LET X AND Y BE CW-COMPLEXES

K_i = SET OF i-CELLS IN X

$$\begin{tikzcd} K_i \times S^{i-1} \arrow[r, "f_i"] \arrow[d] & X^{i-1} \arrow[d] \\ K_i \times D^i \arrow[r] & X^i \end{tikzcd}$$

f_i = iTH ATTACHING FOR X

L_i = SET OF i-CELLS IN Y

$$\begin{tikzcd} L_i \times S^{i-1} \arrow[r, "g_i"] \arrow[d] & Y^{i-1} \arrow[d] \\ L_i \times D^i \arrow[r] & Y_i \end{tikzcd}$$

g_i = iTH ATTACHING MAP FOR Y

WILL CONSTRUCT SIMILAR DATA FOR $X \times Y$ $\{M_k, h_k\}$

M_n = SET OF n-CELLS IN $X \times Y$

$$= \coprod_{0 \le i \le n} K_i \times L_{n-i}$$

$$\begin{tikzcd} M_n \times S^{n-1} \arrow[r, "h_n"] \arrow[d] & (X \times Y)^{n-1} \arrow[d] \\ M_n \times D^n \arrow[r] & (X \times Y)^n \end{tikzcd} \qquad (X\times Y)^{n-1} = \bigcup_{0 \le i \le n} X^i \times Y^{n-1-i}$$

TO DESCRIBE $h_n \mid K_i \times L_{n-i} \times S^{n-1}$

RECALL $S^{n-1} = \underbrace{S^{i-1} \times D^{n-i}} \cup \underbrace{D^i \times S^{n-i-1}}$

MANIFOLDS BOUNDED BY $S^{i-1} \times S^{n-i-1}$

$$K_i \times S^{i-1} \times L_{n-i} \times S^{n-i-1}$$

$$
\begin{tikzcd}
K_i \times S^{i-1} \times L_{n-i} \times D^{n-i} \arrow[d, "f_i \times L_{n-i} \times D^{n-i}"'] \arrow[dr] & & K_i \times D^i \times L_{n-i} \times S^{n-i-1} \arrow[dl] \arrow[d, "K_i \times D^i \times g_{n-i}"] \\
X^{i-1} \times (L_{n-i} \times D^{n-i}) \arrow[d] & K_i \times L_{n-i} \times S^{n-1} \arrow[dd, "\text{PART OF } h_n"'] & K_i \times D^i \times Y^{n-i-1} \\
X^{i-1} \times Y^{n-i} \arrow[dr] & & X^i \times Y^{n-i-1} \\
 & \, &
\end{tikzcd}
$$

TAKE THE UNION OF THE ABOVE FOR $0 \leq i \leq n$ TO DEFINE h_n.

THESE DEFINITIONS IMPLY

$$C(X \times Y) \simeq C(X) \otimes C(Y)$$

QED

THE EULER CHARACTERISTIC

THEOREM LET X BE A FINITE CW-COMPLEX WITH c_i CELLS IN DIMENSION i THEN THE NUMBER

$$\sum_{i=0} (-1)^i c_i = \sum_{i=0} (-1)^i \operatorname{rank} H_i(X; k) =: \chi(X)$$

FOR ANY FIELD k \chi(X)

REMARKS

1. THE LEFT SIDE IS INDEPENDENT OF THE CHOICE of k
2. THE RIGHT SIDE IS A TOPOLOGICAL INVARIANT INDEPENDENT OF THE CW STRUCTURE
3. THIS # IS THE <u>EULER CHARACTERISTIC</u> OF X.

CLASSICAL EXAMPLE

THE PLATONIC SOLIDS ARE CW-STRUCTURES ON S^2

	c_0	c_1	c_2	χ
TETRAHEDRON	4	6	4	2
CUBE	8	12	6	2
OCTAHEDRON	6	12	8	2
DODECAHEDRON	20	30	12	2
ICOSAHEDRON	12	30	20	2

$$H_i(S^2; k) = \begin{cases} k & \text{FOR } i = 0, 2 \\ 0 & \text{ELSE} \end{cases}$$

$$\sum (-1)^i \operatorname{rank} H_i(S^2) = 2$$

RELATED EXAMPLE

THE ANTIPODAL MAP PRESERVES THE CW-STRUCTURE FOR 4 OF THE 5 CASES ABOVE, EXCLUDING THE TETRAHEDRON

HENCE 4 CW STRUCTURES ON

$\mathbb{R}P^2$. IN EACH THE #S C_i ARE HALF OF THE VALUE ABOVE $\rightsquigarrow$ $\chi(\mathbb{R}P^2) = 1$

THERE IS A CW-STRUCTURE ON $\mathbb{R}P^2$ WITH A SINGLE IN DIMENSIONS 0, 1 AND 2

$$\begin{array}{ccccc} \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\ \| & & \| & & \| \\ C_0 & \xleftarrow{0} & C_1 & \xleftarrow{2} & C_2 \end{array}$$

$$H_i(RP^2) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2 & i=1 \\ 0 & i>1 \end{cases}$$

USING THE UCT WE FIND

1) FOR $k = \mathbb{Z}/2$

$$\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2$$

$$H_i(RP^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{FOR } i = 0,1,2 \\ 0 & \text{FOR } i > 2 \end{cases}$$

$$\Sigma(-1)^i \operatorname{rank} H_i(RP^2; \mathbb{Z}/2) = 1$$

2) IF $\operatorname{char}(k) \neq 2$, CHAIN CX BECOMES

$$k \xleftarrow{0} k \xleftarrow[\simeq]{2} k$$

$$H_i(RP^2; k) = \begin{cases} k & i=0 \\ 0 & i>0 \end{cases}$$

$$\sum (-1)^i \operatorname{rank}(H_i) = 1$$

PROOF OF THEOREM

LET $C(X)$ = CELLULAR CHAIN CX OF X

CHOOSE A FIELD k

THEN $C(X) \otimes k$ HAS HOMOLOGY $H_*(X;k)$. LET h_i DENOTE rank $H_i(X;k)$

c_i = # OF i-CELLS = RANK OF THE FREE ABELIAN GR $C_i(X)$

WANT TO SHOW

$$\sum_{i \geq 0} (-1)^i c_i = \sum_{i \geq 0} (-1)^i h_i$$

MORE NOTATION

z_i = RANK OF $\ker\left(C_i(X)_{\otimes k} \xrightarrow{d_i} C_{i-1}(X)_{\otimes k}\right)$

= RANK OF VECTOR SPACE OF i-CYCLES

b_i = RANK OF $\operatorname{im}\left(C_{i+1}(X) \xrightarrow{d_{i+1}} C_i(X)\right)$

= RANK OF VECTOR SPACE OF i-BOUNDARIES

SINCE $H_i(C \otimes k) = \ker(d_i \otimes k)/\operatorname{im}(d_{i+1} \otimes k)$,

$$h_i = z_i - b_i$$

FOR EACH i THERE IS A SES

$$0 \to \underset{\substack{\| \\ \ker d_i}}{Z_i} \to C_i \to \underset{\substack{\| \\ \operatorname{im} d_i}}{B_{i-1}} \to 0$$

SO $\boxed{C_i = Z_i + B_{i-1}}$

$$\sum_{i \geq 0} (-1)^i h_i = \sum_{i \geq 0} (-1)^i Z_i - \sum (-1)^i B_i$$

$$= \sum_{i \geq 0} (-1)^i Z_i + \sum_{i \geq 0} (-1)^i B_{i-1} \qquad B_{-1} = 0$$

$$= \sum_{i \geq 0} (-1)^i (Z_i + B_{i-1})$$

$$= \sum_{i \geq 0} (-1)^i C_i \qquad \text{QED}$$

COROLLARY

LET X AND Y BE FINITE CW-COMPLEXES.

$$\chi(X \times Y) = \chi(X)\, \chi(Y)$$

PROOF: LET

k_i = # OF i-CELLS IN X

l_i = 〃 Y

LET $k(t) = \sum k_i t^i$. $\quad t$ = DUMMY VARIABLE

$l(t) = \sum l_j t^j$

THEN $k(-1) = \sum (-1)^i k_i = \chi(X)$

$$\ell(-1) = \sum (-1)^j \ell_j = \chi(Y)$$

LET m_i = # of i-CELLS IN $X \times Y$

WE KNOW

$$m_n = \sum_{0 \le i \le n} k_i \, \ell_{n-i}$$

LET $m(t) := \sum_{i=0} m_i t^i$

SO $m(t) = k(t) \, \ell(t)$

$$m(-1) = k(-1) \, \ell(-1)$$

$$\chi(X \times Y) = \chi(X) \chi(Y)$$ QED

COROLLARY SUPPOSE $\tilde{X} \xrightarrow{p} X$ IS A FINITE COVERING OF DEGREE d FOR X A FINITE CW COMPLEX. THEN $\tilde{X}$ IS ALSO A FINITE CW COMPLEX WITH $\chi(\tilde{X}) = d \, \chi(X)$

(EXERCISE)