5-LEMMA. SUPPOSE WE HAVE A COMM. DIAGRAM OF ABELIAN GROUPS

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \varepsilon\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

WITH EXACT ROWS. THEN IF α, β, δ AND ε ARE ISOMORPHISMS, SO IS γ. THE PROOF IS IN THE BOOK AND ONLINE.

<u>COR</u> SUPPOSE WE A MAP

$$(X, A) \xrightarrow{f} (Y, B) \text{ OF TOP PAIRS}$$

IN f INDUCES ISOMORPHISMS

$$H_*(A) \xrightarrow[\cong]{f_*} H_*(B) \quad \text{AND} \quad H_*(X, A) \xrightarrow[\cong]{f_*} H_*(Y, B)$$

THEN $f_*: H_* X \longrightarrow H_* Y$ IS AN ISOMORPHISM

PROOF

$$\begin{array}{ccccccccc} H_{n+1}(X,A) & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) \\ f_*^1\downarrow & & f_*^2\downarrow & & f_*^3\downarrow & & f_*^4\downarrow & & f_*^5\downarrow \\ H_{n+1}(Y,B) & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) \end{array}$$

f_*^1, f_*^2, f_*^4 AND f_*^5 ARE ISOS BY HYPOTHESES, SO f_*^3 IS AN ISO BY THE FIVE LEMMA. QED.

THE HOMOTOPY AXIOM

THEOREM

a) ABSOLUTE FORM: IF $f, g : X \longrightarrow Y$ ARE HOMOTOPIC, THEN $f_* = g_* : H_* X \longrightarrow H_* Y$

b) RELATIVE FORM: IF $f, g : (X, A) \longrightarrow (Y, B)$ ARE HOMOTOPIC, THEN

$$f_* = g_* : H_*(X, A) \longrightarrow H_*(Y, B)$$

PROOF a): WE HAVE CHAIN MAPS

$$S(f), S(g) : S(X) \longrightarrow S(Y)$$

WE KNOW THAT IF THEY ARE CHAIN HOMOTOPIC, THEY INDUCE THE SAME MAP IN $H_*(-)$.

RECALL TWO CHAIN $\alpha, \beta : C \longrightarrow C'$ ARE CHAIN HOMOTOPIC IF THERE ARE HOMOMORPHISMS

$D_n : C_n \longrightarrow C'_{n+1}$ SUCH THAT

$$\begin{tikzcd}
C_{n-1} \arrow[dr, "D_{n-1}"'] & C_n \arrow[l, "d_n"'] \arrow[d, shift right, "\alpha_n"'] \arrow[d, shift left, "\beta_n"] \arrow[dr, "D_n"] & \\
& C'_n & C'_{n+1} \arrow[l, "d'_{n+1}"]
\end{tikzcd}$$

$$D_{n-1} d_n + d'_{n+1} D_n = \alpha_n - \beta_n$$

$$\begin{tikzcd}
S_{n-1}(X) \arrow[dr, dashed, "D_{n-1}"'] & S_n(X) \ni \sigma_n \arrow[l, "d_n"'] \arrow[d, shift right, "S_n(f)"'] \arrow[d, shift left, "S_n(g)"] \arrow[dr, dashed, "D_n"] & \\
& S_n(Y) & S_{n+1}(Y) \ni D_n(\sigma) \arrow[l, "d'_{n+1}"]
\end{tikzcd}$$

WE NEED SUITABLE MAPS D_n AND D_{n-1} AS ABOVE

GIVEN $\Delta^n \xrightarrow{\sigma} X$ WE NEED AN ELEMENT IN $S_{n+1}(Y)$, I.E. A LINEAR COMBINATION OF MAPS

$$\Delta^{n+1} \longrightarrow Y$$

WE HAVE A HOMOTOPY

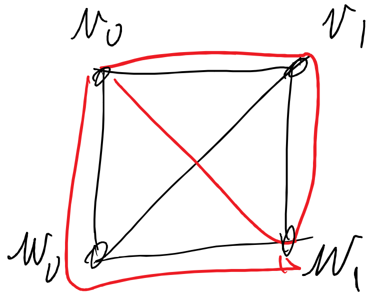
$$\Delta^{n+1} \xrightarrow{g_i} \Delta^n \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{h} Y$$

$0 \le i \le n$

PRISM

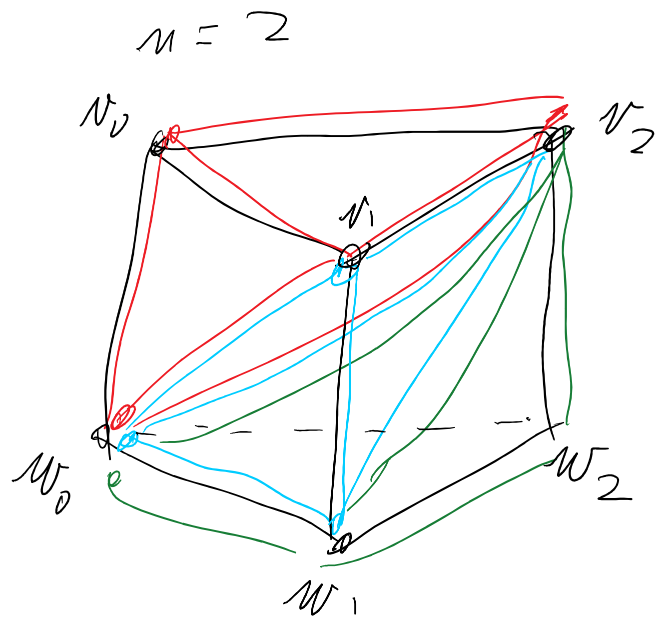
WILL DESCRIBE $\Delta^n \times I$ AS A UNION $(n+1)$ COPIES OF Δ^{n+1}

$n=1$



$\Delta^1 \times I$ AS UNION OF 2 COPIES OF Δ^2

$n=2$



g_0 $\quad \Delta^3_0$ HAS VERTICES $\{w_0, v_0, v_1, v_2\}$

g_1 $\quad \Delta^3_1$ " $\{w_0, w_1, v_1, v_2\}$

g_2 $\quad \Delta^3_2$ " $\{w_0, w_1, w_2, v_2\}$

ANY 4 OF THE SIX VERTICES OF $\Delta^2 \times I$ (15 CHOICES) DEFINES A COPY OF Δ^3. CLAIM THE CLOSURE OF THE (DISJOINT) UNION OF THEIR INTERIORS IS THE WHOLE PRISM.

$$D_n \sigma = \sum_{0 \le i \le n} h(\sigma \times I) g_i \in S_{n+1}(Y)$$

ONE CAN CHECK THAT THIS IS THE DESIRED CHAIN HOMOTOPY. SEE HATCHER FOR DETAILS. THIS COMPLETES THE ABSOLUTE CASE.

b) RELATIVE CASE

$$S_n(X, A) = S_n(X) / S_n(A)$$

A SIMILAR ARGUMENT CONVERTS THE RELATIVE HOMOTOPY TO THE DESIRED CHAIN HOMOTOPY.

QED

EXCISION AXIOM: GIVEN

$$Z \subset A \subset X$$

WITH CLOSURE $(Z) \subseteq$ INTERIOR (A)

THEN

$$H_*(X-Z, A-Z) \cong H_*(X,A)$$

EQUIVALENT FORMULATION:

GIVEN $A, B \subset X$ WITH

$$\mathrm{INT}(A) \cup \mathrm{INT}(B) = X$$

THEN

$$H_*(A, A\cap B) \xrightarrow[\cong]{\lambda} H_*(X,A)$$

TO SEE THE EQUIVALENCE

LET $B = X - Z$, SO $Z = X - B$

THEN THE TWO STATEMENTS ARE THE SAME.

WILL PROVE THE AXIOM IN THE A B FORMULATION

LET $S(A+B) \overset{\lambda}{\hookrightarrow} S(X)$

BE THE SUB CHAIN COMPLEX GENERATED BY MAPS

$$\Delta^n \longrightarrow A \longrightarrow X$$

AND $\Delta^n \longrightarrow B \longrightarrow X$

CLAIM: THE MAP λ IS A

CHAIN HOMOTOPY EQUIVALENCE,

SO $H_* S(A+B) = H_* X$

THERE A SES OF CHAIN CXS

$$0 \to S(A \cap B) \longrightarrow S(A) \oplus S(B) \longrightarrow S(A+B) \to 0$$
$$\alpha \oplus \beta \longmapsto \alpha - \beta$$

$$\begin{array}{ccc} A & & B \\ \sigma_1 \uparrow & & \sigma_2 \uparrow \\ \Delta^n & & \Delta^n \end{array}$$

$$\sigma_1 \oplus \sigma_2 \longmapsto \sigma_1 - \sigma_2$$

THE DEFINITIONS IMPLY THIS IS A SES OF CHAIN CXS

THEREFORE WE HAVE A LES

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to \underbrace{H_n(S(A+B))}_{\overset{\text{BY CLAIM}}{=} H_n(X)} \to H_{n-1}(A \cap B) \to \cdots$$

THIS IS THE MAYER-VIETORIS SEQUENCE.

TO PROVE THE CLAIM RECALL FOR $W \subset Y$,

$$S(Y, W) = S(Y)/S(W)$$

RELATIVE SINGULAR CHAIN CX

$$\begin{tikzcd}
0 \arrow[r] & S(A \cap B) \arrow[r] \arrow[d] & S(B) \arrow[r] \arrow[d] & S(B, A \cap B) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & S(A) \arrow[r] & S(X) \arrow[r] & S(X, A) \arrow[r] & 0 \\
0 \arrow[r] & S(A) \arrow[r] \arrow[u, equal, "\alpha"] & S(A+B) \arrow[r] \arrow[u, "\beta"] & S(A+B, A) \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

$$S(A+B, A) = S(A+B)/S(A)$$

AM

THE ROWS ARE EXACT

β IS A CHE BY THE a

BY THE FIVE LEMMA, $H_*(\gamma)$

IS AN ISOMORPHISM.

TO BE CONTINUED.