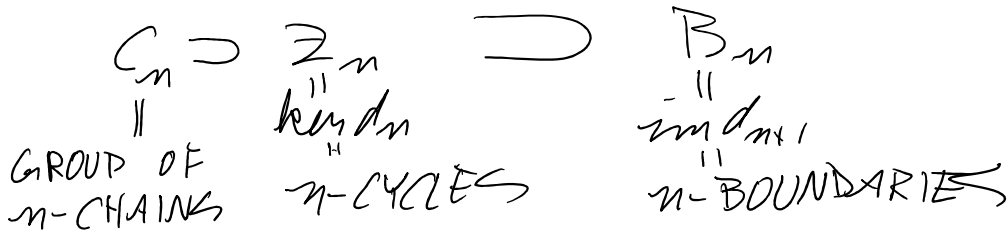


A CHAIN COMPLEX  $C$  IS A DIAGRAM OF ABELIAN GROUPS

$$C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 \xleftarrow{\dots} \dots$$

WITH  $d_i d_{i+1} = 0$  FOR ALL  $i \geq 0$

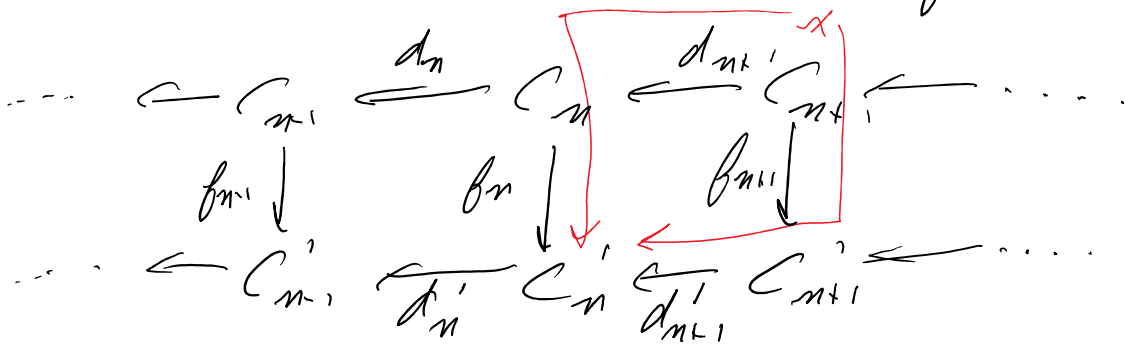


$$H_n(C) := \mathbb{Z}_n / B_n = \text{ } n\text{TH HOMOLOGY GROUP}$$

REMARK  $\mathbb{Z} = \text{INTEGERS}$

$\mathbb{Z}/n = \text{INTEGERS MOD } n$   
(NOT  $\mathbb{Z}_n$  OR  $\mathbb{Z}_n$ )

DEF A MAP  $f: C \rightarrow C'$  OF CHAIN COMPLEXES IS A COLLECTION OF HOMS  $f_n: C_n \rightarrow C'_n$



WITH  $d'_n f_n = f_{n-1} d_n$  FOR ALL  $n > 0$

PROP, SUCH AN  $f: C \rightarrow C'$  INDUCE MAPS  $H_n(C) \rightarrow H_n(C')$

PROOF:  $H_n(C) = \mathbb{Z}_n / B_n = \text{kernel } d_n / \text{image } d_{n+1}$

$$H_n(C') = Z_n' / B_n' = \ker d_n' / \text{im } d_{n+1}'$$

NOTE for  $B_n \subset B_n'$  AND for  $Z_n \subset Z_n'$   
 SO WE GET  $Z_n / B_n \xrightarrow{f} Z_n' / B_n'$   
 $H_n(C) \xrightarrow{f} H_n(C')$

RECALL A SEQUENCE OF ABELIAN GRPS

$$\dots \rightarrow A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} \dots$$

IS EXACT IF  $\ker a_n = \text{im } a_{n-1}$

A SHORT EXACT SEQUENCE (SES)

IS ONE OF THE FORM

$$0 \rightarrow A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \rightarrow 0$$

WHERE  $a_1$  IS 1-1.

$$\ker a_2 = \text{im } a_1$$

$a_3$  IS ONTO

EXAMPLES

$$0 \rightarrow A_1 \rightarrow A_1 \oplus A_3 \rightarrow A_3 \rightarrow 0$$

THIS IS A SPLIT SES

NOT SPLIT

$$\left. \begin{array}{l} 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0 \\ 0 \rightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0 \end{array} \right\}$$

SPLIT

$$0 \rightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \oplus \mathbb{Z}/2 \xrightarrow{\beta} \mathbb{Z}/4 \rightarrow 0$$

$\gamma \xrightarrow{\quad} \begin{matrix} a & b \\ \hline 2a + b \end{matrix}$

DEF A SES OF CHAIN COMPLEX

$$0 \rightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \rightarrow 0$$

WHERE IN

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 0 & \rightarrow & C'_n & \xrightarrow{f_n} & C_n & \xrightarrow{g_n} & C''_n \rightarrow 0 \\
 & & \downarrow d'_n & & \downarrow d_n & & \downarrow d''_n \\
 0 & \rightarrow & C'_{n-1} & \xrightarrow{f_{n-1}} & C_{n-1} & \xrightarrow{g_{n-1}} & C''_{n-1} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

EACH SQUARE COMMUTES AND

EACH ROW IS A SES OF ABELIAN GRPS, i.e.  $\text{ker } g_n = \text{im } f_n$  FOR ALL  $n$ .

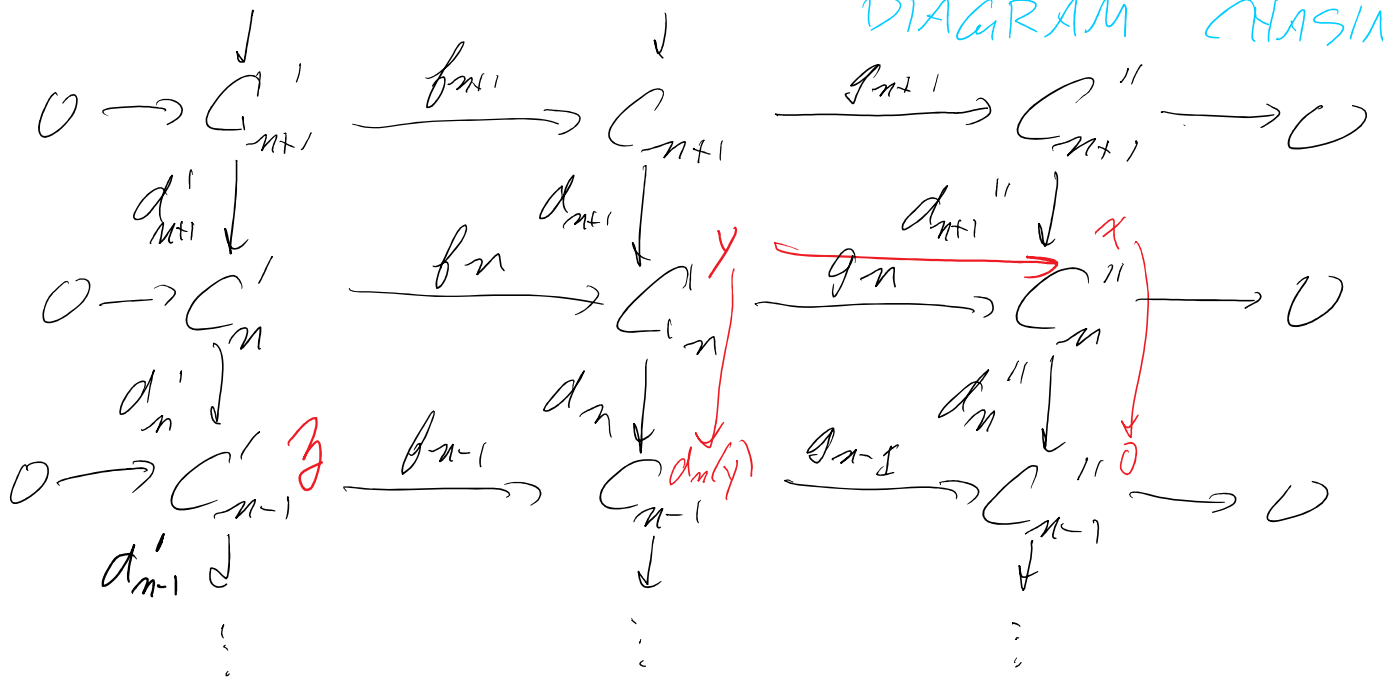
THEOREM GIVEN THE ABOVE THERE IS A LONG EXACT SEQUENCE

$$\begin{array}{c}
 \dots \xrightarrow{d_{n+1}} H_n(C') \xrightarrow[\partial_n]{H_n(f)} H_n(C) \xrightarrow{H_n(g)} H_n(C'') \\
 \xrightarrow{\quad} H_{n-1}(C') \xrightarrow{\partial_{n-1}} H_{n-1}(C) \rightarrow \dots
 \end{array}$$

$$\hookrightarrow H_{n-1}(C') \xrightarrow{f_x} H_{n-1}(C) \rightarrow \dots$$

$\mathcal{Z}_n$  IS THE CONNECTING HOMO MORPHISM

PROOF CONSIDER THE FOLLOWING COMMUTATIVE DIAGRAM WITH EXACT ROWS  
 DIAGRAM CHASING



TO DEFINE  $\mathcal{Z}_n : H_n(C'') \rightarrow H_{n-1}(C')$ ,

LET  $d \in H_n(C'')$  BE REPRESENTED BY

$$x \in \ker d_n'' \subset C_n''$$

THIS A CHOICE

SINCE  $g_n$  IS ONTO,  $\exists y \in C_n$  WITH  $g_n(y) = x$

AND  $\exists! z \in C'_{n-1}$  WITH  $f_{n-1}(z) = d_n(y)$

ANOTHER CHOICE

AND  $d'_{n-1}(z) = 0$ . SO  $z$  REPRESENTS

A CLASS IN  $H_{n-1}(C')$ . CALL  $\mathcal{Z}_n(d)$ .

NEED TO CHECK THAT  
 $\mathcal{D}_n(x)$  IS INDEPENDENT OF  
 THE CHOICES OF  $x$  AND  $y$   
 ONCE THIS DONE, WE HAVE  
 A WELL DEFINED HOM

$$\mathcal{D}_n: H_n(C'') \longrightarrow H_n(C')$$

WE NEED TO CHECK EXACTNESS  
 AT EACH STAGE, TETDIOUS  
 BUT EASY.

QED

BOOKS ON HOMOLOGICAL ALGEBRA

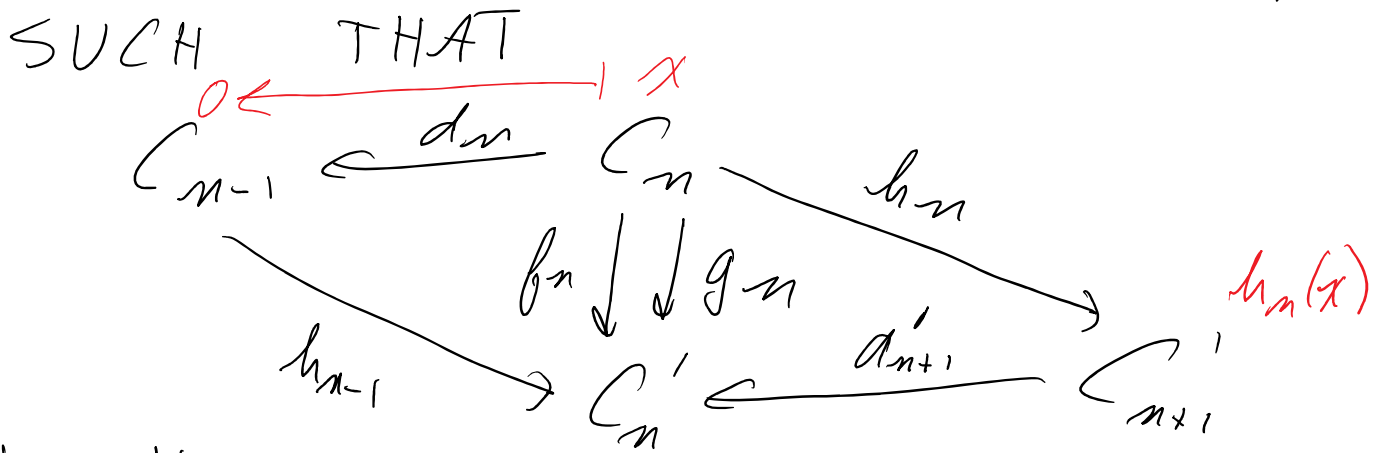
CARTAN - EILENBERG	1956
MAC LANE	1970?
HILTON - STAMMBACH	1975?
WEIBEL	1990?

DEF TWO CHAIN MAPS

$$C \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} C' \quad \text{ARE CHAIN HOMOTOPIC}$$

IF  $\exists$  HOMS  $h_n: C_n \longrightarrow C'_{n+1}$

SUCH THAT             $\alpha$



WITH  $h_{n-1} d_n + d'_{n+1} h_n = f_n - g_n$

THEOREM IF  $f$  AND  $g$  ARE CHAIN HOMOTOPIC, THEN  $H_*(f) = H_*(g)$ .

PROOF: IT SUFFICES  $H_*(f-g) = 0$

LET  $\alpha \in H_n(C)$  BE REPRESENTED BY  $x \in C_n$ . THEN

$$\begin{aligned}
 & (h_{n-1} d_n + d'_{n+1} h_n) x \\
 &= h_{n-1} d_n(x) + d'_{n+1} h_n(x) \\
 &= d'_{n+1} h_n(x) \\
 &= (f_n - g_n)(x)
 \end{aligned}$$

SO  $(f_n - g_n) x$  IS A BOUNDARY AND THEREFORE REPRESENTS

AND THEREFORE REPRESENTS  
0 IN  $H_n \mathbb{C}!$  (GET)

---

BIG PICTURE

WANT TO DEFINE  $H_*(X)$

AND  $H_*(X, A)$  FOR EACH SPACE  
 $X$  AND EACH PAIR  $(X, A)$ ,

WE DO THIS VIA CHAIN COMPLEXES.

THE HOMOTOPY AXIOM SAYS

IF  $f, g: X \rightarrow Y$  ARE HOMOTOPIC

THEN  $H_*(f) = H_*(g): H_* X \rightarrow H_* Y$ .

WE WILL SHOW THAT THE INDUCED  
MAP BETWEEN (YET TO BE DEFINED)

CHAIN COMPLEXES ARE CHAIN

HOMOTOPIC. WILL THE THE

HOMOTOPY BETWEEN  $f$  AND  $g$

TO CONSTRUCT THE DESIRED

CHAIN HOMOTOPY