LIMITS & COLIMITS IN A (BIG) CATEGORY

A DIAGRAM IN A CATEGORY $\mathcal{C}$ IS A FUNCTOR F TO IT FROM SOME SMALL CATEGORY J

e.g. $\mathcal{C} = \mathcal{S}et$, $\mathcal{A}b$, $\mathcal{S}paces$

$$\mathbb{N} \quad \underset{0}{\bullet} \to \underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet} \to \underset{4}{\bullet} \to \underset{5}{\bullet} \to \cdots$$

$$\mathbb{N}^{op} \quad \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$$

$$P \quad \begin{tikzcd} & \bullet \\ \bullet \arrow[ur] \arrow[r] & \bullet \end{tikzcd} \qquad P^{op} \quad \begin{tikzcd} & \bullet \arrow[dl] \\ \bullet & \bullet \arrow[l] \end{tikzcd}$$

PUSHOUTS PULL BACKS

EMPTY CATEGORY : NO OBJECTS, NO MORPHISMS

GIVEN A FUNCTOR $J \xrightarrow{F} \mathcal{C}$ WE CAN ASK FOR OBJECTS $\mathrm{colim}_J F$ AND $\lim_J F$ IN $\mathcal{C}$ WITH UNIVERSAL PROPERTIES

FOR COLIMIT : SUPPOSE WE AN OBJECT W AND MAPS $F(j) \xrightarrow{g_j} W$ FOR EACH $j \in J$

S.T. FOR EACH MORPHISM $j \xrightarrow{f} j'$ IN J, THE TRIANGLE

$$\begin{tikzcd} F(j) \arrow[rr, "F(f)"] \arrow[dr, "g_j"'] & & F(j') \arrow[dl, "g_{j'}"] \\ & W & \end{tikzcd}$$

COMMUTES

$$\begin{tikzcd} & W & \arrow[l, "g_{j'}"'] \\ \arrow[ur, "g_j"] & & \end{tikzcd}$$

$\exists!$ MAP $h : \operatorname{colim}_J F \longrightarrow W$ S.T.

$$\begin{tikzcd} & \operatorname{colim}_J F & \\ F(j) \arrow[ur] \arrow[rr, "F(f)"] & & F(j') \arrow[ul] \end{tikzcd}$$

SUCH A COLIMIT MAY OR MAY NOT EXIST DEPENDING ON $\mathcal{C}$

FOR LIMITS WE HAVE MAPS

$$\begin{tikzcd} & \lim_J F \arrow[dl] \arrow[dr] & W \arrow[l] \\ F(j) \arrow[rr, "F(f)"] & & F(j'). \end{tikzcd}$$

EXAMPLE FROM LAST TIME

$\mathcal{C} = \mathcal{A}b$ = ABELIAN GPS

$J = \mathbb{N}$ F IS GIVEN

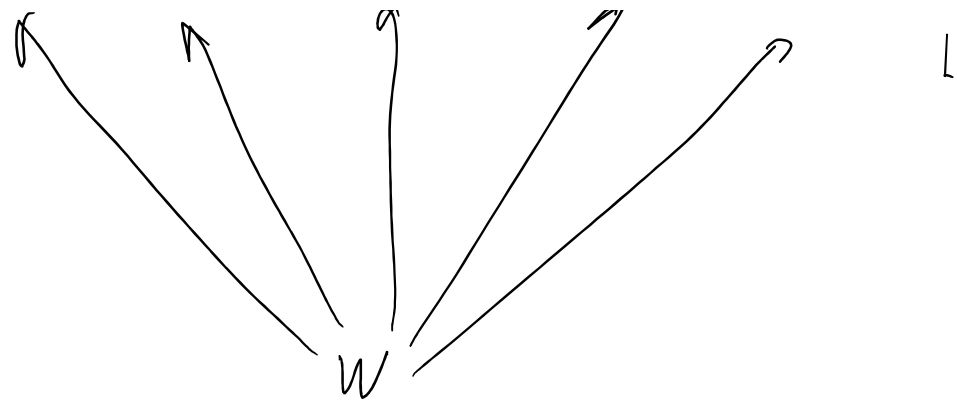
$$\begin{tikzcd}[column sep=small] \overset{0}{\mathbb{Z}} \arrow[r, "1"] \arrow[drrr, "1"'] & \overset{1}{\mathbb{Z}} \arrow[r, "2"] \arrow[drr, "1"] & \overset{2}{\mathbb{Z}} \arrow[r, "3"] \arrow[dr, "1/2"] & \overset{3}{\mathbb{Z}} \arrow[r, "4"] \arrow[d, "\frac{1}{6}"] & \overset{4}{\mathbb{Z}} \arrow[r] \arrow[dl, "\frac{1}{4!}"] & \cdots \\ & & & \mathbb{Q} & & \end{tikzcd}$$

EXAMPLE OF A LIMIT

$\mathcal{C} = \mathcal{A}b$, $J = \mathbb{N}^{op}$

PICK A PRIME p

$$0 \leftarrow \mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \cdots$$



W COULD BE $\mathbb{Z}$

OR $\mathbb{Z}_{(p)}$ (p-LOCAL INTEGERS)

= SUBRING OF $\mathbb{Q}$ OF #S WITH DENOMINATOR PRIME TO p.

THE LIMIT ABOVE IS NEITHE[R] OF THESE. IT IS $\mathbb{Z}_p$, THE p-ADIC INTEGER.

$\mathbb{Z}_p =$ " $\mathbb{Z}[[p]]$ "

e.g. $x = 1 + p + p^2 + p^3 + p^4 + \cdots$

$y = 1 + 2p + 3p^2 + 4p^3 + 5p^4 + \cdots$

EXERCISES SHOW $x^2 = y$

$x(1-p) = 1$ SO x "=" $\frac{1}{1-p}$

$1/p \notin \mathbb{Z}_p$.

THE p-ADIC METRIC

$$d(m, n) \leq p^k \quad \text{IF} \quad p^k \mid (m-n)$$

"COMPLETING" $\mathbb{Z}$ W.R.T THIS METRIC GIVES $\mathbb{Z}_p$.

$\mathbb{Z}_5 \ni x$ WITH $x^2 = -1$

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IT IS THE "LIMIT" (AS IN CALCULUS) OF THE SEQUENCE

$$2,\ \underset{\substack{\| \\ 32}}{2^5},\ 2^{25},\ 2^{(5^3)},\ 2^{(5^4)},\ \ldots\ldots$$

$2 \equiv 32$ MOD 5

$32 \equiv 2^{25}$ MOD 25

LEMMA IF $m \equiv n$ MOD p^k

THEN $m^p \equiv n^p$ MOD p^{k+1}

GOOGLE p-ADIC ANALYSIS

RECALL PUSHOUTS

$$\begin{tikzcd} & B_1 \arrow[dr, dashed] & \\ A \arrow[ur] \arrow[dr] & & C = \operatorname{colim}_P F \\ & B_2 \arrow[ur, dashed] & \end{tikzcd}$$

$$\begin{tikzcd} & & B_1 \\ L \arrow[urr, "f_1"] \arrow[r, "\lambda"] \arrow[drr, "f_2"'] & {} & \\ & & B_2 \end{tikzcd}$$

$\lim_P F = A$ NOT INTERESTING

<u>DEF</u> A CATEGORY $\mathcal{C}$ IS <u>COMPLETE</u> (COCOMPLETE) IF ALL LIMITS (COLIMITS)

IF ALL LIMITS (COLIMITS) EXIST IN IT. IF IT HAS BOTH IT IS BICOMPLETE.

CONSIDER THE CATEGORY

$$J \quad \bullet \rightrightarrows \bullet$$

A J-SET IS A DIAGRAM

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B$$

THE LIMIT IS $\{a \in A : f_1(a) = f_2(a)\}$

THE COLIMIT IS A QUOTIENT OF B BY $f_1(a) \sim f_2(a) \ \forall a \in A$.

THE EQUALIZER IS THE LIMIT

COEQUALIZER " COLIMIT.

THEOREM A CATEGORY IS (CO) COMPLETE IF IT HAS ALL (CO) EQUALIZERS.

LET E BE THE EMPTY CATEGORY

$\exists !$ FUNCTOR $E \longrightarrow \mathcal{C}$, THE EMPTY DIAGRAM.

WHAT ABOUT ITS LIMIT / COLIMIT?

THE COLIMIT C HAS A UNIQUE MAP TO EVERY OBJECT

W IN $\mathcal{C}$. THIS IS CALLED AN <u>INITIAL OBJECT</u> IN $\mathcal{C}$,

e.g. $\emptyset \in \mathcal{S}et$

THE LIMIT OF THE EMPTY DIAGRAM IS A <u>TERMINAL OBJECT</u>, ONE THAT RECEIVES A UNIQUE MORPHISM FROM EVERY OTHER OBJECT.

e.g. SINGLETON SET.

LET G BE A GROUP

BG = ONE OBJECT CATEGORY WHOSE MORPHISM SET IS G.

A FUNCTOR $BG \xrightarrow{F} \mathcal{C}$ IS AN OBECT X IN $\mathcal{C}$ EQUIPPED WITH AN ACTION OF G.

WHAT ARE ITS LIMIT AND COLIMIT?

FOR A G-SET X WE WANT A MAP

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[dr, bend right, "f"'] & X \arrow[d, "g"] \\ & X \end{tikzcd} \qquad g \in X$$

$$W = \{ x \in X : g(x) = x \ \forall g \in G \}$$

$$= X^G = \text{FIXED POINT SET OF } X$$

THE COLIMIT IS THE ORBIT SET $X_G = X/G$