Wednesday, September 28, 2022 1:55 PM

RECALL WE HAVE REDUCED

$\pi_1(S^1) \cong \mathbb{Z}$ TO THE FOLLOWING

GIVEN MAP i, j, g, p AS SHOWN

$$\begin{tikzcd}
(X, x_0) \times \{0\} \arrow[r, "j"] \arrow[d, "i"'] & (\mathbb{R}, 0) \arrow[d, "p"] \\
(X, x_0) \times (I, 0) \arrow[r, "g"] \arrow[ur, dashed, "h"] & (S^1, 1)
\end{tikzcd}$$

$\exists! \; h : X \times I \to \mathbb{R}$

WITH

$ph = g$ AND

$hi = j$

X IS PATH CONNECTED

WE CAN REPLACE p BY ANY

COVERING $\tilde{Y} \to Y$

FOR $y_0 \in Y$, CHOOSE A NBD U WHICH

IS EVENLY COVERED BY p (I.E.

$p^{-1}U \cong U \times K$ FOR K DISCRETE

$$\begin{tikzcd}
(x,0) \quad X \times \{0\} \arrow[r, "j"] \arrow[d, "i"'] & \mathbb{R} \arrow[d, "p"] & \tilde{U} \ni j(x,0) \arrow[d, "\cong"] \\
x,0 \quad X \times I \supset g^{-1}(U) \arrow[r, "g"] \arrow[ur, dashed, "h"] & S^1 & U \ni y
\end{tikzcd}$$

CHOOSE $x \in X$ LET $y = g(x,0)$

CHOOSE EVEN COVERED NBD U of y

$p^{-1}(U) \cong$ MANY DISJOINT COPIES OF U

CHOOSE THE ONE CONTAINING $j(x)$

CALL IT $\tilde{U}$

WE CAN DEFINE h ON $g^{-1}(U)$

CHOOSE ANOTHER pt x_1 IN $g^{-1}(U)$,
THEN CHOOSE AN EVENLY
COVERED NBD U_1 OF $g(x_1)$
AND EXTEND TO $g^{-1}(U_1)$
PROCEEDING IN THIS WE CAN
EXTEND h TO ALL OF X.
SEE HATCHER FOR MORE
DETAILS

QED.

$\pi_1(S^1) \cong \mathbb{Z}$, AN INTEGER $n \in \mathbb{Z}$
IS REPRESENTED BY

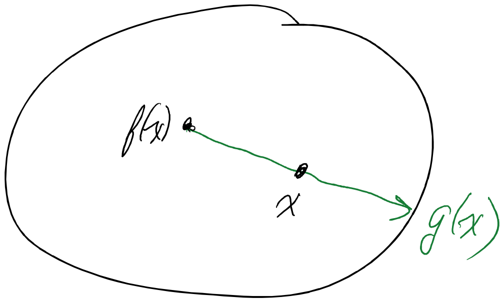
$$\mathbb{C} \supset S^1 \xrightarrow{[n]} S^1 \subset \mathbb{C}$$
$$z \longmapsto z^n$$

<u>COR</u> THE IDENTITY MAP
$S^1 \longrightarrow S^1$ IS NOT HOMOTOPIC
TO THE CONSTANT MAP.

<u>SOME CONSEQUENCES</u>

BROUWER FIXED POINT THEOREM
(1910?) ANY MAP $D^2 \xrightarrow{f} D^2$
HAS A FIXED POINT, i.e.
$\exists\, x \in D^2$ WITH $f(x) = x$.
PROOF: ASSUME THERE IS NO
FIXED POINT. DEFINE

$D^2 \xrightarrow{g} S^1 = \partial D^2$ AS FOLLOWS



NOTE

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[rr, bend right, "\text{IDENTITY}"'] & D^2 \arrow[r, "g"] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} S^1 \times I \arrow[r] \arrow[rr, bend left, "h"] & D^2 \arrow[r, "g"] & S^1 \end{tikzcd}$$

$$(x, t) \longmapsto (tx)$$

$h(x, 1) = x$ AND $h(x, 0) = g(0)$

h IS A HOMOTOPY THAT IS EXCLUDED BY THE COROLLARY.

CONTRADICTION. QED

HIGHER DIMENSIONAL GENERALZN

REPLACE D^2 BY D^n FOR $n > 2$

WE GET A MAP $D^n \to \partial D^n = S^{n-1}$

AND HENCE A FORBIDDEN HOMOTOPY AS ABOVE.

NEED TO KNOW $S^{n-1} \xrightarrow{\text{ID}} S^{n-1}$

IS ESSENTIAL, I.E. NOT HOMOTOPIC TO A CONSTANT MAP

FUNDAMENTAL THEOREM OF ALGEBRA (GAUSS ~1800)

LET $p(z) \in \mathbb{C}[z]$ BE A NONCONSTANT POLYNOMIAL.

THEN $\exists z_0$ WITH $p(z_0)=0$,

so $p(z) = (z-z_0)q(z)$

TOPOLOGICAL PROOF: ASSUME $\nexists z_0$

$$\begin{tikzcd}
 & & \mathbb{C}-\{0\} \arrow[r] \arrow[d] & S^1 \\
S^1 \arrow[r] \arrow[rrru, bend right=60, "f_r"'] & \mathbb{C} \arrow[r, "p"] \arrow[ur, dashed, "\hat{p}"] & \mathbb{C} &
\end{tikzcd}$$

$z \longmapsto \dfrac{z}{|z|}$

$z \longmapsto rz$

$r = 0$

$$p(z) = \sum_{0 \le i \le n} p_i z^i \qquad p_0 \neq 0$$

$n \neq 0$

$p_n \neq 0$

f_r IS CONSTANT WITH VALUE $\dfrac{p_0}{|p_0|}$

FOR $r \gg 0$

THEN FOR $|z| = 1$

$$p(rz) = p_n r^n z^n + \text{SMALL TERMS}$$

$$\frac{p_n z^n}{|p_n|} = \frac{p_n r^n z^n}{|p_n| r^n + \ldots}$$

SO f_r IS HOMOTOPIC TO nTH POWER MAP

THIS IS A CONTRADICTION.

QED

CAN ASSUME $p_n = 1$,

POLYNOMIAL IS MONIC.

HAM SANDWICH THEOREM

CAN CUT A SANDWICH SO THAT HAM, CHEESE + BREAD ARE BISECTED.

GIVEN 3 COMPACT SUBSETS OF $\mathbb{R}^3$, $\exists$ A PLANE THAT BISECTS ALL 3.

WILL DERIVE IT FROM BORSUK-ULAM THEOREM.

GIVEN A MAP $S^2 \xrightarrow{f} \mathbb{R}^2$ $\exists\ x \in S^2$ SUCH THAT $f(x) = f(-x)$.

"ADVENTURES OF A MATHEMATICIAN" MOVIE ABOUT ULAM.