G-CW COMPLEXES

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FOR AN ORDINARY CW-COMPLEX WE HAVE

1. SKELETA $X^0 \subset X^1 \subset X^2 \subset \cdots$ CONSTRUCTED INDUCTIVELY
2. SETS $X^0 = K_0, K_1, K_2 \ldots$
3. ATTACHING MAPS

$$\begin{tikzcd}
K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, "K_n \times i_n"'] & X^{n-1} \arrow[d] \\
K_n \times D^n \arrow[r] & X^n
\end{tikzcd}$$

X^{n-1} PREVIOUSLY CONSTRUCTED

X^n = PUSHOUT

FOR A FINITE GP G, SUPPOSE EACH K_n IS A G-SET AND f_n IS AN G-EQUIVARIANT MAP WE GET A G-ACTION ON EACH X^n.

G ACTS ON X BY PERMUTING CELLS IN EACH DIMENSION.

THIS IS A CW-COMPLEX WITH G-ACTION OF A CERTAIN TYPE.

WILL STUDY $S^n_{\pm}$, MEANING S^n

WITH ANTIPODAL ACTION OF C_2 WE GET A CELLULAR CHAIN COMPLEX $C(X)$ AS BEFORE WITH $C_n(X)$ = FREE AB GP GENERATED BY K_n

THE G-ACTION ON K_n MAKES $C_n(X)$ A MODULE OVER THE GROUP RING $\mathbb{Z}[G]$.

DEF $\mathbb{Z}[G]$ IS THE FREE AB GP ON THE SET G. FOR $\gamma \in G$, LET $[\gamma] \in \mathbb{Z}[G]$ BE THE GENERATOR CORRESPONDING TO γ. THE MULTIPLICATION IS DEFINED BY

$$[\gamma_1][\gamma_2] = [\gamma_1\gamma_2] \quad \text{FOR } \gamma_1, \gamma_2 \in G$$

MULTIPLICATION IN G

EXAMPLE $G = C_2 = \{e, \gamma\}$, $e = 1$

NONTRIVIAL ELEMENT

IDENTITY

$$\mathbb{Z}[G] = \mathbb{Z}\{1, \gamma\} \text{ AS ABELIAN GP}$$
$$\cong \mathbb{Z}[x]/(x^2-1) \qquad x = [\gamma]$$

NOTE $\gamma^2 - 1 = (\gamma+1)(\gamma-1) = 0$

HENCE $\mathbb{Z}[G]$ IS NOT AN INTEGRAL DOMAIN.

EXAMPLE $X = S^n_{\pm} = S^n$ WITH ANTIPODAL G-

EXAMPLE $X = S^n_\pm = S^n$ WITH ANTIPODAL C_2-ACTION.

$K_i = C_2$ FOR $0 \le i \le n$

$X^0 = S^0_\pm = 2$ POINTS PERMUTED BY C_2

$$\begin{tikzcd}
C_2 \times S^{i-1} \arrow[r, "f_i"] \arrow[d] & X^{i-1} = S^{i-1}_\pm \arrow[d] \\
C_2 \times D^i \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^i_\pm
\end{tikzcd}$$

f_i IS IDENTITY ON ONE COPY OF S^{i-1} AND ANTIPODAL MAP ON THE OTHER

THE CELLULAR CHAIN COMPLEX OF $\mathbb{Z}[C_2]$-MODULES

$$C_i(X) = \mathbb{Z}[C_2] \quad \text{FOR } 0 \le i \le n$$

THEOREM THE BOUNDARY OPERATOR

$$\begin{array}{ccc} d_i : C_i(S^n_\pm) & \longrightarrow & C_{i-1}(S^n_\pm) \\ \| & & \| \\ \mathbb{Z}[C_2] & & \mathbb{Z}[C_2] \end{array}$$

IS MULTIPLICATION BY

$$[e] + (-1)^i [\gamma] = 1 + (-1)^i x.$$

CONSEQUENCES

$$\begin{aligned} d_{i-1} d_i &= (1 + (-1)^{i-1} x)(1 + (-1)^i x) \\ &= 1 + x\left((-1)^{i-1} + (-1)^i\right) + (-1)^{2i-1} x^2 \\ &= 1 - x^2 = 0 \end{aligned}$$

THE CHAIN COMPLEX IS

$$0 \leftarrow \overset{0}{\mathbb{Z}[G]} \xleftarrow[d_1]{1-x} \overset{1}{\mathbb{Z}[G]} \xleftarrow[d_2]{1+x} \overset{2}{\mathbb{Z}[G]} \xleftarrow{1-\gamma} \overset{3}{\mathbb{Z}[G]} \leftarrow \cdots$$

$H_0 = \mathbb{Z}[G]/(1-x) = \mathbb{Z}_+ = \mathbb{Z}$ WITH TRIVIAL C_2-ACTION

TO FIND H_1 NOTE

$\ker d_1 = \{a+bx : a=b\} = (1+x)$

$= \operatorname{im} d_2$

$$(a+bx)(1-x) = (a-b) + x(b-a)$$

HENCE $H_1 = 0$ (FOR $n>1$)

SIMILARLY WE FIND

$H_i = 0$ FOR $0 < i < n$.

NOTE $\ker(1-\gamma) = (1+\gamma)$ $\gamma = x$

AND $\ker(1+\gamma) = (1-\gamma)$

$(a+b\gamma)(1+\gamma) = a + (a+b)\gamma + b\gamma^2$

$= (a+b)(1+\gamma)$

$= 0$ WHEN $a=-b$.

WHAT ABOUT H_n?

$$H_n = \begin{cases} \mathbb{Z}_+ & \text{FOR } n \text{ ODD} \\ \mathbb{Z}_- & \text{FOR } n \text{ EVEN} \end{cases}$$

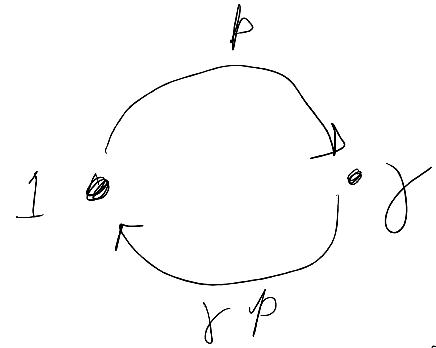
$\mathbb{Z}$ WITH C_2 ACTING AS MULTIPLICATION BY (-1)

WE HAVE COMPUTED

$H_*(S^n_\pm)$ AS A $\mathbb{Z}[C_2]$-MODULE

PROOF OF THEOREM:

FOR $i=1$



$d_1(p) = 1-\gamma$

FOR $i>1$, USE INDUCTION AS FOLLOWS

WE KNOW $H_i(S^n)=0$ FOR $0<i<n$

SO $H_i(S^n_\pm)=0$ "

d_i MUST SATISFY $\mathrm{im}\, d_i = \ker d_{i-1}$

THIS IMPLIES $d_i = 1+(-1)^i\gamma$.

QED

WILL USE THIS TO COMPUTE $H_*(\mathbb{R}P^n)$. $\mathbb{R}P^n = S^n_\pm / C_2$

= ORBIT SPACE

OUR G-CW STRUCTURE ON $S^n_\pm$ LEADS TO ONE ON $\mathbb{R}P^n$ WITH $C(\mathbb{R}P^n)$ = THE ORBIT CHAIN CX

$C(S^n_\pm)/C_2$

SINCE $C_i(S^n_\pm) \cong \mathbb{Z}[G]$,

$C_i(\mathbb{RP}^n) = \mathbb{Z}[G]/G = \mathbb{Z}$

$\mathbb{RP}^n$ HAS A CW-STRUCTURE WITH A SINGLE CELL IN EACH DIMENSION 0 THRU n.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\text{DOUBLE COVER}"] \arrow[d] & \mathbb{RP}^{n-1} \arrow[d] \\ D^n \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

$$\begin{array}{lccccccccc} & 0 & & 1 & & 2 & & 3 & & \\ C(S^n_\pm) & \mathbb{Z}[G] & \xleftarrow{1-\gamma} & \mathbb{Z}[G] & \xleftarrow{1+\gamma} & \mathbb{Z}[G] & \xleftarrow{1-\gamma} & \mathbb{Z}[G] & \leftarrow & \cdots \\ C(\mathbb{RP}^n) & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \leftarrow & \cdots \end{array}$$

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & i = 2 \\ \vdots & \end{cases}$$

$$= \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i \text{ ODD AND } 0 \le i < n) \\ 0 & i \text{ EVEN } i > 0 \\ \mathbb{Z} & i = n, \ n \text{ ODD} \end{cases}$$

ANOTHER EXAMPLE

$G = C_m$ FOR $m > 2$

$X = S^{k-1} \subset \mathbb{C}^k$

ACTION IS MULTIPLICATION BY $e^{2\pi i/m}$

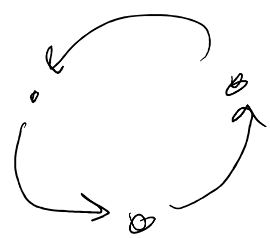
$$\mathbb{Z}[G] = \mathbb{Z}[x]/(x^m - 1) = \mathbb{Z}[y]/(y^m - 1)$$

THERE IS A G-CW STRUCTURE IN WHICH

$$K_i = \begin{cases} C_m & \text{FOR } 0 \le i \le 2k-1 \\ 0 & \text{FOR } i > 2k-1 \end{cases}$$

WITH ATTACH MAPS

$$\begin{tikzcd}
C_m \times S^0 = K_1 \times S^0 \arrow[r, "f_1"] \arrow[d] & X^0 = K_0 = C_m \arrow[d] \\
K_1 \times D^1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X^1
\end{tikzcd}$$

FOR $m = 3$ $X^1 =$ 

$$\begin{tikzcd}
C_m \times S^1 = K_2 \times S^1 \arrow[r, "f_2"] \arrow[d] & X^1 \arrow[d] \\
K_2 \times D^2 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X^2
\end{tikzcd}$$

f_2 CONSISTS OF ROTATION $S^1 \to S^1$

CLAIM $X^3 = S^3$ WITH C_m ACTION

EACH ODD SKELETON IS A SPHERE

$X^2 = S^1$ WITH m COPIES OF D^2 ATTACHED.

$X^{2i} = S^{2i-1}$ WITH m COPIES OF D^{2i} ATTACHED

S^{2k-1}/C_m IS CALLED A <u>LENS SPACE</u>. L^{2k-1}

$$d_i = \begin{cases} 1-\gamma & \text{FOR } i \text{ ODD} \\ 1+\gamma+\gamma^2 \cdots \gamma^{m-1} & \text{FOR } i \text{ EVEN} \end{cases}$$

NEXT TOPIC: CUP PRODUCTS

3:30 TOMORROW