ADJOINT FUNCTORS (TITLE OF PAPER BY DAN KAN 1958)

SUPPOSE WE HAVE CATEGORIES AND FUNCTOR

$$X \in \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \ni Y \qquad (\perp)$$

F AND G ARE <u>ADJOINT</u> IF FOR ANY OBJECTS X IN $\mathcal{C}$ AND Y IN $\mathcal{D}$ THERE IS A NATURAL ISOMORPHISM

$$\star \qquad \mathcal{C}(X, GY) \xrightarrow{\cong} \mathcal{D}(FX, Y)$$

EXAMPLE — LEFT ADJOINT $F \dashv G$ RIGHT ADJOINT

(1) $\mathcal{C} = \mathcal{S}et$, $\mathcal{D} = \mathcal{A}b$

$F : \mathcal{S}et \longrightarrow \mathcal{A}b$ IS THE FREE ABELIAN GROUP FUNCTOR

$U : \mathcal{A}b \longrightarrow \mathcal{S}et$ IS THE FORGETFUL FUNCTOR

LET X BE A SET AND A AN ABELIAN GROUP

$$\mathcal{A}b(FX, A) \cong \mathcal{S}et(X, UA)$$

A HOM $FX \longrightarrow A$ IS DETERMINED (AND DETERMINES) A MAP OF

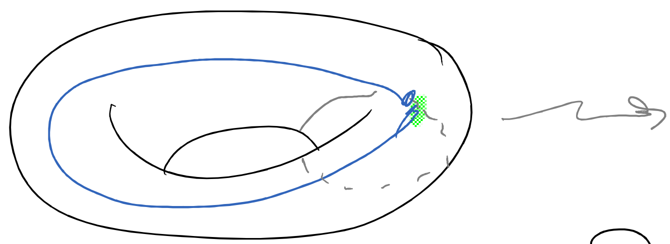
SETS $\quad X \longrightarrow UA$

(2) $\mathcal{C} = \mathcal{D} =$ POINTED SPACE Top_0

SUSPENSION $\quad \Sigma : \mathrm{Top}_0 \circlearrowleft$

$$(X, x_0) \longmapsto (S^1, s_0) \times (X, x_0) / (S^1 \times x_0 \cup s_0 \times X)$$

FOR $X = S^1$

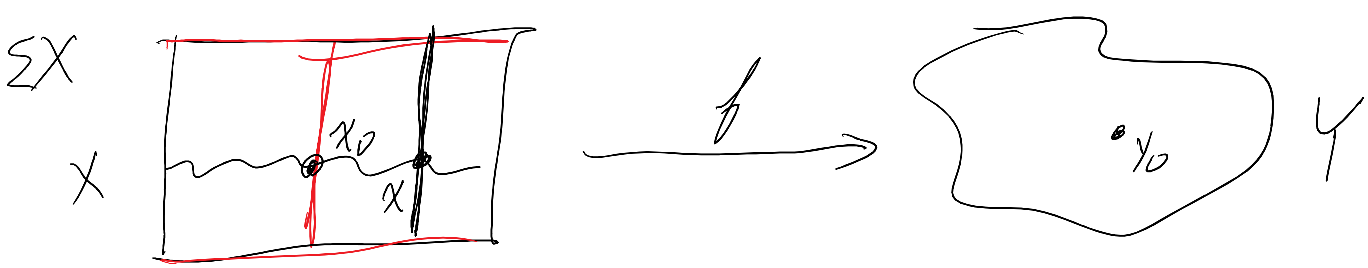


$\rightsquigarrow S^1$

LOOP SPACE FUNCTOR $\quad \Omega : \mathrm{Top}_0 \circlearrowleft$

$(X, x_0) \longmapsto$ SPACE OF CLOSED PATHS IN (X, x_0)

$$\mathrm{Top}_0(\Sigma X, Y) \cong \mathrm{Top}(X, \Omega Y)$$



FOR EACH $x \in X$, f DETERMINES A CLOSED (LOOP) IN Y BY RESTRICTION TO THE VERTICAL LINE.

(3) J = SMALL CATEGORY

$\mathcal{C}$ = CATEGORY (BIG)

$\mathcal{C}^J$ = CATEGORY OF J-SHAPED

DIAGRAMS IN $\mathcal{C}$,
I.E OF FUNCTOR $J \to \mathcal{C}$

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$$ DIAGONAL FUNCTOR

$X \longmapsto$ CONSTANT X-VALUED FUNCTOR $J \to \mathcal{C}$

IN FAVORABLE CIRCUMSTANCES A FUNCTOR $J \xrightarrow{F} \mathcal{C}$ HAS A LIMIT AND A COLIMIT.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\Delta" description] & & \mathcal{C}^J \arrow[ll, bend right, "\mathrm{colim}"'] \arrow[ll, bend left, "\lim"] \arrow[ll, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift left=3, "\perp"]
\end{tikzcd}$$

colim AND lim ARE THE LEFT AND RIGHT ADJOINTS OF Δ.

THIS IS A GOOD EXERCISE

<u>WARNING</u>: FUNCTORS NEED NOT LEFT OR RIGHT ADJOINTS BUT WHEN ADJOINTS EXIST THEY ARE UNIQUE.

BACK TO EARTH

THEOREM $\pi_1(S^1) \cong \mathbb{Z}$

PROOF: THINK S^1 AS THE UNIT CIRCLE IN $\mathbb{C}$ WITH BASE PT. 1

$$I \xrightarrow{w_n} \mathbb{R} \xrightarrow{p} S^1 \subset \mathbb{C}$$

$$A \longmapsto e^{2\pi i t}$$

$$p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$$

FOR ANY PATH w_n IN $\mathbb{R}$ FROM 0 TO $n \in \mathbb{Z}$, $p w_n$ IS A CLOSED PATH IN S^1 GOING AROUND n TIMES COUNTERCLOCKWISE

WE CAN DEFINE A HOM.

$$\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1)$$

$$n \longmapsto p w_n$$

THE HOMOTOPY CLASS OF $p w_n$ DEPENDS ONLY ON n.

WANT TO SHOW Φ IS 1-1 AND ONTO.

<u>CLAIM A</u>

$$\{0\} \xrightarrow{b} (\mathbb{R}, 0)$$

CLAIM A

$$\begin{tikzcd} \{0\} \arrow[r, "0"] \arrow[d] & (\mathbb{R}, 0) \arrow[d, "p"] \\ (I, 0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (S^1, 1) \end{tikzcd}$$

FOR ANY PATH

$\exists!\ \tilde{f}$ AS SHOWN

$p\tilde{f} = f$

$\tilde{f}$ IS A <u>LIFTING</u> OF f.

THIS IMPLIES Φ IS ONTO.

ANY $\alpha \in \pi_1(S^1)$ IS REPRESENTED BY A <u>CLOSED</u> PATH f

THEN $p\tilde{f}(1) \in p^{-1}(1) = \mathbb{Z}$

so $\alpha = \Phi\,(p^{-1}(1))$

CLAIM B

LIFTING HOMOTOPIES

$$\begin{tikzcd} (I, 0) \arrow[rr, "\text{CONSTANT}"] \arrow[d] & & (\mathbb{R}, 0) \arrow[d, "p"] \\ (I^2, \{0\} \times I) \arrow[rru, dashed, "\tilde{h}"] \arrow[rr, "h"] & & (S^1, 1) \end{tikzcd}$$

h IS A HOMOTOPY BETWEEN PATHS IN S^1 STARTING AT 1

$\exists!$ LIFTING $\tilde{h} : (I^2, -) \longrightarrow (\mathbb{R}, 0)$ AS SHOWN.

THIS IMPLIES Φ IS 1-1.

BECAUSE h COULD BE A HOMOTOPY BETWEEN CLOSED PATH IN S^1

BOTH CLAIMS ARE SPECIAL CASES OF

<u>CLAIM C</u>

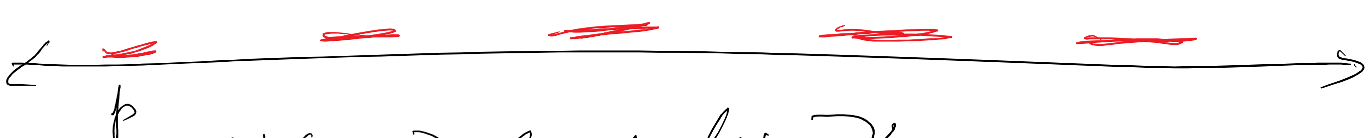
$$\begin{tikzcd}
(X, x_0) \times \{0\} \arrow[r, "g"] \arrow[d] & (\mathbb{R}, 0) \arrow[d, "p"] & (\tilde{Y}, \tilde{y}_0) \arrow[d, "p"] \\
(X, x_0) \times (I, 0) \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & (S^1, 1) & (Y, y_0)
\end{tikzcd}$$

$\exists !\ \tilde{g}$ WITH $p\tilde{g} = g$

$X = pt$ IS CLAIM A
$X = I$ " B

WE COULD REPLACE $p : \mathbb{R} \longrightarrow S^1$ BY ANY <u>COVERING</u> $\tilde{Y} \xrightarrow{p} Y$,

i.e. A MAP IN WHICH EACH $y \in Y$ HAS A NBD U S.T. $p^{-1}(U) \approx U \times K$ FOR SOME DISCRETE K INDEPENDENT OF y.



$p^{-1}(\text{RED}) \approx \text{red} \times \mathbb{Z}$