SOME MORE TOPOLOGY

Wednesday, September 14, 2022 1:29 PM

DEF A CW-COMPLEX X IS A SPACE CONSTRUCTED AS FOLLOWS

$$X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow X^3 \hookrightarrow \cdots\cdots X$$

X^0 = DISCRETE SPACE

X^n IS OBTAINED FROM X^{n-1} — THERE IS A SET K_n AND A MAP

n-SKELETON

$$\begin{tikzcd} K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d] & X^{n-1} \arrow[d] \\ K_n \times D^n \arrow[r] & X^n \end{tikzcd}$$

1

S^{n-1} = UNIT SPHERE IN $\mathbb{R}^n$

$S^{n-1} \subset D^n$ = UNIT BALL IN $\mathbb{R}^n$

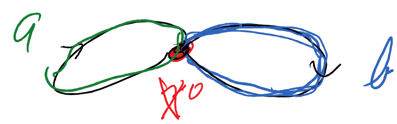
X^n IS OBTAINED FROM X^{n-1} BY ATTACHING SOME n-CELLS

EXAMPLES

1) $X^0 = pt, \cdots = X^{n-1}$

ATTACHING AN n-BALL VIA $S^{n-1} \longrightarrow pt$ GIVES US S^n

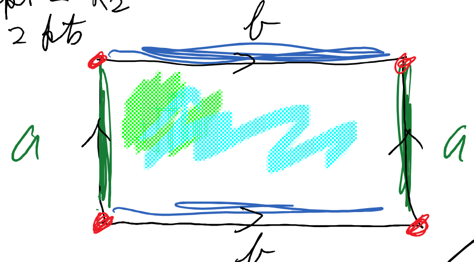
2) $X^0 = pt$. X^1



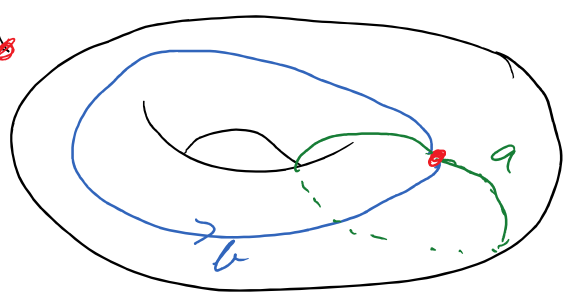
X^1

$K_0 = pt = K_2$

$K_1 = 2\ pts$



X^2 = TORUS



3) n-DIMENSIONAL REAL PROJECTIVE SPACE $\mathbb{R}P^n$

:= SET OF LINES THRU ORIGIN

IN $\mathbb{R}^{n+1}$

FACTS: THERE IS A MAP

$S^n \xrightarrow{p} \mathbb{RP}^n$ TWO TO ONE

$p^{-1}(\ell) = \pm x$

$\mathbb{R}^{n+1} \ni \underset{\neq 0}{x} \longmapsto$ LINE THRU x AND 0.

$\mathbb{RP}^2$ IS ILLUSTRATED BY BOY'S SURFACE

$\mathbb{RP}^1$ IS S^1

MÖBIUS STRIP

FUN FACT: ATTACHING A D^2 ALONG THE BOUNDARY RESULTS IN $\mathbb{RP}^2$

WE WILL SEE LATER THAT

$$SO(3) \cong \mathbb{RP}^3.$$

<u>PROP</u> $\mathbb{RP}^n$ IS A CW-COMPLEX IN WHICH THE k-SKELETON IS $\mathbb{RP}^k$, THERE IS A SINGLE $(k+1)$-CELL ATTACHED BY THE MAP $S^k \longrightarrow \mathbb{RP}^k$ AS ABOVE

$$\begin{tikzcd} S^k \arrow[r] \arrow[d] & \mathbb{RP}^k \arrow[d] \\ D^{k+1} \arrow[r] & \mathbb{RP}^{k+1} \end{tikzcd}$$

PROOF: EXERCISE

4) n-DIMENSION COMPLEX

PROJECTIVE SPACE $\mathbb{C}P^n$
IS THE SET OF COMPLEX LINES THRU ORIGIN IN $\mathbb{C}^{n+1}$

THERE IS A MAP

$$\begin{array}{ccc} S^{2n-1} & \longrightarrow & \mathbb{C}P^{n-1} \quad \forall n \\ \cap & & \\ \mathbb{R}^{2n} \cong \mathbb{C}^n & & \end{array}$$

CONSIDER THE CASE $n=2$

$$S^3 \xrightarrow{\eta} \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$$

$$\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$$

A SUBSET CONTAINING ∞ IS OPEN IFF ITS COMPLEMENT IS COMPACT (HENCE CLOSED) IN $\mathbb{C}$ OR $\mathbb{R}$.

TO DEFINE η $\quad z_1, z_2 \in \mathbb{C}$

$$\mathbb{C}^2 \supset S^3 \ni (z_1, z_2) \longmapsto \begin{cases} z_1/z_2 & \text{IF } z_2 \neq 0 \\ \infty & \text{IF } z_2 = 0 \end{cases}$$

THIS MAP IS CONTINUOUS

THE PREIMAGE OF $y \in \mathbb{C}P^{n-1}$ IS A CIRCLE

HIGHLY RECOMMENDED EXERCISE

FOR $n=2$, ANY TWO SUCH CIRCLES ARE LINKED

THIS MAP WAS FIRST STUDIED IN 1930 BY HEINZ HOPF

HE PROVED IT IS <u>NOT</u> HOMOTOPIC TO A CONSTANT MAP

HE SHOWED IT REPRESENTS

A GENERATOR OF $\pi_3(S^2) \cong \mathbb{Z}$.

HE CONSTRUCTED SIMILARS

$S^7 \longrightarrow S^4$ USING QUATERNIONS

$S^{15} \longrightarrow S^8$ USING OCTONIONS AKA CAYLEY #S.

WE WILL SEE MORE OF $\mathbb{RP}^n$ AND $\mathbb{CP}^n$

(1) IS AN EXAMPLE OF A PUSHOUT

SUPPOSE WE HAVE A DIAGRAM OF SETS

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d] \\ B_2 \arrow[r] & C \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

PUSHOUT SYMBOLS

$$C = B_1 \amalg B_2 \Big/ \big(f_1(a) = f_2(a) \ \forall\, a \in A \big)$$

EXAMPLE : $A = \emptyset$ THEN $C = B_1 \amalg B_2$

$A = \{1\}$ $C =$ "ONE POINT UNION" OF B_1 AND B_2

UNIVERSAL PROPERTY OF A PUSHOUT

$$\begin{tikzcd} A \arrow[r, "f_1"] & B_1 \end{tikzcd}$$

SUPPOSE WE HAVE

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B_1 \arrow[d, "e_1"] \arrow[ddr, bend left, "g_1"] & \\
B_2 \arrow[r, "e_2"] \arrow[drr, bend right, "g_2"'] & C \arrow[dr, dashed, "h"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & W
\end{tikzcd}$$

SUPPOSE WE HAVE W, g_1 AND g_2 WITH $g_1 f_1 = g_2 f_2 : A \longrightarrow W$

THEN $\exists!\ h : C \longrightarrow W$ WITH $g_1 = h e_1$ AND $g_2 = h e_2$