WHAT IS BEHIND THE BELT AND COFFEE CUP TRICKS.

LET $SO(3)$ BE THE SET 3×3 ORTHOGONAL MATRICES

THE MOTION OF THE COFFEE CUP DETERMINE A CLOSED PATH IN $SO(3) \subset \mathbb{R}^9$

AS DOES THE BELT

IN THE BELT TRICK WE ARE CONTINUOUSLY DEFORMING THE "2 TWIST PATH" TO THE "NO TWIST PATH"

THE COFFEE CUP PATH IN $SO(3)$ CAN CONTINUOUSLY DEFORMED TO A CONSTANT.

DEFINITIONS: $I := [0,1]$ = CLOSED UNIT INTERVAL

A <u>PATH</u> IN A TOP. SPACE X IN A CONTINUOUS MAP $\phi: I \longrightarrow X$

THE PATH STARTS AT $\phi(0) \in X$ AND ENDS AT $\phi(1) \in X$

AND ENDS AT $p(1) \in X$

p IS CLOSED IF $p(0) = p(1)$

DEF: TWO MAPS $f_0, f_1 : X \to Y$ ARE HOMOTOPIC IF THERE IS A MAP $h : I \times X \to Y$, $\boxed{f_0 \simeq f_1}$ SUCH $h(0, x) = f_0(x)$ $h(1, x) = f_1(x)$

h IS CALLED A HOMOTOPY BETWEEN f_0 AND f_2

NOTE FOR $t \in I$ WE GET A MAP $f_t : X \to Y$ DEFINED BY

$$f_t(x) = h(t, x).$$

VARIATIONS

LET $x_0 \in X$ AND $y_0 \in Y$ (BASE POINTS)

A BASE POINT PRESERVING MAP IS $f : X \to Y$ WITH $f(x_0) = y_0$

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

LET $A \subset X$ AND $B \subset Y$ BE SUBSPACES

$h : (X, A) \longrightarrow (Y, B)$ DENOTES

$f : (X,A) \longrightarrow (Y,B)$ DENOTES A MAP $X \xrightarrow{f} Y$ WITH $f(A) \subset B$.

LET $\boxed{\partial} I = \{0,1\} \subset I$

BOUNDARY

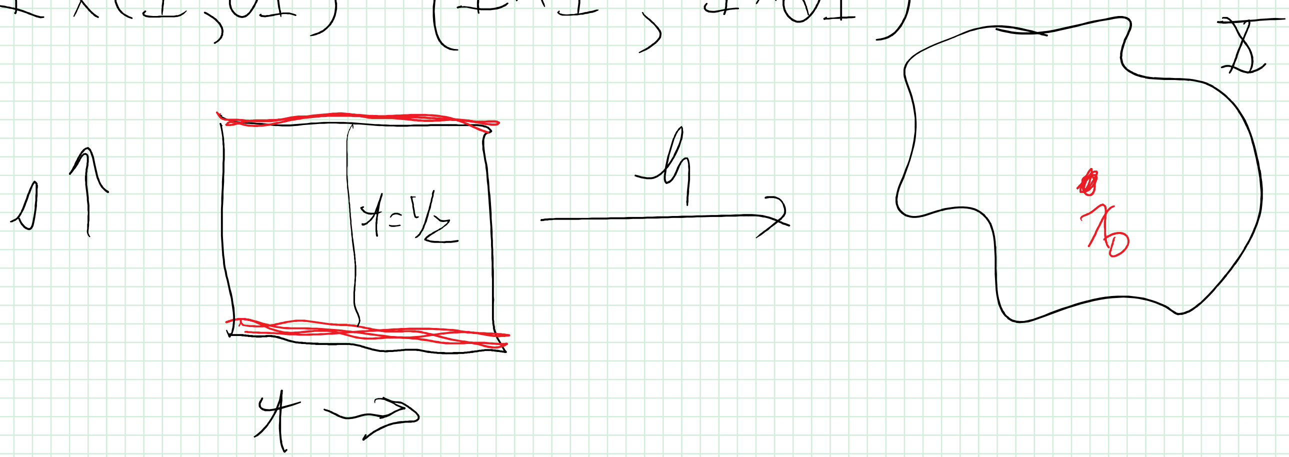
A <u>CLOSED PATH</u> IN X AT $x_0 \in X$ IS A MAP $(I, \partial I) \xrightarrow{\phi} (X, x_0)$

TWO SUCH PATHS ϕ_0, ϕ_1 ARE <u>HOMOTOPIC</u> IF $\exists$

$$I \times (I, \partial I) \xrightarrow{h} (X, x_0)$$

HOMOTOPY PARAMETER t PATH PARAMETER s

$$I \times (I, \partial I) = (I \times I, I \times \partial I)$$



HOMOTOPY DEFINES AN

HOMOTOPY DEFINES AN EQUIVALENCE RELATION AMONG

$$X \to Y \quad \text{OR} \quad (X,A) \to (Y,B),$$

SO THERE IS A SET OF EQUIVALENCE CLASSES.

$$[X,Y] \quad \text{OR} \quad [(X,A),(Y,B)]$$

IN PARTICULAR

$$[(I,\partial I),(X,x_0)] =: \pi_1(X,x_0)$$

= SET OF HOMOTOPY CLASSES OF CLOSED PATHS IN X AT x_0

SPOILER

$$[(I^n,\partial I^n),(X,x_0)] =: \pi_n(X,x_0)$$

THEOREM

THE SET $\pi_1(X,x_0)$ HAS A NATURAL GROUP STRUCTURE.

THE WORD "NATURAL" MEANS

GIVEN $(X,x_0) \xrightarrow{f} (Y,y_0)$

INDUCES $\pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$

CLOSED PATH h IN X

$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

CLOSED PATH IN X

$$\begin{tikzcd}(I, \partial I) \arrow[r, "p"] \arrow[rr, bend right, "\text{CLOSED PATH IN } Y"'] & (X, x_0) \arrow[r, "f"] & (Y, y_0)\end{tikzcd}$$

WHICH IS A GROUP HOMOMORPHISM

TO GET A GROUP STRUCTURE WE NEED A BINARY OPERATION ON $\pi_1(X, x_0)$.

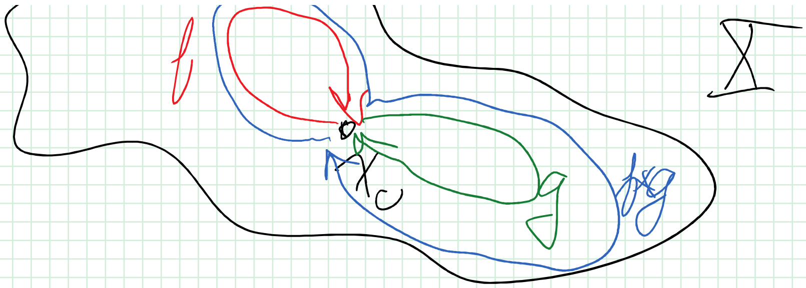
LET $f, g : (I, \partial I) \to (X, x_0)$

THEN $(f * g)(s) = \begin{cases} f(2s) & \text{FOR } 0 \leq s \leq 1/2 \\ g(2s-1) & \text{FOR } 1/2 \leq s \leq 1 \end{cases}$

NOTE $(f*g)(0) = f(0) = x_0$

$(f*g)(1) = g(1) = x_0$

$(f*g)(1/2) = \begin{cases} f(1) = x_0 \\ g(0) = x_0 \end{cases}$



EASY THAT IF $f \simeq f'$ AND $g \simeq g'$
THEN $f*g \simeq f'*g'$
SO WE HAVE A BINARY OPERATION
ON $\pi_1(X,x_0)$ AS WELL AS
ON $Map((I,\partial I),(X,x_0))$,
THE SET OF ALL CLOSED PATHS

THE IDENTITY ELEMENT IS
THE HOMOTOPY CLASS OF THE
CONSTANT PATH $(I,\partial I) \xrightarrow{e} (X,x_0)$
NEED TO SHOW
$e*f \simeq f \simeq f*e$ EASY EXERCISE
(NOTE $e*f \neq f \neq f*e$)

INVERSES

GIVEN $f:(I,\partial I) \to (X,x_0)$

GIVEN $f : (I, \partial I) \to (X, x_0)$

DEFINE $\bar{f} : (I, \partial I) \to (X, x_0)$

BY $\bar{f}(s) = f(1-s)$

NEED TO SHOW

$$\bar{f} * f \simeq e \simeq f * \bar{f}$$

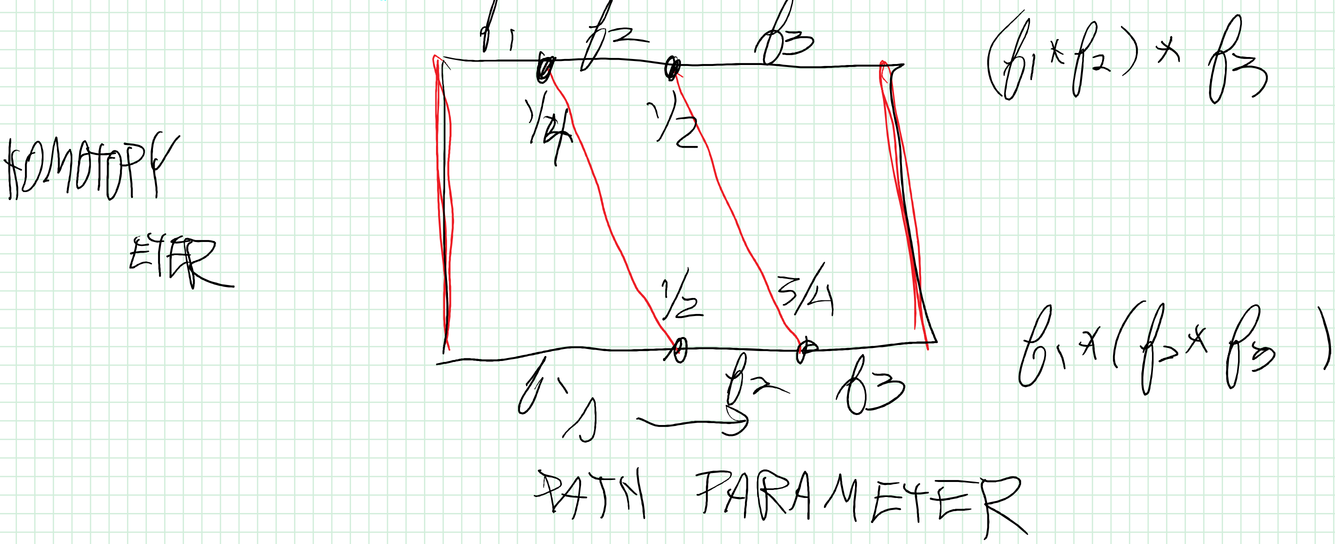
EASY EXERCISE

ASSOCIATIVITY

LET $f_1, f_2, f_3 : (I, \partial I) \to (X, x_0)$

NEED TO SHOW

$$(f_1 * f_2) * f_3 \simeq f_1 * (f_2 * f_3)$$



FINAL REMARK

WILL SEE $\pi_1 SO(2) = \pi_1(S^1) \cong \mathbb{Z}$

CIRCLE

INTEGERS

AND $\pi_1(SO(3)) \cong \mathbb{Z}/2$

INTEGERS MOD 2

THE INCLUSION

$$SO(2) \longrightarrow SO(3)$$

INDUCES $\pi_1(SO(2)) \longrightarrow \pi_1 SO(3)$

$$\begin{array}{ccc} \pi_1(SO(2)) & \longrightarrow & \pi_1 SO(3) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow{\text{ONTO}} & \mathbb{Z}/2 \end{array}$$