THEOREM. IF X IS A SPACE WHICH IS

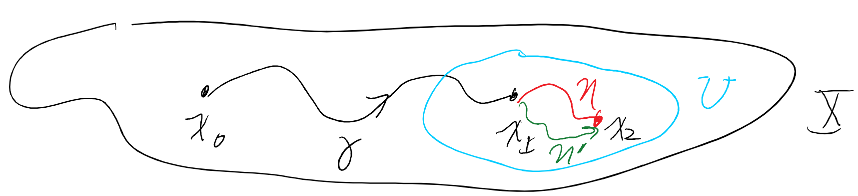
i) PATH CONNECTED
ii) LOCALLY PATH CONNECTED
iii) SEMI-LOCALLY SIMPLY CONNECTED (SLSC)

THEN $\exists$ COVERING $\tilde{X} \xrightarrow{p} X$ WITH $\tilde{X}$ SATISFYING (i)-(iii) AND $\pi_1(\tilde{X}) = 0$

PROOF: CHOOSE A BASE POINT $x_0 \in X$. LET $\tilde{X} = \{(x_1, [\gamma])\}$ WHERE $x_1 \in X$, $[\gamma]$ IS A HOMOTOPY CLASS OF PATHS FROM x_0 TO x_1. DEFINE $p: \tilde{X} \to X$ BY

$$p(x_1, [\gamma]) = x_1$$

A TOPOLOGY ON $\tilde{X}$ CAN BE DEFINED IN TERMS OF NBS U OF x_1 WHICH ARE PC, LPC AND $\pi_1(U) \longrightarrow \pi_1(X)$ IS ZERO.



LET $U_{[\gamma]} = \{[\gamma * \eta] : \eta$ IS A PATH IN U STARTING AT $x_1 = \gamma(1)\}$

HATCHER (pp 64-65) SHOWS THE COLLECTION $\mathcal{U}$ OF SUCH SUBSETS OF $\tilde{X}$ IS A BASIS FOR A TOPOLOGY ON $\tilde{X}$.

WE HAVE A MAP $\tilde{X} \xrightarrow{p} X$

$$(x_1, [\gamma]) \longmapsto x_1$$

WHICH IS CONTINUOUS

REMARKS

1) p IS ONTO BECAUSE X IS PC

2) p SENDS $U_{[\gamma]}$ TO U BECAUSE U IS PATH CONNECTED

3) THE MAP $U_{[\gamma]} \longrightarrow U$ IS 1-1 BECAUSE $\pi_1 U \longrightarrow \pi_1 X$ IS TRIVIAL.

WHY $p : \tilde{X} \xrightarrow{p} X$ IS A COVERING

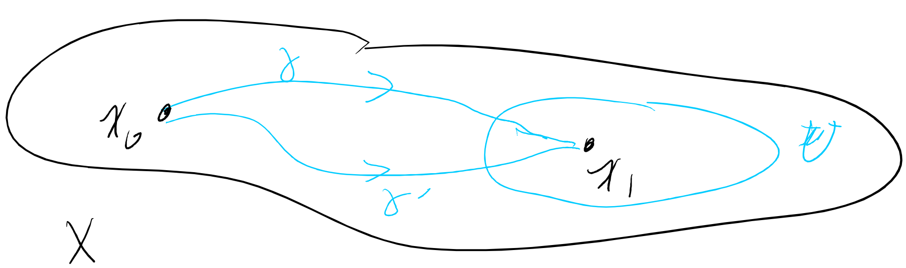
FOR U AS ABOVE,

$$p^{-1}(U) = \bigcup_{\gamma} U_{[\gamma]}$$

WHERE THE UNION IS OVER ALL PATHS γ FROM x_0 TO A GIVEN $x_1 \in U$.

THIS PREIMAGE IS HOMEO TO $U \times D$ (FOR SOME DISCRETE D)

BECAUSE IF γ' IS ANOTHER PATH FROM x_0 TO x_1, TH



THEN $[\gamma * \bar{\gamma}'] \in \pi_1 X$, SO THE SET OF SUCH $[\gamma']$ IS ISO TO $\pi_1(X, x_0)$. THIS IS OUR D_0

WHY $\tilde{X}$ IS SIMPLY CONNECTED

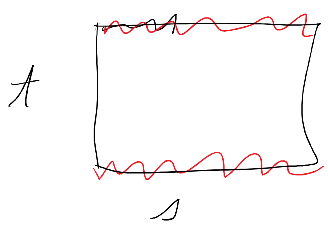
THE BASE POINT OF $\tilde{X}$ IS $(x_0, [\text{CONSTANT PATH}]) =: \tilde{x}_0$

A CLOSED PATH ζ IN $\tilde{X}$ IS A MAP $(I, \partial) \xrightarrow{\zeta} (\tilde{X}, \tilde{x}_0)$ IS EQUIVALENT TO A MAP

$$I^2 \xrightarrow{\hat{\zeta}} X$$

$$(s, t) \longmapsto \zeta(t)(s)$$

$\zeta(t)$ IS A PATH IN X



RED GOES TO x_0

WE WANT TO SHOW THAT ANY SUCH ζ IS HOMOTOPIC TO THE CONSTANT PATH, WE WANT A HOMOTOPY h IN $\tilde{X}$, CORRESPONDING TO A MAP

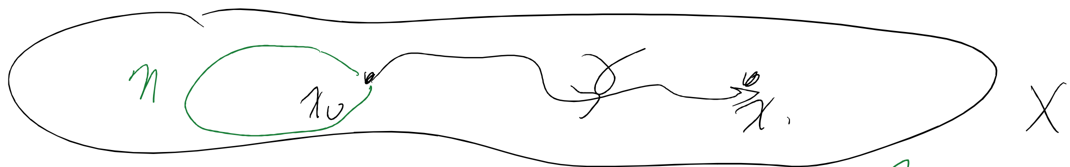
$$I^3 \xrightarrow{\hat{h}} X$$

$\hat{h}(s,t,u) = \hat{\gamma}(s, ut) = \gamma(s)(ut)$

THIS IS THE DESIRED HOMOTOPY MAKING $\pi_1(\tilde{X}) = 0$. QED

REMARKS

THE GROUP $G = \pi_1(X, x_0)$ ACTS ON $\tilde{X}$ IS AN OBVIOUS WAY



$\eta \in \pi_1(X) \quad [\gamma] \mapsto [\eta * \gamma] \in \tilde{X}$

THIS ACTION IS FREE, AND $\tilde{X}/G = X$ BY CONSTRUCTION.

FOR EACH SUBGP $H \subset G$, $\hat{X}/H \longrightarrow \hat{X}/G = X$ IS ALSO A COVERING OF X.

EACH PATH CONN COVERING IS ONE OF THESE.

NEW TOPIC (THE MAIN TOPIC)

HOMOLOGY.

FOR A POINTED SPACE (X, x_0) WE HAVE HOMOTOPY GROUPS $\pi_k(X, x_0)$, ABELIAN FOR $k \geq 2$.

THESE ARE EASY TO DEFINE

BUT HARD TO COMPUTE

WILL DEFINE ABELIAN GROUP $H_k(X)$ AND $H_k(X, A)$ FOR $A \subset X$ THAT ARE MUCH EASIER TO COMPUTE.

$k = 0, 1, 2, 3 \ldots$

WE HAVE FUNCTORS

$$\begin{tikzcd} X \arrow[d, mapsto] & \mathrm{Top} \arrow[r, "H_k"] \arrow[d] & Ab \\ (X, \emptyset) & \mathrm{Pairs} \arrow[ur, "H_k"'] & \end{tikzcd}$$

: Spaces

(X, A)

Ab = ABELIAN GROUPS

A CONTINUOUS MAP $X \xrightarrow{f} Y$ OR $(X, A) \xrightarrow{f} (Y, B)$

INDUCES HOMOMORPHISMS $H_k(X) \xrightarrow[f_*]{H_k(f)} H_k(Y)$

$H_k(X, A) \longrightarrow H_k(Y, B)$

THESE WILL BE SHOWN TO SATISFY FOUR AXIOMS

1) <u>HOMOTOPY AXIOM</u> IF $f, g : X \to Y$ WITH $f \simeq g$, THEN $H_k(f) = H_k(g)$

2) <u>EXACTNESS AXIOM</u>

GIVEN $A \xhookrightarrow{i} X$ . . .

$$(A,\emptyset) \xrightarrow{i} (X,\emptyset) \xrightarrow{j} (X,A)$$

WE GET

MYSTERY

$$\cdots \xrightarrow{\partial_{k+1}} H_k(A) \xrightarrow{H_k(i)} H_k(X) \xrightarrow{H_k(j)} H_k(X,A) \xrightarrow{\partial_k} H_{k-1}(A) \to \cdots$$

CONNECTING HOMOMORPHISM

$H_k(A) = H_k(A,\emptyset)$, $H_k(X) = H_k(X,\emptyset)$

THIS SEQUENCE IS EXACT

I.E. THE KERNEL OF EACH HOM IN SIGHT IS EQUAL TO THE IMAGE OF THE INCOMING HOM.

3) EXCISION AXIOM

LET $B \subset A \subset X$ WITH

CLOSURE $(B) \subseteq$ INTERIOR (A)

CLOSURE (B) = INTERSECTION OF ALL CLOSED SUPERSETS OF B

INTERIOR (A) = UNION OF ALL OPEN SUBSETS OF A

THEN

$$H_k(X-B, A-B) \longrightarrow H_k(X,A)$$

IS AN ISOMORPHISM

4) DIMENSION AXIOM

$$H_k(pt) = \begin{cases} \mathbb{Z} & \text{IF } k=0 \\ 0 & \text{IF } k>0 \end{cases}$$

SOME ALGEBRAIC INFRA STRUCTURE

DEF A CHAIN COMPLEX C IS A DIAGRAM

$$C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 \xleftarrow{d_4} C_4 \leftarrow \cdots$$

WHERE EACH C_i IS AN AB. GP,
EACH d_i IS A HOMOMORPHISM
WITH $d_i d_{i+1} = 0$ $\forall i$.

C_i = CHAIN GROUP
$C_i \ni x$ IS A CHAIN
d_i = BOUNDARY HOM.

IF $x \in C_i$ AND $d_i(x) = 0$, WE SAY x IS A CYCLE.
IF $x \in C_i$ IS $d_{i+1}(y)$ FOR SOME $y \in C_{i+1}$ WE SAY x IS A BOUNDARY

HENCE $\ker d_i \supset \operatorname{im} d_{i+1}$

$$\ker d_i / \operatorname{im} d_{i+1} = H_i(C)$$

= iTH HOMOLOGY GROUP OF C.