RECALL A PROJECTIVE RESOLUTION OF AN R-MODULE M IS A LONG EXACT SEQUENCE

$$0 \leftarrow M \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2 \xleftarrow{d_3} \cdots$$

WHERE EACH P_i IS PROJECTIVE.

PROP. A. EVERY M HAS A PROJECTIVE RESOLUTION.

PROOF: THERE IS A PROJECTIVE MODULE P_0 MAPPING ONTO M. CONSIDER THE SET UNDERLYING M, AND P_0 BE THE FREE GENERATED BY IT. d_0 SENDS AN R-GENERATOR $[m]$ OF P_0 CORRESPONDING TO $m \in M$ TO m.

LET M' BE THE KERNEL OF d_0, SO WE HAVE A SES

$$0 \leftarrow M \leftarrow P_0 \leftarrow M' \leftarrow 0$$

LET P_1 BE A PROJ MODULE MAPPING ONTO M'

$$d_n = \beta_{n-1}\alpha_n$$

$$0 \leftarrow M \xleftarrow{\alpha_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} P_2$$

$\nwarrow \beta_0 \quad \alpha_1 \swarrow \quad \nwarrow \beta_1 \quad \alpha_2 \swarrow \quad \nwarrow \beta_2$

$$\begin{tikzcd}
0 & M \arrow[l] & P_0 \arrow[l] & & P_1 \arrow[ll] \arrow[dl, "\alpha_1"'] & & P_2 \arrow[ll] \arrow[dl, "\alpha_2"'] & \\
& & & M' \arrow[ul, "\beta_0"'] \arrow[dl] & & M'' \arrow[ul, "\beta_1"'] \arrow[dl] & & M''' \arrow[ul, "\beta_2"'] \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & 0 \arrow[u]
\end{tikzcd}$$

DO THE SAME FOR M'', AND SO ON. FOR EACH $n \geq 0$ WE HAVE SES

$$0 \leftarrow M^{(n)} \xleftarrow{\alpha_n} P_n \xleftarrow{\beta_n} M^{(n+1)} \leftarrow 0$$

THESE CAN BE SPLICED TOGETHER AS ABOVE TO GET THE DESIRED RESOLUTION. QED

EXERCISE: FIND A PROJECTIVE OF $\mathbb{Q}$ OVER $\mathbb{Z}$.

LEMMA LET P AND P' BE TWO PROJECTIVE RESOLUTIONS OF M. THEY ARE CHAIN HOMOTOPICALLY EQUIVALENT.

IT FOLLOWS THAT $P \otimes_R N$ AND $P' \otimes_R N$ ARE ALSO CHE AND HENCE HAVE THE SAME H_1. HENCE $\mathrm{Tor}_1^R(M,N)$ IS WELL DEFINED.

PROOF OF LEMMA:

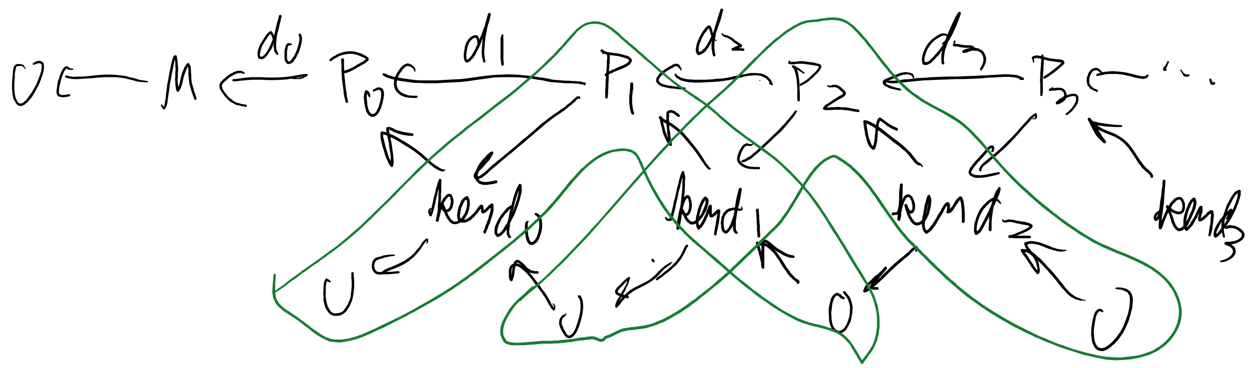
$$\begin{tikzcd}
0 & M \arrow[l] & P_0 \arrow[l, "d_0"'] & P_1 \arrow[l, "d_1"'] & P_2 \arrow[l, "d_2"'] & \cdots \arrow[l, "d_3"'] \\
0 & M \arrow[l] \arrow[u, equal] & P_0' \arrow[l, "d_0''"] \arrow[u, dashed, "g_0"] \arrow[ul] & P_1' \arrow[l, "d_1'"] \arrow[u, dashed, "g_1"] & P_2' \arrow[l, "d_2'"] \arrow[u, dashed, "g_2"] & \cdots \arrow[l, "d_3'"]
\end{tikzcd}$$

RECALL P IS PROJECTIVE IF

$$\begin{tikzcd}
 & P \arrow[dl, dashed] \arrow[d] & \\
N \arrow[r, "\text{ONTO}"] & N' \arrow[r] & 0
\end{tikzcd}$$

$\exists\, g_0$ BECAUSE P_0' IS PROJECTIVE

WE CAN REVERSE THE PROCESS USED TO PROVE PROP. A



SUPPOSE WE HAVE CONSTUCTED

$g_{n-1} : P'_{n-1} \longrightarrow P_{n-1}$

$$\begin{tikzcd}
P_{n-2} & P_{n-1} \arrow[l, "d_{n-1}"'] & & P_n \arrow[ll, "d_n"'] \arrow[dl, "\text{ONTO}"] \\
& & \ker d_{n-1} \arrow[ul] & \\
P'_{n-2} \arrow[uu] & P'_{n-1} \arrow[l, "d'_{n-1}"] \arrow[uu, "g_{n-1}"] \arrow[ur, "0"] & & P'_n \arrow[ll] \arrow[ul] \arrow[uu, dotted, "g_n"'] \arrow[dl] \\
& & \ker d'_{n-1} \arrow[ul] \arrow[uu, "g'_{n-1}"] &
\end{tikzcd}$$

WE GET A MAP $P'_n \longrightarrow \ker d_{n-1}$

AND HENCE $g_n : P'_n \longrightarrow P_n$ SINCE

P'_n IS PROJECTIVE

$$\begin{tikzcd}
0 & M \arrow[l] & P'_0 \arrow[l] & P'_1 \arrow[l] & P'_2 \arrow[l] & \arrow[l] \\
0 & M \arrow[l] \arrow[u, equal] & P_0 \arrow[l] \arrow[u, "f_0"] & P_1 \arrow[l] \arrow[u, "f_1"] & P_2 \arrow[l] \arrow[u, "f_2"] & \cdots \arrow[l] \\
0 & M \arrow[l] \arrow[u, equal] & P'_0 \arrow[l] \arrow[u, "g_0"] & P'_1 \arrow[l] \arrow[u, "g_1"] & P'_2 \arrow[l] \arrow[u, "g_2"] & \arrow[l] \\
0 & M \arrow[l] \arrow[u, equal] & P_0 \arrow[l] \arrow[u, "f_0"] & P_1 \arrow[l] \arrow[u, "f_1"] & P_2 \arrow[l] \arrow[u, "f_2"] & 0 \arrow[l]
\end{tikzcd}$$

WE NEED TO SHOW

$gf : P \longrightarrow P$ IS CHAIN

HOMOTOPIC TO THE IDENTITY AND SIMILARLY FOR

$$fg : P' \longrightarrow P'$$

THE FIRST CHAIN HOMOTOPY IS A COLLECTION OF MAPS

$P_n \xrightarrow{D_n} P_{n+1}$ WITH CERTAIN PROPERTIES

$$\begin{tikzcd}
P_{i-1} \arrow[dr, "D_{i-1}"'] & P_i \arrow[l, "d_i"'] \arrow[d, dotted] \arrow[dr, "D_i"] & \\
 & P_i & P_{i+1} \arrow[l, "d_{i+1}"]
\end{tikzcd}$$

WITH $d_{i+1} D_i + D_{i-1} d_i = 1_{P_i} - g_i f_i$

THESE CAN BE CONSTRUCTED BY INDUCTION ON i USING PROJECTIVITY OF P_i. DETAILS OMMITED. QED

SINCE P AND P' ARE CHE,

SO ARE $P \otimes_R N$ AND $P' \otimes_R N$, SO THEY HAVE THE SAME HOMOLOGY.

NOW CONSIDER $\mathrm{Hom}_R(P, N)$

$$(*) \quad 0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

GIVEN TWO R-MODULES M AND N, THE GROUP $\mathrm{Hom}_R(M, N)$ OF R-MODULE HOMOMORPHISMS IS ITSELF AN R-MODULE

APPLY THE CONTRAVARIANT FUNCTOR $\mathrm{Hom}_R(-, N)$ TO $(*)$ AND GET

$$(**) \quad 0 \to \mathrm{Hom}_R(M,N) \xrightarrow{d_0^*} \mathrm{Hom}(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}(P_1, N) \xrightarrow{d_2^*} \cdots$$

COCHAIN COMPLEX

$$\begin{tikzcd}
P_i \arrow[d, "f"'] & P_{i+1} \arrow[l, "d_{i+1}"'] \arrow[dl] \\
N &
\end{tikzcd}$$

$$\overset{\text{IV}}{f \in \operatorname{Hom}_R(P_i, N)} \xrightarrow{d_{i+1}^*} \operatorname{Hom}_R(P_{i+1}, N)$$

DEF $\operatorname{Ext}_R^i(M, N)$ IS THE iTH COHOMOLOGY GROUP OF THE COCHAIN COMPLEX (**).

$$H^i\left(\operatorname{Hom}_R(P, N)\right)$$

THIS GROUP IS INDEPENDENT OF CHOICE OF $P_\bullet$ AS BEFORE THE FUNCTOR $\operatorname{Hom}_R(-, N)$ PRESERVES CHAIN HOMOTOPY EQUIVALENCE.

$$\operatorname{Ext}_R^0(M, N) = \operatorname{Hom}_R(M, N)$$

IF R IS A PID, THEN

$\operatorname{Ext}_R^i(M, N) = 0$ FOR $i \geq 1$.

THEOREM LET

$$0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

BE A SES OF R-MODULES. THEN THERE ARE LONG EXACT SEQUENCES

$$\cdots \to \mathrm{Tor}_i^R(M',N) \to \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M'',N) \to \mathrm{Tor}_{i-1}^R(M',N) \to \cdots$$

AND

$$\cdots \leftarrow \mathrm{Ext}_R^i(M',N) \leftarrow \mathrm{Ext}_R^i(M,N) \leftarrow \mathrm{Ext}_R^i(M'',N) \leftarrow \mathrm{Ext}^{i-1}(M',N) \leftarrow \cdots$$

PROOF LATER.

REMARKS

1. IF R IS A PID, EACH LES ABOVE HAS ONLY 6 TERMS
2. THERE IS A SIMILAR STATEMENT FOR A SES

$$0 \to N' \to N \to N'' \to 0$$

NOTE $\mathrm{Hom}_R(-,-)$

IS CONTRAVARIANT (ARROW REVERSING) IN THE FIRST VARIABLE AND COVARIANT IN THE SECOND VARIABLE

GIVEN

$$\begin{tikzcd} B \arrow[r, "f"] & B' \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$\mathrm{Hom}_R(A, B) \xrightarrow{f_*} \mathrm{Hom}_R(A, B')$$