

CATEGORY THEORY

Monday, September 12, 2022 4:03 PM

INVENTED IN 1945 BY
SAMUEL EILENBERG + SAUNDERS
MAC LANE.

REVOLUTION LED BY JACOB LURIE

RECALL THE FUNDAMENTAL GROUP

$\pi_1(X, x_0)$ IS A GROUP
ASSOCIATED WITH A POINTED
SPACE (X, x_0) SUCH THAT

FOR ANY MAP $(X, x_0) \xrightarrow{f} (Y, y_0)$
WE GET GROUP HOMOMORPHISM

$\pi_1(X, x_0) \xrightarrow{\pi_1(f)} \pi_1(Y, y_0)$
 $\pi_1(-)$ IS DEFINED IN TERMS OF MAP

$(I, \partial I) \xrightarrow{f} (X, x_0) \xrightarrow{g} (Y, y_0)$

$f \circ p$ IS A CLOSED PATH IN Y .

DEFINITION A CATEGORY \mathcal{C}

CONSISTS OR CLASS

1) A COLLECTION OF OBJECTS

(WHICH MAY OR MAY NOT BE
A SET) (EXAMPLE $\mathcal{C} = \text{Set}$)

2) FOR EACH PAIR OF OBJECTS

X, Y THERE IS A SET OF

MORPHISMS $X \rightarrow Y$.

(EXAMPLE MAPS BETWEEN SETS)

DENOTED BY $\mathcal{C}(X, Y)$

SUCH THAT

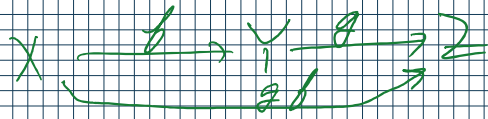
a) GIVEN 3 OBJECTS

X, Y, Z THERE IS A

COMPOSITION PAIRING

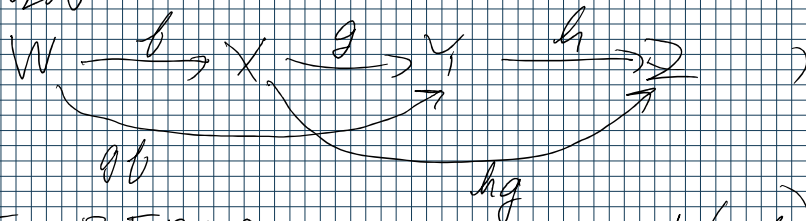
$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z)$$

EXAMPLE: X, Y, Z ARE SETS



$$f \in \mathcal{C}(X, Y) \quad g \in \mathcal{C}(Y, Z)$$

b) THIS PAIRING IS ASSOCIATIVE, I.E. GIVEN



WE REQUIRE $(hg) \circ f = h \circ (gf)$

c) EACH OBJECT X HAS AN IDENTITY MORPHISM

$$1_X \in \mathcal{C}(X, X) \text{ s.t.}$$

GIVEN ANY MORPHISMS

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

$$\begin{aligned}
 g \circ 1_X &= g \\
 \text{AND } 1_X \circ f &= f
 \end{aligned}$$

EXAMPLES

1) $\text{Set} =$ CATEGORY OF SETS
 OBJECTS ARE SETS
 MORPHISMS ARE MAPS
 BETWEEN SETS

2) $\text{Top} =$ CATEGORY OF TOPOLOGICAL SPACES
 OBJECTS ARE SPACES

MORPHISMS ARE CONTINUOUS MAPS

3) Grp = CATEGORY OF GROUPS

OBJECTS ARE GROUPS

MORPHISMS ARE GROUP

HOMOMORPHISM

4) Ab = CATEGORY OF ABELIAN GRPS

OBJECTS ARE ABELIAN GRPS

MORPHISMS ARE GP HOMOMORPHISMS

5) THE CATEGORIES OF FIELDS, RINGS,
MODULES OVER A GIVEN RING R

6) EMPTY CATEGORY
NO OBJECTS

7) TRIVIAL CATEGORY
ONE OBJECT WITH IDENTITY
MORPHISM

8) LET G BE A GROUP

$\mathcal{B}G$ HAS ONE OBJECT X

$\mathcal{B}G(X, X) = \text{SET OF MORPHISMS } X \rightarrow X$

(DEF. A MONOID M IS A SET
WITH A BINARY OPERATION
WHICH ASSOCIATIVE AND THERE
IS AN IDENTITY ELEMENT)

IN WHICH EACH MORPHISM
IS INVERTIBLE.

9) LET Y BE A SET ACTED
BY A GROUP G .

$\mathcal{B}G(Y) = \text{CATEGORY}$

OBJECTS ARE ELTS IN Y
FOR EACH $y \in Y$ AND EACH
 $g \in G$ THERE IS A

MORPHISM $y \rightarrow x(y) \in Y$

(10) WALKING ARROW CATEGORY
TWO OBJECTS A, B

\exists ONE MORPHISM $A \rightarrow B$
 \exists NO MORPHISMS $B \rightarrow A$

MANY OTHER EXAMPLES

NEW DEFINITION

LET \mathcal{C} AND \mathcal{D} BE CATEGORIES
A FUNCTOR $F: \mathcal{C} \rightarrow \mathcal{D}$ CONSISTS

1) FOR EACH OBJECT X IN \mathcal{C} ,
THERE IS AN OBJECT
 $F(X)$ IN \mathcal{D}

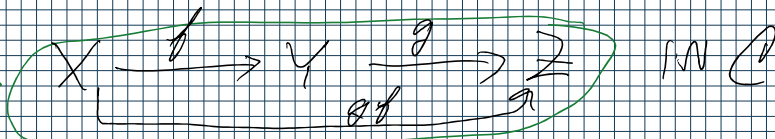
(EXAMPLE: $\mathcal{C} =$ CATEGORY OF POINT SPACES
 $\mathcal{D} = \text{Set}$)

$\mathbb{T}_1: \mathcal{C} \rightarrow \mathcal{D}$ IS A FUNCTOR

2) FOR EACH MORPHISM
 $X \rightarrow Y$ IN \mathcal{C} , WE GET
A MORPHISM

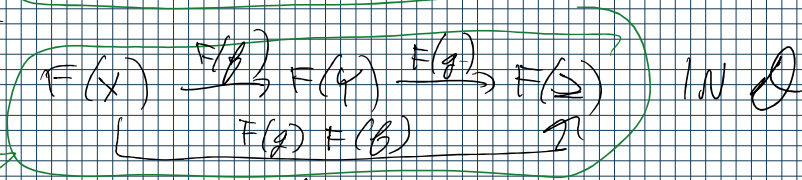
$F(X) \xrightarrow{F(f)} F(Y)$ IN \mathcal{D}

FOR S.T.



WE GET

DIAGRAMS



WITH

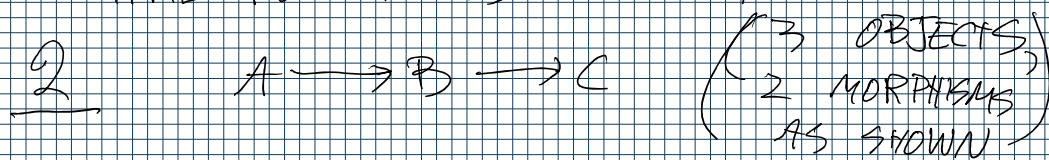
$$F(gf) = F(g)F(f)$$

$$F(I_x) = I_{F(x)} \text{ in } \mathcal{C}$$

OBSERVATION:

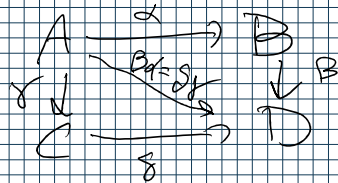
DIAGRAMS IN A CATEGORY \mathcal{C}
 CAN BE REGARDED AS
 FUNCTORS TO \mathcal{C} FROM
 A SUITABLE SMALL
 CATEGORY (MEANING ONE IN
 WHICH THE COLLECTION OF
 OBJECTS IS A SET)

EXAMPLE: THE TWO DIAGRAMS
 ABOVE (CIRCLED) IN GREEN
 ARE FUNCTORS FROM



EXAMPLE

$$\underline{1} \times \underline{1}$$



WITH
 $\beta\alpha = \delta\gamma$

EXAMPLE

A FUNCTOR $\mathbb{B}G \rightarrow \mathcal{C}$

DEFINES AN OBJECT IN \mathcal{C}
 WITH AN ACTION OF THE
 GROUP G .