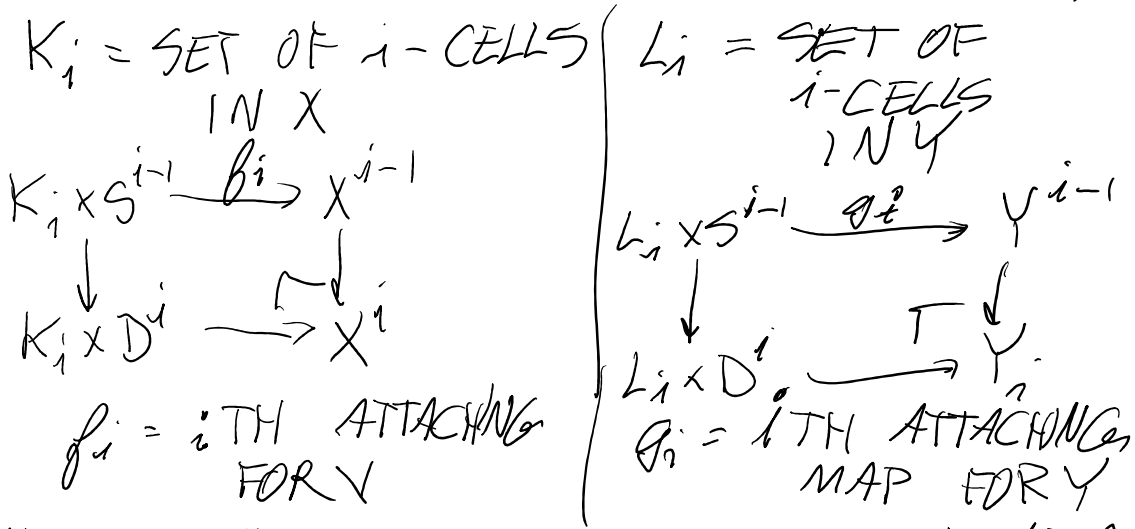


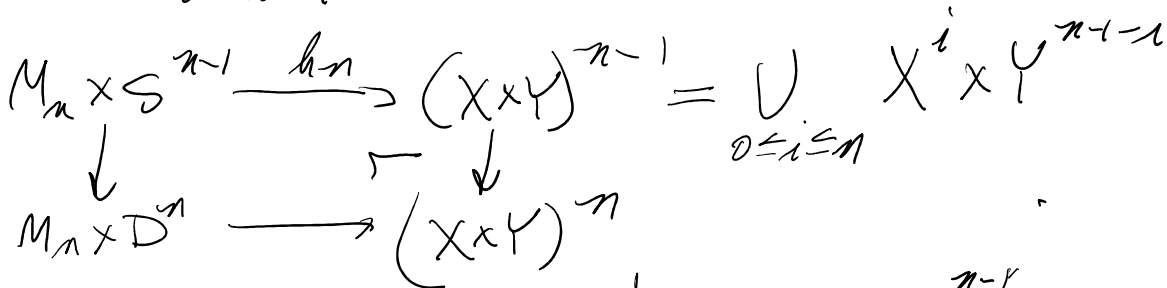
LET X AND Y BE CW-COMPLEXES



WILL CONSTRUCT SIMILAR DATA FOR $X \times Y$ $\{M_n, h_n\}$

$$M_n = \text{SET OF } n\text{-CELLS IN } X \times Y$$

$$= \coprod_{0 \leq i \leq n} K_i \times L_{n-i}$$



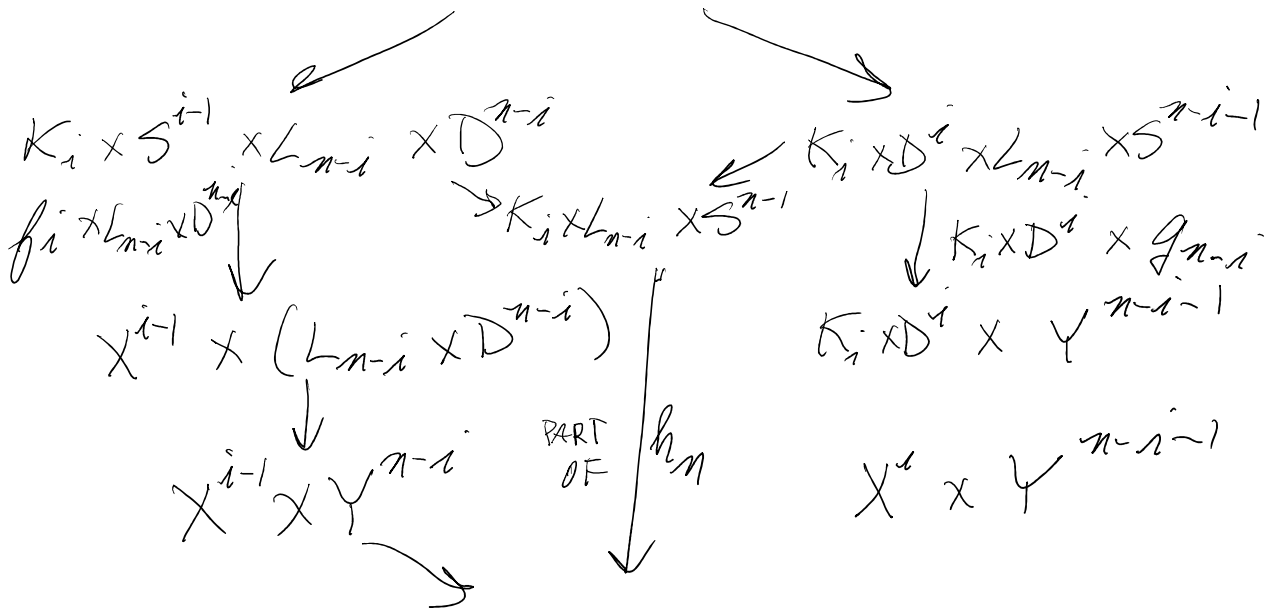
TO DESCRIBE $h_n | K_i \times L_{n-i} \times S^{n-1}$

RECALL $S^{n-1} = \underbrace{S^{i-1} \times D^{n-i}}_A \cup \underbrace{D^i \times S^{n-i-1}}_B$

MANIFOLDS BOUNDED BY $S^{i-1} \times S^{n-i-1}$

$$K_i \times S^{i-1} \times L_{n-i} \times S^{n-i-1}$$

↙ ↘



TAKE THE UNION OF THE ABOVE FOR $0 \leq i \leq n$ TO DEFINE h_n . THESE DEFINITIONS IMPLY $C(X \times Y) \cong C(X) \otimes C(Y)$

QED

THE EULER CHARACTERISTIC

THEOREM LET X BE A FINITE CW-COMPLEX WITH C_i CELLS IN DIMENSION i . THEN THE NUMBER

$$\sum_{i=0}^n (-1)^i C_i = \sum_{i=0}^n (-1)^i \text{rank } H_i(X; k) =: \chi(X)$$

FOR ANY FIELD k $\chi(X)$

REMARKS

- ① THE LEFT SIDE IS INDEPENDENT OF THE CHOICE of k
- ② THE RIGHT SIDE IS A TOPOLOGICAL INVARIANT INDEPENDENT OF THE CW STRUCTURE
- ③ THIS # IS THE EULER CHARACTERISTIC OF X .

CLASSICAL EXAMPLE

THE PLATONIC SOLIDS ARE CW-STRUCTURES ON S^2

	C_0	C_1	C_2	χ
TETRAHEDRON	4	6	4	2
CUBE	8	12	6	2
OCTAHEDRON	6	12	8	2
DODECAHEDRON	20	30	12	2
ICOSAHEDRON	12	30	20	2

$$H_i(S^2; k) = \begin{cases} k & \text{FOR } i=0, 2 \\ 0 & \text{ELSE} \end{cases}$$

$$\sum (-1)^i \text{rank } H_i(S^2) = 2$$

RELATED EXAMPLE

THE ANTIPODAL MAP PRESERVES THE CW-STRUCTURE FOR 4 OF THE 5 CASES ABOVE, EXCLUDING THE TETRAHEDRON, HENCE 4 CW STRUCTURES ON

$\mathbb{R}P^2$. IN EACH THE #S C_i
 ARE HALF OF THE VALUE
 ABOVE $\implies \chi(\mathbb{R}P^2) = 1$

THERE IS A CW-STRUCTURE
 ON $\mathbb{R}P^2$ WITH A SINGLE
 IN DIMENSIONS 0, 1 AND 2

$$\begin{array}{ccccc} \cong & & \cong & & \cong \\ \parallel & & \parallel & & \parallel \\ C_0 & \xleftarrow{0} & C_1 & \xleftarrow{0} & C_2 \end{array}$$

$$H_i(\mathbb{R}P^2) = \begin{cases} \cong \mathbb{Z} & i=0 \\ \cong \mathbb{Z}/2 & i=1 \\ 0 & i>1 \end{cases}$$

USING THE UCT WE FIND

1) FOR $k = \mathbb{Z}/2$

$$\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2$$

$$H_i(\mathbb{R}P^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{FOR } i=0,1,2 \\ 0 & \text{FOR } i>2 \end{cases}$$

$$\sum (-1)^i \text{rank } H_i(\mathbb{R}P^2; \mathbb{Z}/2) = 1$$

2) IF $\text{char}(k) \neq 2$, CHAIN CX
 BECOMES

$$k \xleftarrow{0} k \xleftarrow{\frac{2}{2}} k$$

$$H_i(\mathbb{R}P^2; k) = \begin{cases} k & i=0 \\ 0 & i>0 \end{cases}$$

- ; 0 ... 1

$$\sum (-1)^i \text{rank}(H_i) = 1$$

PROOF OF THEOREM

LET $C(X) = \text{CELLULAR CHAIN}$
 $CX \text{ OF } X$

CHOOSE A FIELD K

THEN $C(X) \otimes K$ HAS HOMOLOGY
 $H_*(X; K)$. LET h_i DENOTE
 RANK $H_i(X; K)$

$C_i = \# \text{ OF } i\text{-CELLS}$
 $= \text{RANK OF THE FREE ABELIAN GR } C_i(X)$

WANT TO SHOW

$$\sum_{i \geq 0} (-1)^i C_i = \sum_{i \geq 0} (-1)^i h_i$$

MORE NOTATION

$z_i = \text{RANK OF } \ker (C_i(X) \xrightarrow{d_i} C_{i-1}(X))$
 $= \text{RANK OF VECTOR SPACE OF } i\text{-CYCLES}$

$b_i = \text{RANK OF } \text{im} (C_{i+1}(X) \xrightarrow{d_{i+1}} C_i(X))$
 $= \text{RANK OF VECTOR SPACE OF } i\text{-BOUNDARIES}$

SINCE $H_i(C \otimes K) = \ker(d_i \otimes K) / \text{im}(d_{i+1} \otimes K)$,

$$h_i = z_i - b_i$$

FOR EACH i THERE IS A $_$ SES

$$0 \rightarrow Z_i \xrightarrow{d_i} C_i \xrightarrow{d_{i-1}} B_{i-1} \rightarrow U$$

\parallel $\text{ker } d_i$ \parallel $\text{im } d_i$

SD $C_i = Z_i + b_{i-1}$

$$\begin{aligned} \sum_{i \geq 0} (-1)^i h_i &= \sum_{i \geq 0} (-1)^i z_i - \sum_{i \geq 0} (-1)^i b_i \\ &= \sum_{i \geq 0} (-1)^i z_i + \sum_{i \geq 0} (-1)^i b_{i-1} \quad b_{-1} = 0 \\ &= \sum_{i \geq 0} (-1)^i (z_i + b_{i-1}) \\ &= \sum_{i \geq 0} (-1)^i C_i \quad \text{QED} \end{aligned}$$

COROLLARY

LET X AND Y BE FINITE CW-COMPLEXES.

$$\chi(X \times Y) = \chi(X) \chi(Y)$$

PROOF:

LET

$k_i = \#$ OF i -CELLS IN X

$l_i = \#$ OF i -CELLS IN Y

LET $k(t) = \sum k_i t^i$ $t =$ DUMMY VARIABLE

$l(t) = \sum l_j t^j$

THEN $k(-1) = \sum (-1)^i k_i = \chi(X)$

$l(-1) = \sum (-1)^i l_i = \chi(Y)$
 LET $m_i = \#$ of i -CELLS IN $X \times Y$

WE KNOW

$$m_n = \sum_{0 \leq i \leq n} k_i l_{n-i}$$

LET $m(t) := \sum_{i \geq 0} m_i t^i$

SO $m(t) = k(t) l(t)$

$$m(-1) = k(-1) l(-1)$$

$$\chi(X \times Y) = \chi(X) \chi(Y) \quad \text{QED}$$

COROLLARY SUPPOSE

$\tilde{X} \xrightarrow{p} X$ IS A FINITE COVERING OF DEGREE d
 FOR X A FINITE CW COMPLEX. THEN \tilde{X} IS ALSO A FINITE CW COMPLEX

WITH $\chi(\tilde{X}) = d \chi(X)$

(EXERCISE)