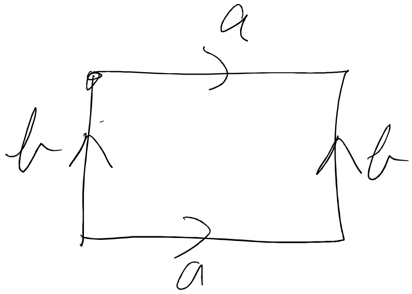
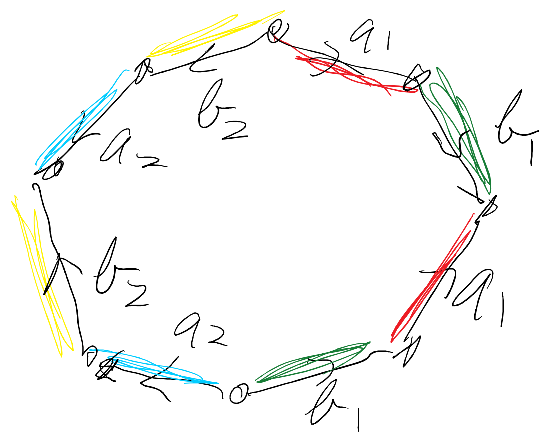
MORE ABOUT THE VAN KAMPEN THEOREM

A SURFACE OF GENUS g CAN BE CONSTRUCTED AS A QUOTIENT OF A $4g$-GON BY IDENTIFYING CERTAIN EDGES

$g=1$



$g=$



THE SURFACE IS A UNIT OF TWO SUBSPACES, ONE $\cong D^2$ AND THE OTHER A WEDGE OF $2g$ CIRCLES. THE VAN KAMPEN DIAGRAM

$$\begin{tikzcd} & \pi_1 D^2 = 0 \arrow[dr] & \\ \mathbb{Z} = \pi_1 S^1 \arrow[ur] \arrow[dr, "\theta"] & & \pi_1(\text{SURFACE}) \\ & \pi_1(\text{WEDGE}) = F_{2g} \arrow[ur] & \end{tikzcd}$$

θ SENDS A GENERATOR TO

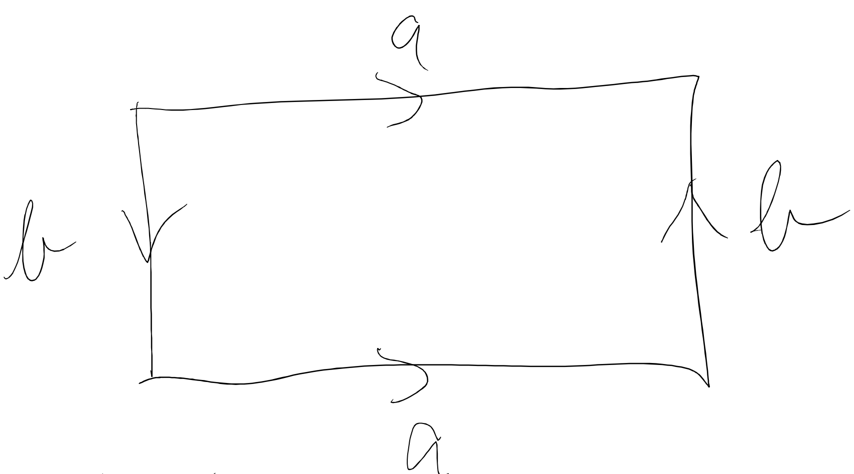
$$[a_1, b_1][a_2, b_2]\cdots[a_g, b_g]$$

$$[a,b] := aba^{-1}b^{-1}$$

WE HAVE LOOKED AT $\mathbb{RP}$ AS UNION OF MOBIUS BAND AND A DISK

$\rightsquigarrow \pi_1 \mathbb{RP}^2 \simeq \mathbb{Z}/2$

KLEIN BOTTLE



π_1 KLEIN BOTTLE

$$\langle a, b : aba^{-1}b = e \rangle$$

$$aba^{-1} = b^{-1}$$

INTERESTING GROUP.

PROOF OF VKT:

$X = A \cap B$ WITH
X, A, B AND $A \cap B$ ARE
PATH CONNECTED

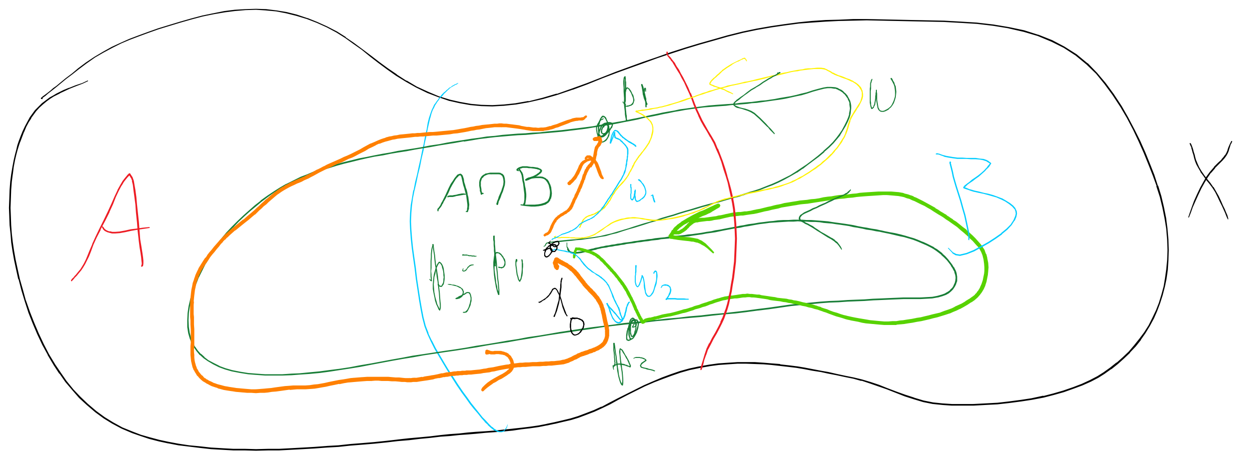
$$\begin{tikzcd}
 & A \arrow[dr] & \\
A \cap B \arrow[ur] \arrow[dr] & & X \\
 & B \arrow[ur] &
\end{tikzcd}$$

PUSHOUT DIAGRAM
IN CAT OF
POINTED SPACES

$$\begin{tikzcd}
 & \pi_1(A) \arrow[d] \arrow[dr, bend left] & & \\
\pi_1(A \cap B) \arrow[ur] \arrow[dr] & P \arrow[r, "\Phi"] & \pi_1(X) \\
 & \pi_1(B) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

DIAGRAM
IN
CATEGORY
OF GROUPS

THEOREM SAYS LOWER DIAGRAM
IS ALSO A PUSHOUT.
LET P BE THE PUSHOUT GROUP. IT
HAS A UNIQUE MAP Φ TO $\pi_1 X$
MAKING THE DIAGRAM COMMUTE.
WANT TO SHOW Φ IS 1-1 AND ONTO.
TO SHOW IT IS ONTO



LET w BE A CLOSED PATH IN X

WE CAN CHOOSE # $0 = t_0 < t_1 < \cdots t_{n-1} < t_n = 1$ SUCH $w[t_{i-1}, t_i]$ LIES IN EITHER A OR B. FOR EACH i WITH $0 < i < n$, CHOOSE A PATH w_i FROM p_0 TO $w(t_i)$.

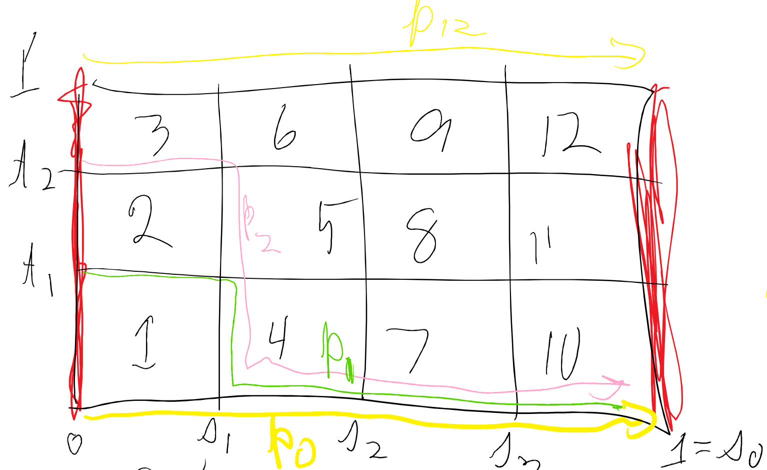
USE THESE PATHS w_i IN $A \cap B$ TO FORM CLOSED PATHS IN A OR B EACH USING A SEGMENT OF w.

CONCLUSION: w REPRESENTS AN ELEMENT IN $\pi_1 X$ THAT IS A PRODUCT OF IMAGES OF $\pi_1 A$ AND $\pi_1 B$. THIS MEANS Φ IS ONTO.

TO SHOW Φ IS 1-1, SUPPOSE WE HAVE A HOMOTOPY BETWEEN CLOSED PATHS, IE A MAP

$$I^2 \xrightarrow{h} X$$

WITH CERTAIN CONDITIONS



s = PATH PARAMETER

t = HOMOTOPY PARAMETER

RED EDGES MAP TO x_0

WE CAN AS $0 = s_0 < s_1 < \cdots < s_m = 1$

AND $0 = t_0 < t_1 < \cdots\cdots t_n = 1$

SUCH h SENDS $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ TO A OR B. NUMBER THE mn RECTANGLES AS SHOWN.

FOR $0 \le k \le mn$, LET p_k BE THE PATH IN I^2 FROM LEFT TO RIGHT THAT ABOVE THE FIRST k RECTANGLE

AND BELOW THE OTHERS.

ϕ_{k-1} AND ϕ_k ARE RELATED BY

A HOMOTOPY COMING FROM EITHER

A OR B

WE ARE DECOMPOSING THE BIG

HOMOTOPY INTO LITTLE ONES

EACH HAPPENING IN A OR B.

THIS MEANS Φ IS 1-1.

QED