BORSUK-ULAM PROOF AGAIN

$$\begin{tikzcd} & & (\mathbb{R}^1, 0) \arrow[d, "p"] \\ (I, 0) \arrow[urr, dashed, "\tilde{h}"] \arrow[r, "\eta"] \arrow[rr, bend right, "h"'] & S^2 \arrow[r, "g"] & (S^1, 1) \end{tikzcd}$$

WITH $g(-x) = -g(x)$ $\forall x \in S^2$

$\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$

$\eta(s + 1/2) = -\eta(s)$ FOR $0 \le s \le 1/2$

$h(s + 1/2) = g(-\eta(s)) = -g\eta(s) = -h(s)$

$p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$

FOR ANY $z \in S^1$, $p^{-1}(z) =$ SOME TRANSLATION OF $\mathbb{Z}$

$p^{-1}(-z) = 1/2 + p^{-1}(z)$

$\tilde{h}(s + 1/2) \in p^{-1}(-h(s)) = 1/2 + p^{-1}(h(s))$

SO $\tilde{h}(s + 1/2) = \tilde{h}(s) + q/2$ FOR ODD INTEGER q.

$\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q/2 + q/2$

$= \tilde{h}(0) + q$

h REPRESENTS A NONZERO ELEMENT IN $\pi_1(S^1) \cong \mathbb{Z}$

HOWEVER η DEFINES A NULL HOMOTOPIC PATH IN S^2

SO $h = g\eta$ IS NULL IN S^1

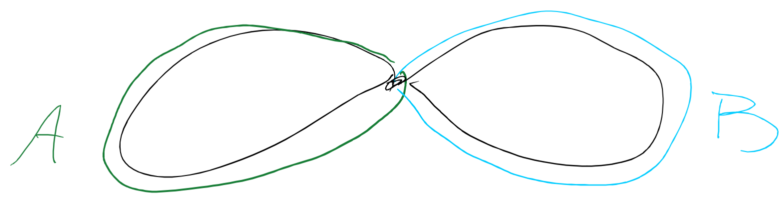
CONTRADICTION

QED

VAN KAMPEN THEOREM

$X = S^1 \vee S^1$

$=$ " ∞ ", OR "8"



WE GET A PUSHOUT DIAGRAM OF GROUPS

$$\begin{tikzcd}
 & \pi_1 A \arrow[dr] & \\
0 = \pi_1(A \cap B) \arrow[ur] \arrow[dr] & & \pi_1(X) \\
 & \pi_1 B \arrow[ur] &
\end{tikzcd}$$

$\mathbb{Z} \ni a$ ($= \pi_1 A$)

$\pi_1 B = \mathbb{Z} \ni b$

THIS IMPLIES $\pi_1 X \cong F_2 =$ FREE GROUP ON 2 GENERATORS.

$$= \left\{ a^{i_1} b^{j_1} a^{i_2} b^{j_2} \cdots a^{i_n} b^{j_n} : i_t, j_t \in \mathbb{Z} \right\}$$

WORD

$$(a^2 b^{-3} a^3 b^0)(a^{-5} b^2) = a^2 b^{-3} a^3 \cancel{b^0} a^{-5} b^2$$

$$= a^2 b^{-3} a^{-2} b^2$$

$$(a^{-5} b^2)^{-1} = b^{-2} a^5 = a^0 b^{-2} a^5 b^0$$

$$ab \overset{?}{\neq} ba$$

$e =$ IDENTITY ELEMENT

$$aba^{-1} \neq baa^{-1} = b$$

$aba^{-1}b^{-1} \neq bb^{-1} = e$

$[a,b]$ THE COMMUTATOR OF a AND b

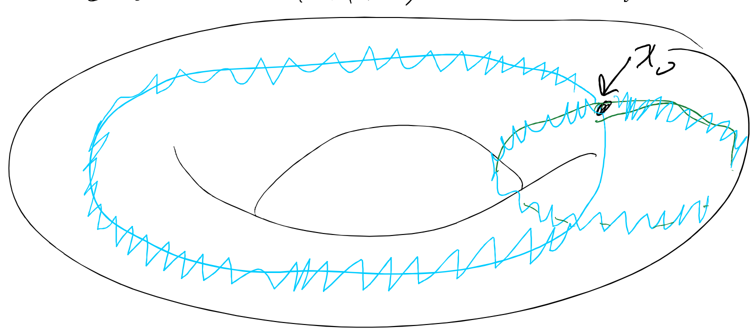
CAN SIMILARLY DEFINE F_n, THE FREE GROUP ON n GENERATORS FOR $n \geq 0$.

<u>SURPRISING FACT</u>

FOR EACH $n \geq 0$, F_2 HAS A SUBGROUP ISO TO F_n.

WILL GIVE A TOPOLOGICAL PROOF OF THIS LATER.

X = TORUS



LET $A = S^1 \vee S^1$ THICKENED

$B = X - \text{INT}(A) \cong D^2$

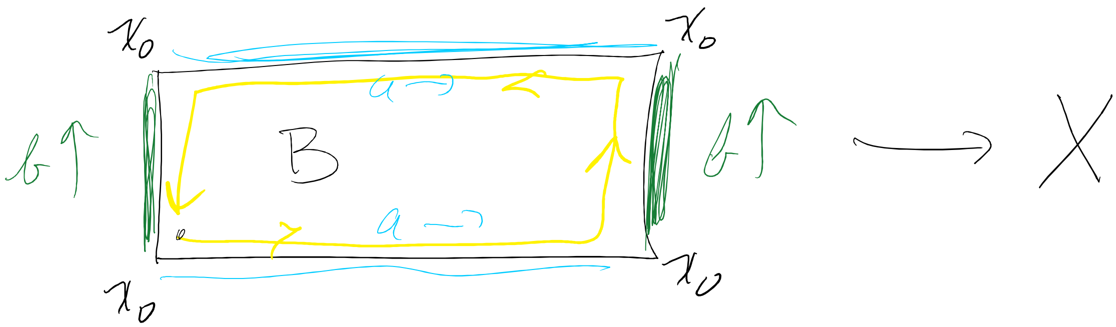
$A \cap B \cong S^1$

VAN KAMPEN DIAGRAM

$$\begin{tikzcd} & \pi_1(A) \cong F_2 \arrow[dr] & \\ \mathbb{Z} = \pi_1(A \cap B) \arrow[ur, "?"] \arrow[rr, bend left, "0"] \arrow[dr] & & \pi_1(X) = ??? \\ & \pi_1(B) = 0 \arrow[ur] & \end{tikzcd}$$

CUTTING X ALONG THE

COLORED CIRCLES LEADS TO A RECTANGLE

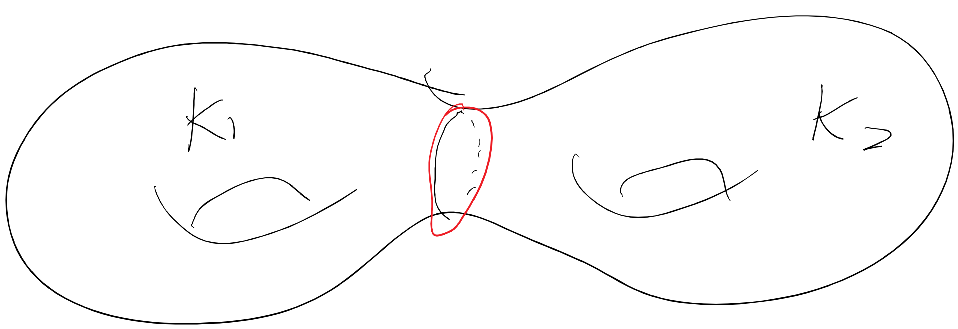


YELLOW ELEMENT $= a b a^{-1} b^{-1} \in F_2$

$\pi_1 X = F_2 \,/\,$ SUBGROUP GENERATED BY $[a, b]$

$\cong \mathbb{Z} \oplus \mathbb{Z}$

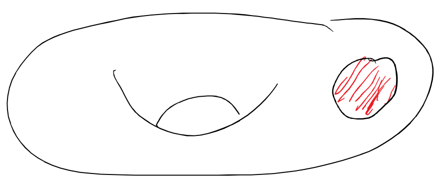
SURFACE OF GENUS 2



$= X$

$A \cap B = S^1$

K_1 = TORUS WITH DISK REMOVED



$\pi_1 K_1 \cong \pi_1 K_2 \cong F_2$

$a_1, b_1 \qquad a_2, b_2$

$F_2 \ni a_1, b_1$

$\parallel$

KNOWN $\rightarrow$ $\pi_1 K_1$

$$
\begin{tikzcd}
 & \pi_1 K_1 \arrow[dr] & \\
\mathbb{Z} = \pi_1(A \cap B) \arrow[ur, "\text{KNOWN}"] \arrow[dr, "\text{KNOWN}"'] & & \pi_1 X \\
 & \pi_1 K_2 \ni a_2, b_2 \arrow[ur, dash] \arrow[d, equal] & \\
 & F_2 &
\end{tikzcd}
$$

$\pi_1 X$ IS GENERATED BY IMAGES OF

a_1, b_1, a_2 AND b_2

WITH $[a_1, b_1] = [a_2, b_2]$

$$[a_1, b_1] = a_1 b_1 a_1^{-1} b_1^{-1} \qquad [a_2, b_2] = a_2 b_2 a_2^{-1} b_2^{-1}$$

SURFACE OF GENUS g

$$\pi_1(-) = \left(a_1, b_1, a_2, b_2 \cdots a_g\ b_g ; \ [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = e \right)$$

PROOF BY INDUCTION ON g.

BACK TO GENUS 2

$$[a_1, b_1] = [a_2, b_2]$$

$$a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1}$$

$$a_1 b_1 a_1^{-1} b_1^{-1} b_2 = a_2 b_2 a_2^{-1}$$

$$\vdots$$

$$a_1 b_1 a_1^{-1} b_1^{-1} b_2 a_2 b_2^{-1} a_2^{-1} = e$$

$[a_1, b_1][b_2, a_2]$