THEOREM: GIVEN A SES OF R-MODULES,

$$0 \to M' \to M \to M'' \to 0,$$

THERE LONG EXACT SEQUENCES

$$\cdots \to \mathrm{Tor}_i^R(M',N) \to \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M'',N) \to \mathrm{Tor}_{i-1}^R(M',N) \to \cdots$$

AND

$$\cdots \leftarrow \mathrm{Ext}_R^i(M',N) \leftarrow \mathrm{Ext}_R^i(M,N) \leftarrow \mathrm{Ext}_R^i(M'',N) \leftarrow \mathrm{Ext}^{i-1}(M',N) \leftarrow \cdots$$

PROOF: WE WILL SHOW THERE IS A SES OF PROJECTIVE RESOLUTIONS

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

OF M', M AND M''. THESE LEAD TO SES'S OF (CO)CHAIN COMPLEXES

$$0 \to P'_\bullet \otimes_R N \to P_\bullet \otimes_R N \to P''_\bullet \otimes_R N \to 0$$

AND

$$0 \leftarrow \mathrm{Hom}_R(P_\bullet', N) \leftarrow \mathrm{Hom}_R(P_\bullet, N) \leftarrow \mathrm{Hom}_R(P_\bullet'', N) \leftarrow 0$$

THESE LEAD TO LONG EXACT SEQUENCES OF Tor AND Ext GROUPS.

SUPPOSE $P_\bullet'$ AND $P_\bullet''$ ARE PROJECTIVE RESOLUTIONS OF M' AND M''.

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] & M \arrow[r, "\beta", "\text{ONTO}"'] & M'' \arrow[r] & 0 \\
0 \arrow[r] & P_0' \arrow[u, "d_0'"] \arrow[r] \arrow[ur, "f_0", "\alpha d_0'"'] & P_0' \oplus P_0'' \arrow[r] \arrow[u, dashed, "d_0"] & P_0'' \arrow[r] \arrow[u, "d_0''"'] \arrow[ul, dashed, "g_0"'] & 0
\end{tikzcd}$$

$\beta g_0 = d_0''$

$$\begin{tikzcd}
0 \arrow[r] & \ker d_0' \arrow[r, "\alpha"] & \ker d_0 \arrow[r, "\beta", "\text{ONTO}"'] & \ker d_0'' \arrow[r] & 0 \\
0 \arrow[r] & P_1' \arrow[u, "\text{ONTO}"] \arrow[r] \arrow[ur, "f_1"] & P_1' \oplus P_1'' \arrow[r] \arrow[u, "d_1"] & P_1'' \arrow[r] \arrow[u, "\text{ONTO}"'] \arrow[ul, dashed, "g_1"'] & 0
\end{tikzcd}$$

$\exists g_0$ BECAUSE P_0'' IS PROJECTIVE

$d_0 = f_0 \oplus g_0$ IS ONTO

$\exists g_1$ BECAUSE P_1'' IS PROJ

CAN REPEAT THIS INFINITELY MANY TIMES AND GET

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & {} \\
0 \arrow[r] & P_\bullet' \arrow[u] \arrow[r] & P_\bullet \arrow[u] \arrow[r] & P_\bullet'' \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

WHERE $P_1 \cong P_1' \oplus P_1''$, BUT

$d_1 \neq d_1' \oplus d_1''$ BECAUSE

$\ker d_{n-1} \neq \ker d'_{n-1} \oplus \ker d''_{n-1}$.

THIS OUR SES OF PROJECTIVE RESOLUTIONS. THE THEOREM FOLLOWS. QED

THE ABOVE WORKS FOR ANY RING R, BUT WE ONLY NEED IT A PID SUCH AS $\mathbb{Z}$, $\mathbb{Q}$, AND $\mathbb{Z}/(n)$.

UNIVERSAL COEFFICIENT THEOREM.

LET C BE A CHAIN COMPLEX OF FREE R-MODULES FOR A P.I.D. R (SUCH AS $\mathbb{Z}$). FOR ANY R-MODULE N, THERE A SHORT EXACT SEQUENCE

$$0 \to \underbrace{H_n(C) \otimes_R N}_{\text{NAIVE GUESS}} \to H_n(C \otimes_R N) \to \underbrace{\mathrm{Tor}_1^R(H_{n-1}(C), N)}_{\text{ERROR TERM}} \to 0$$

THIS SES IS SPLIT, I.E

$$H_n(C \otimes_R N) \cong H_n(C) \otimes_R N \oplus \mathrm{Tor}_1^R(\dots)$$

REMARKS

① FOR $C = S(X)$ WE GET A DESCRIPTION OF $H_*(X; A)$ — $R = \mathbb{Z}$, $N = A$ = ABELIAN GP

② EXAMPLE:

$R = \mathbb{Z}$, $C_0 = C_1 = \mathbb{Z}$, $C_i = 0$ FOR $i > 1$

$$0 \leftarrow \underset{C_0}{\mathbb{Z}} \xleftarrow{2} \underset{C_1}{\mathbb{Z}} \leftarrow 0$$

$$H_i(C) = \begin{cases} \mathbb{Z}/2 & i = 0 \\ 0 & i = 1 \\ 0 & i > 1 \end{cases}$$

$A = \mathbb{Z}/2$

$C \otimes \mathbb{Z}/2$

$$0 \leftarrow \underset{C_0 \otimes \mathbb{Z}_2}{\mathbb{Z}/2} \xleftarrow{0} \underset{C_1 \otimes \mathbb{Z}/2}{\mathbb{Z}/2} \leftarrow 0 \leftarrow \cdots$$

$$H_i(C \otimes \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & i > 0 \end{cases}$$

$\ne$

$H_*(C) \otimes \mathbb{Z}/2$ UCT SAYS

$$H_1(C \otimes \mathbb{Z}_2) = (H_1(C) \otimes \mathbb{Z}/2) \oplus \mathrm{Tor}^1(H_0(C), \mathbb{Z}/2)$$

$$= (0 \otimes \mathbb{Z}/2) \oplus \mathrm{Tor}^1(\mathbb{Z}/2, \mathbb{Z}/2)$$

$$= 0 \oplus \mathbb{Z}/2 = \mathbb{Z}/2$$

PROOF: IN ANY CHAIN COMPLEX

$$\cdots \leftarrow C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1} \leftarrow \cdots$$

$C_n \supset Z_n = \ker d_n$ = GROUP OF n-CYCLES

$\parallel$ . .

$B_n = \mathrm{im}\, d_{n+1}$ = GROUP n-BOUNDARIES

$H_n(C) := Z_n / B_n$

FOR EACH n $\exists$ A SES

$$0 \to Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

LET Z DENOTE THE CHAIN COMPLEX WHOSE nTH CHAIN GP IS $Z_n = \ker d_n$, WITH TRIVIAL BOUNDARY OPERATOR

LET B DENOTE THE CHAIN CX WHOSE nTH GROUP IS $B_{n-1} = C_n / Z_n$, WITH TRIVIAL BOUNDARY OPERATOR

THIS MEANS WE HAVE A SES OF CHAIN COMPLEXES

(1) $$0 \longrightarrow Z \longrightarrow C \longrightarrow B \longrightarrow 0$$

THE LONG EXACT SEQ IN H_*

$$\cdots \to Z_n \xrightarrow{0} H_n(C) \longrightarrow B_{n-1} \xrightarrow{\text{INCLUSION MAP}} Z_{n-1} \xrightarrow{\text{ONTO}} H_{n-1}(C) \xrightarrow{0} \cdots$$

NOTE $H_{n-1}(C) = Z_{n-1} / B_{n-1}$

BY DEFINITION, OUR LES DECOMPOSES INTO SESs

(2) $0 \to B_n \to Z_n \to H_n(C) \to 0$

WANT TO TENSOR (2) WITH N. WE GET A LES OF Tor GROUPS. SINCE B_n AND Z_n ARE FREE R-MODULES, $\mathrm{Tor}_i^R(B_n, N)$ AND $\mathrm{Tor}_i^R(Z_n, N)$ VANISH FOR $i > 0$. OUR LES REDUCES TO

$$0 \to \mathrm{Tor}_1^R(H_n(C), N) \to B_n \otimes N \xrightarrow{d_{n+1} \otimes N} Z_n \otimes N \to H_n(C) \otimes N \to 0$$

HENCE $\mathrm{Tor}_1^R(H_n(C), N) = \ker(d_{n+1} \otimes N)$

AND $\mathrm{Tor}_0^R(H_n(C), N) = \operatorname{coker}(d_{n+1} \otimes N)$

$$\mathrm{Tor}_0^R(H_n(C), N) = H_n(C) \otimes_R N$$

SINCE (1) IS A SES OF CHAIN COMPLEXES OF FREE R-MODULES, TENSOR IT WITH N PRESERVES EXACTNESS

WE GET

$$0 \to Z \otimes_R N \longrightarrow C \otimes_R N \longrightarrow B \otimes_R N \to 0$$

WE KNOW $H_n(Z \otimes_R N) = Z_n \otimes_R N$

AND $H_n(B \otimes N) = B_{n-1} \otimes_R N$

THE LES IN H_* IS

MYSTERY

$$\cdots \to Z_n \otimes_R N \xrightarrow{d_n \otimes N} \boxed{H_n(C \otimes_R N)} \to B_{n-1} \otimes N \to Z_{n-1} \otimes_R N \to H_{n-1}(C \otimes_R N) \to \cdots$$

IT FOLLOW THAT $\exists$ SES

$$0 \to \operatorname{coker} d_{n+1} \otimes N \to H_n(C \otimes_R N) \to \ker d_n \otimes N \to 0$$

$$\operatorname{coker} d_{n+1} \otimes N = H_{n+1}(C) \otimes N \qquad \ker d_n \otimes N = \operatorname{Tor}_1^R(H_n C, N)$$

QED

MAKEUP CLASSES

THURSDAY 12/15 2:00 } ZOOM
FRIDAY 12/16 3:30 } ZOOM

FINAL 12/19 MONDAY 4:00
IN 1104