NEXT TOPIC: CUP PRODUCTS

3:30 TOMORROW

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CUP PRODUCTS

$H_* X = H_* S(X)$

$S(X)$ = SINGULAR CHAIN CX OF X

$\mathrm{Hom}(S(X), A)$ $\quad$ $A$ = ABELIAN GP OR RING

THIS IS A COCHAIN CX

ITS COHOMOLOGY IS

$H^*(X; A)$ USUALLY

$A = \mathbb{Z}, \mathbb{Z}/p$ OR $\mathbb{Q}$

WILL CONSTRUCT MAPS

$$H^i(X;A) \otimes H^j(X;A) \to H^{i+j}(X;A)$$

THIS MAKES

$H^*(X;A)$ INTO A <u>GRADED RING</u>. THIS STRUCTURE IS USEFUL.

<u>HOW WE GET IT</u>.

$$X \xrightarrow{\Delta} X \times X \qquad \text{DIAGONAL MAP}$$

$$x \longmapsto (x, x)$$

\cup

$$\begin{tikzcd}
\alpha \cup \beta & H^*(X) & H^*(X \times X) \arrow[l, "\Delta^*"'] \\
& & \alpha \otimes \beta \in H^*(X) \otimes H^*(X) \arrow[u] \arrow[llu, mapsto]
\end{tikzcd}$$

$\alpha \cup \beta \in H^{i+j}(X)$

$\alpha \in H^i(X) \quad \beta \in H^j(X)$

FORMAL PROPERTIES OF CUP PRODUCT

1) LINEAR IN EACH FACTOR

$$(\alpha_1 + \alpha_2)\beta = \alpha_1 \beta + \alpha_2 \beta$$

$$\alpha(\beta_1 + \beta_2) = \alpha \beta_1 + \alpha \beta_2$$

2) ASSOCIATIVE

$$(\alpha \beta)\gamma = \alpha(\beta \gamma)$$

3) COMMUTATIVE UP TO SIGN

FOR $\alpha \in H^i$ AND $\beta \in H^j$,

$$\beta\alpha = (-1)^{ij}\alpha\beta = \begin{cases} \alpha\beta & \text{IF } i \text{ OR } j \text{ IS EVEN} \\ -\alpha\beta & \text{IF } i \text{ AND } j \text{ ARE ODD} \end{cases}$$

4) NATURALITY. GIVEN $f: X \to Y$.

$$\begin{tikzcd}
H^i(X) \otimes H^j(X) \arrow[d] & H^i(Y) \otimes H^j(Y) \arrow[l, "f^* \otimes f^*"'] \arrow[d] \\
H^{i+j}(X) & H^{i+j}(Y) \arrow[l, "f^*"']
\end{tikzcd}$$

COMMUTES

# EXAMPLES

1) $X = \mathbb{R}P^n$. WE HAVE SEEN

$$H^i(\mathbb{R}P^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \le i \le n \\ 0 & i > n \end{cases}$$

LET $0 \neq x \in H^1$. THEN $x^i$ GENERATES $H^i(\mathbb{R}P^n;\mathbb{Z}/2)$. HENCE AS A GRADED RING

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{n+1})$$

TRUNCATED POLYNOMIAL RING OF HEIGHT $n$.

2) $$H^i(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{FOR } i \text{ EVEN AND } 0 \le i \le 2n \\ 0 & \text{ELSE} \end{cases}$$

LET $x$ BE A GENERATOR OF $H^2$. THEN $x^i$ GENERATE $H^{2i}$ FOR $0 \le i \le n$.

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$$

3) LET $k$ BE A FIELD. THEN

$$H^*(X \times Y; k) \cong H^*(X;k) \otimes H^*(Y;k)$$

AS GRADED RINGS

COROLLARY

$\mathbb{C}P^2$ AND $S^2 \vee S^4$ ARE

NOT HOMOTOPY EQUIVALENT.

PROOF: $H_*(CP^2)$ IS ISO TO $H_*(S^2 \vee S^4)$ AS ABELIAN GPS BUT NOT AS RINGS.

$H^*(CP^2;\mathbb{Z}) = \mathbb{Z}[x]/(x^3)$

THE GENERATOR OF $H^4$ IS THE SQUARE OF THAT OF $H^2$.

FOR $H^*(S^2 \vee S^4)$ WE HAVE $x_1 \in H^2$ AND $x_2 \in H^4$

$$\begin{tikzcd} S^2 \arrow[r] \arrow[rr, bend right, "\text{IDENTITY}"'] & S^2 \vee S^4 \arrow[r] & S^2 \end{tikzcd}$$

$$\begin{array}{ccccc} H^2(S^2) & \longleftarrow & H^2(S^2 \vee S^4) & \longleftarrow & H^2 S^2 \\ & & & & \rotatebox{90}{$\in$} \\ & & x_1 & & x_1 \\ & & x_1^2 = 0 & \longleftarrow\!\shortmid & x_1^2 \in H^4(S^2) = 0 \end{array}$$

QED

BOTH SPACES HAVE CW STRUCTURES WITH A SINGLE CELL IN DIMENSIONS 0, 2, AND 4.

IN $S^2 \vee S^4$, THE ATTACHING MAP FOR THE 4-CELL $S^3 \to S^2$ IN NULL

IN $\mathbb{C}P^2$, IT IS THE HOPF MAP $S^3 \xrightarrow{\eta} S^2$

IN FACT $\eta$ GENERATES $\pi_3(S^2) \cong \mathbb{Z}$. PROVED BY HOPF IN 1930

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CLOCK PROBLEM

HOW MANY TIMES IN A 12 HOUR PERIOD DO THE MINUTE AND HOUR HANDS POINT IN THE SAME DIRECTIONS.

ANSWER: 11 (ELEMENTARY)

TOPOLOGICAL APPROACH:

THE POSITION OF EACH HAND IS A POINT ON $S^1$

TOGETHER THEY DEFINE A POINT IN $S^1 \times S^1$, THE TORUS.

THEIR MOTION OVER 12 HOURS DETERMINES A CERTAIN CLOSED PATH IN $S^1 \times S^1$.

CALL THIS THE <u>CLOCK PATH</u>

THERE IS ALSO A <u>DIAGONAL PATH</u> $S^1 \longrightarrow S^1 \times S^1$

$$\boxed{\text{AT HOW MANY POINTS DO THESE PATHS INTERSECT?}}$$

## BIG IDEA: POINCARE DUALITY

PD THEOREM LET $M$ BE A CLOSED ORIENTED $n$-DIMENSIONAL MANIFOLD

e.g. $M = S^1 \times S^1$ FOR $n=2$.

THEN THERE ISOMORPHISMS

$$H^i(M) \cong H_{n-i}(M).$$

SUPPOSE $\alpha \in H^i$, $\beta \in H^j$ WITH $i+j \leq n$

$$\alpha\beta \in H^{i+j}$$

$\alpha$ AND $\beta$ ARE POINCARÉ DUAL TO CLASSES $a \in H_{n-i}(M)$ AND $b \in H_{n-j}(M)$

$\alpha\beta$ IS DUAL TO SOME CLASS

$$c \in H_{n-i-j}(M).$$

HOW IS $c$ RELATED TO $a$ AND $b$.

IF $N$ IS - CONNECTED ORIENTED CLOSED $k$-MANIFOLD THEN $H_k(N) = \mathbb{Z}$ GENERATED BY A CLASS $[N]$.

SUPPOSE THERE ARE MANIFOLDS $A$ OF DIMENSION $n-i$ AND

B $\quad \shortparallel \quad$ $n-j$

AND MAPS $A \hookrightarrow M \hookleftarrow B$

$[A] \longmapsto a$

$b \longleftarrow [B]$

$a \in H_{n-i} M \qquad b \in H_{n-j}(M)$

SUPPOSED THAT A AND B ARE SUCH THAT $A \cap B$ IS SUBMANIFOLD OF M OF DIMENSION $n-i-j$.

WE GET A CLASS

$$[A \cap B] \in H_{n-i-j}(M) \cong H^{i+j}(M) \ni \alpha\beta$$

$$[A] = a \in H_{n-i}(M) \cong H^i M \ni \alpha$$

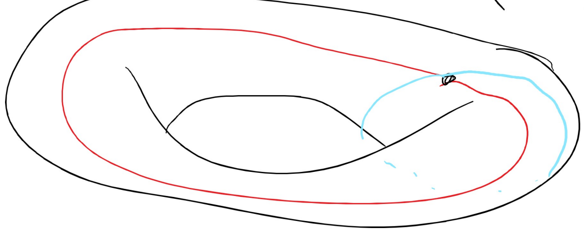
$$[B] = b \in H_{n-j}(M) \cong H^j M \ni \beta$$

**CUP PRODUCTS ARE POINCARE DUAL TO INTERSECTIONS**

## APPLICATION TO CLOCK PROBLEM:

$$H_i(S^1 \times S^1) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} \oplus \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & \text{ELSE} \end{cases}$$

$T = S^1 \times S^1$



$x \quad y \in H_1(T)$

CLOCK PATH

$x + 12y$

DIAGONAL PATH

$x + y.$

LET $\alpha, \beta \in H^1 T$ BE THE POINCARE DUALS OF $x$ AND $y$.

CLOCK PATH IS DUAL $\alpha + 12\beta$

DIAGONAL " $\alpha + \beta$

CAN SHOW $\alpha^2 = 0$, $\beta^2 = 0$ AND $\alpha\beta$ GENERATES $H^2 T$.

WE WANT TO COMPUTE

$$(\alpha + \beta)(\alpha + 12\beta)$$

$$= \cancel{\alpha^2} + 12\alpha\beta + \beta\alpha + \cancel{12\beta^2}$$

$$= 12\alpha\beta - \alpha\beta = 11\alpha\beta$$

INTERSECTION NUMBER.

IF WE DIDN'T KNOW THAT $\beta\alpha = -\alpha\beta$, WE WOULD $13\alpha\beta$ AS OUR ANSWER.

THIS ILLUSTRATES THE NECESSITY OF THE SIGN IN THE CUP PRODUCT.

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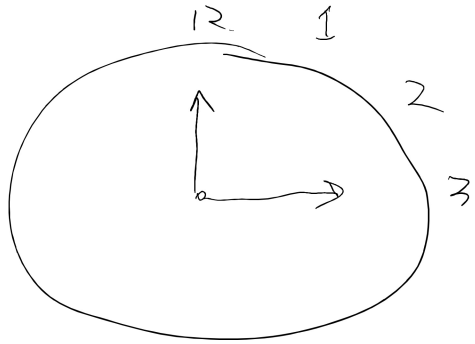
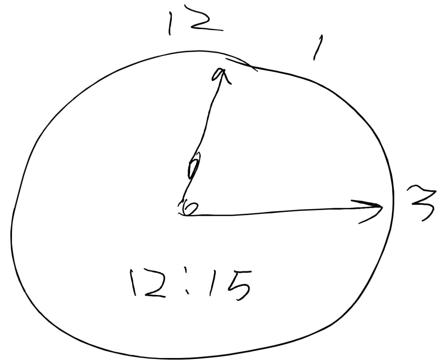
ANOTHER CLOCK PROBLEM.

SUPPOSE THE MINUTE AND HOUR HANDS WE IDENTICAL

3:00




3:01 ?

12:16 ?

AMBIGUOUS MOMENT

HOW MANY SUCH MOMENTS ARE THERE ?

THE TORUS HAS THE CLOCK PATH $x+12y$

AND THE ANTICLOCK PATH $12x+y$

WHERE THEY INTERSECT, WE CANNOT TELL WHICH IS WHICH.

EMAIL ME THE ANSWER

FINAL EXAM
4:00 PM MONDAY 12/19
IN ROOM 1104.