So far we have seen two elementary examples of model categories: $\mathbf{Top}$, the category of topological spaces, and $\mathbf{Ch}_R$, the category of chain complexes of R-modules. In both of them all objects are fibrant while only some objects (CW-complexes and chain complexes of projective modules respectively) are cofibrant.

Let **Δ** be the category whose objects are the finite ordered sets [**n**] = {0,1,...,n} ($n \geq 0$) and whose morphisms are the order-preserving maps between these sets. (Here "order-preserving" means that $f(i) \geq f(j)$ whenever $i \geq j$). It is an easy exercise to show that any such map can be written as a composite of the following ones:

- the face maps $d_i: [\boldsymbol{n-1}] \to [\boldsymbol{n}]$ for $0 \leq i \leq n$, where d_i is the order preserving monomorphism that does not have i in its image and
- the degeneracy maps s_i: $[\boldsymbol{n+1}] \to [\boldsymbol{n}]$ for $0 \leq i \leq n$, where s_i is the order preserving epimorphism sending i and $i+1$ to i.

These satisfy the simplicial identities:

(i) $d_i\, d_j = d_{j-1}d_i$ for $i < j$
(ii) $d_i s_j = s_{j-1}d\_i$ for $i < j$
(iii) $d_i s_j = id$ for $i = j$ and $i = j + 1$
(iv) $d_i s_j = s_j\, d_{i-1}$ for $i > j + 1$
(v) $s_i s_j = s_{j+1}\, s_i$ for $i \leq j$.

The category s**Set** of *simplicial sets* is defined to be the category of functors $\mathbf{\Delta}^{\mathrm{op}}$ → **Set**; the morphisms, as usual, are natural transformations. Recall that a functor Δ^{op}→ Set is the same as a contravariant functor **Δ** → **Set**. If X is a simplicial set it is customary to denote the set $X([\boldsymbol{n}])$ by X_n and call it the set of n-simplices of X.

Let Δ^n denote the standard topological n-simplex, considered as the space of formal convex linear combinations of the points in the set [**n**], i.e.,

$$\{(t_0, t_1, \dots, t_n) \in \boldsymbol{R}^{n+1}: t_i \geq 0 \text{ and } \textstyle\sum_i t_i = 1\}.$$

It is homeomorphic to the n-disk D^n. It boundary ∂D^n is the set of points with at least one coordinate equal to 0; it is homeomorphic to S^{n-1}. The ith face Δ_i^n for $0 \leq i \leq n$ is the set of points with $t_i = 0$, it is homeomorphic to D^{n-1}. The ith horn Λ_i^n is the complement of the ith face in the boundary, the set of points with at least one vanishing coordinate and with $t_i > 0$. It is also homeomorphic to D^{n-1}.

If *Y* is a topological space, it is possible to construct an associated simplicial set $\mathrm{Sing}(Y)$ by letting the set of *n*-simplices $\mathrm{Sing}(Y)_n$ be the set of all continuous maps $\Delta^n \to Y$; this is a set-theoretic precursor of the singular chain complex of Y. The functor Sing : **Top** → s**Set** has a left adjoint, which sends a simplicial set *X* to a space $|X|$ called the geometric realization of X; this construction is a generalization of the geometric realization construction for simplicial complexes. See Wikipedia for more information.

A map $f: X \to Y$ of simplicial sets is a *Kan fibration* if for for each diagram of the form

$$\begin{array}{ccc} & s & \\ \Lambda^n_k & \rightarrow & X \\ i\downarrow & & \downarrow f \\ \Delta^n & \rightarrow & Y \\ & t & \end{array}$$

there is a lifting $g: \Delta^n \to X$ such that $fg = t$ and $gi = s$. X is a *Kan complex* if the map $X \to *$ is a Kan fibration.

We can define a model structure on s**Set** by saying that a morphism f is

(i) a weak equivalence if $|f|$ is a weak homotopy equivalence of topological spaces,

(ii) a cofibration if each map $f_n: X_n \to Y_n$ is a monomorphism for each $n \geq 0$, and

(iii) a fibration if it is a Kan fibration as above.

Hence every object is cofibrant, but only Kan complexes are fibrant. We will give an example of a simplicial set that is not a Kan complex below.

Let $\mathcal{C}$ be a small category. There is a simplicial set associated with it called $N(\mathcal{C})$, the nerve of $\mathcal{C}$, defined as follows. The set of n-simplices is the set of diagrams in $\mathcal{C}$ of the form

$$X_0 \to X_1 \to \cdots \to X_n.$$

The face operators contract such a diagram by composing two morphisms within it, and the degeneracy operators enlarge it by inserting an identity morphism. It is known that $\mathrm{N}(\mathcal{C})$ is a Kan complex iff $\mathcal{C}$ is a groupoid, meaning that all morphisms are invertible. When $\mathcal{C}$ is the one object category associated with a group G, the geometric realization $|\mathrm{N}(\mathcal{C})|$ is the classifying space BG.

We can construct a simplicial set that is not a Kan complex by taking the nerve of a small category that is not a groupoid. The simplest example is the two object category with one nonidentity morphism, $\{a \to b\}$. In its nerve the horn Λ^2_0 corresponding to the diagram

$$\begin{array}{ccc} b & & \\ \uparrow & & \\ a & \rightarrow & a \end{array}$$

cannot be extended to a map from Δ^2 because there is no morphism $b \to a$.

For an arbitrary small category $\mathcal{C}$ the horn Λ^2_1 of the 2-simplex of $N(\mathcal{C})$ associated with the diagram $a \to b \to c$ is the diagram

$$\begin{array}{ccc} a & & \\ \downarrow & & \\ b & \rightarrow & c. \end{array}$$

It can always be extended uniquely to Δ^2 using the composite morphism $a \to c$. More generally each horn Λ^n_i for $0 < i < n$ (the so called inner horns) extends uniquely to an n-simplex Δ^n. A small category is determined by its nerve, so a small category is the same thing as a simplicial set in which each inner horn Λ^n_i extends uniquely to an n-simplex Δ^n.

Now suppose we have a simplicial set X in which each inner horn Λ^n_i extends (possibly in more than one way) to an n-simplex Δ^n. Then we have something like a category in that it has a set of objects X_0

and a set of morphisms X_1, each with an object and source identified by face operators. Each object or 0-simplex has an identity morphism identified by the degeneracy operator $X_0 \to X_1$. However each composable pair of morphisms has not just one composite but a nonempty set of them. *This is the definition of an ∞-category.* For introductions to this subject, see papers by Caramena and Groth.

Next topic: What is a (commutative or not) ring spectrum R? It should have a map $m: R \wedge R \to R$ with certain properties. There should be a unit map $i: S^{-0} \to R$ with $m(i \wedge R) = m(R \wedge i) = R$. The associativity condition is $m(R \wedge m) = m(m \wedge R)$.

For commutativity, let $t: R \wedge R \to R \wedge R$ be the twist map,ie the natuaral isomorphism given by the symmetric monoidal strcuture. We want $mt = m$. Equivalently, m should factor through $Sym^2 R$.

We say that R is an associative (commutative) algebra if it satisfies these conditions. We get subcategories $\mathcal{Assoc}$ and $\mathcal{Comm}$ of the category of G-spectra. We need model structures for them. Recall that S^{-0} is not cofibrant.