < s_1 < \cdots < s_m = 1$ PATH PARAMETER

AND $0 = t_0 < t_1 < \cdots < t_n = 1$ HOMOTOPY PARAMETER

SUCH THAT h SENDS EACH RECTANGLE $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ TO EITHER A OR B.

PICTURE WITH $m = n = 4$



WE GET mn RECTANGLES

FOR EACH $0 \le k \le mn$ LET p_k BE THE LINE WITH THE FIRST k RECTANGLES BELOW IT AND THE OTHERS ABOVE IT.

p_0 = BOTTOM EDGE OF I^2

p_{mn} = TOP EDGE

THEY REPRESENT HOMOTOPIC CLOSED PATH IN X

THE HOMOTOPY BETWEEN p_{k-1} AND p_k LIES IN k th RECTANGLE, WHICH MAPS TO A OR B

HENCE EACH HOMOTOPY BETWEEN CLOSED PATHS IN X IS

THE "COMPOSITE" OF HOMOTOPIES IN A AND B.

THIS MEANS Φ IS 1-1.

QED

NEXT TOPIC: COVERINGS

FOR A "NICE" SPACE X, FOR EACH SUBGROUP

$H \subset G = \pi_1 X$

THERE IS A COVERING $\hat{X}_H \to X$ WITH $\pi_1(\hat{X}_H) \cong H$

(THE MAP $\pi_1 \hat{X} \to \pi_1 X$ FOR ANY COVERING $\hat{X} \to X$ IS 1-1)

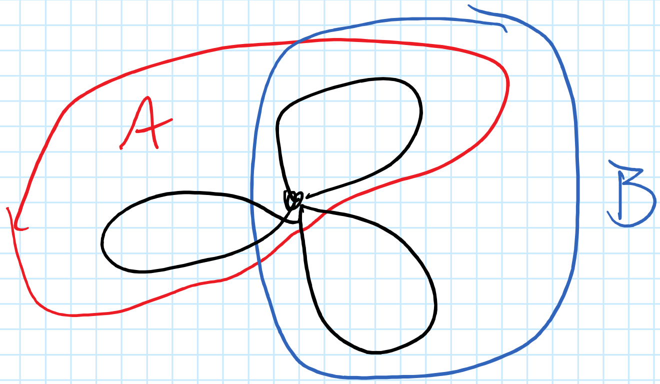
$\exists$ BIJECTION BETWEEN PATH CONNECTED COVERING OF THE PATH CONNECTED SPACE X

AND SUBGROUPS $H \subset \pi_1 X$.

EXAMPLE $X = S^1 \vee \underbrace{S^1 \vee S^1}_{B}$

EXAMPLE $X = \underbrace{S^1 \vee S^1}_{A} \vee S^1$

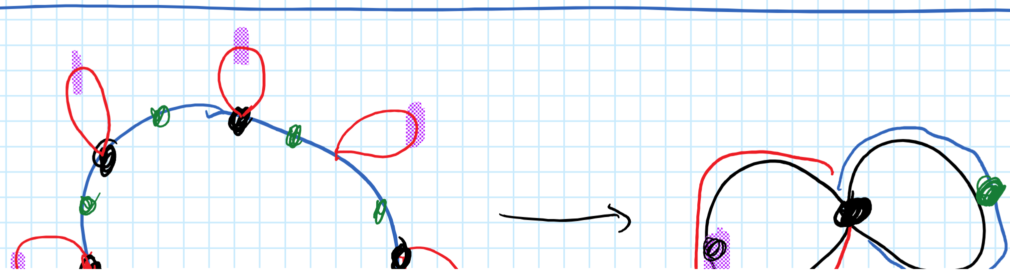
$A \cap B = S^1$

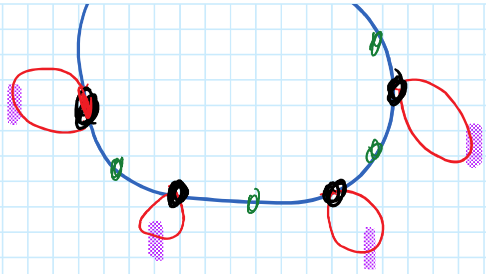


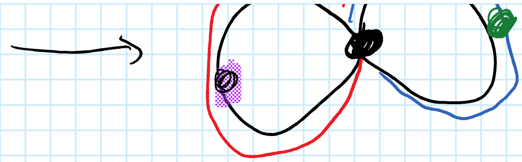
$A \cap B = S^1$

$A, B \simeq S^1 \vee S^1$

$$\begin{tikzcd}
 & \pi_1(S^1 \vee S^1) = F_2 \arrow[dr] & \\
\pi_1 S^1 = \mathbb{Z} \arrow[ur] \arrow[dr] & & \pi_1 X = \pi_1(S^1 \vee S^1 \vee S^1) \\
 & F_2 = \pi(S^1 \vee S^1) \arrow[ur] &
\end{tikzcd}$$







$\tilde{X} \longrightarrow X$