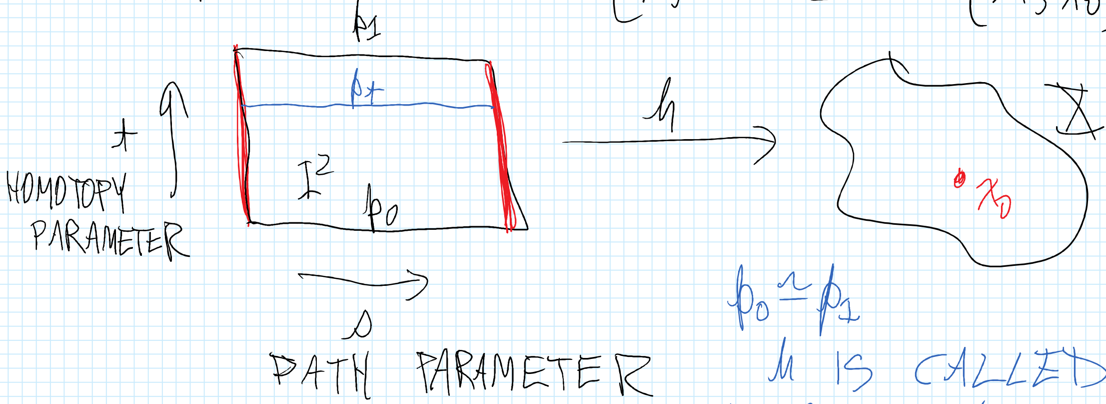
RECALL WE DEFINED THE FUNDAMENTAL GROUP $\pi_1(X, x_0)$ FOR $x_0 \in X$ (THE BASE POINT). AS A SET IT IS THAT OF HOMOTOPY CLASSES OF CLOSED PATHS IN X STARTING/ENDING AT $x_0$

FOR $I := [0,1]$, $\partial I = \{0,1\}$, SUCH A CLOSED PATH IS A MAP

$$(I, \partial I) \xrightarrow{p} (X, x_0)$$

TWO SUCH PATHS $p_0$ AND $p_1$ ARE <u>HOMOTOPIC</u> IF THERE IS A MAP $(I, \partial I) \times I \xrightarrow{h} (X, x_0)$




$p_0 \simeq p_1$

$h$ IS CALLED A <u>HOMOTOPY</u> BETWEEN $p_0$ AND $p_1$.

HOMOTOPY DEFINES AN EQUIVALENCE RELATION AMONG SUCH CLOSED PATHS THERE A BINARY OPERATION ON THE SET OF EQUIV CLASSES WHICH MAKES IT A GROUP.

GIVEN PATHS $f$ AND $g$, DEFINE $f * g : (I, \partial I) \to (X, x_0)$ BY

$$① \quad (f*g)(s) = \begin{cases} f(2s) & \text{FOR } 0 \le s \le 1/2 \\ g(2s-1) & \text{FOR } 1/2 \le s \le 1 \end{cases}$$

GENERALIZATION TO HIGHER DIMENSIONS

CONSIDER MAPS

$$(I^n, \partial I^n) \xrightarrow{f} (X, x_0)$$

WHERE $\partial I^n$ = BOUNDARY OF $I^n$

$= \{(s_1, \cdots s_n) \in I^n : \text{SOME } s_i \text{ IS } 0 \text{ OR } 1\}$

CAN DEFINE HOMOTOPIES BETWEEN SUCH MAPS AS BEFORE.

BINARY OPERATION: FOR SUCH MAPS $f$ AND $g$, DEFINE $f*g$ BY

$$(f*g)(s_1, \cdots s_n) = \begin{cases} f(2s_1, s_2, \cdots s_n) & \text{FOR } 0 \le s_1 \le 1/2 \\ g(2s_1 - 1, s_2, \cdots s_n) & \text{FOR } 1/2 \le s_1 \le 1 \end{cases} \quad (2)$$

NOTE FOR $s_1 = 1/2$, $f(2s_1, s_2 \cdots s_n) = f(1, s_2, \cdots s_n) = x_0$

AND $g(2s_1 - 1, s_2, \cdots s_n) = g(0, s_2, \cdots s_n) = x_0$

WE CAN SHOW THIS BINARY OPERATIONS LEADS TO A GROUP STRUCTURE ON THE SET $[(I^n, \partial I^n), (X, x_0)]$ OF HOMOTOPY CLASSES OF SUCH MAPS. THIS GROUP IS DENOTED BY $\pi_n(X, x_0)$, THE <u>nTH HOMOTOPY GROUP</u> OF X.

WILL SHOW LATER

$\pi_1(S^1, x_0) \cong \mathbb{Z}$ = INTEGERS UNDER +

CIRCLE = SET OF UNIT VECTORS

IN $\mathbb{R}^2$:

$\pi_n(S^1,x_0)=0$ FOR $n>1$.

LET $S^m$ BE THE SET OF UNIT VECTORS IN $\mathbb{R}^{m+1}$. WE KNOW

$$\pi_n(S^m,x_0)=\begin{cases}0 & \text{FOR } n<m\\ \mathbb{Z} & \text{FOR } n=m\\ ??? & \text{FOR } n>m\end{cases}$$

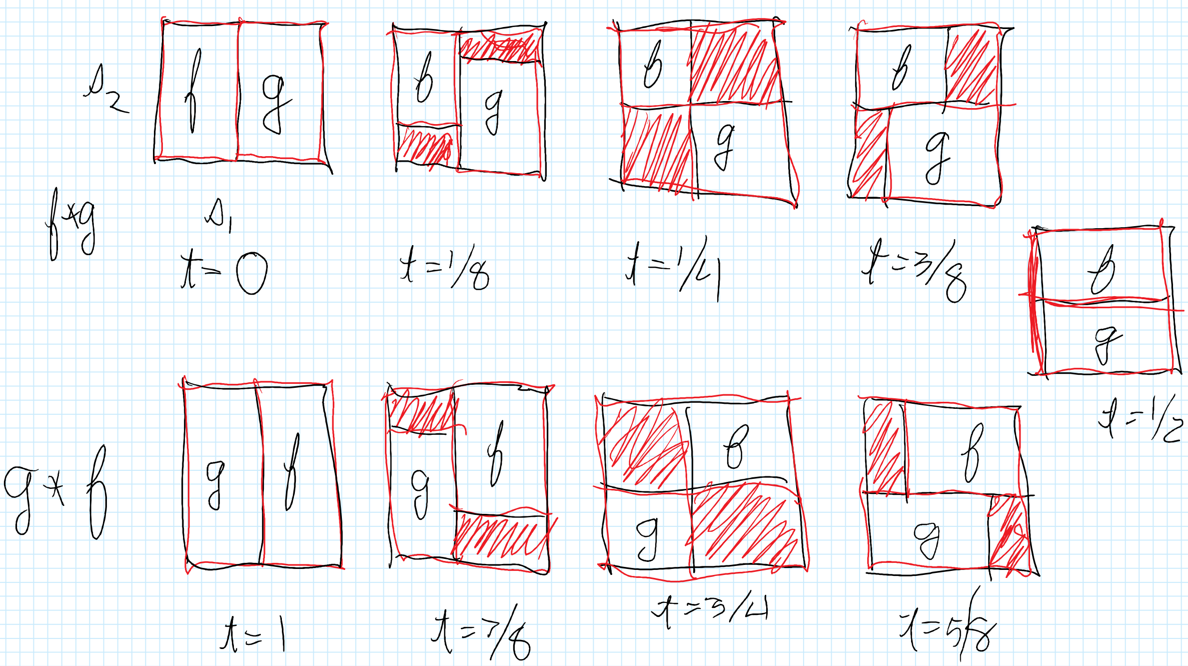
THEOREM $\pi_n(X,x_0)$ FOR $n\geq 2$ IS ABELIAN. (NOT TRUE FOR $n=1$)

PROOF FOR $n=2$:

LET $f,g:(I^2,\partial I)\longrightarrow(X,x_0)$

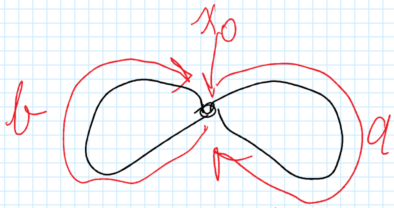
HAVE DEFINED $f*g:(I^2,\partial I^2)\longrightarrow(X,x_0)$ IN 2. NEED TO SHOW $f*g\simeq g*f$

WILL ILLUSTRATE SUCH HOMOTOPY FOR VARIOUS VALUES OF THE HOMOTOPY PARAMETER $t$




QED

EXAMPLES OF SPACES WITH NONABELIAN $\pi_1$ (MORE ABOUT THIS LATER)

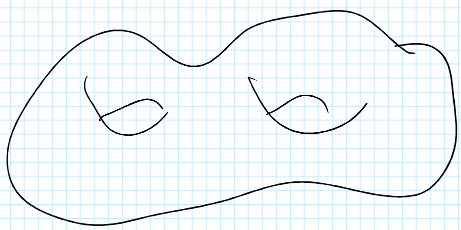
i) 


$a * b \neq b * a$

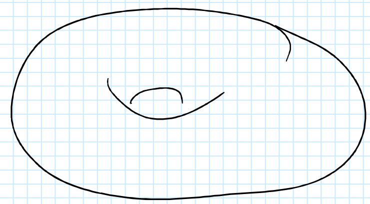
$S^1 \vee S^1$

ONE POINT UNION OF TWO CIRCLES.

SPOILER THE VAN KAMPEN THEOREM DESCRIBES $\pi_1(A \vee B)$ IN TERMS OF $\pi_1(A)$ AND $\pi_1(B)$.

ii) 

$\pi_1$ NONABELIAN.

FOR $X = S^1 \times S^1$ 

$\pi_1 X = \mathbb{Z} \oplus \mathbb{Z}$

EXAMPLE $X = \mathbb{R}^k$

THEN $\pi_n(X, 0) \approx 0$ FOR ALL $n$.

PROOF: LET $f : (I^n, \partial I^n) \longrightarrow (\mathbb{R}^k, 0)$

IT IS HOMOTOPIC TO CONSTANT MAP

$(I^n, \partial I^n) \times I \xrightarrow{h} (\mathbb{R}^k, 0)$

$$h(s_1, \cdots s_n, t) = (1-t)\, f(s_1, \cdots s_n)$$

$$= \begin{cases} f(s_1, \cdots s_n) & \text{IF } t = 0 \\ 0 & \text{IF } t = 1 \end{cases}$$

$\rightsquigarrow \pi_n(\mathbb{R}^k, 0) = 0 \quad \forall\, k, n.$

<u>DEF</u> TWO (POINTED) SPACES $X, Y$ ARE <u>HOMOTOPY EQUIVALENT</u> IF THERE ARE MAPS $X \underset{g}{\overset{f}{\rightleftarrows}} Y$

$\xrightarrow{f} (Y, y_0)$

$$\underbrace{X \xrightarrow{f} Y \xrightarrow{g} X}_{gf} \qquad \underbrace{Y \xrightarrow{g} X \xrightarrow{f} Y}_{fg}$$

CALCULUS CONVENTION

WANT $gf \simeq 1_X$ AND $fg \simeq 1_Y$

$1_X$ = IDENTITY MAP ON $X$

SUCH MAPS $f$ AND $g$ ARE CALLED <u>HOMOTOPY EQUIVALENCES</u>.

IF $gf \simeq 1_X$ BUT $fg \simeq ???$, WE SAY X IS A <u>RETRACT</u> OF Y.

<u>EXAMPLES</u> $X = \mathbb{R}^k$, $Y = pt$

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
\mathbb{R}^k \arrow[r, "\text{UNIQUE}"'] \arrow[rr, bend right=40, "gf \simeq 1_X"'] & pt \arrow[r, "0"] \arrow[rr, bend right=40, "fg = 1_Y"'] & \mathbb{R}^k \arrow[r, "\text{UNIQUE}"'] & pt
\end{tikzcd}
$$

MORE TERMINOLOGY. IF X AND Y ARE HOMOTOPY EQUIVALENT, THEY HAVE THE <u>SAME HOMOTOPY TYPE</u>.

A SPACE HOMOTOPY EQUIV TO A POINT IS <u>CONTRACTIBLE</u>.

<u>PROP</u> A HOMOTOPY INDUCES AN ISOMORPHISM $\pi_n(-)$ FOR ALL $n$.