Monday, November 7, 2022 2:01 PM

SIMPLICIAL SETS

SOURCE PURPLE BOOK 3.4.1

Δ = SIMPLICAL CATEGORY

OBJECTS ARE FINITE ORDERED SETS e.g. $[n] = \{0, \dots, n\}$

<u>FACE MAPS</u> $d_i : [n-1] \longrightarrow [n]$ $\quad 0 \le i \le n$

i IS NOT IN THE IMAGE

e.g. $n = 2$

$$d_1 : [1] \to [2], \quad 0 \mapsto 0, \quad 1 \mapsto 2 \qquad\qquad d_0 : [1] \to [2], \quad 0 \mapsto 1, \quad 1 \mapsto 2$$

<u>DEGENERACY MAPS</u>

$s_i : [n+1] \longrightarrow [n]$ $\quad 0 \le i \le n$

WHERE TWO ELEMENTS MAP TO i

THESE GENERATE ALL MORPHISMS IN Δ, AND THEY SATISFY CERTAIN IDENTITIES.

<u>DEF</u> 3.4.1 A <u>SIMPLICIAL SET</u> X IS A FUNCTOR $\Delta^{op} \xrightarrow{X} Sets$

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$$X([n]) =: X_n = \text{SET OF } n\text{-SIMPLICES}$$

FACE MAPS

$d_i \quad [n-1] \longrightarrow [n]$ INDUCES

$$X_{n-1} \longleftarrow X_n$$

DEGENERACY MAPS

$s_i \quad [n] \longrightarrow [n-1]$ INDUCES

$$X_n \xleftarrow{s_i'} X_{n-1}$$

AN n-SIMPLEX $x \in X_n$ IS <u>NONDEGENERATE</u> IF IT IS <u>NOT</u> IN THE IMAGE OF ANY s_i

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DEF 3.4.2 THE <u>STANDARD</u> n-SIMPLEX

a) $\Delta^n = \left\{ (t_0, t_1, \dots t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1 \right\}$

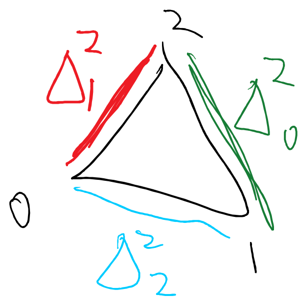
Δ^2 = TRIANGE IN $\mathbb{R}^3$ WITH VERTICES $(1,0,0)$, $(0,1,0)$ AND $(0,0,1)$

$\Delta^n \approx D^n$ = n-DIMENSIONAL BALL

$$\Delta^n \supset \partial\Delta^n = \text{BOUNDARY} = \{x \in \Delta^n : t_i = 0 \ \exists i\}$$
$$\approx S^{n-1}$$

$$\Lambda^n_i = i\text{TH HORN} = \text{COMPLEMENT IN } \partial\Delta^n \text{ int}(\Delta^n_i)$$

$n=2$



Δ^2

$$\Lambda^2_0 = \Delta^2_1 \cup \Delta^2_2$$
$$\Lambda^2_1 = \Delta^2_0 \cup \Delta^2_2$$
$$\Lambda^2_2 = \Delta^2_0 \cup \Delta^2_1$$

$$\Lambda^n_i \approx D^{n-1}$$

IT IS AN <u>INNER</u> HORN IF $0 < i < n$
<u>OUTER</u> IF $i = 0$ OR n.

b) $\Delta[n]$ IS A SIMPLICIAL SET DEFINED BY

$$\Delta[n]_k = \boxed{\Delta([k],[n])} \quad k \geq 0$$

NOTE: IN ANY CATEGORY $\mathcal{C}$,

THE SET $\mathcal{C}(X,Y)$ IS CONTRAVARIANT IN X AND COVARIANT IN Y

I.E

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & X' \arrow[d, "g"] \\ & Y \end{tikzcd} \qquad \mathcal{C}(X,Y) \xleftarrow{f^*} \mathcal{C}(X',Y)$$

AND $X \xrightarrow{f} Y \xrightarrow{h} Y'$ $\quad \mathcal{C}(X,Y) \xrightarrow{h_*} \mathcal{C}(X,Y')$

THUS WE GET A FUNCTOR

$$\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{S}ets$$
$$(X,Y) \longmapsto \mathcal{C}(X,Y)$$

WE CAN ASSOCIATE TO A SIMP. SET $\underline{X}$ A TOPOLOGICAL SPACE $|\underline{X}|$, THE <u>GEOMETRIC REALIZATION</u> OF $\underline{X}$.

e.g $|\Delta[n]| = \Delta^n$

$$|X| = \coprod_{n \geq 0} \left(X_n \times \Delta^n \right) / \sim$$

SET SPACE

THESE PIECES GET GLUED TOGETHER IN A CERTAIN WAY

EXAMPLE: ① THE SIMPLICIAL SET $\Delta[0]$

$$\Delta[0]_k = \Delta\!\!\!\Delta([k],[0])$$

$= $ SINGLETON $\forall k$.

WE HAVE DEGENERACY MAPS

$$\Delta[0]_0 \to \Delta[0]_1 \to \Delta[0]_2 \to \cdots$$

THE ONLY NONDEGEN. SIMPLEX IS IN DIMENSION 0

② $\Delta[1]_k = \Delta\!\!\!\Delta([k],[1])$

WHICH HAS $k+1$ ELEMENTS

$$\Delta[1]_0 \overset{d_0}{\underset{d_1}{\rightrightarrows}} \Delta[1]_1 \;\substack{\to\\ \to\\ \to}\; \Delta[1]_2 \cdots$$

2 elements 3 elements 4 elements

CLAIM THERE ARE

2 NON DEGEN SIMPLICES IN DIM 0

1 1 1

NONE IN DIMENSION > 1

IN $\Delta[n]$ THERE

$\binom{n+1}{k+1}$ NONDEGEN SIMPLICES

IN DIMENSION k

$$\rightsquigarrow |\Delta[n]| = \Delta^n \subset \mathbb{R}^{n+1}$$

COENDS

NOTE $X : \Delta^{op} \longrightarrow \mathcal{S}ets$

$$\Delta \longrightarrow \mathcal{S}paces$$

$$[n] \longmapsto \Delta^n$$

THIS MEANS WE HAVE A FUNCTOR

$$\Delta^{op} \times \Delta \xrightarrow{H} \mathcal{S}paces$$

$$([k], [n]) \longmapsto X_k \times \Delta^n$$

WILL CONSIDER FUNCTORS

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{H} \mathcal{D}$$ = CATEGORY WITH COLIMITS

$$\Delta^{op} \times \Delta \longrightarrow \mathcal{S}paces$$

$\mathcal{C}$ = SMALL CATEGORY

WILL ASSOCIATE TO H AN OBJECT IN $\mathcal{D}$.

THIS WILL SPECIALIZE TO GEOMETRIC REALIZATION

TO BE CONTINUED.