FOUR AXIOMS FOR HOMOLOGY

✓ (1) EXACTNESS
(2) HOMOTOPY
(3) EXCISION
✓ (4) DIMENSION

WE HAVE PROVED (1) AND (4) AND WILL ASSUME (2) AND (3)

THEOREM

$$H_i S^n = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{FOR } i=0,\ n=0 \\ 0 & \text{FOR } i>0,\ n=0 \\ \mathbb{Z} & \text{FOR } i=0,\ n>0 \\ & \text{FOR } i=n,\ n>0 \\ 0 & \text{FOR } i\neq 0, n \end{cases}$$

PROOF FOR $n=0$.

LET $X = X_1 \amalg X_2$ WHERE X_1 AND X_2 ARE BOTH OPEN (AND HENCE) CLOSED IN X, IE BOTH ARE "CLOPEN"

THEN $S(X) = S(X_1) \oplus S(X_2)$ AS CHAIN COMPEXES.

SINCE ANY MAP $\Delta^n \to X$ HAS IMAGE IN X_1 OR IN X_2

IT FOLLOWS THAT $H_*(X) \cong H_*(X_1) \oplus H_*(X_2)$

LET $X = S^0 = pt \amalg pt$.

$H_i S^0 = H_i(pt) \oplus H_i(pt)$

$$H_i S^0 = H_i(pt) \oplus H_i(pt)$$
$$= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{FOR } i = 0 \\ 0 & \text{FOR } i > 0 \end{cases}$$

FOR $n > 0$, ARGUE BY INDUCTION ON n. ASSUME $H_* S^{n-1}$ IS AS CLAIMED. CONSIDER THE

PAIR $S^n \supset D^n$ = NORTHERN HEMISPHERE

THE HOMOTOPY AXIOM IMPLIES THAT HOMOTOPY EQUIVALENT SPACES HAVE THE SAME HOMOLOGY

HENCE $H_* D^n \cong H_*(pt)$.

THE EXCISION AXIOM SAYS

GIVEN $B \subset A \subset X$ WITH

CLOSURE$(B) \subset$ INTERIOR OF A

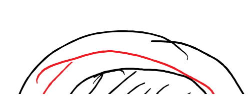
THEN $H_*(X-B, A-B) \longrightarrow H_*(X, A)$ IS AN ISOMORPHISM.

$$\begin{array}{ccccc} B & \hookrightarrow & A & \hookrightarrow & X \\ \| & & \| & & \| \\ (D^n)' & \hookrightarrow & D^n & \hookrightarrow & S^n \\ \| & & & & \end{array}$$

SMALLER DISK

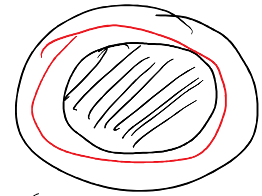
$$H_*(S^n - (D^n)', D^n - (D^n)') \xrightarrow{\cong} H_*(S^n, D^n)$$

$$D^n - (D^n)' \simeq S^{n-1}$$



$D^n - (D^n)' \simeq S^{n-1}$

$S^n - (D^n)' \simeq pt.$



THE PAIR ON THE LEFT LEADS A LES IN H_*

$$\underbrace{H_i(D^n - (D^n)')}_{=\begin{cases} \mathbb{Z} & \text{FOR } i=0,n-1 \\ 0 & \text{ELSE}\end{cases},\ 0} \to \underbrace{H_i(S^n - (D^n)')}_{=\begin{cases} \mathbb{Z} & \text{FOR } i=0 \\ 0 & \text{FOR } i>0\end{cases},\ 0} \to \underbrace{H_i(\text{pair})}_{\text{1-1 ONTO}} \to$$

$i = n > 0$

$$\to \underbrace{H_{i-1}(D^n - (D^n)')}_{\mathbb{Z},\ =\begin{cases} \mathbb{Z} & \text{FOR } i=0,n \\ 0 & \text{ELSE}\end{cases}} \to \underbrace{H_{i-1}(S^n - (D^n)')}_{0}$$

CLAIM

$$H_i(\text{pair}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

FOR $i = n$, THE CONNECTING HOM IS AN ISOMORPHISM.

FOR $i \neq n$ AND $i > 0$, THE SAME ARGUMENT SHOWS $H_i(\text{pair}) = 0$

EXCISION AXIOM

$$H_i(S^n - (D^n)', D^n - (D^n)') \simeq H_i(S^n, D^n)$$

$= \mathbb{Z}$ IF $i = n$

$$= \begin{cases} \mathbb{Z} & \text{IF } i = n \\ 0 & \text{IF } i \neq 0 \end{cases}$$

LOOK AT LES

$$\underset{H_i(pt)}{H_i(D^n)} \longrightarrow H_i(S^n) \longrightarrow \overset{\text{KNOWN}}{H_i(S^n, D^n)} \longrightarrow \underset{\underset{H_{i-1}(pt)}{\shortparallel}}{H_{i-1}(D^n)} \longrightarrow \cdots$$

$\rightsquigarrow$ $H_*(S^n)$ IS AS CLAIMED.

QED

MAYER-VIETORIS SEQUENCE
1930

WALTER MAYER 1887–1948
LEOPOLD VIETORIS 1891–2002

__THEOREM__ LET $X = A \cup B$ WITH MILD HYPOTHESIS ON $A \cap B$.

$$\text{CLOSURE}(A \cap B) \subset \begin{matrix} \text{INT}(A) \\ \text{INT}(B) \end{matrix}$$

THEN THERE IS A LONG EXACT SEQUENCE

SEQUENCE

$$\begin{tikzcd} & A \arrow[dr, "j_1"] & \\ A\cap B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X \\ & B \arrow[ur, "j_2"'] & \end{tikzcd}$$

$$x \longmapsto i_1(x) \oplus i_2(x)$$

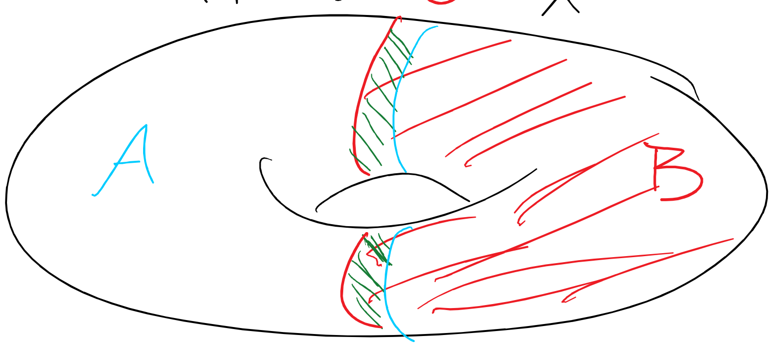
$$\cdots \to H_i(A\cap B) \longrightarrow H_i(A)\oplus H_i(B) \longrightarrow H_i(X) \to$$

$$(a, b) \longmapsto j_1(a) - j_2(b)$$

$$\to H_{i-1}(A\cap B) \to \cdots$$

PROOF LATER

EXAMPLE ① X = TORUS



$A \simeq S^1 \times I \simeq B$

$A\cap B = 2$ CYLINDER $\simeq S^1 \sqcup S^1$

$$\underset{\overset{\|}{0}}{H_2(A)\oplus H_2(B)} \longrightarrow \underset{\overset{\|}{\mathbb{Z}}}{H_2(X)} \to$$

$$\to \underset{\mathbb{Z}\oplus\mathbb{Z}}{\overset{\|}{H_1(A\cap B)}} \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(A)\oplus H_1(B)} \longrightarrow \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(X)} \to$$

$$\to \underset{\mathbb{Z}\oplus\mathbb{Z}}{\overset{\|}{H_0(A\cap B)}} \xrightarrow{\text{rank } 1} \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_0(A)\oplus H_0(B)} \longrightarrow \underset{\mathbb{Z}}{\overset{\|}{H_0(X)}} \to 0$$

$\alpha \quad \beta \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \gamma \quad \delta$

$\alpha \quad \beta \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \gamma \quad \delta \qquad \Sigma$

$m\gamma \oplus n\delta \longmapsto m-n$

$\alpha \longmapsto \gamma \oplus \delta$ IMAGE IS SUBGP

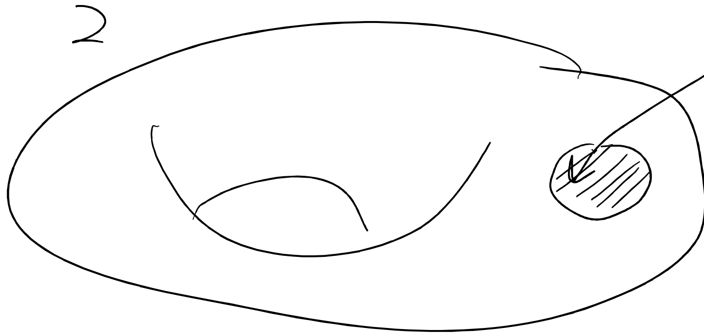
$\beta \longmapsto \gamma \oplus \delta$ GENERATED BY

$\alpha - \beta \longmapsto 0$ $\quad 1 \oplus 1$

$$H_i X = \begin{cases} \mathbb{Z} & \text{FOR } i = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{FOR } i = 1 \\ 0 & \text{FOR } i > 2 \end{cases}$$

EXAMPLE 2

$Y =$



$X = Y \cup D^2 \qquad D^2 \cap Y = S^1$

MVS

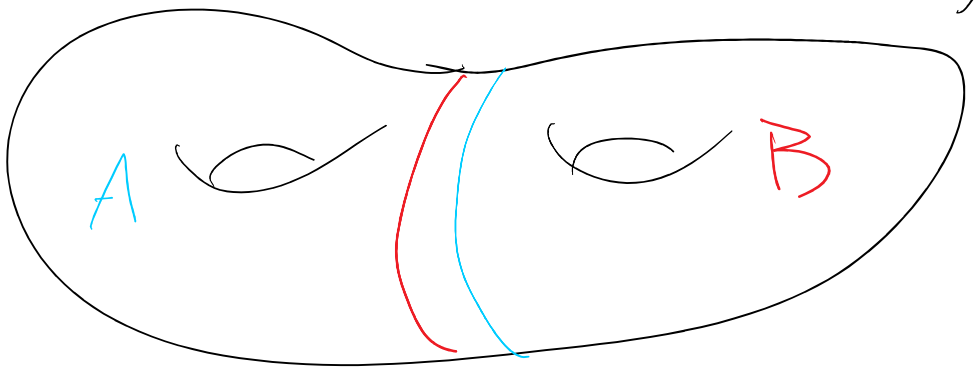
$$\to H_i(S^1) \longrightarrow H_i(D^2) \oplus H_i(Y) \longrightarrow H_i X \to H_{i-1}(S^1) \longrightarrow \cdots$$

$$\leadsto H_i(Y) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ 0 & i > 1 \end{cases}$$

EXAMPLE 3

$A \simeq Y \simeq B$

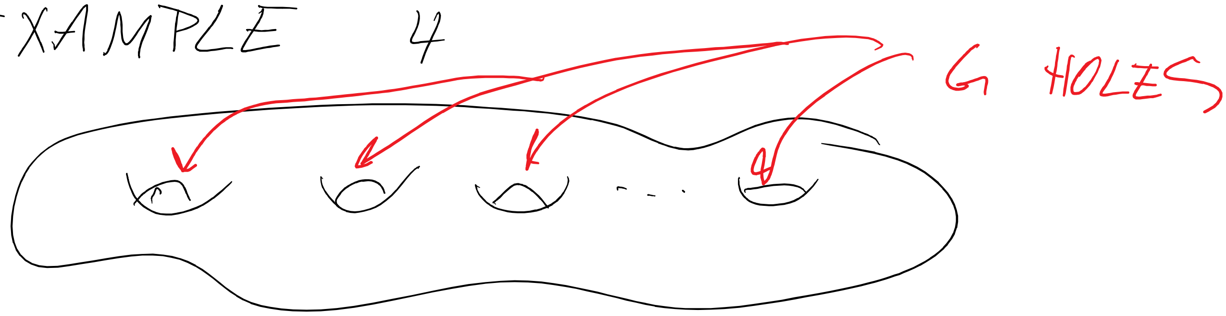
$A \cap B \simeq S^1$



$W = $ GENUS 2 SURFACE

$$\rightsquigarrow H_i W = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & i = 1 \\ 0 & i > 2 \end{cases}$$

EXAMPLE 4

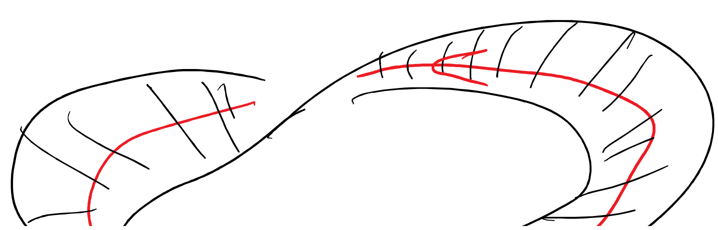


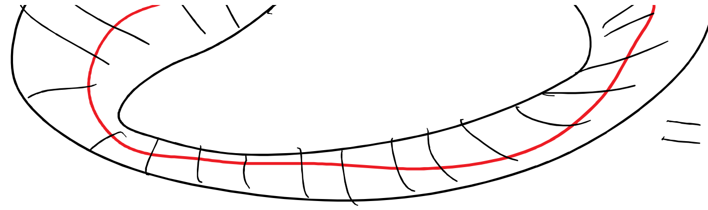
M_g SURFACE OF GENUS g

$$H_i M_g = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & i > 0 \end{cases}$$

EXAMPLE 5

$X = \mathbb{RP}^2$



 $= A \simeq S^1$

$A \cap B = S^1$

$pt \simeq D^2 = B$

$$0 \longrightarrow \overset{0}{\overset{\shortparallel}{H_2 RP^2}}$$

$$\longrightarrow \overset{\mathbb{Z}}{\overset{\shortparallel}{H_1(A\cap B)}} \xrightarrow{1 \mapsto 2} \overset{\mathbb{Z}}{H_1(A)} \oplus \underset{0}{H_1(B)} \longrightarrow \overset{\mathbb{Z}/2}{\overset{\shortparallel}{H_1 RP^2}}$$

$$\longrightarrow H_0(A\cap B) \longrightarrow H_0(A) \oplus H_0(B) \longrightarrow H_0(RP^2) \longrightarrow 0$$

$$\mathbb{Z} \xrightarrow{\text{1-1}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{ONTO}} \mathbb{Z} \longrightarrow 0$$

TO BE CONTINUED.

FOR PATH CONNECTED X

$H_1(X)$ = ABELIANIZATION OF $\pi_1(X)$

FOR ANY PATH CONN X

$$\pi_n(X) \xrightarrow{h_n} H_n(X)$$

$$\pi_n(X) \ni \alpha$$

$$S^n \xrightarrow{\alpha} X$$

$$\mathbb{Z} = H_n(S^n) \longrightarrow H_n(X)$$
$$\text{gen} \longmapsto h(\alpha)$$

HUREWICZ THEOREM

IF X IS $(n-1)$-CONNECTED
($\pi_i X = 0$ FOR $i < n$)
THEN h_n IS AN ISOMORPHISM.