Monday, September 19, 2022 12:01 PM

MORE ABOUT PUSHOUTS

A DIAGRAM $B_1 \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B_2$

IN A SUITABLE CATEGORY $\mathcal{C}$

(*Set*, *Top*, *Gp*, *Ab*, *Rings*, ...)

LEADS TO A PUSHOUT OBJECT WITH A UNIVERSAL PROPERTY DESCRIBED ABOVE (ON CHALK BOARD)

THIS DIAGRAM IS A FUNCTOR TO $\mathcal{C}$ FROM THE CATEGORY $\mathbb{P}$

$$\bullet \leftarrow \bullet \rightarrow \bullet$$

GENERALIZATION:

REPLACE $\mathbb{P}$ BY <u>ANY</u> SMALL CATEGORY J (ALTER THE SHAPE OF THE DIAGRAM) AND LOOK FOR AN OBJECT IN $\mathcal{C}$ WITH A UNIVERSAL PROPERTY.

EXAMPLE

J = "NATURAL NUMBERS"

$$0 \to 1 \to 2 \to 3 \to 4 \to \cdots$$

$$\Big\downarrow$$

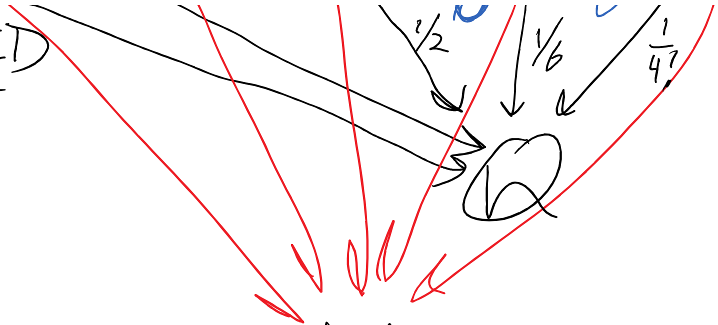
IN $\mathcal{C}$: $X_0 \to X_1 \to X_2 \to X_3 \to \cdots$

$\mathcal{C} = Ab$ $\quad \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{5} \cdots$

THE DESIRED

$1/2$ 0 $1/6$ 0 $\frac{1}{4!}$

THE DESIRED OBJECT IS $\mathbb{Q}$



W = ABELIAN GP

THE MAPS TO W FROM THE VARIOUS $\mathbb{Z}$'s DETERMINE A MAP

$$\mathbb{Q} \longrightarrow W.$$

GENERAL SET UP:

WE HAVE A FUNCTOR

$$J \xrightarrow{F} \mathcal{C}$$

WE WANT AN OBJECT $G \in \mathcal{C}$ SUCH THAT ANY COLLECTION OF MAPS

$$\underset{\mathcal{C}}{\underset{\cap}{F(j)}} \xrightarrow{w_j} \underset{\mathcal{C}}{\underset{\cap}{W}} \qquad j \in J$$

s.t. $\forall\ j \xrightarrow{f} j'$ IN J

THE TRIANGLE

$$\begin{tikzcd}
F(j) \arrow[rr, "F(f)"] \arrow[dr, "w_j"'] & & F(j') \arrow[dl, "w_{j'}"] \\
& W &
\end{tikzcd}$$

COMMUTES IN $\mathcal{C}$

$$w_{j'} \circ F(f) = w_j$$

THEN $\exists!\ G \xrightarrow{h} W$ SUCH THAT ....

WE SAY C IS THE <u>COLIMIT</u> OF F

DUAL DISCUSSION (ARROWS REVERSED)

GIVEN A DIAGRAM OF SETS

$$B_1 \xrightarrow{f_1} A \xleftarrow{f_2} B_2$$

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THE <u>PULLBACK</u>

$$\hat{C} = \{(b_1, b_2) \in B_1 \times B_2 : f_1(b_1) = f_2(b_2) \in A\}$$

PULL BACK SYMBOL

$$\begin{tikzcd}
\hat{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B_2 \arrow[d] \\
B_1 \arrow[r] & A
\end{tikzcd}$$

e.g. IF $A = \{*\}$, PULLBACK IS $B_1 \times B_2$

UNIVERSAL PROPERTY

GIVEN ANY W WITH MAPS g_2 AND g_1 S.T. $f_1 g_1 = f_2 g_2 : W \to A$

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& C \arrow[r, "j_2"] \arrow[d, "j_1"'] & B_2 \arrow[d, "f_2"] \\
& B_1 \arrow[r, "f_1"'] & A
\end{tikzcd}$$

$\exists! \, h : W \to C$ S.T. $j_1 h = g_1$ AND $j_2 h = g_2$

(3) IS EQT TO A FUNCTOR

$$\underbrace{\bullet \to \bullet \leftarrow \bullet}_{P^{op}} \xrightarrow{F} \mathcal{C}$$

DEF FOR ANY CATEGORY $\mathcal{C}$, $\mathcal{C}^{op}$ IS THE CATEGORY WITH THE SAME OBJECTS BUT WITH MORPHISMS IN OPPOSITE DIRECTION

$$\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X)$$

WE CAN REPLACEMENT P^{op} (PULLBACK CATEGORY) BY ANY SMALL CATEGORY J AND ASK FOR AN OBJECT IN $\mathcal{C}$ DETERMINED BY

$$J \xrightarrow{F} \mathcal{C}$$

HAVING A SUITABLE UNIVERSAL PROPERTY.

ELEMENTARY EXAMPLE

$\mathcal{C} = \mathcal{A}b$ = CATEGORY OF ABELIAN

GROUPS

$J = \mathbb{N}^{op}$ $\quad 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$

CHOOSE A PRIME p.

$$\begin{tikzcd}
0 & \mathbb{Z}/p \arrow[l] & \mathbb{Z}/p^2 \arrow[l] & \mathbb{Z}/p^3 \arrow[l] & \cdots \arrow[l] \\
 & & \mathbb{Z} \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

THIS IS NOT THE UNIVERSAL CHOICE

$\mathbb{Z}_p$ = p-ADIC INTEGERS.