A CHAIN COMPLEX C IS A DIAGRAM OF ABELIAN GROUPS

$$C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 \longleftarrow \cdots$$

WITH $d_i d_{i+1} = 0$ FOR ALL $i > 0$

$$\underset{\substack{\text{GROUP OF} \\ n\text{-CHAINS}}}{C_n} \supset \underset{\substack{\ker d_n \\ n\text{-CYCLES}}}{Z_n} \supset \underset{\substack{\operatorname{im} d_{n+1} \\ n\text{-BOUNDARIES}}}{B_n}$$

$H_n(C) := Z_n / B_n$

$=$ n TH HOMOLOGY GROUP

REMARK $\mathbb{Z}$ = INTEGERS

$\mathbb{Z}/n$ = INTEGERS MOD n

(NOT Z_n OR $\mathbb{Z}_n$)

<u>DEF</u> A MAP $f: C \to C'$ OF CHAIN COMPLEXES IS A COLLECTION OF HOMS $f_n : C_n \to C_n'$

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l] \arrow[d, "f_{n-1}"] & C_n \arrow[l, "d_n"'] \arrow[d, "f_n"] & C_{n+1} \arrow[l, "d_{n+1}"'] \arrow[d, "f_{n+1}"] & \cdots \arrow[l] \\
\cdots & C'_{n-1} \arrow[l] & C'_n \arrow[l, "d'_n"] & C'_{n+1} \arrow[l, "d'_{n+1}"] & \cdots \arrow[l]
\end{tikzcd}$$

WITH $d_n' f_n = f_{n-1} d_n$ FOR ALL $n > 0$

PROP. SUCH AN $f: C \to C'$ INDUCE MAPS $H_n(C) \longrightarrow H_n(C')$

PROOF: $H_n(C) = Z_n / B_n = \ker d_n / \operatorname{im} d_{n+1}$

$H_n(C') = Z_n'/B_n' = \ker d_n' / \operatorname{im} d_{n+1}'$

NOTE $f_n B_n \subset B_n'$ AND $f_n Z_n \subset Z_n'$

SO WE GET $Z_n/B_n \xrightarrow{f} Z_n'/B_n'$

$$\begin{array}{ccc} & & \| \\ {}_n(C) & & H_n(C') \end{array}$$

RECALL A SEQUENCE OF ABELIAN GRPS

$$\cdots \longrightarrow A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \xrightarrow{a_3} \cdots$$

IS <u>EXACT</u> IF $\ker a_n = \operatorname{im} a_{n-1}$

A <u>SHORT</u> <u>EXACT SEQUENCE</u> (SES) IS ONE OF THE FORM

$$0 \longrightarrow A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} A_3 \longrightarrow 0$$

WHERE a_1 IS 1-1.

$\ker a_2 = \operatorname{im} a_1$

a_3 IS ONTO

EXAMPLES

$$0 \longrightarrow A_1 \longrightarrow A_1 \oplus A_3 \longrightarrow A_3 \longrightarrow 0$$

THIS IS A <u>SPLIT</u> SES

NOT SPLIT

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

SPLIT

$$0 \to \mathbb{Z}/2 \xrightarrow{\alpha} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \to 0$$

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \to 0$$

$a \quad b$

$x \longmapsto 2a + b$

DEF A SES OF CHAIN COMPLEXES

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

WHERE IN

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C'_n \arrow[r, "f_n"] \arrow[d, "d'_n"'] & C_n \arrow[r, "g_n"] \arrow[d, "d_n"'] & C''_n \arrow[r] \arrow[d, "d''_n"'] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "f_{n-1}"'] \arrow[d] & C_{n-1} \arrow[r, "g_{n-1}"'] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

EACH SQUARE COMMUTES AND EACH ROW IS A SES OF ABELIAN GRPS, i.e. $\ker g_n = \operatorname{im} f_n$ FOR ALL n.

<u>THEOREM</u> GIVEN THE ABOVE THERE IS A LONG EXACT SEQUENCE

$$\cdots \xrightarrow{\partial_{n+1}} H_n(C') \xrightarrow[\partial_n]{H_n(f)} H_n(C) \xrightarrow{H_n(g)} H_n(C'')$$

$$\longrightarrow H_{n-1}(C') \xrightarrow{f_*} H_{n-1}(C) \longrightarrow \cdots$$

$\hookrightarrow H_{n-1}(C') \xrightarrow{f_*} H_{n-1}(C) \longrightarrow \cdots$

∂_n IS THE CONNECTING HOMOMORPHISM

PROOF CONSIDER THE FOLLOWING COMMUTATIVE DIAGRAM WITH EXACT ROWS

DIAGRAM CHASING

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & & \\
0 \arrow[r] & C'_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d'_{n+1}"'] & C_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "d_{n+1}"'] & C''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"'] & 0 \\
0 \arrow[r] & C'_n \arrow[r, "f_n"] \arrow[d, "d'_n"'] & C_n \; y \arrow[r, "g_n"] \arrow[d, "d_n"'] & C''_n \; x \arrow[r] \arrow[d, "d''_n"'] & 0 \\
0 \arrow[r] & C'_{n-1} \; z \arrow[r, "f_{n-1}"] \arrow[d, "d'_{n-1}"'] & C_{n-1} \; d_n(y) \arrow[r, "g_{n-1}"] \arrow[d] & C''_{n-1} \; 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

TO DEFINE $\partial_n : H_n(C'') \longrightarrow H_{n-1}(C')$,

LET $\alpha \in H_n C''$ BE RERESENTED BY

$x \in \ker d''_n \subset C''_n$ THIS A CHOICE

SINCE g_n IS ONTO, $\exists y \in C_n$ WITH $g_n(y) = x$

ANOTHER CHOICE

AND $\exists!\, z \in C'_{n-1}$ WITH $f_{n-1}(z) = d_n(y)$

AND $d'_{n-1} z = 0$. SO z REPRESENTS A CLASS IN $H_{n-1} C'$. CALL $\partial_n(\alpha)$.

NEED TO CHECK THAT $\partial_n(\alpha)$ IS INDEPENDENT OF THE CHOICES OF x AND y

ONCE THIS DONE, WE HAVE A WELL DEFINED HOM

$$\partial_n : H_n C'' \longrightarrow H_{n-1} C'$$

WE NEED TO CHECK EXACTNESS AT EACH STAGE, TEDIOUS BUT EASY.

QED

BOOKS ON HOMOLOGICAL ALGEBRA

CARTAN - EILENBERG	1956
MAC LANE	1970?
HILTON - STAMMBACH	1975?
WEIBEL	1990?

DEF TWO CHAIN MAPS $C \overset{f}{\underset{g}{\rightrightarrows}} C'$ ARE <u>CHAIN HOMOTOPIC</u>

IF $\exists$ HOMS $h_n : C_n \longrightarrow C'_{n+1}$

SUCH THAT

SUCH THAT

$$\begin{tikzcd}
0 & x \arrow[l, mapsto, red] & \\
C_{n-1} \arrow[dr, "h_{n-1}"'] & C_n \arrow[l, "d_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dr, "h_n"] & {\color{red} h_n(x)} \\
& C_n' & C_{n+1}' \arrow[l, "d_{n+1}'"']
\end{tikzcd}$$

WITH $h_{n-1} d_n + d_{n+1}' h_n = f_n - g_n$

<u>THEOREM</u> IF f AND g ARE CHAIN HOMOTOPIC, THEN $H_*(f) = H_*(g)$.

<u>PROOF:</u> IT SUFFICES $H_*(f-g) = 0$

LET $\alpha \in H_n(C)$ BE REPRESENTED $x \in C_n$. THEN

$$(h_{n-1} d_n + d_{n+1}' h_n) x$$
$$= h_{n-1} d_n(x) + d_{n+1}' h_n(x)$$
$$= d_{n+1}' h_n(x)$$
$$= (f_n - g_n)(x)$$

SO $(f_n - g_n)x$ IS A BOUNDARY AND THEREFORE REPRESENTS

AND THEREFORE REPRESENTS 0 IN $H_n C'$. QED

BIG PICTURE

WANT TO DEFINE $H_*(X)$ AND $H_*(X,A)$ FOR EACH SPACE X AND EACH PAIR (X,A).

WE DO THIS VIA CHAIN COMPLEXES.

THE HOMOTOPY AXIOM SAYS IF $f, g : X \longrightarrow Y$ ARE HOMOTOPIC THEN $H_*(f) = H_*(g) : H_* X \longrightarrow H_* Y$.

WE WILL SHOW THAT THE INDUCED MAP BETWEEN (YET TO BE DEFINED) CHAIN COMPLEXES ARE CHAIN HOMOTOPIC. WILL THE THE HOMOTOPY BETWEEN f AND g TO CONSTRUCT THE DESIRED CHAIN HOMOTOPY