LOOSE END IN PROOF OF EXCISION AXIOM.

LET $X = A \cup B$

$$S(A+B) \subset S(X)$$

$\parallel$

SUBCOMPLEX OF $S(X)$ GENERATED BY MAPS $\Delta^n \xrightarrow{\sigma} X$ WHOSE IMAGE LIES IN A OR B.

CLAIM $S(A+B) \longrightarrow S(X)$ IS A CHAIN HOMOTOPY EQUIVALENCE

GENERIZATION

LET $\mathcal{U} = \{U_1, U_2, \dots\}$ BE AN OPEN COVERING OF X, IE A COLLECTION OF OPEN SUBSETS WHOSE UNION IS X.

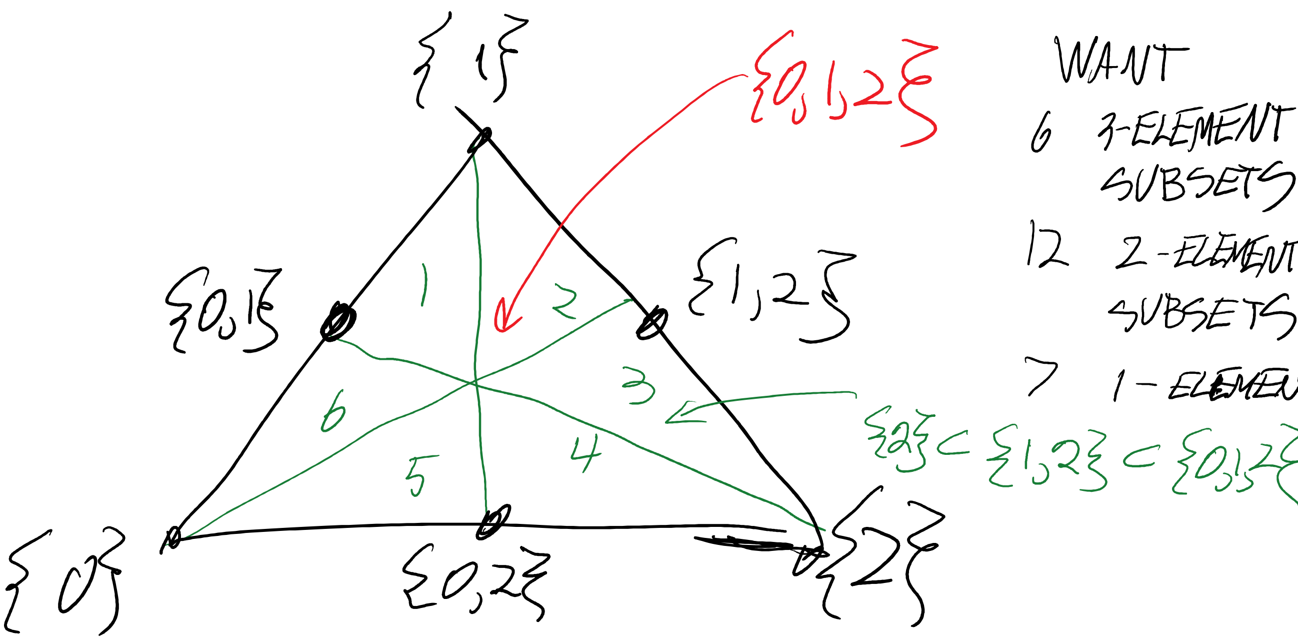
LET $S^{\mathcal{U}}(X) \subset S(X)$ BE THE SUB CHAIN COMPLEX GENERATED BY MAPS $\Delta^n \longrightarrow X$ WHOSE IMAGE LIES IN SOME U_α

CLAIM $S^{\mathcal{U}}(X) \longrightarrow S(X)$ IS ALSO A CHAIN HOMOTOPY EQUIVALENCE

CLAIM $S^{\mathcal{U}}(X) \to S(X)$ IS ALSO A CHAIN HOMOTOPY EQUIVALENCE

<u>DEF</u> A SIMPLEX $\Delta^n \xrightarrow{\sigma} X$ IS <u>SMALL</u> WITH RESPECT TO $\mathcal{U}$ IF ITS IMAGE LIES IN SOME U_i

TECHNICAL TOOL FOR PROVING THESE CLAIMS:
<u>BARYCENTRIC SUBDIVISION</u>



WANT
6 3-ELEMENT SUBSETS
12 2-ELEMENT SUBSETS
7 1-ELEMENT

<u>DEF</u> A SIMPLICIAL COMPEX A IS A SET B ALONG WITH A COLLECTION C OF FINITE

SUBSETS SUCH IF $L \subset K \in \mathcal{C}$ THEN $L \in \mathcal{C}$. THESE FINITE SUBSETS ARE CALLED SIMPLICES. THERE IS A SPACE FOR SUCH DATA THAT INVOLVES EMBEDDING B INTO BIG EUCLIDEAN SPACE SUCH THE POINTS A LINEARLY INDEPENDENT, AND SO ON.

EXAMPLE:

1. THE SIMP. CX ASSOCIATED WITH Δ^n CONSISTS THE SET $[n] = \{0, 1, \dots n\}$ WITH ALL OF ITS SUBSETS
2. THE SUBDIVIDED Δ^n THE BIG SET A IS THE COLLECTION OF NONEMPTY SUBSETS OF $[n] = \{0, \dots n\}$ B CONSISTS OF CHAINS

$B_0 \subset B_1 \subset \cdots B_k \quad k \leq n \quad$ FOR $B_i \subset [n]$

THIS MAKES Δ^n A UNION OF $(n+1)!$ SMALLER Δ^ns. THIS IS THE SUBDIVIDED STANDARD n-SIMPLEX.

WHY DO THIS?

IF WE REPLACE Δ^n BY THE UNION OF $((n+1)!)^k$ SMALLER n-SIMPLICES (FOR SOME $k<\infty$), EACH ONE WILL MAP TO SOME U_i. THE BIG SIMPLEX σ CAN BE "REPLACED" BY COLLECTION OF SMALL ONES.

WE WANT TO SHOW THE MAP

$$S^{\mathcal{U}}(X) \xrightarrow[\text{INCLUSION}]{L} S(X) \xrightarrow{\exists P} S^{\mathcal{U}}(X)$$

IS A CHE

LEMMA (SEE HATCHER pp 121-124)

IF THERE IS A MAP P AS

IF THERE IS A MAP P AS ABOVE SUCH THAT LP AND PL ARE CHAIN HOMOTOPIC TO THE IDENTITY MAPS ON $S(X)$ AND $S^{\mathcal{U}}(X)$, THEN —.

GIVEN $\sigma : \Delta^n \longrightarrow X$, WE CAN SUBDIVIDE IT $k(\sigma)$ TIMES FOR SOME $k(\sigma) < \infty$ TO MAKE ALL THE LITTLE SIMPLICES SMALL. CHOOSE THE SMALLEST POSSIBLE VALUE OF $k(\sigma)$.

DEFINE $S(X) \xrightarrow{P} S^{\mathcal{U}}(X)$

$\sigma \longmapsto$ SUM OF ALL LITTLE SIMPLICES IN σ

THEN

$$\begin{tikzcd} S^{\mathcal{U}}(X) \arrow[r, "L"] \arrow[rr, bend right, "\text{IDENTITY}"'] & S(X) \arrow[r, "P"] & S^{\mathcal{U}}(X) \end{tikzcd}$$

THE OTHER COMPOSITE

$S(X) \xrightarrow{P} S^{\mathcal{U}}(X) \xrightarrow{L} S(X)$

$\sigma \longmapsto$ SUM OF OTHER

SMALL SIMPLICES

HATCHER DESCRIBES THE CHAIN HOMOTOPY BETWEEN LP AND $\mathbb{1}_{S(X)}$.

QED

AS EXPLAINED BEFORE, THE CHAIN HOMOTOPY EQUIVALENCE FOR $\mathcal{U}=\{A,B\}$ GIVES THE MVS AND THE EXCISION AXIOM.

WE HAVE SHOWN THAT THE FUNCTORS

$H_*(X) := H_* S(X)$ AND

$H_*(X,A) := H_*(S(X)/S(A))$

SATISFY THE FOUR EILENBERG-STEENROD AXIOMS.

FUTURE DIRECTIONS:

(1) CAN WE DESCRIBE

$H_*(X \times Y)$ IN TERMS OF $H_*(X)$ AND $H_*(Y)$?

NAIVE GUESS

$$H_*(X \times Y) = H_*(X) \otimes H_*(Y)$$

I.E. $$H_n(X \times Y) = \bigoplus_{0 \leq i \leq n} H_i(X) \otimes H_{n-i}(Y)$$

NOT TRUE IN GENERAL

(2) GIVEN CHAIN COMPLEXES C' AND C'', DESCRIBE $H_*(C' \otimes C'')$ IN TERMS OF $H_*(C')$ AND $H_*(C'')$. SAME NAIVE GUESS IS WRONG.

(3) GIVEN A CHAIN COMPLEX C AND AN ABELIAN A, DESCRIBE $H_*(C \otimes A)$ IN TERMS OF $H_*(C)$ AND A.

NAIVE GUESS

$$H_*(C \otimes A) = H_*(C) \otimes A$$

EXAMPLE

$$C: \; 0 \xleftarrow{} \mathbb{Z} \xleftarrow[d]{2} \mathbb{Z} \xleftarrow[d]{} 0 \cdots$$

$$C: \; 0 \xleftarrow{d_0} \underset{\substack{\parallel \\ C_0}}{\mathbb{Z}} \xleftarrow[d_1]{2} \underset{\substack{\parallel \\ C_1}}{\mathbb{Z}} \xleftarrow{d_2} 0 \cdots$$

$$H_0(C) = \ker d_0 / \operatorname{im} d_1$$
$$= \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2$$

$$H_1(C) = \ker d_1 / \operatorname{im} d_2$$
$$= 0/0 = 0$$

$$A = \mathbb{Z}/2$$

$$C \otimes A \; : \; \underset{\substack{\parallel \\ C_0 \otimes A}}{\mathbb{Z}/2} \xleftarrow[0]{2} \underset{\substack{\parallel \\ C_1 \otimes A}}{\mathbb{Z}/2}$$

$$H_0(C \otimes A) = \mathbb{Z}/2$$
$$H_1(C \otimes A) = \mathbb{Z}/2 \neq H_1(C) \otimes A = 0$$

TO BE CONTINUED