FOR A SIMPLICIAL SET X, THERE IS A TOP. SPACE $|X|$, ITS <u>GEOMETRIC REALIZATION</u>, WHICH IS A CERTAIN TOPOLOGICAL QUOTIENT OF

$$\coprod_{n \geq 0} X_n \times \Delta^n = W$$

SET OF n-SIMPLICES IN X

STANDARD n-SIMPLEX $\subset \mathbb{R}^{n+1}$

THIS WILL BE A CO-EQUALIZER FOR

$$Y \overset{\alpha}{\underset{\beta}{\rightrightarrows}} W \dashrightarrow |X|$$

TO DEFINE Y:

FOR MORPHISM $[k] \xrightarrow{f} [\ell]$ IN Δ WE GET A DIAGRAM

$$\begin{tikzcd}
X_\ell \times \Delta^k \arrow[r, "f^* \times \Delta^k"] \arrow[d, "X_\ell \times f_*"'] \arrow[dr, phantom, "\text{DOES NOT COMMUTE}"] & X_k \times \Delta^k \arrow[d] \\
X_\ell \times \Delta^\ell \arrow[r] & \coprod_{n \geq 0} X_n \times \Delta^n
\end{tikzcd}$$

THIS LEADS TO

$$W' := \coprod_{f:[k]\to[\ell]} X_\ell \times \Delta^k \underset{\text{VERTICAL}}{\overset{\text{HORIZONTAL}}{\rightrightarrows}} \underbrace{\coprod_{n\geq 0} X_n \times \Delta^n}_{= W}$$

SUFFICES TO CONSIDER ONLY THE FACE + DEGENERACY MAPS IN Δ.

THE COEQUALIZER OF H AND V IS $|X|$ BY DEFINITION.

EXAMPLE $\quad \big|\Delta[n]\big| = \Delta^n$.

THIS DEFINES A FUNCTOR

$$\mathcal{S}et_\Delta \xrightarrow{|\cdot|} \mathcal{S}paces$$

CATEGORY OF SIMPLICIAL SETS

WHAT IS $\mathcal{S}et_\Delta$? IT IS THE CATEGORY WHOSE

OBJECTS ARE SIMPLICIAL SETS

A MORPHISM $X \xrightarrow{f} Y$ OF SIMPLICIAL SETS IS A COLLECTION OF MAPS

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d, "d_i"'] & Y_n \arrow[d, "d_i"] \\
X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "s_j"'] & Y_{n-1} \arrow[d, "s_j"] \\
X_n \arrow[r, "f_n"] & Y_n
\end{tikzcd}$$

THERE IS ANOTHER FUNCTOR

$$\mathrm{Spaces} \xrightarrow{\mathrm{Sing}} \mathrm{Set}_\Delta$$

LET X BE A SPACE, LET

$\mathrm{Sing}(X)_n$ = THE SET OF ALL CONTINUOUS MAPS

$$\Delta^n \xrightarrow{\sigma} X$$

$$\begin{tikzcd}
\mathrm{Set}_\Delta \arrow[r, "|\cdot|"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Spaces}
\end{tikzcd}$$

$$\begin{tikzcd} \mathit{Set}_\Delta \arrow[r, phantom, "\perp"] & \mathit{Spaces} \arrow[l, "\mathrm{Sing}"] \end{tikzcd}$$

THESE ARE ADJOINT TO EACH OTHER

FOR A SIMP. SET X AND TOP. SPACE Y

$$\mathit{Set}_\Delta(X, \mathit{Sing}(Y))$$

AND $\mathit{Map}(|X|, Y)$

ARE NATURALLY ISOMORPHIC.

THE INDUCE AN EQUIVALENCE OF CATEGORIES.

THERE A FUNCTOR

$C : \mathit{Set}_\Delta \longrightarrow CC =$ CATEGORY OF CHAIN COMPLEXES

LET X BE A SIMP. SET

$C_n(X)$ = FREE ABELIAN GP ON THE SET X_n

WE NEED A BOUNDARY OPERATOR

$$C_n(X) \longrightarrow C_{n-1}(X)$$

IN Δ WE HAVE FACE MAPS

$$[n-1] \xrightarrow{d_i} [n] \qquad 0 \le i \le n$$

$$\{0, 1, \cdots n-1\} \longrightarrow \{0, 1, \cdots n\}$$

$[n] \ni i$ IS NOT IN THE IMAGE

$$X_{n-1} \xleftarrow{d_i} X_n$$

$$C(X_{n-1}) \xleftarrow{d_i} C(X_n) \qquad 0 \le i \le n$$

$$C_{n-2}(X) \xleftarrow[0 \le j \le n-1]{\sum (-1)^j d_j} C_{n-1}(X) \xleftarrow[0 \le i \le n]{\sum (-1)^i d_i} C_n(X)$$

THE SIMPICIAL IDENTITIES IMPLY THAT THIS COMPOSITE HOMOMORPHISM IS ZERO.

GIVEN A TOP SPACE X, . . .

WE HAVE A SIMPLICIAL $Sing(X)$ AND A CHAIN COMPLEX

$$C(Sing(X)) =: S(X),$$

THE <u>SINGULAR CHAIN COMPLEX</u> OF X.

WE DEFINE $H_*(X) = H_*(S(X))$

HOW DO WE ~~DEAL~~ WITH THIS ???

EXAMPLE:

$X = pt$ = SINGLE POINT

$$S_n(X) = \mathbb{Z}$$

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & 3 & & \\ \mathbb{Z} & \leftarrow & \mathbb{Z} & \leftarrow & \mathbb{Z} & \leftarrow & \mathbb{Z} & \leftarrow & \cdots \end{array}$$

A FACE MAP $[n-1] \xrightarrow{d_i} [n]$ INDUCES

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "d_i"] \arrow[dr, "\sigma d_i"'] & \Delta^n \arrow[d, "\sigma"] \\ & X = pt \end{tikzcd}$$

$$\begin{tikzcd} & {} \arrow[d, "\sigma"] \\ {} \arrow[r, bend left, "\sigma d_i"] & X \arrow[d, equal] \\ & pt \end{tikzcd}$$

THIS INDUCES

$$\begin{tikzcd} S_n(pt) \arrow[r] \arrow[d, equal] & S_{n-1}(pt) \arrow[d, equal] \\ \mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \end{tikzcd}$$

$$\underset{0}{0 \leftarrow \mathbb{Z}} \xleftarrow{0} \underset{1}{\mathbb{Z}} \xleftarrow{1} \underset{2}{\mathbb{Z}} \xleftarrow{0} \cdots \underset{n-1}{\mathbb{Z}} \leftarrow \underset{n}{\mathbb{Z}}$$

$$\sum_{0 \le i \le n} (-1)^i = \begin{cases} 0 & \text{IF } n \text{ IS ODD} \\ 1 & \text{IF } n \text{ IS EVEN} \end{cases}$$

$H_0 = \mathbb{Z}$ $\quad H_1 = 0$ $\quad H_2 = 0$ $\quad \ldots$

CONCLUSION

$$H_n(pt) = \begin{cases} \mathbb{Z} & \text{FOR } n = 0 \\ 0 & \text{FOR } n > 0 \end{cases}$$

THIS IS THE DIMENSION AXIOM

WE ALSO NEED TO DEFINE

$H_*(X, A)$, FOR $A \subset X$, CALLED RELATIVE HOMOLOGY.

$$0 \to S(A) \xrightarrow{1-1} S(X) \to S(X)/S(A) \to 0$$

LET $H_*(X, A) := H\left(S(X)/S(A)\right)$

THIS LEADS TO A LONG EXACT SEQUENCE

$$\cdots \to H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \to H_{n-1}(A) \to \cdots$$

THIS THE EXACTNESS AXIOM.