Monday, November 28, 2022 2:17 PM

MAKEUP CLASSES 12/15 2:00
12/16 4:00

WE NEED MORE ALGEBRA TO PROCEED FURTHER.

QUESTIONS

1. HOW TO DESCRIBE $H_*(X \times Y)$ IN TERMS OF $H_*(X)$ AND $H_*(Y)$.
2. GIVEN CHAIN COMPLEXES C' AND C'' DESCRIBE $H_*(C' \otimes C'')$ IN TERMS OF $H_*(C')$ AND $H_*(C'')$
3. GIVEN A CHAIN COMPLEX C AND AN ABELIAN GROUP A, DESCRIBE $H_*(C \otimes A)$ IN TERMS OF $H_*(C)$ AND A.

NAIVE ANSWERS

1. $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$

   i.e. $H_n(X \times Y) = \bigoplus_{0 \le i \le n} H_i(X) \otimes H_{n-i}(Y)$
2. $H_*(C' \otimes C'') \cong H_*(C') \otimes H_*(C'')$
3. $H_*(C \otimes A) = H_*(C) \otimes A$

ALL ARE INCORRECT.

ALL ARE INCORRECT.

FUNDAMENTAL DIFFICULTY: TENSORING DOES <u>NOT</u> PRESERVE EXACTNESS.

SUPPOSE WE HAVE A SHORT EXACT SEQUENCE OF ABELIAN GROUPS

$$0 \to M' \xrightarrow[1-1]{\alpha} M \xrightarrow[\text{ONTO}]{\beta} M'' \to 0$$

$\ker \beta = \operatorname{im} \alpha$

THEN

$$0 \to M' \otimes A \to M \otimes A \to M'' \otimes A \to 0$$

NEED NOT BE EXACT.

EXAMPLE

$$0 \to \mathbb{Z} \xrightarrow[1-1]{2} \mathbb{Z} \xrightarrow[\text{ONTO}]{} \mathbb{Z}/2 \to 0$$

$A = \mathbb{Z}/2$

$$0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow[1-1]{\text{ONTO}} \mathbb{Z}/2 \to 0$$

NOT EXACT HERE

IF WE TENSOR A SURJECTION WITH SOME A, IT IS STILL ONTO, BUT NOT SO WITH INJECTIONS.

WITH INJECTIONS.

THIS LEADS TO HOMOLOGICAL ALGEBRA.

DEF AN R-MODULE P IS PROJECTIVE IF FOR ANY SURJECTION f

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ & & \uparrow & & \\ & & P & & \end{array}$$

AND ANY R-HOM α, $\exists\ \tilde{\alpha}$ WITH $f\tilde{\alpha} = \alpha$.

EXAMPLE $P = R$ = RING WITH UNIT

α IS DETERMINED BY $\alpha(1) \in N$
SINCE f IS ONTO, $\exists\ m \in M$ WITH $f(m) = \alpha(1)$. LET $\tilde{\alpha}(1) = m$.

A SIMILAR ARGUMENT SHOWS ANY FREE R-MODULE IS PROJECTIVE.

FOR SOME RING (e.g. $\mathbb{Z}$) EVERY PROJECTIVE MODULE IS FREE.

EXAMPLE $R = \mathbb{Z}[\sqrt{-5}]$

$= \mathbb{Z}[x]/(x^2+5)$

IT IS KNOWN TO HAVE PROJECTIVE MODULES THAT ARE NOT FREE (SUBTLE NUMBER THEORY)

<u>DEF</u> A <u>PROJECTIVE RESOLUTION</u> OF AN R-MODULE M IS A LONG EXACT SEQUENCE

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

WHERE EACH P_i IS PROJECTIVE.

EXAMPLES

① R = FIELD K   KORP

ALL MODULES (VECTOR SPACES) ARE FREE AND HENCE PROJECTIVE.

$$0 \leftarrow M \xleftarrow{\cong} \underset{\substack{\parallel \\ M}}{P_0} \leftarrow 0$$

② SUPPOSE R IS A PRINCIPAL

IDEAL DOMAIN (PID), i.e. AN INTEGRAL DOMAIN IN WHICH EVERY IDEAL IS PRINCIPAL (GENERATED BY A SINGLE ELEMENT). $\mathbb{Z}$ IS A P.I.D., AS IS ANY SUBRING OF $\mathbb{Q}$.

$$
\begin{tikzcd}
0 & \mathbb{Z}/2 \arrow[l] \arrow[d, equal] & \mathbb{Z} \arrow[l] \arrow[d, equal] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, equal] & 0 \arrow[l] \\
& M & P_0 & P_1 &
\end{tikzcd}
$$

THIS IS A PROJECTIVE RESOLUTION OF <u>LENGTH 1</u>

ANY MODULE OVER A PID HAS A PROJECTIVE RESOLUTION OF LENGTH 1.

(3) K = FIELD $\quad R = K[x]/(x^2)$

$= \{a + bx : a, b \in K\}$

$M = K$ IS AN R-MODULE IN WHICH MULT BY x IS 0.

$0 \longleftarrow\!\shortmid x$

$$
\begin{tikzcd}
0 & M \arrow[l] & P_0 \arrow[l] & P_1 \arrow[l, "x"'] & P_2 \arrow[l, "x"'] & P_3 \arrow[l, "x"'] & \cdots \arrow[l]
\end{tikzcd}
$$

$$\begin{array}{ccccccc} 0 \leftarrow M \leftarrow \cdots & & & & & & \\ \| & \| & & \| & & \| & \| \\ K & R & \xleftarrow{x} & R & & R & R \end{array}$$

$$0 = x^2 \leftarrow\!\shortmid\, x$$

THIS PROJECTIVE RESOLUTION HAS INFINITE LENGTH.

(4) $R = K[x]$ IS A PID

$R = K[x_1, \dots x_n]$. WE KNOW. EACH MODULE HAS A PROJ RESOLN OF LENGHT $\leq n$.

SUPPOSE M AND N ARE R-MODULES AND WE HAVE A PROJECTIVE RESOLN OF M

$$0 \leftarrow M \leftarrow P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \xleftarrow{d_3} \cdots$$

(THESE ALWAYS EXIST)

CONSIDER

$$P_0 \otimes_R N \xleftarrow{d_1 \otimes N} P_1 \otimes_R N \xleftarrow{d_2 \otimes N} P_2 \otimes_R N \xleftarrow{d_3 \otimes N} \cdots$$

THIS IS A CHAIN COMPLEX SINCE $(d_i \otimes N)(d_{i+1} \otimes N) = d_i d_{i+1} \otimes N$
$= 0$

BUT MAYBE NOT EXACT.

CONSIDER $H_i(P_* \otimes N)$

IF WE HAD EXACTNESS, WE WOULD HAVE

$$H_i(P_* \otimes N) = \begin{cases} M \otimes N & \text{FOR } i = 0 \\ 0 & \text{FOR } i \neq 0 \end{cases}$$

<u>FUNDAMENTAL THEOREM OF HOM. ALGEBRA</u>

$H_i(P_* \otimes N)$ IS INDEPENDENT OF THE CHOICE OF RESOLUTION.

THIS ALLOWS US TO DEFINE

$$\mathrm{Tor}_i^R(M, N) = H_i(P_* \otimes N).$$

REMARKS

1. $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$
2. IF R IS A PID, $\mathrm{Tor}_i^R(M, N) = 0$ FOR $i > 1$.

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/\gcd(m, n)$$

$$\mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/n) = 0$$

$$\mathrm{Tor}_1(\mathbb{Z}/n, \mathbb{Z}) = 0$$