COVERINGS

Wednesday, September 30, 2020 1:53 PM

COVERING SPACE

BASE SPACE

DEF. A MAP $\tilde{X} \xrightarrow{p} X$ IS A <u>COVERING</u> IF $\forall x \in X$ $\exists$ NBD $U \ni x$. SUCH THAT $p^{-1}(U) \approx U \times D$ WITH D DISCRETE.

IF X IS PATH CONNECT, WE GET THE SAME D FOR EACH POINT.

MAIN THEOREM SAYS FOR "NICE" X THERE IS A PATH CONNECTED COVERING $\tilde{X}$ WITH $\pi_1(\tilde{X}) = 0$. LET $G = \pi_1(X)$. THEN G ACTS FREELY ON $\tilde{X}$ WITH ORBIT SPACE X. FOR EACH SUBGROUP $H \subset G$, $\tilde{X}/H$ IS ALSO A COVERING SPACE OF X. EVERY PATH CONNECTED COVERING OF X IS ISOMORPHIC TO $\tilde{X}/H$ FOR SOME $H \leq G$.

COMPARE THIS TO GALOIS THEORY. LET $K \subset L$ BE A GALOIS EXTENSION WITH

GALOIS GROUP G. G ACTS ON BY FIELD AUTOMORPHISMS FIXING K. HENCE

$$K = L^G = \{\lambda \in L : \gamma(\lambda) = \lambda \ \forall \gamma \in G\}$$

EACH INTERMEDIATE FIELD $K \subset L' \subset L$ IS L^H FOR SOME SUBGROUP $H \subset G$.

$$\text{SUBGROUPS OF } G \xleftrightarrow{\text{BIJECTION}} \text{INTERMEDIATE FIELDS}$$

$$\text{SUBGROUPS OF } G = \pi_1 X \longleftrightarrow \text{PATH CONNECTED COVERINGS OF } X.$$

IF $H \subset G$ IS NORMAL, $L' = L^H$ IS ALSO A GALOIS EXTENSION OF K WITH GROUP G/H.

$\tilde{X}_H$ HAS AN ACTION BY G/H WITH ORBIT SPACE X.

WHAT IS A "NICE" SPACE?

DEF A SPACE IS LOCALLY GREEN IF EVERY NBD OF EVERY POINT HAS A GREEN SUBNEIGHBORHOOD.

EXAMPLE: THE CANTOR SET IS NOT LOCALLY PATH CONNECTED.

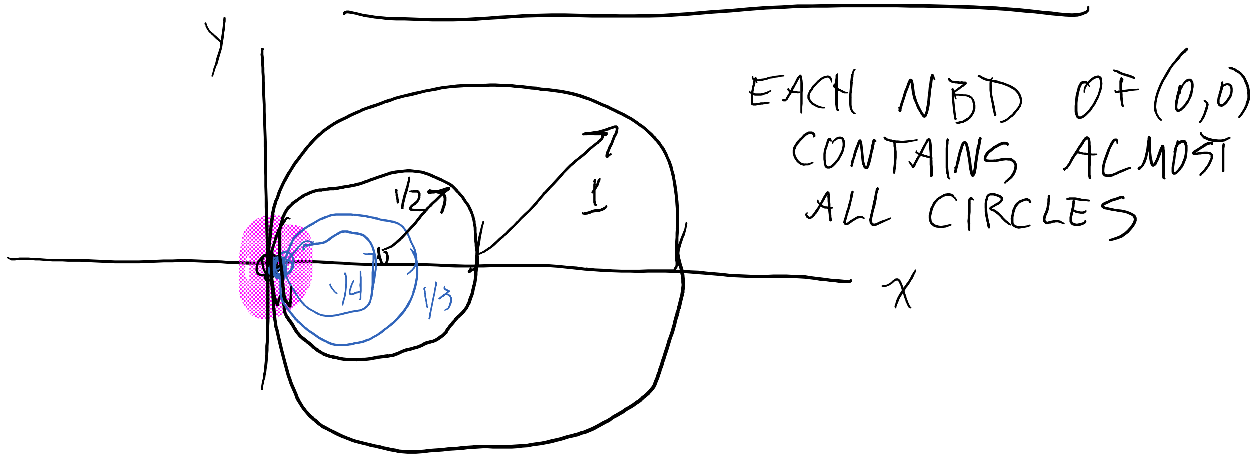
DEF A SPACE IS SEMI-LOCALLY GREEN IF (SOMETHING WEAKER THAN ABOVE)

EXAMPLE

DEF. A SPACE X IS SEMI-LOCALLY SIMPLY CONNECTED (SLSC) IF EACH $x \in X$ HAS A NBD U SUCH THAT THE MAP

$\pi_1 U \to \pi_1 X$ IS TRIVIAL.

DEF. THE HAWAIIAN EARRING $H \subset \mathbb{R}^2$



H IS THE OF CIRCLE THRU (0,0)

WITH CENTER $(1/n, 0)$ FOR $n = 1, 2, 3, 4, \ldots$

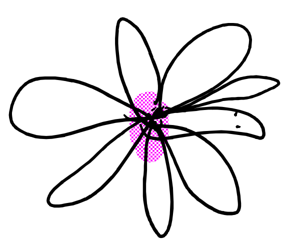
NOTE H IS <u>NOT</u> THE SAME AS $W = S^1 \vee S^1 \vee S^1 \vee \cdots\cdots S^1 \vee \cdots$

= INFINITE WEDGE OF CIRCLES WITH COMMON POINT x_0

EVEN THOUGH H $\exists$ $W \xrightarrow{f} H$

f SENDS THE nTH CIRCLE OF W TO THE CIRCLE CENTERED AT $(1/n, 0)$.

W HAS MORE OPEN SETS THAN H.



$\exists$ NEIGHBORHOOD OF x_0 CONTAINING NONE OF THE CIRCLES

IT IS SIMPLY CONNECTED

W IS LSC

H IS NOT SLSC.

THEOREM. LET X BE A SPACE THAT IS

a) PATH CONNECTED

b) LOCALLY PATH CONNECTED

c) SLSC

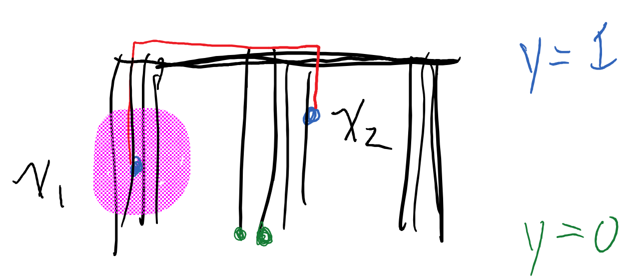
NICE

THEN THERE IS A PATH CONNECTED SIMPLY CONNECTED COVERING $\hat{X}$ OF X.

EXAMPLE OF A PATH CONNECTED X THAT IS NOT LPC.

LET C = CANTOR SET $\subset [0,1]$

$\mathbb{R}^2 \supset X = C \times [0,1] \cup [0,1] \times \{1\}$



EACH SMALL NBD OF x_1

FAILS TO HAVE A PATH CONNECTED SUBNEIGHBORHOOD.

WE WILL SEE $\boxed{\tilde{X} = \text{UNIVERSAL COVERING OF } X.}$

a) $G = \pi_1(X)$ ACTS FREELY ON $\tilde{X}$ WITH ORBIT SPACE X.

(AN ACTION OF A GROUP Γ ON A SPACE Y IS FREE $\forall \gamma \neq e \in \Gamma$ AND EACH $y \in Y$, $\gamma(y) \neq y$.)

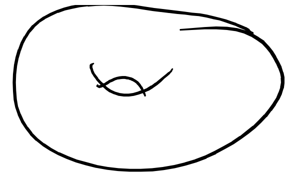
b) IF $Y \to X$ IS ANOTHER PATH CONNECTED COVERING, THEN Y IS ALSO NICE AND $\tilde{Y} \cong \tilde{X}$

c) THE MAP $\underset{H}{\pi_1(Y)} \to \underset{G}{\pi_1(X)}$ IS 1-1.

d) IF $H \subset G$ IS NORMAL, THEN G/H ACTS FREELY ON Y WITH ORBIT SPACE X.

EXAMPLE

$$X = S^1 \times S^1 = \text{TORUS}$$



$$G = \mathbb{Z} \oplus \mathbb{Z} \ni (m, n) = \gamma$$

$\tilde{X} = \mathbb{R}^2$ WITH FREE ACTION

$$\gamma(x, y) = (x + m, y + n)$$