GOAL FOR TODAY:
DESCRIBE $H_*(X \times Y)$ IN TERM OF $H_* X$ AND $H_* Y$.

UNIVERSAL COEFFICIENT THEOREM

LET C BE A CHAIN CX OF FREE ABELIAN GROUPS, AND LET A BE ANY ABELIAN GROUP. THEN THERE IS A SHORT EXACT SEQUENCE

$$0 \to \underbrace{\boxed{H_n(C) \otimes A}}_{\text{NAIVE GUESS}} \longrightarrow H_n(C \otimes A) \longrightarrow \underbrace{\boxed{\mathrm{Tor}_1(H_{n-1}(C), A)}}_{\text{ERROR TERM}} \to 0$$

KÜNNETH THEOREM

LET C' AND C'' BE CHAIN CXS OF FREE ABELIAN GPS. THEN THERE IS SES

$$0 \to \bigoplus_{0 \le i \le n} H_i(C') \otimes H_{n-i}(C'') \to H_n(C' \otimes C'') \to \bigoplus \mathrm{Tor}_1\left(H_i(C'), H_{n-i-1}(C'')\right) \to 0$$

SUPPOSE $C' = S(X)$ = SINGULAR CHAIN CX FOR X

$C'' = S(Y) =$ "

THEN WE HAVE A DESCRIPTION

THEN WE HAVE A DESCRIPTION OF $H_*(S(X)\otimes S(Y))$ IN TERMS OF $H_*(X)$ AND $H_*(Y)$.

NOTE
$$\begin{tikzcd} S(X\times Y) \arrow[r, phantom, "\neq"] & S(X)\otimes S(Y) \arrow[l, bend left, "\phi"'] \end{tikzcd}$$

THERE IS A MAP AS INDICATED WHICH IS KNOWN TO BE A CHAIN HOMOTOPY EQUIVALENCE (EILENBERG-ZILBER THEOREM).

I WILL NOT PROVE IT - SEE HATCHER.

TO DESCRIBE THE MAP ϕ, LET

$$\Delta^i \xrightarrow{\sigma'} X \quad \text{AND} \quad \Delta^j \xrightarrow{\sigma''} Y$$

$v_0, v_1, \dots v_i \leftarrow$ VERTICES $\rightarrow w_0, \dots w_j$

WANT A MAP

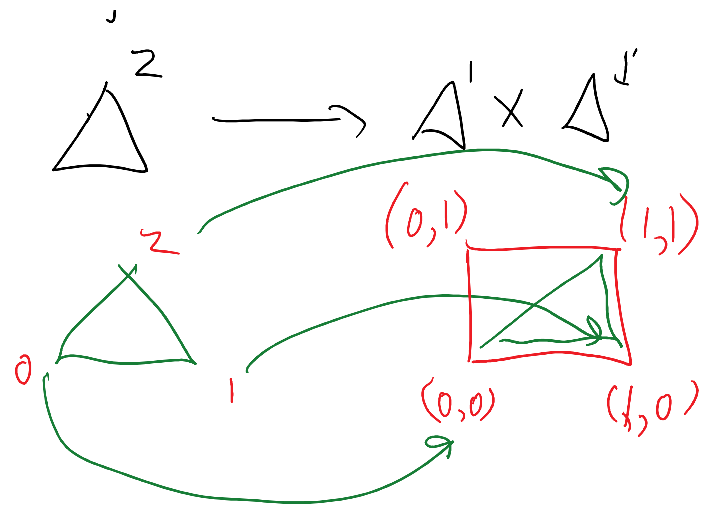
$$\Delta^{i+j} \xrightarrow{\phi_{i,j}} \Delta^i \times \Delta^j \xrightarrow{\sigma' \times \sigma''} X \times Y$$

$u_0, \dots u_{i+j}$ VERTICES

$$u_k \longmapsto \begin{cases} (v_k, w_0) & \text{FOR } 0 \le k \le i \\ (v_i, w_{k-i}) & \text{FOR } i \le k \le i+j. \end{cases}$$

FOR $i = j = 1$

$$\Delta^2 \quad \to \quad \Delta^1 \times \Delta^1$$



THIS LEADS TO A CHAIN MAP

$$S(X) \otimes S(Y) \longrightarrow S(X \times Y)$$

INSTEAD OF PROVING THE E-Z THEOREM, SUPPOSE X AND Y ARE CW-COMPLEXES.

DEF A CW-COMPLEX IS A SPACE X CONSTRUCTED AS A UNION OF SUBSPACES (CALLED SKELETA)

$$X^0 \to X^1 \to X^2 \to \cdots \to X$$

WHERE

① X^0 IS DISCRETE

② FOR $n > 0$ X^n IS OBTAINED FROM X^{n-1} AS FOLLOWS.

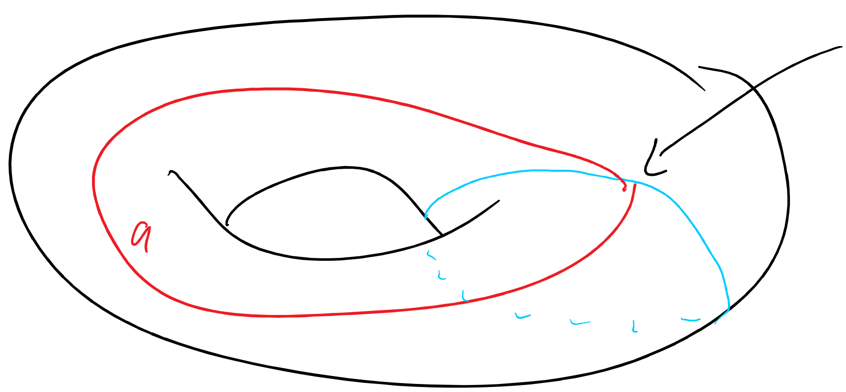
THERE IS A SET K_n $(K_0 = X^0)$ AND A DIAGRAM

$$\begin{tikzcd}
K_n \times S^{n-1} \arrow[r, "f_n", "\text{nTH ATTACHING MAP}"'] \arrow[d, "K_n \times i_n"'] & X^{n-1} \arrow[d] \\
K_n \times D^n \arrow[r] & X_n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \quad X_n = \text{PUSHOUT}$$

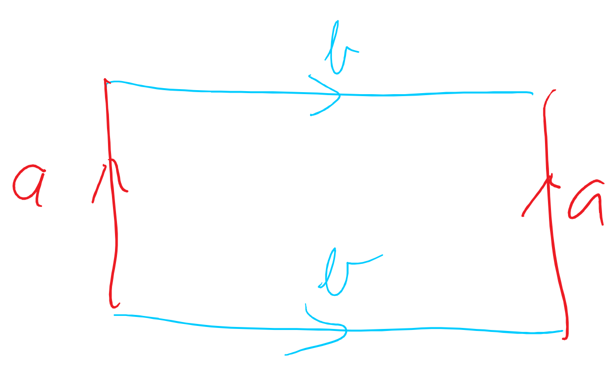
ATTACHING n-CELLS

$i_n : S^{n-1} \to D^n$ IS INCLUSION OF BOUNDARY

EXAMPLE $X =$ TORUS



$X^1 =$ UNION OF RED + BLUE CIRCLES



ONE 0-CELL X^0

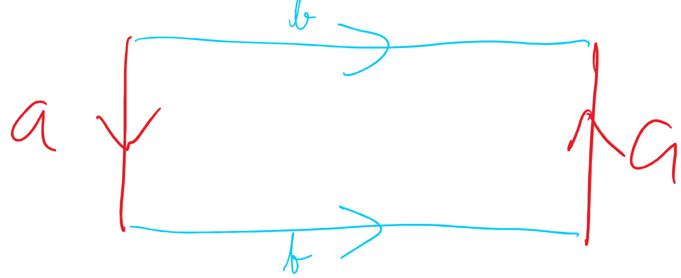
TWO 1-CELLS (RED + BLUE CIRCLES)

ONE 2-CELL

(RECTANGLE)

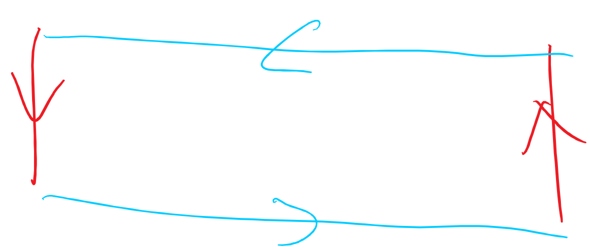
EXAMPLE: KLEIN BOTTLE

SAME 1-SKELETON AS ABOVE



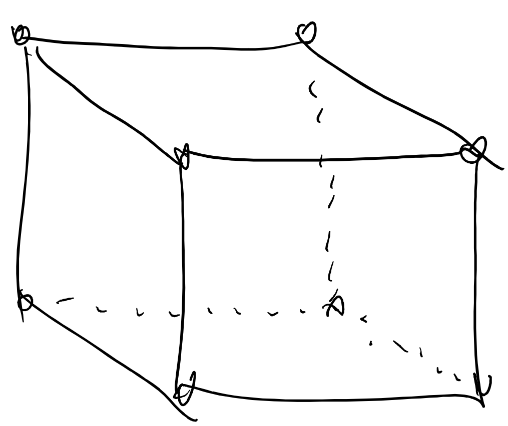
EXAMPLE: PROJECTIVE $\mathbb{R}P^2$

SAME 1-SKELETON



EXAMPLE S^2 AS A CW-CX

$S^2 \cong$



$X^0 = 8$ VERTICES

$X^1 = X^0 \cup 12$ EDGES

$X^2 = X^1 \cup 6$ FACES

TO EACH CW-COMPLEX X

WE HAVE A CELLULAR CHAIN

COMPLEX $C(X)$ WITH

$$H_*(C(X)) \cong H_*(S(X)) =: H_*(X)$$

$C_n(X)$ = FREE ABELIAN GP ON THE K_n OF n-CELLS

TO DEFINE BOUNDARY OPERATOR

$$\begin{array}{ccc} C_n(X) & \xrightarrow{d_n} & C_{n-1}(X) \\ \| & & \| \\ \mathbb{Z}[K_n] & & \mathbb{Z}[K_{n-1}], \end{array}$$

CONSIDER THE SPACES

$$\begin{array}{ccc} X^n/X^{n-1}, & X^{n-1}/X^{n-2}, & \text{AND } X^n/X^{n-2} \\ \cong & \cong & \uparrow \\ \bigvee_{K_n} S^n & \bigvee_{K_{n-1}} S^{n-1} & \end{array}$$

ONE 0-CELL

K_{n-1} $(n-1)$-CELLS

K_n n-CELLS

$$X^{n-1}/X^{n-2} \longrightarrow X^n/X^{n-2} \longrightarrow X^n/X^{n-1}$$

FACTS

$$H_{n-1}(X^{n-1}/X^{n-2}) = C_{n-1}(X)$$

$$H_n(X^n/X^{n-1}) = C_n(X)$$

$$C_n(X^n/X^{n-2}) = C_n(X)$$

$$C_n(X^n/X^{n-2}) \to C_{n-1}(X)$$

THERE A LONG EXACT OF HOMOLOGY GROUPS OF THESE THREE SPACES (EASY TO PROVE)

$$\begin{tikzcd}
H_n(X^{n-1}/X^{n-2}) \arrow[r] \arrow[d, equal] & H_n(X^n/X^{n-2}) \arrow[r] \arrow[d, equal] & H_n(X^n/X^{n-1}) \arrow[d, equal] \arrow[ddll, bend left, "d_n"'] \\
0 & ? & C_n(X) \\
H_{n-1}(X^{n-1}/X^{n-2}) \arrow[r] \arrow[d, equal] & H_{n-1}(X^n/X^{n-2}) \arrow[r] \arrow[d, equal] & H_{n-1}(X^n/X^{n-1}) \arrow[d, equal] \\
C_{n-1}(X) & ? & 0
\end{tikzcd}$$

THIS DEFINES d_n

<u>THEOREM</u> $H_*(C(X)) \cong H_*(X)$

PROOF SKETCH: SHOW BY INDUCTION ON n THAT

$$H_* C(X^n) \cong H_*(X^n)$$

USING LESS FOR THE PAIR (X^n, X^{n-1}).

SUPPOSE X AND Y ARE CW-COMPLEXES. THEN $X \times Y$ HAS A CW-STRUCTURE AS FOLLOWS

$$(X \times Y)^n = \bigcup_{0 \le i \le n} X^i \times Y^{n-i}$$

IT TURNS OUT THAT

$$C(X \times Y) \cong C(X) \otimes C(Y)$$

THE KÜNNETH THEOREM RELATES $H_*(C(X \times Y)) = H_*(X \times Y)$ TO $H_*(C(X)) = H_* X$ AND $H_*(C(Y)) = H_* Y$

TO PROVE THAT $X \times Y$ HAS A CW-STRUCTURE AS CLAIMED, NOTE

(1) $D^n \cong D^i \times D^{n-i}$ FOR $0 \le i \le n$

(2) SUPPOSE X AND Y ARE MANIFOLDS WITH BOUNDARIES ∂X AND ∂Y, THEN

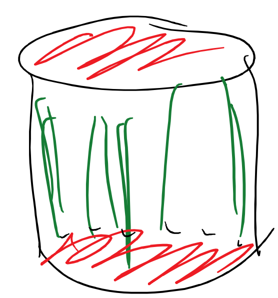
$$\partial(X \times Y) = \underbrace{(\partial X)\times Y} \cup \underbrace{X \times (\partial Y)}$$

EACH BOUNDED BY $(\partial X)\times(\partial Y)$

$$\partial(D^m \times D^n) = (\partial D^m)\times D^n \cup D^m \times (\partial D^n)$$

$$S^{m+n-1} = S^{m-1}\times D^n \cup D^m \times S^{n-1}$$

$m=1$, $n=2$ $\quad D^1\times D^2$



$$\partial(D^1\times D^2) = (\partial D^1)\times D^2 \cup D^1\times \partial D^2$$

$$= \underline{(S^0\times D^2)} \cup \underline{(D^1\times S^1)}$$