ARE THERE OTHER PLATONIC SOLIDS?
i.e. ARE THERE OTHER CW-STRUCTURES ON S^2 IN WHICH EACH VERTEX HAS q EDGES AND EACH FACE HAS p EDGES (SIDES) FOR SOME p AND q.

$V = c_0 = \#$ OF VERTICES OR 0-CELLS
$E = c_1 = \#$ OF EDGES OR 1-CELLS
$F = c_2 = \#$ " FACES " 2-CELLS

$2E = qV$ AND $2E = pF$

$$\chi = V - E + F = \boxed{E\left(\frac{2}{q} - 1 + \frac{2}{p}\right) = 2}$$

HENCE $\boxed{\frac{2}{q} + \frac{2}{p} - 1 > 0}$ WITH $p, q \geq 2$

THE ONLY (p,q) FOR WHICH THIS HOLDS ARE THE ONES WE KNOW ABOUT, NAMELY

$(p,q) = (3,3)$ TETRAHEDRON
$(p,q) = (4,3)$ CUBE

$(p,q) = (3,4)$ OCTAHEDRON

$(p,q) = (3,5)$ ICOSAHEDRON

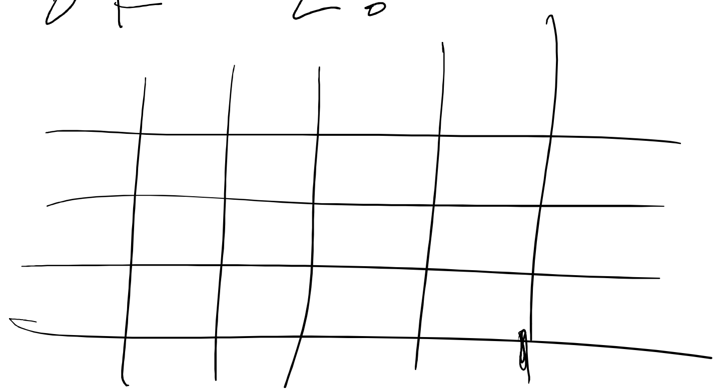
$(p,q) = (5,3)$ DODECAHEDRON

VARIATION: REPLACE S^2 BY THE TORUS. THEN WE FIND $\frac{2}{p} + \frac{2}{q} = 1$ FOR $p,q \geq 3$

$(p,q) = (4,4), (6,3)$ OR $(3,6)$

IN EACH THERE ARE INFINITELY MANY POSSIBLE VALUES OF E.

$(p,q) = (4,4)$



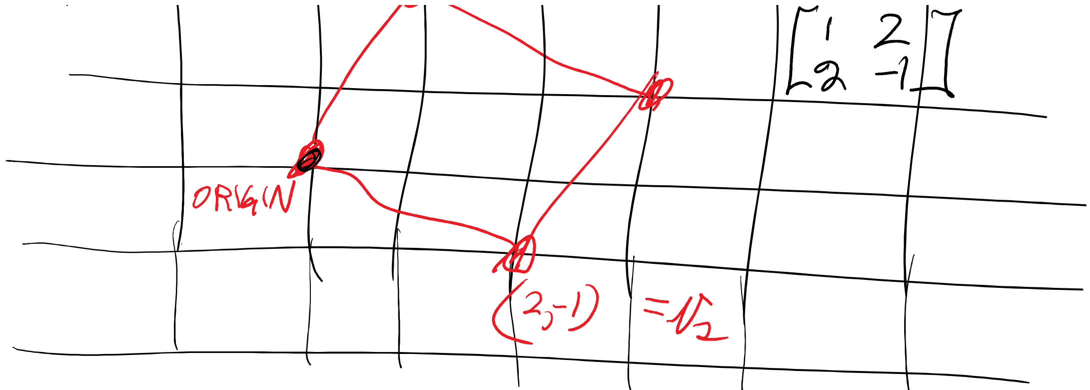
WE CAN COVER ALL OF $\mathbb{R}^2$ THIS WAY.

CHOOSE A SUBGP $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$ THAT PRESERVES THIS TESSELATION

EXAMPLE

$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

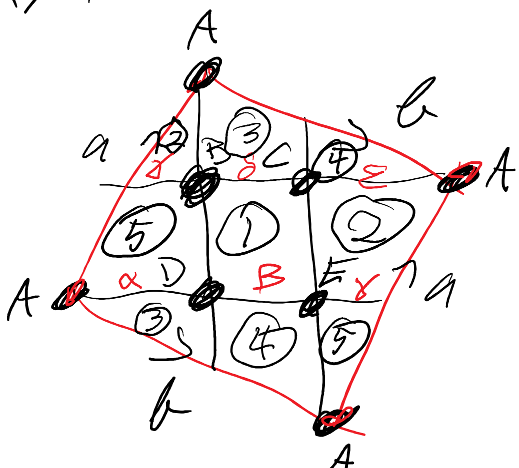
EXAMPLE



$\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

α, β GENERATE SUCH A SUBGROUP Γ OF $\mathbb{R}^2$. Γ ACTS FREELY BY TRANSLATION AND

$\mathbb{R}/\Gamma$ IS A TORUS



WE GET CW-STR ON TORUS

5 VERTICES

10 EDGES

$\alpha, \beta, \gamma, \delta, \varepsilon$

$\zeta, \eta, \theta, \iota, \kappa$

5 FACES

1, 2, 3, 4, 5

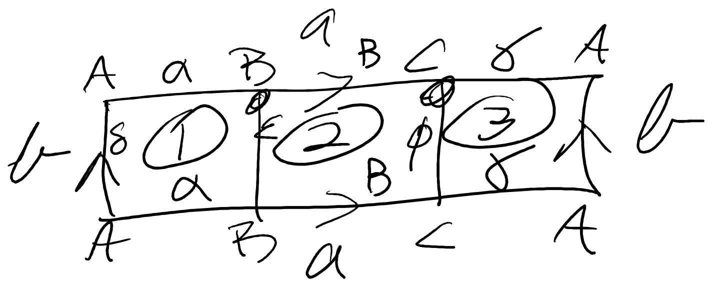
$\chi = 5 - 10 + 5 = 0$

THERE INFINITELY MANY SUCH SUBGRPS

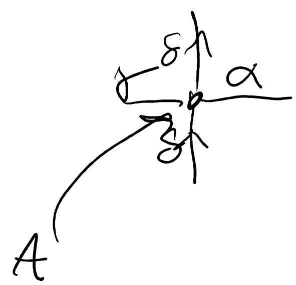
ANOTHER EXAMPLE

$v_1 = (3,0)$, $v_2 = (0,1)$

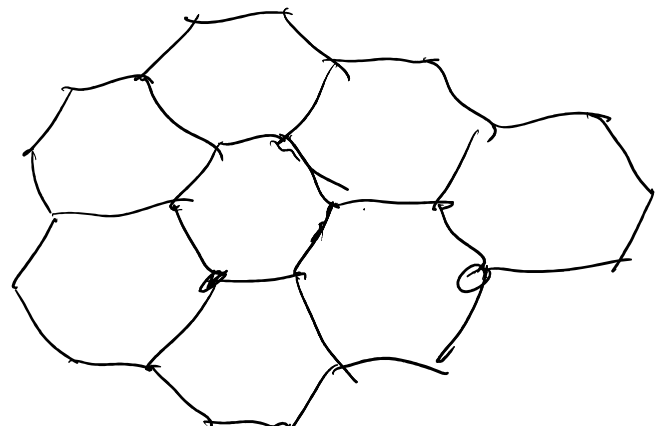
a . . . 3 VERTICES



3 VERTICES
6 EDGES
3 FACES



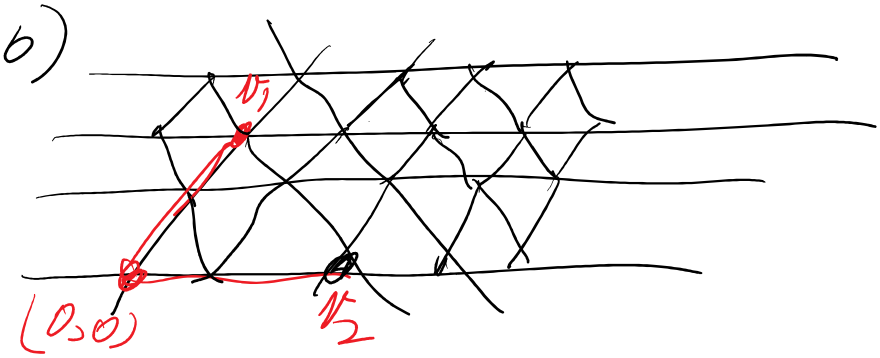
$(p,q) = (6,3)$

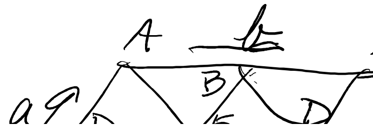


HONEYCOMB

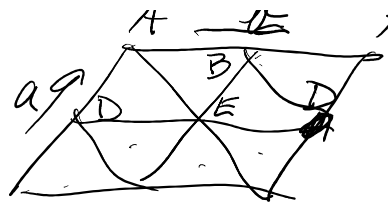
FIND FREE ABELIAN RANK 2
SUBGRS OF R^2 THAT
PRESERVE THIS TESSELATION

$(p,q) = (3,6)$





4 VERTICES
12 EDGES



4 VERTICES
12 EDGES
8 TRIANGLE

WE COULD DO SIMILAR THINGS ON SURFACE OF GENUS > 1

FOR GENUS g, $\chi = 2 - 2g$

$$E\left(\frac{2}{p} + \frac{2}{q} - 1\right) = 2 - 2g$$

THERE ARE PICTURES FOR $(p, q) = (3, 7)$

HYPERBOLIC GEOMETRY

$$\frac{2}{3} + \frac{2}{7} - 1 = \frac{14 + 6 - 21}{21} = -\frac{1}{21}$$

THERE IS CW-STRUCTURE WITH 42 EDGES ON A SURFACE OF GENUS 2.

$\{p, q\}$ IS THE SCHÄFFLI SYMBOL.

SEE "REGULAR POLYTOPES" BY HSM COXETER.

NEW TOPIC: LET G BE A FINITE GROUP.

A <u>G-SPACE</u> X IS A TOP. SPACE WITH AN ACTION OF G.

<u>EXAMPLE</u> $X = S^n$, $G = C_2$ ACTING BY ANTIPODAL MAP.

<u>DEF</u> A G-CW COMPLEX IS A G-SPACE X CONSTRUCTED AS FOLLOWS.

$K_0 = X^0 =$ DISCRETE SET WITH A G-ACTION

$$\begin{array}{ccc} K_n \times S^{n-1} & \xrightarrow{f_n} & X^{n-1} \\ \downarrow & & \downarrow \\ K_n \times D^n & \longrightarrow & X^n \end{array}$$

EACH K_n IS A G-SET

X^{n-1} IS A G-SPACE

f_n IS G-EQUIVARIANT MAP, i.e.

FOR EACH $\gamma \in G$

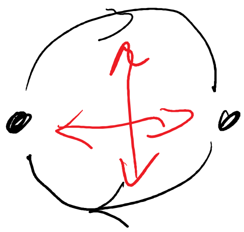
$$\begin{array}{c} K_n \\ \gamma \downarrow \\ K_n \end{array} \qquad \begin{array}{ccc} K_n \times S^{n-1} & \xrightarrow{f_n} & X^{n-1} \\ \gamma \downarrow & & \downarrow \gamma \\ K_{n-1} \times S^{n-1} & \xrightarrow{f_n} & X^{n-1} \end{array}$$

DIAGRAM COMMUTES

THIS IMPLIES THE PUSHOUT X_n IS ALSO A G-SPACE.

EXAMPLE RELATED TO S^n WITH ANTIPODAL ACTION OF C_2:

K_i FOR $0 \leq i \leq n$ IS THE SET C_2 ITSELF



$S^1_{\pm}$

ATTACHING 2 D^2's GIVES $S^2_{\pm}$, ETC.

RECALL A CW STRUCTURE ON X LEADS A CELLULAR CHAIN COMPLEX $C(X)$ with $H_*(C(X)) = H_*(X)$

A G-CW COMPLEX X HAS A CELLULAR CHAIN COMPLEX $C_G(X)$. IT IS A CHAIN COMPLEX OF $\mathbb{Z}[G]$-MODULES

WHERE $\mathbb{Z}[G]$ DENOTES THE GROUP RING OF G,

THE FREE ABELIAN GROUP GENERATED BY THE SET G. FOR $\gamma \in G$, THE CORRESPONDING IS DENOTED BY $[\gamma] \in \mathbb{Z}[G]$

THE MULTIPLICATION IS DEFINED BY

$$[\gamma_1][\gamma_2] = [\gamma_1 \gamma_2]$$

MULTIPLICATION IN G.

MEET AGAIN ON ZOOM 2:00 THURSDAY AND 3:30 FRIDAY

HOMEWORK DUE BEFORE MONDAY!