BORSUK-ULAM THEOREM

GIVEN A MAP $f : S^2 \to \mathbb{R}^2$, $\exists\, x \in S^2$ WITH $f(x) = f(-x)$.

PROOF: ASSUME THERE IS NO SUCH POINT x. LET $g : S^2 \to S^1$

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \in S^1$$

NOTE $g(-x) = -g(x)$

$$\begin{tikzcd} S^2 \arrow[r, "g"] \arrow[d, "{[-1]}"'] & S^1 \arrow[d, "{[-1]}"] \\ S^2 \arrow[r, "g"] & S^1 \end{tikzcd}$$

WILL SHOW THAT NO MAP $g : S^2 \to S^1$ HAS THIS PROPERTY.

$$\begin{tikzcd} & & (\mathbb{R}, 0) \arrow[d, "p"] \\ ([0,1], 0) \arrow[r, "h"] \arrow[urr, dashed, "\tilde{h}"] & (S^2, *) \arrow[r, "g"] & (S^1, 1) \\ & \mathbb{R}^3 \arrow[u, phantom, "\cap"] & \mathbb{C} \arrow[u, phantom, "\cap"] \end{tikzcd}$$

WHERE $h(s) = (\cos 2\pi s, \sin 2\pi s, 0)$ FOR $0 \le s \le 1$, gh IS A CLOSED PATH IN S^1

$$h(1/2) = (-1, 0, 0)$$

SO $\tilde{h}(1/2) = \frac{2n+1}{2}$ FOR SOME $n \in \mathbb{Z}$

$$\tilde{h}(1) = 2n+1$$

$s \overset{h}{\longmapsto} (\cos 2\pi s, \sin 2\pi s, 0)$

$1/2 \longmapsto (-1, 0, 0)$

$s + 1/2 \longmapsto -h(s) \longrightarrow -g\,h(s)$

h

$$-g\,h(s) = g(-h(s)) = g(-\cos 2\pi s, -\sin 2\pi s, 0)$$

CAN SHOW THE DEGREE OF THE CLOSED PATH gh IS ODD.

BUT THE CLOSED PATH h IN S^2 IS NULL HOMOTOPIC SO THE DEGREE OF gh MUST BE ZERO.

CONTRADICTION

QED.

$$\begin{tikzcd}
 & & & \mathbb{R} \arrow[d, "p"] \\
I \arrow[urrr, dashed, "\tilde{h}"] \arrow[rr, "h"] & & S^2 \arrow[r, "g"] \arrow[d, "{[-1]}"'] & S^1 \arrow[d, "{[-1]}"] \\
 & & S^2 \arrow[r, "g"] & S^2
\end{tikzcd}$$

$\tilde{h}(\frac{1}{2}) = \frac{2n+1}{2}$

so

$\tilde{h}(1) = 2n+1$

$p\tilde{h} = gh$

HAS ODD DEGREE

$h(s) = (\cos 2\pi s, \sin 2\pi s, 0)$

$h(s + 1/2) = (-\cos 2\pi s, -\sin 2\pi s, 0) = [-1]\, h(s)$

$gh(s + 1/2) = g(-h(s)) = -g(h(s))$

$s=0: \quad gh(1/2) = -g(h(0))$
$s=1/2 \quad gh(1) = -g(h(1/2)) = g(-h(1/2))$

HAM SANDWICH THEOREM

GIVEN THREE COMPACT SUBSETS $K_1, K_2, K_3 \subset \mathbb{R}^3$,
$\exists$ A PLANE WHICH BISECTS ALL THREE.

PROOF: GIVEN A UNIT VECTOR $(x \in S^2)$ $x \in \mathbb{R}^3$. $\exists$ PLANE $\perp$ TO v THAT BISECTS K_1.

IT MAY FAIL TO BISECT K_2 AND K_3, IN EACH WE CAN SUBTRACT THE VOLUME BELOW THE PLANE FROM THAT ABOVE IT.

THIS DEFINES A MAP

$$S^2 \xrightarrow{f} \mathbb{R}^2$$

$$x \longmapsto \text{PLANE BISECTING } K_1 \left(\begin{pmatrix} \text{VOLUME OF } K_2 \text{ ABOVE} \\ - \text{VOLUME BELOW} \end{pmatrix}, \begin{matrix} \text{SAME} \\ \text{FOR} \\ K_3 \end{matrix} \right)$$

HST FOLLOWS IF $(0,0) \in \operatorname{Im} f$.

$$f(-x) = -f(x)$$

BORSUK-ULAM SAYS $\exists$ x

WITH $f(-x) = f(x)$

HENCE $f(x) = -f(x)$ AT THE BORSUK-ULAM POIN

SO $f(x) = 0$

THIS GIVES US THE DESIRED PLANE. QED

VAN KAMPEN THEOREM

SUPPOSE $X = A \cup B$, WHERE X, A, B AND $A \cap B$ ARE PATH CONNECTED

NOTE

$$\begin{tikzcd} & A \arrow[dr] & \\ x_0 \in A \cap B \arrow[ur, "a"] \arrow[dr, "b"'] & & X \\ & B \arrow[ur] & \end{tikzcd}$$

IS A PUSHOUT DIAGRAM IN $\mathcal{T}op$

THEN THE DIAGRAM

$$\begin{tikzcd} & \pi_1(A) \arrow[dr] & \\ \pi_1(A \cap B) \arrow[ur] \arrow[dr] & & \pi_1(X) \\ & \pi(B) \arrow[ur] & \end{tikzcd}$$

IS A PUSHOUT DIAGRAM IN THE CATEGORY OF GROUPS.

IN PRACTICE THIS MEANS WE CAN COMPUTE $\pi_1 X$ IF WE KNOW THE OTHER THREE GROUPS AND MAPS BETWEEN THEM.

EXAMPLE

$M = A =$ MOBIUS STRIP, A SURFACE BOUNDED BY

A CIRCLE

$D^2 = B$ = A DISK BOUNDED BY ∂A

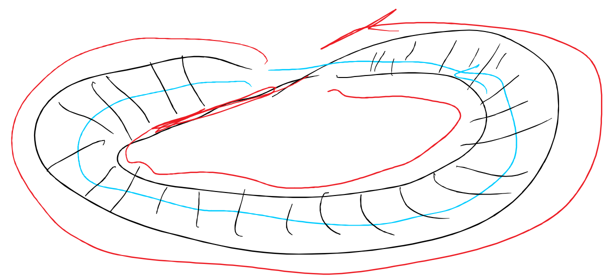
$S^1 = A \cap B$ = BOUNDING CIRCLE

X = UNION $\simeq \mathbb{RP}^2$

$$\begin{tikzcd}
 & \mathbb{Z} = \pi_1(M) \arrow[dr] & \\
\mathbb{Z} = \pi_1(S^1) \arrow[ur, "2"] \arrow[dr] & & \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2 \\
 & \pi_1(D_2) = 0 \arrow[ur] &
\end{tikzcd}$$

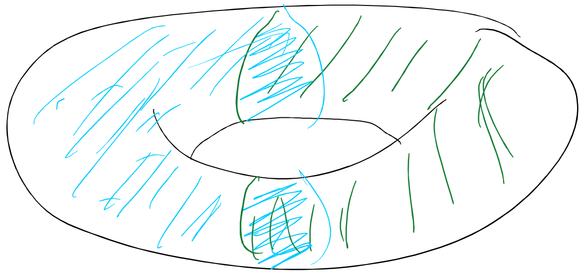
BOUNDARY

BLUE CIRCLE $\simeq M$



$$\begin{tikzcd}
 & A = \text{"Y-SHAPED MOBIUS STRIP"} \\
\partial A = \partial B \arrow[ur] \arrow[dr] & \\
 & B = D^2 \text{ BOUNDED BY } \partial A
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dr] & \\
\mathbb{Z} \arrow[ur, "3"] \arrow[dr] & & \pi_1(A \cup B) = \mathbb{Z}/3 \\
 & 0 \arrow[ur] &
\end{tikzcd}$$



TORUS $= X$

$A =$ CYLINDER

$B =$ CYLINDER

$A \cap B =$ TWO CYLINDERS

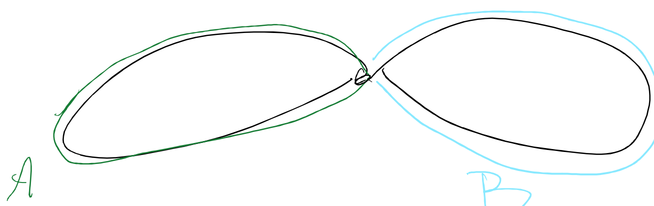
$X = A \cap B$

VKT DOES NOT APPLY

$X =$ GENUS 2 SURFAC

$A, B \simeq$ TORUS - DISK

$A \cap B \simeq S^1$



$A, B \simeq S^1$

$A \cap B = pt$

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 0 \arrow[ur] \arrow[dr] & & ? \\ & 2 \arrow[ur] & \end{tikzcd}$$