CATEGORY THEORY

INVENTED IN 1945 BY
SAMUEL EILENBERG + SAUNDERS
MAC LANE.

REVOLUTION LED BY JACOB LURIE

RECALL THE FUNDAMENTAL GROUP
$\pi_1(X, x_0)$ IS A GROUP
ASSOCIATED WITH A POINTED
SPACE (X, x_0) SUCH THAT
FOR ANY MAP $(X, x_0) \xrightarrow{f} (Y, y_0)$
WE GET GROUP HOMOMORPHISM

$$\pi_1(X, x_0) \xrightarrow{\pi_1(f)} \pi_1(Y, y_0)$$

$\pi_1(-)$ IS DEFINED IN TERMS OF MAPS

$$(I, \partial I) \xrightarrow{p} (X, x_0) \xrightarrow{f} (Y, y_0)$$

fp IS A CLOSED PATH IN Y.

DEFINITION A **CATEGORY** $\mathcal{C}$
CONSISTS
1) A COLLECTION OR CLASS OF OBJECTS
(WHICH MAY OR MAY NOT BE
A SET) (EXAMPLE $\mathcal{C} = \mathcal{S}et$)
2) FOR EACH PAIR OF OBJECTS
X, Y THERE A SET OF
MORPHISMS $X \longrightarrow Y$.
(EXAMPLE MAPS BETWEEN SETS)
DENOTED BY $\mathcal{C}(X, Y)$
SUCH THAT
a) GIVEN 3 OBJECTS
X, Y, Z THERE IS A
COMPOSITION PAIRING

$$\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)$$

EXAMPLE: X, Y, Z ARE SETS

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$f \in \mathcal{C}(X,Y)$ $\quad g \in \mathcal{C}(Y,Z)$

b) THIS PAIRING IS ASSOCIATIVE, I.E.

GIVEN

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & X \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & Y \arrow[r, "h"] & Z \end{tikzcd},$$

WE REQUIRE $(hg)f = h(gf)$

c) EACH OBJECT X HAS AN IDENTITY MORPHISM

$1_X \in \mathcal{C}(X,X)$ S.T.

GIVEN ANY MORPHISMS

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

$$g\, 1_X = g$$

AND $1_X\, f = f$

EXAMPLES

1) $\mathcal{S}et$ = CATEGORY OF SETS

OBJECTS ARE SETS

MORPHISMS ARE MAPS BETWEEN SETS

2) $\mathcal{T}op$ = CATEGORY OF TOPOLOGICAL SPACES

OBJECTS ARE SPACES

MORPHISMS ARE CONTINUOUS MAPS

3) $\mathcal{G}rp$ = CATEGORY OF GROUPS

OBJECTS ARE GROUPS
MORPHISMS ARE GROUP HOMOMORPHISM

4) $\mathcal{A}b$ = CATEGORY OF ABELIAN GPS

OBJECTS ARE ABELIAN GPS
MORPHISMS ARE GP HOMOMORPHISMS

5) THE CATEGORIES OF FIELDS, RINGS, MODULES OVER A GIVEN RING R

6) EMPTY CATEGORY
NO OBJECTS

7) TRIVIAL CATEGORY
ONE OBJECT WITH IDENTITY MORPHISM

8) LET G BE A GROUP

BG HAS ONE OBJECT X

$BG(X, X)$ = SET OF MORPHISMS $X \to X$

(DEF. A MONOID M IS A SET WITH A BINARY OPERATION WHICH ASSOCIATIVE AND THERE IS AN IDENTITY ELEMENT)

IN WHICH EACH MORPHISM IS INVERTIBLE.

9) LET Y BE A SET ACTED BY A GROUP G.

$BG(Y)$ = CATEGORY

OBJECTS ARE ELTS IN Y

FOR EACH $y \in Y$ AND EACH $\gamma \in G$ THERE IS A

MORPHISM $y \longrightarrow \gamma(y) \in Y$

(10) WALKING ARROW CATEGORY

TWO OBJECTS A, B

$\underline{1}$ $A \longrightarrow B$ ONE MORPHISMS

$\nexists$ NO MORPHISMS $B \longrightarrow A$.

MANY OTHER EXAMPLES

NEW DEFINITION

LET $\mathcal{C}$ AND $\mathcal{D}$ BE CATEGORIES

A FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$ CONSISTS

1) FOR EACH OBJECT X IN $\mathcal{C}$, THERE IS AN OBJECT $F(X)$ IN $\mathcal{D}$

(EXAMPLE : $\mathcal{C}$ = CATEGORY OF POINT SPACES

$\mathcal{D} = \mathcal{G}rp$

$\pi_1 : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FUNCTOR)

2) FOR EACH MORPHISM $X \xrightarrow{f} Y$ IN $\mathcal{C}$, WE GET A MORPHISM

$$F(X) \xrightarrow{F(f)} F(Y) \quad \text{IN } \mathcal{D}$$

S.T.

FOR
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd} \quad \text{IN } \mathcal{C}$$

WE GET
$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(g)F(f)"'] & F(Y) \arrow[r, "F(g)"] & F(Z) \end{tikzcd} \quad \text{IN } \mathcal{D}$$

DIAGRAMS

$F(gf)$

WITH $F(gf) = F(g)\,F(f)$.

$$F(1_X) = 1_{F(X)} \text{ in } \mathcal{D}$$

OBSERVATION:

DIAGRAMS IN A CATEGORY $\mathcal{C}$ CAN REGARDED AS FUNCTORS TO $\mathcal{C}$ FROM A SUITABLE <u>SMALL</u> CATEGORY (MEANING ONE IN WHICH THE COLLECTION OF OBJECTS IS A SET)

EXAMPLE: THE TWO DIAGRAMS ABOVE CIRCLED IN GREEN ARE FUNCTORS FROM

$$\underline{2} \qquad A \longrightarrow B \longrightarrow C \qquad \left(\begin{array}{c}3 \text{ OBJECTS,} \\ 2 \text{ MORPHISMS} \\ \text{AS SHOWN}\end{array}\right)$$

EXAMPLE

$$\underline{1} \times \underline{1} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, "\beta\alpha = \delta\gamma"] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"'] & D \end{tikzcd} \qquad \text{WITH } \beta\alpha = \delta\gamma$$

<u>EXAMPLE</u> A FUNCTOR $\mathcal{B}G \longrightarrow \mathcal{C}$ DEFINES AN OBJECT IN $\mathcal{C}$ WITH A ACTION OF THE GROUP G.