COVERINGS

Thursday, October 13, 2022 8:06 AM

DEF A MAP $\tilde{X} \xrightarrow{p} X$ IS A <u>COVERING</u> IF EACH $x \in X$ HAS A NBD U SUCH THAT $p^{-1}(U) \cong U \times D$ FOR DISCRETE D

SUCH A NBD IS SAID TO BE <u>EVENLY COVERED</u>.

EXAMPLE ① $\mathbb{R} \xrightarrow{p} S^1 \subset \mathbb{C}$

$t \longmapsto e^{2\pi i t}$

FOR ANY $U \subset S^1$, $p^{-1}(U) \cong \mathbb{Z} \times U$

② $\mathbb{C} \supset S^1 \xrightarrow{[n]} S^1$

$z \longmapsto z^n$

IN BOTH CASES ONE HAS $\pi_1(p)$ IS 1-1

<u>MAIN THEOREM</u> LET X BE A "NICE" PATH CONNECTED SPACE WITH $\pi_1 X \cong G$. THEN THERE IS COVERING $p: \hat{X} \longrightarrow X$ WITH

a) $\tilde{X}$ IS PATH CONNECTED WITH $\pi_1 \tilde{X} = 0$,

b) G ACTS FREELY ON $\tilde{X}$ SO THAT

THE ORBIT SPACE $\tilde{X}/G$ IS X

c) FOR EACH SUBGP $H \subset G$, THE MAP $\tilde{X}/H \longrightarrow \tilde{X}/G = X$ IS A COVERING

d) ANY PATH CONNECTED COVERING OF X IS ISOMORPHIC TO ONE OF THE ABOVE

e) IF $H \subset G$ IS NORMAL, THE ACTION OF G ON $\tilde{X}$ INDUCES A FREE ACTION OF $G/H = K$ ON $\tilde{X}/H$ WITH $(\tilde{X}/H)/K = X$.

THERE IS A 1-1 CORRESPONDANCE BETWEEN PATH CONN COVERINGS OF X AND SUBGROUPS OF $\pi_1 X$.
FOR $X = S^1$, THESE ARE SHOWN ABOVE
FOR $X = S^1 \vee S^1$ $\quad \pi_1 X = F_2$

ANALOGY WITH GALOIS THEORY:

LET $K \subset L$ BE A GALOIS EXTENSION, I.E. $\exists G$ ACTING ON L SUCH THE K = THE FIXED POINT SET e.g. $\mathbb{R} \subset \mathbb{C} \circlearrowleft$ CONJUGATION

$G = C_2$

$\exists$ 1-1 CORRESPONDENCE BETWEEN SUBGPS $H \subset G$ AND INTERMEDIATE FIELDS

$$\begin{array}{ccccc} K & \subset & K' & \subset & L \\ \| & & \| & & \\ L^G & & L^H & & \end{array}$$

IF H IS NORMAL IN G THEN G/H ACTS ON K' FIXING K.

$$\begin{array}{ccccc} L^G & & L^H & & \\ \| & & \| & & \\ K & \subset & K' & \subset & L \end{array}$$

$$\begin{array}{ccccc} X & \leftarrow & X' & \leftarrow & \tilde{X} \\ \| & & \| & & \\ \tilde{X}/G & & \tilde{X}/H & & \end{array}$$

$\tilde{X}$ = UNIVERSAL COVER OF X

DEF A SPACE X IS LOCALLY ANGRY IF EVERY NBD OF EACH $x \in X$ HAS AN ANGRY SUB-NEIGHBORHOOD.

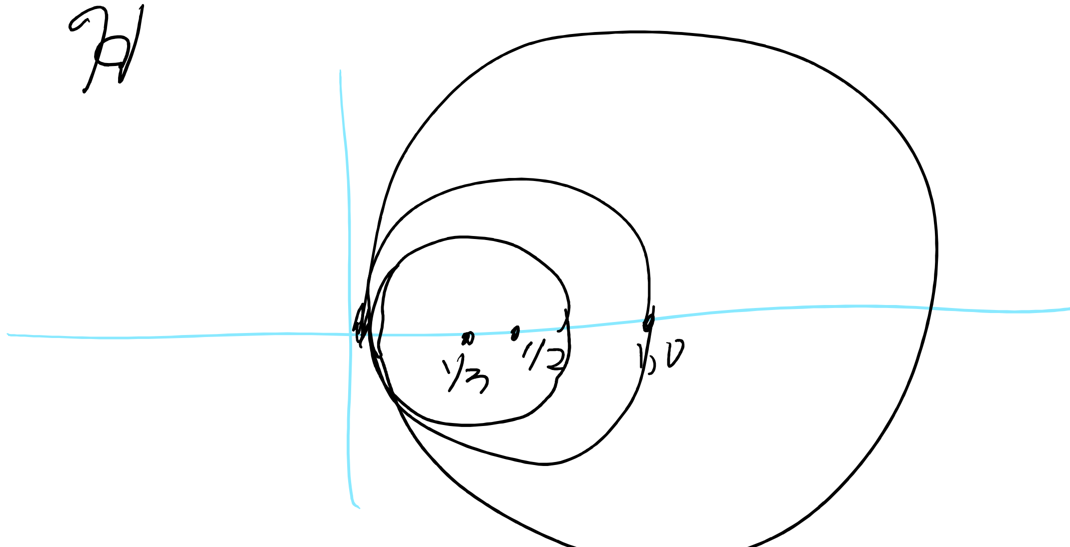
DEF A SPACE IS SEMI-LOCALLY ANGRY IF CERTAIN CONDITIONS

ARE MET.

DEF A SPACE IS SEMI-LOCALLY SIMPLY CONNECTED (SLSC) IF EACH $x \in X$ HAS A NBD U SUCH THAT $\pi_1(U) \to \pi_1(X)$ IS TRIVIAL

COUNTER EXAMPLE HAWAIIAN EARRING

$\mathcal{H}$



$\mathcal{H}$ = UNION OF CIRCLES THRU $(0,0)$ CENTERED AT $\{(1/n, 0) : n = 1, 2, 3 \ldots\}$

EACH NBD U OF $(0,0)$ HAS

$$0 \neq \pi_1(U) \xrightarrow{1-1} \pi_1(\mathcal{H})$$

$\mathcal{H}$ IS NOT SLSC

REMARK

$W = S^1 \vee S^1 \vee S^1 \vee S^1 \vee \cdots$

THERE IS A MAP $W \xrightarrow{f} \mathcal{H}$

f IS 1-1 AND ONTO nTH $S^1 \to S^1$ ABOUT $(1/n, 0)$

f^{-1} IS WELL DEFINED

BUT <u>NOT</u> CONTINUOUS.

W HAS MORE OPEN SETS THAN $\mathcal{H}_0$

<u>MAIN THEOREM</u> LET X BE A SPACE

THAT IS

a) PATH CONNECTED

b) LOCALLY PATH CONNECTED (LPC)

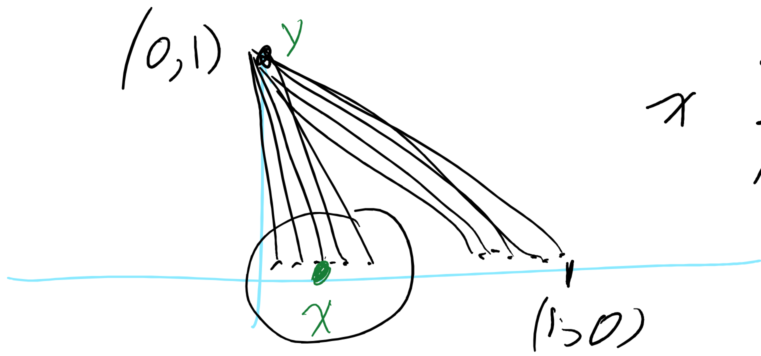
c) SLSC.

THEN X HAS A SIMPLY CONNECTED PATH CONNECTED COVERING SPACE.

EXAMPLE OF A PATH CONNECTED SPACE THAT IS NOT LPC.

LET $K \subset [0,1]$ BE THE CANTOR SET

$\hat{K}$ = CONE ON K



NO NBD OF x NOT CONTAINING y IS PATH CONNECTED

PROOF: CHOOSE BASEPT $x_0 \in X$

CONSIDER PAIRS (x_1, ω) FOR $x_1 \in X$ AND ω IS A PATH FROM x_0 TO x_1. LET $[\omega]$ DENOTE THE HOMOTOPY CLASS OF ω

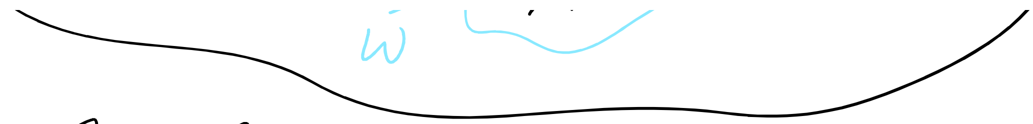
$\hat{X} = \{ (x_1, [\omega]) \}$. WE HAVE A MAP

$\hat{X} \xrightarrow{p} X \qquad (x_1, [\omega]) \longmapsto x_1$

WE NEED A TOPOLOGY ON $\hat{X}$ SUCH THAT p IS CONTINUOUS, $\hat{X}$ IS PATH CONNECTED AND SIMPLY CONNECTED.

WILL DEFINE A BASIS FOR THE TOPOLOGY ON $\hat{X}$.

TO DEFINE NBDS OF $(x_1, [\omega]) \in \hat{X}$ WE NEED "SUITABLE" NBDS U OF $x_1 \in X$ ← TO BE DEFINED LATER



LET $\tilde{U} \subset \hat{X} = \left\{ (x_2, [w * w']) : \begin{array}{l} x_2 \in U \\ w' \subset U \end{array} \right\}$

THE NBD U MUST BE PATH CONN WITH $\pi_1(U, x_1) \longrightarrow \pi_1(X, x_1)$ TRIVIAL.

THESE SUBSETS OF $\hat{X}$ CAN BE SHOWN TO BE A BASIS FOR A TOPOLOGY, I.E. ANY FINITE INTERSECTION OF THESE SUBSETS IS ALSO A UNION OF SUCH SUBSETS.