DETECTING β ELEMENTS IN ITERATED ALGEBRAIC K-THEORY

GABRIEL ANGELINI-KNOLL

ABSTRACT. The Lichtenbaum–Quillen conjecture (LQC) relates special values of zeta functions to algebraic K-theory groups. The Ausoni–Rognes red-shift conjectures generalize the LQC to higher chromatic heights in a precise sense. In this paper, we propose an alternate generalization of the LQC to higher chromatic heights and give evidence for it at height two. In particular, if the n-th Greek letter family is detected by a commutative ring spectrum R, then we conjecture that the $n+1$-st Greek letter family will be detected by the algebraic K-theory of R. We prove this in the case $n = 1$ for $R = \mathrm{K}(\mathbb{F}_q)$ modulo (p, v_1) where $p \geq 5$ and $q = \ell^k$ is a prime power generator of the units in $\mathbb{Z}/p^2\mathbb{Z}$. In particular, we prove that the commutative ring spectrum $\mathrm{K}(\mathrm{K}(\mathbb{F}_q))$ detects the part of the p-primary β-family that survives mod (p, v_1). The method of proof also implies that these β elements are detected in iterated algebraic K-theory of the integers. Consequently, one may relate iterated algebraic K-theory groups of the integers to integral modular forms satisfying certain congruences.

1. Introduction

Following F. Waldhausen [44, §4], the famous Lichtenbaum–Quillen conjecture (LQC) [26, 35] states that the map

$$(1.1) \qquad \mathrm{K}_n(A; \mathbb{Z}/p\mathbb{Z}) \to \mathrm{K}_n^{\text{ét}}(A; \mathbb{Z}/p\mathbb{Z})$$

from algebraic K-theory to étale K-theory is an isomorphism for n sufficiently large for suitable rings A when $p > 2$ is a prime. Here by suitable ring A, we mean that A is a regular Noetherian commutative ring with $A = A[p^{-1}]$, which is finitely generated as a $\mathbb{Z}$-algebra [26, §9 p.175]. Since algebraic K-theory satisfies Nisnevich descent and étale K-theory satisfies étale descent, the question can be translated into the question of whether the map from motivic cohomology to étale cohomology is an isomorphism in a range. In this way, the conjecture was resolved by work of M. Rost and V. Voevodsky [43] as a consequence of their proof of the Bloch–Kato conjecture (see [19] for a complete proof and [45, Historical Remark 4.4] for further discussion).

In [40], R.W. Thomason showed that $\mathrm{K}_n^{\text{ét}}(A; \mathbb{Z}/p\mathbb{Z}) \cong \beta^{-1}\mathrm{K}_n(A; \mathbb{Z}/p\mathbb{Z})$ under the same conditions on A where β is the Bott element in $\mathrm{K}_2(A; \mathbb{Z}/p\mathbb{Z})$. From the perspective of homotopy theory, we may therefore view the map (1.1) as the map

1

on π_n induced by the map of spectra

$$S/p \wedge \mathrm{K}(A) \to v_1^{-1} S/p \wedge \mathrm{K}(A),$$

where S/p is the cofiber of multiplication by p. Here the map $v_1 \colon \Sigma^{2p-2} S/p \to S/p$ is a v_1-self map, which means it is non-nilpotent, and we define $v_1^{-1} S/p$ as the homotopy colimit of the diagram

$$S/p \xrightarrow{v_1} \Sigma^{-2p+2} S/p \xrightarrow{v_1} \Sigma^{-4p+4} S/p \xrightarrow{v_1} \dots$$

of spectra. The effect of inverting the Bott element is the same as the effect of inverting v_1 by work of V. Snaith [39] as interpreted by F. Waldhausen [44, §4].

The original motivation of the Lichtenbaum–Quillen conjecture was to relate algebraic K-theory groups to special values of zeta functions. For A the ring of integers in a totally real number field F and p an odd prime, Wiles proved that quotients of étale cohomology groups of $A[1/p]$ recover special values of the Dedekind zeta function ζ_F [46]. The Lichtenbaum–Quillen conjecture then gives a correspondence between algebraic K-theory groups and special values of Dedekind zeta functions, see [35, Conjecture 2.4]. Notably these special values correspond to the v_1-periodic part of $S/p_* \mathrm{K}(A)$ because they are detected in $v_1^{-1} S/p_* \mathrm{K}(A)$.

Fix an odd prime p throughout this section and fix a prime power q that generates the units in $\mathbb{Z}/p^2 \mathbb{Z}$. As another specific example, consider the algebraic K-theory of finite fields $\mathbb{F}_q$. D. Quillen [34] computed $\mathrm{K}_n(\mathbb{F}_q)$ for all n and after localizing at p, there is an isomorphism

$$\mathrm{K}_{2(p-1)k-1}(\mathbb{F}_q; \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p^{\nu_p(k)+1} \mathbb{Z},$$

where $\nu_p(k)$ is the p-adic valuation of k. The order of the group $\mathrm{K}_{2s-1}(\mathbb{F}_q; \mathbb{Z}_{(p)})$ corresponds exactly to the p-adic valuation of the denominator of $B_s/2s$ where B_s is the s-th Bernoulli number. Recall that Bernoulli numbers are the coefficients in the Taylor series

$$\frac{x}{e^x - 1} = \sum_{s \geq 0} B_s \frac{x^s}{s!}$$

and the special values of the Riemann zeta function satisfy $\zeta(-s) = (-1)^s B_s/(s+1)$ for $s \geq 0$.

This example is intimately tied to stable homotopy theory as well. J. F. Adams showed that the image of the J-homomorphism from the homotopy groups of the stable orthogonal group to the stable homotopy groups of spheres is highly nontrivial and the classical Bott periodicity in the homotopy groups of the stable orthogonal group corresponds to periodicity in the stable homotopy groups of spheres [1]. In fact, the p-local image of the J-homomorphism exactly corresponds to the image of the map $\pi_{2(p-1)k-1} S_{(p)} \to \mathrm{K}_{2(p-1)k-1}(\mathbb{F}_q; \mathbb{Z}_{(p)})$ when p is an odd prime. The image of J therefore bridges the fields of homotopy theory and number theory. The spectrum $H\mathbb{F}_q$ detects v_0-periodicity in the sense that $p^k = v_0^k$ is nontrivial in the image of the Hurewicz map $\pi_* S \to \pi_* H\mathbb{F}_q$. Therefore, we have observed an instance where algebraic K-theory of a spectrum that detects v_0-periodic elements detects v_1-periodic elements. One goal of this introduction is to formulate a precise conjecture about a higher chromatic height generalization of this phenomenon. The main theorem of this paper is evidence for this conjecture at a higher chromatic height.

In chromatic stable homotopy theory, we study periodic families of elements in the homotopy groups of spheres. The first such family, due to J. F. Adams [1] and H. Toda [41], is the α-family, which consists of maps α_k defined as the composites

$$\alpha_k \colon \Sigma^{(2p-2)k} S \xrightarrow{i_0} \Sigma^{(2p-2)k} S/p \xrightarrow{v_1^k} S/p \xrightarrow{\delta_0} \Sigma S \ ,$$

where p is an odd prime. The elements α_k are p-torsion elements in the groups $\pi_{2(p-1)k-1} S$. These elements are in the image of the J-homomorphism at odd primes p and as discussed earlier they are also detected in algebraic K-theory of finite fields of order q, under our hypotheses on p and q. In particular, they correspond to certain special values of the Riemann zeta function. Now, consider the cofiber of the periodic self-map $v_1 \colon \Sigma^{2p-2} S/p \to S/p$ denoted $V(1)$. When $p \geq 5$, there exists a periodic self-map $v_2 \colon \Sigma^{2p^2-2} V(1) \to V(1)$ and there is an associated periodic family of elements in the homotopy groups of spheres called the β-family. In particular, L. Smith [38] proved that the maps

$$\beta_k \colon \Sigma^{(2p^2-2)k} S \xrightarrow{i_1 i_0} \Sigma^{(2p^2-2)k} V(1) \xrightarrow{v_2^k} V(1) \xrightarrow{\delta_0 \delta_1} \Sigma^{2p} S$$

are nontrivial.

In the language of chromatic homotopy theory, the α-family is a periodic family of height one and the β-family is a periodic family of height two. There is a family of homology theories $K(n)_*$ called Morava K-theory which is useful for detecting periodicity of chromatic height n in the homotopy groups of spheres. The coefficients of Morava K-theory are $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ for $n \geq 1$ and $K(0)_*$ is rational homology. We say a p-local finite cell S-module V has type n if the groups $K(n)_* V \neq 0$ and the groups $K(n-1)_* V$ vanish. By the celebrated periodicity theorem of Hopkins–Smith [23], any p-local finite spectrum V of type n admits a periodic self-map

$$v_n^m \colon \Sigma^{(2p^n-2)m} V \to V.$$

We can therefore define $v_n^{-1} V$ in the same way that we defined $v_1^{-1} S/p$. We can also construct the n-th Greek letter family by including into the bottom cell, iterating v_n^m k-times, and then projecting onto the top cell. However, it is highly non-trivial to prove that Greek letter elements that are constructed in this way are actually nonzero.

The study of Greek letter family elements was significantly expanded by the groundbreaking work of Miller–Ravenel–Wilson [28] using the chromatic spectral sequence

$$(1.2) \quad E_1^{*,*} = \bigoplus_{i \geq 0} \mathrm{Ext}^{*,*}_{BP_* BP}(BP_*, v_i^{-1} BP_*/(p^\infty, v_1^\infty, \dots, v_{i-1}^\infty))$$

$$\implies \mathrm{Ext}^{*,*}_{BP_* BP}(BP_*, BP_*)$$

which converges to the input of the BP-Adams spectral sequence. If the class

$$v_n^k/p^{i_0} v_1^{i_1} \dots v_{n-1}^{i_{n-1}} \in \mathrm{Ext}^0_{BP_* BP}(BP_*, v_n^{-1} BP_*/(p^\infty, v_1^\infty, \dots, v_{n-1}^\infty))$$

in the E_1-page of (1.2) is a permanent cycle in the chromatic spectral sequence, we will write

$$(1.3) \qquad \overline{\alpha}^{(n)}_{k/(i_{n-1}, i_{n-2}, \dots i_0)} \in \mathrm{Ext}^{*,*}_{BP_* BP}(BP_*, BP_*)$$

for its image in the abutment of the chromatic spectral sequence. We will refer to the collection of all such elements for a fixed n as the n-th divided (algebraic) Greek letter family and when any of the i_j for $0 \leq j \leq n-1$ are 1 we omit them from the notation. If the elements $\overline{\alpha}_k^{(n)}$ are permanent cycles in the BP-Adams spectral sequence, then we will write $\alpha_k^{(n)}$ for the corresponding elements in the stable homotopy groups of spheres and refer to the collection as the *n-th Greek letter family*. The advantage of this approach is that the elements in the input of the chromatic spectral sequence always exist. The question of whether or not certain Greek letter elements exist in homotopy can then be approached by determining whether certain elements in the chromatic spectral sequence and the BP-Adams spectral sequence are permanent cycles.

We will say that a ring spectrum R *detects* the n-th Greek letter family $\{\alpha_k^{(n)}\}$ in the homotopy groups of spheres if each element $\alpha_k^{(n)}$ is non-trivial in the image of the unit map
$$\pi_* S \longrightarrow \pi_* R.$$
We conjecture the following higher chromatic height analogue of the Lichtenbaum–Quillen conjecture, which is in the same spirit as the red-shift conjectures of Ausoni–Rognes [6]. For Conjecture 1.1, suppose the n-th and the $n+1$-st Greek letter family are nontrivial elements in $\pi_* S$ for a given prime p.

Conjecture 1.1. *If R is a commutative ring spectrum that detects the n-th Greek letter family, then $\mathrm{K}(R)$ detects the $n+1$-st Greek letter family.*

When p is sufficiently small Conjecture 1.1 is both more difficult to verify and more subtle, for example at the prime $p = 3$ only the β elements β_i for $i \equiv 0, 1, 3, 5, 6$ mod 9 are known to exist by [12].

We can now state the main theorem of this paper. As discussed earlier, the spectrum $\mathrm{K}(\mathbb{F}_q)_p$ detects the α-family for $p \geq 3$. The main theorem of this paper is a proof of Conjecture 1.1 in the case $n = 1$ where $R = \mathrm{K}(\mathbb{F}_q)_p$, with the following caveat: Since the approach we take is to detect the β-family in the image of the unit map
$$S_* \rightarrow V(1)_* \mathrm{K}(\mathrm{K}(\mathbb{F}_q)_p),$$
we only detect the part of the p-primary β-family that is detected in $V(1)_*$.[1]

Theorem 1.2. *The commutative ring spectrum $\mathrm{K}(\mathrm{K}(\mathbb{F}_q)_p)$ detects the p-primary β-family $\{\beta_k : k \not\equiv 0 \mod p\}$ for all $p \geq 5$.*

In particular, the method of proof also provides the following higher Lichtenbaum–Quillen-type result about iterated algebraic K-theory of the integers.

Corollary 1.3. *The commutative ring spectrum $\mathrm{K}(\mathrm{K}(\mathbb{Z}))$ detects the p-primary β-family $\{\beta_k : k \not\equiv 0 \mod p\}$ for all $p \geq 5$.*

In [10, 11, 25], M. Behrens and G. Laures describe an explicit relationship between the β-family and certain integral modular forms satisfying certain congruences. From this point of view, our main result may be viewed as a higher chromatic height version of the Lichtenbaum–Quillen conjecture. It is therefore a step towards the larger program of understanding the arithmetic of commutative ring spectra.

---

[1]Note that the elements β_{pk} are not p-divisible by [28, Theorem 2.12] and they are p-torsion by construction, however they are v_1-divisible in $\pi_* S/p$ by [31, Theorem A] and [32, Theorem AII] and consequently they have trivial Hurewicz image in $V(1)_*$, see [37] for a survey.

The β elements β_k that we detect only agree with the divided β-family elements $\beta_{k/i,j}$ of M. Behrens [10] when $i = j = 1$ and $k \not\equiv 0 \mod p$. The explicit modular forms associated to these beta elements are also computed in [24]. It would be desirable to detect the entire *divided* β-family in iterated algebraic K-theory of finite fields to detect more information about integral modular forms.

1.1. **Conventions.** Let Sp be the category of symmetric spectra in pointed simplicial sets with the positive flat stable model structure. Most of the results here can also be proven for other models of the stable homotopy category since they depend only on the homotopy category, but the proof relies on [2], which uses this model structure. We write CommSp for the category of commutative monoids in Sp, which we call commutative ring spectra, and $\mathsf{Assoc\,Sp}$ for the category of associative monoids in Sp, which we call associative ring spectra. By a ring spectrum, we mean a homotopy associative monoid in Sp.

Throughout, we write $H_*(-)$ for $\pi_*(- \wedge H\mathbb{F}_p)$; i.e., homology with $\mathbb{F}_p$-coefficients and for a general spectrum E we write $\mathsf{E}_* X = \pi_*(X \wedge \mathsf{E})$. We also write $H_*(-; d)$ for homology of a chain complex with differential d and dually we write $H^*(-; d)$ for the cohomology of a cochain complex with differential d. We write $\otimes$ for tensor over $\mathbb{F}_p$ if not otherwise specified. We write $\mathrm{HH}_*(A)$ for the Hochschild homology over $\mathbb{F}_p$ of a bi-graded $\mathbb{F}_p$ algebra A. We also fix the convention that $E_R(x_1, \ldots, x_n)$ denotes the graded exterior algebra over a ring R with indecomposable algebra generators $x_1, \ldots, x_n$, $P_R(y_1, \ldots, y_n)$ denotes the graded polynomial algebra over a ring R with indecomposable algebra generators $y_1, \ldots, y_n$, and $\Gamma_R(z_1, \ldots z_n)$ denotes the graded divided power algebra with indecomposable algebra generators $z_1, \ldots, z_n$. When $R = \mathbb{F}_p$ we omit R from the notation.

A comodule M over a Hopf algebroid $(\mathsf{E}_*, \mathsf{E}_* \mathsf{E})$ associated to a ring spectrum E will always be considered with left co-action

$$\psi_M^{\mathsf{E}} \colon M \to \mathsf{E}_* \mathsf{E} \otimes_{\mathsf{E}_*} M$$

and we will simply write ψ when the module M and the Hopf algebroid $(\mathsf{E}, \mathsf{E}_* \mathsf{E})$ is understood from the context. The main examples of interest are $\mathsf{E} = H\mathbb{F}_p$, where $\mathsf{E}_* \mathsf{E}$ is the dual Steenrod algebra $\mathcal{A}_*$, and $\mathsf{E} = BP$. We write

$$\Delta_{\mathsf{E}} \colon \mathsf{E}_* \mathsf{E} \to \mathsf{E}_* \mathsf{E} \otimes_{\mathsf{E}_*} \mathsf{E}_* \mathsf{E}$$

for the co-product of the Hopf-algebroid $\mathsf{E}_* \mathsf{E}$ or simply Δ when E is understood from the context. When $\mathsf{E} = H\mathbb{F}_p$, this is the co-product in the dual Steenrod algebra $\mathcal{A}_* \cong P(\bar{\xi}_i \mid i \geq 1) \otimes E(\bar{\tau}_i \mid i \geq 0)$ which is defined on each algebra generator by the formulas

$$\Delta(\bar{\xi}_n) = \sum_{i+j=n} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i},$$

$$\Delta(\bar{\tau}_n) = 1 \otimes \bar{\tau}_n + \sum_{i+j=n} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}.$$

Here, by $\bar{x}$ we mean χx where $\chi \colon \mathcal{A}_* \to \mathcal{A}$ is the antipode structure map of the Hopf-algebra $\mathcal{A}_*$. When $\mathsf{E} = BP$, the co-product on elements of $BP_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \otimes \mathbb{Z}_{(p)}[t_1, t_2, \ldots]$ is defined by the formula

$$\Delta(t_n) = \sum_{i+j=n}^{F} t_i \otimes t_j^{p^i},$$

where F is the formal group law of BP associated to the complex orientation $MU \to BP$, which equips BP with the universal p-typical formal group law.

We will write X_p for the p-completion of a spectrum X, which agrees with the Bousfield localization $L_{S/p}X$ at the mod p Moore spectrum S/p. We will write $\doteq$ to indicate that an equality holds up to multiplication by a unit in $\mathbb{F}_p$. Given a spectral sequence with signature $E_n^{*,*} \implies A_*$, we use the following terminology:

(1) An element $x \in E_n^{*,*}$ is an *infinite cycle* if $d_r(x) = 0$ for all $r \geq n$.
(2) An element $x \in E_n^{*,*}$ is a *permanent cycle* if $d_r(x) = 0$ for all $r \geq n$ and there does not exist a y such that $d_r(y) = x$ for any $r \geq n$.

Also, throughout the paper, fix a prime $p \geq 5$ and let q be a prime power that topologically generates $\mathbb{Z}_p^\times$, or equivalently q generates the units in $\mathbb{Z}/p^2\mathbb{Z}$.

2. OVERVIEW OF THE TOOLKIT

2.1. **The Hochschild–May spectral sequence.** We recall necessary results and definitions from the author's paper [3] and the author's joint paper with A. Salch [2] since they will be cited later. We write for $\mathbb{N}^{\mathrm{op}}$ for the category with objects the natural numbers $n \geq 0$ and morphisms defined by

$$\mathrm{Hom}_{\mathbb{N}^{\mathrm{op}}}(n, m) = \begin{cases} * & \text{if } n \geq m, \\ \emptyset & \text{if } n < m. \end{cases}$$

Definition 2.1. A filtered commutative ring spectrum I is a cofibrant object in $\mathrm{CommSp}^{\mathbb{N}^{\mathrm{op}}}$ where $\mathrm{CommSp}^{\mathbb{N}^{\mathrm{op}}}$ has the model structure created by the forgetful functor to $\mathsf{Sp}^{\mathbb{N}^{\mathrm{op}}}$ and $\mathsf{Sp}^{\mathbb{N}^{\mathrm{op}}}$ has the projective model structure.[2] (See [2, Sec. 4.1] for a discussion of why these model structures exist and have the desired properties.) We write I_i for I evaluated on the natural number i. The associated graded of I is a commutative ring spectrum $E_0^* I$ and it is defined so that, after forgetting the commutative monoid structure, it is the spectrum $\bigvee_{i \geq 0} I_i/I_{i+1}$ where I_i/I_{i+1} is the cofiber of the map $I_{i+1} \to I_i$. Note that by abuse of notation we may also regard the associated graded $E_0^* I = (I_0/I_1, I_1/I_2, \dots)$ as a commutative monoid in $\mathsf{Sp}^{\mathbb{N}^\delta}$ with respect to the Day convolution symmetric monoidal product where $\mathbb{N}^\delta$ is the natural numbers regarded as a discrete category. We can therefore consider topological Hochschild homology of $E_0^* I$ in the category $\mathsf{Sp}^{\mathbb{N}^\delta}$ so that $\mathrm{THH}(E_0^* I)$ is an object in $\mathsf{Sp}^{\mathbb{N}^\delta}$. Applying a generalized homology theory E levelwise, we still have an object $\mathsf{E} \wedge \mathrm{THH}(E_0^* I)$ in $\mathsf{Sp}^{\mathbb{N}^\delta}$. Applying homotopy groups induces a functor $\pi_s^{\mathbb{N}^\delta} \colon \mathsf{Sp}^{\mathbb{N}^\delta} \longrightarrow \mathsf{Ab}^{\mathbb{N}^\delta}$ and then we write

$$(2.1) \qquad \mathsf{E}_{s,t}\mathrm{THH}(E_0^* I) = [\pi_s^{\mathbb{N}^\delta}(\mathsf{E} \wedge \mathrm{THH}(E_0^* I))](t).$$

Example 2.2. By [2, Thm 4.2.1], an example of a filtered commutative ring spectrum associated to a connective commutative ring spectrum R is the Whitehead filtration

$$\cdots \to \tau_{\geq 2} R \to \tau_{\geq 1} R \to \tau_{\geq 0} R$$

which is equipped with structure maps $\rho_{i,j} \colon \tau_{\geq i} R \wedge \tau_{\geq j} R \to \tau_{\geq i+j} R$. Here $\tau_{\geq s} R$ is a spectrum with $\pi_i(\tau_{\geq s} R) \cong 0$ for $i < s$ that is equipped with a map $\tau_{\geq s} R \to R$ that induces an isomorphism on homotopy groups π_i for $i \geq s$. We write simply $\tau_{\geq \bullet} R$

---

[2]This definition differs slightly from that in [2, Def 3.1.2], but a cofibrant object in $\mathrm{CommSp}^{\mathbb{N}^{\mathrm{op}}}$ is always a decreasingly filtered commutative monoid in Sp in the sense of loc. cit.

for the filtered commutative ring spectrum constructed in loc. cit. as a cofibrant object in $\mathsf{Comm}\mathsf{Sp}^{\mathbb{N}^{\mathrm{op}}}$.

Theorem 2.3 ([2, Theorem 3.4.8]). *Let I be a filtered commutative ring spectrum and let E be a connective spectrum. There is a spectral sequence with signature*

$$E^1_{s,t} = \mathsf{E}_{s,t}\mathrm{THH}(E^*_0 I) \implies \mathsf{E}_s\mathrm{THH}(I_0),$$

which we call the E-Hochschild–May spectral sequence, where the bi-grading on the input is explained in Equation (2.1).

Remark 2.4. When $I = \tau_{\geq \bullet} R$ we simply write $H\pi_* R$ for $E^*_0 I$. It is a generalized Eilenberg-MacLane spectrum so whenever $\pi_k R$ is a finitely generated abelian group for all k and $\mathsf{E} = S/p$, $H\mathbb{F}_p$, $V(1)$, or $BP \wedge V(1)$, then $\mathsf{E}_* \mathrm{THH}(H\pi_* R)$ is a graded $H\mathbb{F}_p$-algebra and we can apply Lemma 2.5 to compute the input.

Lemma 2.5 is a consequence of the fact that all $H\mathbb{F}_p$-modules are equivalent to a wedge of suspensions of $H\mathbb{F}_p$ and an Adams spectral sequence argument (cf. [7, Lem. 4.1]).

Lemma 2.5. *Let M be an $H\mathbb{F}_p$-algebra. Then M is equivalent to a wedge of suspensions of $H\mathbb{F}_p$, and the Hurewicz map*

$$\pi_* M \longrightarrow H_* M$$

induces an isomorphism between $\pi_* M$ and the subalgebra of $\mathcal{A}_*$-comodule primitives contained in $H_* M$.

Using the lemma above, one can compute the E^1-page of the $H\mathbb{F}_p$-Hochschild–May spectral sequence, modulo an explicit description of $\mathrm{HH}_*(S/p_*(H\pi_*\mathrm{K}(\mathbb{F}_q)_p))$ given in [3, p. 271-272]. We will argue in the same way as [3]: there is an algebraic Hochschild–May spectral sequence which can be used to compute the E^2-page of the E-Hochschild–May spectral sequence, for $\mathsf{E} = H\mathbb{F}_p$, $\mathsf{E} = H\mathbb{F}_p \wedge V(1)$ and $\mathsf{E} = V(1)$, and therefore at the E^2-page we have a simpler description. We give the description of both the E^1-page and the E^2-page. Note that the E^1-page of the $H\mathbb{F}_p$-Hochschild–May spectral sequence associated to the Whitehead filtration of $\mathrm{K}(\mathbb{F}_q)_p$ is

$$E^1_{*,*} = H_*(\mathrm{THH}(H\pi_*\mathrm{K}(\mathbb{F}_q)_p)).$$

Proposition 2.6 ([3, Proposition 3.10, Proposition 3.11]). *There is an isomorphism of $\mathcal{A}_*$-comodule algebras*

$$E^1_{*,*} = H_*(\mathrm{THH}(H\pi_*\mathrm{K}(\mathbb{F}_q)_p)) \cong \mathcal{A}//E(0)_* \otimes E(\sigma\bar{\xi}_1) \otimes P(\sigma\bar{\tau}_1) \otimes N,$$

where $N = \mathrm{HH}_*(S/p_*(H\pi_*\mathrm{K}(\mathbb{F}_q)_p))$ and where the $\mathcal{A}_*$-co-action on $\mathcal{A}//E(0)_*$ is the usual one, that is the restriction of the co-product in $\mathcal{A}_*$, the coaction on $\sigma\bar{\xi}_1$ is

$$\psi(\sigma\bar{\xi}_1) = 1 \otimes \sigma\bar{\xi}_1$$

and the co-action on $\sigma\bar{\tau}_1$ is

$$\psi(\sigma\bar{\tau}_1) = 1 \otimes \sigma\bar{\tau}_1 + \bar{\tau}_0 \otimes \sigma\bar{\xi}_1.$$

The E^2-page of the $H\mathbb{F}_p$-Hochschild–May spectral sequence associated to the Whitehead filtration of $\mathrm{K}(\mathbb{F}_q)_p$ is

$$E^2_{*,*} = \mathcal{A}//E(0)_* \otimes E(\lambda_1) \otimes P(\mu_1) \otimes E(\alpha_1) \otimes P(v_1) \otimes E(\sigma v_1) \otimes \Gamma(\sigma\alpha_1)$$

as an $\mathcal{A}_*$-comodule, where $\lambda_1 = \sigma\overline{\xi}_1$, $\mu_1 = \sigma\overline{\tau}_1 - \overline{\xi}_1\sigma\overline{\tau}_0$, with the usual $\mathcal{A}_*$-co-action on $\mathcal{A}//E(0)_*$ given by restricting the co-product on $\mathcal{A}_*$. The elements $\alpha_1 v_1^k$, v_1^k, λ_1, μ_1, and $\sigma\alpha_1$ are $\mathcal{A}_*$-comodule primitives. Multiplicatively, the E^2-page is isomorphic to

$$E_{*,*}^2 = \mathcal{A}//E(0)_* \otimes E(\lambda_1) \otimes P(\mu_1) \otimes P(x_i : i \geq 1)/(x_i x_j : i,j \geq 1) \otimes E(\sigma v_1) \otimes \Gamma(\sigma\alpha_1)$$

so that $x_{2i} = v_1^i$, $x_{2i-1} = \alpha_1 v_1^{i-1}$ for $i \geq 1$. The bidegrees of the indecomposable algebra generators at the E^2-page are given by

$$|\sigma\overline{\xi}_1| = (2p-1, 0),$$
$$|\sigma\overline{\tau}_1| = (2p, 0),$$
$$|v_1^k| = ((2p-2)k, (2p-2)k - 1),$$
$$|\sigma v_1| = (2p-1, 2p-2),$$
$$|\gamma_k(\sigma\alpha_1)| = ((2p-2)k, (2p-3)k),$$
$$|\alpha_1 v_1^{k-1}| = ((2p-2)k - 1, (2p-2)k - 1) \text{ for } k \geq 1.[3]$$

The products $v_1^{k+1} \cdot v_1^{j+1} = 0$, $\alpha_1 v_1^k \cdot v_1^{j+1} = 0$, and $\alpha_1 v_1^k \cdot \alpha_1 v_1^j = 0$ for $j, k \geq 0$ hold for bi-degree reasons as made clear by [3, Figure 1].

By [38], we know that

$$H_*(V(1)) = E(\tau_0, \tau_1)$$

with $|\tau_0| = 1$ and $|\tau_1| = 2p - 1$ with $\mathcal{A}_*$-coaction defined by

$$(2.2) \qquad\qquad \psi(\tau_0) = 1 \otimes \tau_0 + \tau_0 \otimes 1,$$
$$(2.3) \qquad\qquad \psi(\tau_1) = 1 \otimes \tau_1 + \tau_1 \otimes 1 + \xi_1 \otimes \tau_0.$$

Note that using the recursive formulas

$$\sum_{i=0}^n \xi_{n-i}^{p^i} \overline{\xi}_i = 0 \quad \text{and} \quad \sum_{i=0}^n \xi_{n-i}^{p^i} \overline{\tau}_i + \tau_n = 0,$$

we can deduce that $\tau_0 = -\overline{\tau}_0$, $\xi_1 = -\overline{\xi}_1$, and consequently $\tau_1 = -\overline{\tau}_1 + \overline{\xi}_1\overline{\tau}_0$ (cf. [29, Lem. 10]). This allows us to rewrite the formulas above in terms of our chosen basis for $\mathcal{A}_*$.

Combining this with Proposition 2.6, we computed the E^1-page and E^2-page of the $V(1)$-Hochschild–May spectral sequence using Lemma 2.5, which we again recall modulo an explicit description of $\mathrm{HH}_*(S/p_*(H\pi_* \mathrm{K}(\mathbb{F}_q)_p))$, which can be found in [3, p. 271-272]. Recall that the E^1-page of the $V(1)$-Hochschild–May spectral sequence associated to the Whitehead filtration of $\mathrm{K}(\mathbb{F}_q)_p$ is

$$E_{*,*}^1 = V(1)_* \mathrm{THH}(H\pi_* \mathrm{K}(\mathbb{F}_q)_p).$$

Proposition 2.7 ([3, Prop. 3.15]). *There is an isomorphism of graded $\mathbb{F}_p$-algebras*

$$E_{*,*}^1 = V(1)_* \mathrm{THH}(H\pi_* \mathrm{K}(\mathbb{F}_q)_p) \cong E(\lambda_1, \epsilon_1) \otimes P(\mu_1) \otimes \mathrm{HH}_*(S/p_*(H\pi_* \mathrm{K}(\mathbb{F}_q)_p))$$

---

[3]There are a few minor typos in the statement of [3, Prop. 3.11]: First, $\sigma\overline{\tau}_2$ should be $\sigma\overline{\tau}_1$. This is clear from [3, Prop. 3.13]. Second, the ideal $\mathfrak{m}$ should be $(x_i x_j : i,j \geq 1)$. This is clear from the additive description. Third, we correct the bidegrees of the elements in the statement, where the correct bidegrees are clear from [3, Figure 2].

and the E^2-page of the $V(1)$-Hochschild–May spectral sequence associated to the Whitehead filtration of $\mathrm{K}(\mathbb{F}_q)_p$ is

$$E^2_{*,*} = E(\lambda_1, \epsilon_1) \otimes P(\mu_1) \otimes E(\alpha_1) \otimes P(v_1) \otimes \Gamma(\sigma\alpha_1) \otimes E(\sigma v_1)$$

as a graded $\mathbb{F}_p$-module where

$$\begin{aligned}
|\epsilon_1| = |\lambda_1| &= (2p-1, 0), \\
|\mu_1| &= (2p, 0), \\
|v_1^k| &= ((2p-2)k, (2p-2)k-1), \\
|\sigma v_1| &= (2p-1, 2p-2), \\
|\gamma_k(\sigma\alpha_1)| &= ((2p-2)k, (2p-3)k), \\
|\alpha_1 v_1^{k-1}| &= ((2p-2)k-1, (2p-2)k-1) \ for \ k \geq 1.
\end{aligned}$$

Here $\epsilon_1 = \tau_1 - \tilde{\tau}_1$ where $\tilde{\tau}_1$ is the indecomposable algebra generator in degree $2p-1$ in $\mathcal{A}//E(0)_* \subset H_*(\mathrm{THH}(H\pi_*\mathrm{K}(\mathbb{F}_q)_p))$ and the multiplicative structure on the E^2-page of the $V(1)$-Hochschild–May spectral sequence associated to the Whitehead filtration of $\mathrm{K}(\mathbb{F}_q)_p$ is

$$E^2_{*,*} = E(\lambda_1, \epsilon_1) \otimes P(\mu_1) \otimes \Gamma(\sigma\alpha_1) \otimes E(\sigma v_1) \otimes P(x_i : i \geq 1)/(x_i x_j : i, j \geq 1),$$

where $x_{2i} = v_1^i$ and $x_{2i-1} = \alpha_1 v_1^{i-1}$ for $i \geq 1$.

Our computations build on the computation of homology of topological Hochschild homology of $\mathrm{K}(\mathbb{F}_q)_p$ due to Angeltveit–Rognes [4].

Theorem 2.8 ([4, Theorem 7.13 and Theorem 7.15])**.** *There is an isomorphism of $\mathcal{A}_*$-comodule algebras*

$$H_*(\mathrm{K}(\mathbb{F}_q)_p) \cong P(\tilde{\xi}_1^p, \tilde{\xi}_2, \bar{\xi}_3, \dots) \otimes E(\tilde{\tau}_2, \bar{\tau}_3, \dots) \otimes E(b) \cong (\mathcal{A}//A(1))_* \otimes E(b),$$

where all the elements in $(\mathcal{A}//A(1))_*$ besides $\tilde{\tau}_2$, $\tilde{\xi}_1^p$, and $\tilde{\xi}_2$, and b have the usual $\mathcal{A}_*$-co-action and the co-action on the remaining elements $\tilde{\tau}_2$, $\tilde{\xi}_1^p$, $\tilde{\xi}_2$, and b are

$$\begin{aligned}
\psi(b) &= 1 \otimes b, \\
\psi(\tilde{\xi}_1^p) &= 1 \otimes \tilde{\xi}_1^p - \tau_0 \otimes b + \bar{\xi}_1^p \otimes 1, \\
\psi(\tilde{\xi}_2) &= 1 \otimes \tilde{\xi}_2 + \bar{\xi}_1 \otimes \tilde{\xi}_1^p + \tau_1 \otimes b + \bar{\xi}_2 \otimes 1, \\
\psi(\tilde{\tau}_2) &= 1 \otimes \tilde{\tau}_2 + \bar{\tau}_1 \otimes \tilde{\xi}_1^p + \bar{\tau}_0 \otimes \tilde{\xi}_2 - \tau_1 \tau_0 \otimes b + \bar{\tau}_2 \otimes 1.
\end{aligned}$$

The degrees of $\tilde{\tau}_2$, $\tilde{\xi}_1^p$, and $\tilde{\xi}_2$, and b are

$$\begin{aligned}
|\tilde{\tau}_2| &= 2p^2 - 1, \\
|\tilde{\xi}_1^p| &= 2p^2 - 2p, \\
|\tilde{\xi}_2| &= 2p^2 - 2, \ and \\
|b| &= 2p^2 - 2p - 1,
\end{aligned}$$

respectively. There is also an isomorphism

$$H_*(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \cong H_*(\mathrm{K}(\mathbb{F}_q)_p) \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2) \otimes P(\sigma\tilde{\tau}_2) \otimes \Gamma(\sigma b)$$

of $\mathcal{A}_*$-comodules and $H_*(\mathrm{K}(\mathbb{F}_q)_p)$-algebras. The $\mathcal{A}_*$-co-action is given by using the formula

$$\psi(\sigma x) = (1 \otimes \sigma) \circ \psi(x)$$

and the previously stated co-actions.

Corollary 2.9. *There is an isomorphism of $\mathcal{A}_*$-comodule algebras*

$$H_*(V(1) \wedge \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \cong E(\tau_0, \tau_1) \otimes H_*(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p))$$

with $\mathcal{A}_*$-co-action on τ_0 given in (2.2), the coaction on τ_1 given in (2.3), and the coaction on $x \in H_*(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p))$ given in Theorem 2.8.

Finally, we recall the computation of $V(1)$-homotopy of topological Hochschild homology of $\mathrm{K}(\mathbb{F}_q)_p$. This was computed by the author in [3, Thm. 1.5] and by Höning in [22, Thm. 7.2] using different methods. [4]

Theorem 2.10. *There is an isomorphism of graded $\mathbb{F}_p$-algebras*

$$V(1)_*\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p) \cong P(\mu_2) \otimes \Gamma(\sigma b) \otimes \mathbb{F}_p\{1, \alpha_1, \lambda_1', \lambda_2\alpha_1, \lambda_2\lambda_1', \lambda_2\lambda_1'\alpha_1\},$$

where $\alpha_1 \cdot (\lambda_2\lambda_1') = \lambda_1' \cdot (\lambda_2\alpha_1) = \lambda_2\lambda_1'\alpha_1$.

2.2. The homotopy fixed point spectral sequence.

In this section, we summarize results from [16, §2-4], which also apply when replacing $H\mathbb{F}_p$ with any spectrum E.

Definition 2.11. We write

$$\mathsf{E}_{s+t}^c(X^{h\mathbb{T}}) := \lim \mathsf{E}_{s+t}F(S(\mathbb{C}^k)_+, X)^{\mathbb{T}},$$

for the *continuous E-homology* of $X^{h\mathbb{T}}$ and we refer to the filtration $F(S(\mathbb{C}^k)_+, X)^{\mathbb{T}}$ as the *skeletal* filtration. When E is a finite spectrum, then $\mathsf{E}_*^c(X^{h\mathbb{T}}) = \mathsf{E}_*(X^{h\mathbb{T}})$ since smashing with a dualizable spectrum commutes with homotopy limits.

Proposition 2.12 (cf. Proposition 2.1 and Proposition 4.1 in [16]). *Let E be a homotopy associative ring spectrum. There is a spectral sequence $\mathsf{E}_*\mathsf{E}$ comodules with signature*

$$E_{*,*}^2 = P_{\mathbb{Z}}(t) \otimes \mathsf{E}_*X \implies \mathsf{E}_*^c(X^{h\mathbb{T}}),$$

which we simply call the homotopy fixed point spectral sequence. *If, in addition, X is a commutative ring spectrum, then this is a spectral sequence of $\mathsf{E}_*\mathsf{E}$-comodule algebras where E_*X has the Pontryagin product.*

Proof. The proof is essentially the same as in [16, Prop. 2.1, Prop. 4.1] so we omit it. $\square$

Lemma 2.13. *The d^2 differentials in the homotopy fixed point spectral sequence associated to $\mathrm{THH}(R)$ are of the form*

$$d^2(x) = t\sigma x,$$

where t is the generator of $H_{gp}^{-*}(\mathbb{T}; \mathbb{Z}) \cong P_{\mathbb{Z}}(t)$ in degree -2 whenever $\eta \in \pi_1 S$ acts trivially on $\mathsf{E}_*\mathrm{THH}(R)$.

Proof. This follows directly from [21, Lemma 1.4.2]. $\square$

---

[4]There is a minor typo in the statement of [3, Thm. 1.5]. The element 1 should also be included. This is evident since $V(1)_*\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)$ is ring.

Definition 2.14. We let $T_k(R)$ denote the spectrum $F(S(\mathbb{C}^{k+1})_+, \mathrm{THH}(R))^{\mathbb{T}}$, which has the property that $\mathrm{holim}_k T_k(R) = \mathrm{THH}(R)^{h\mathbb{T}}$. We write

$$\mathrm{TC}^-(R) := \mathrm{THH}(R)^{h\mathbb{T}}$$

following the convention of [20] and we also write

$$\mathsf{E}_*^c(\mathrm{TC}^-(R)) := \mathsf{E}_*^c(\mathrm{THH}(R)^{h\mathbb{T}}).$$

Note that $T_0(R) = \mathrm{THH}(R)$.

Remark 2.15. Note that there is a *truncated homotopy fixed point spectral sequence* with signature

$$E_{*,*}^2 = P_{\mathbb{Z}}(t)/t^{k+1} \otimes \mathsf{E}_* \mathrm{THH}(R) \implies \mathsf{E}_* T_k(R)$$

that is constructed in exactly the same way as the $\mathbb{T}$-homotopy fixed point spectral sequence. The d^2 differential is the same as 2.13 whenever $\eta \in \pi_1 S$ acts trivially on $\mathsf{E}_* \mathrm{THH}(R)$.

3. DETECTING β ELEMENTS IN ITERATED ALGEBRAIC K-THEORY

3.1. **The E-Adams spectral sequence.**

The following discussion applies to any homotopy commutative and homotopy associative ring spectrum E such that $(\mathsf{E}_*, \mathsf{E}_* \mathsf{E})$ is a flat Hopf algebroid satisfying some mild additional hypotheses [36, Assumption 2.2.5], but for simplicity we assume $\mathsf{E} = H\mathbb{F}_p$ or $\mathsf{E} = BP$ (cf. [36, Prop. 2.2.6]). We write $\Delta \colon \mathsf{E}_* \mathsf{E} \to \mathsf{E}_* \mathsf{E} \otimes_{\mathsf{E}_*} \mathsf{E}_* \mathsf{E}$ for the co-product and $\eta_R \colon \mathsf{E}_* \to \mathsf{E}_* \mathsf{E}$ for the right unit. Given a spectrum X we write $\psi \colon \mathsf{E}_* X \to \mathsf{E}_* \mathsf{E} \otimes_{\mathsf{E}_*} \mathsf{E}_* X$ for the left $(\mathsf{E}_*, \mathsf{E}_* \mathsf{E})$-coaction on $\mathsf{E}_* X$. We also assume X is bounded below and $\mathsf{E}_* X$ is finite type for simplicity.

We briefly recall the construction of the E-Adams spectral sequence. For more details, we refer the reader to [36, Ch. 2 §2]. The main examples we will consider are $\mathsf{E} = H\mathbb{F}_p$ and $\mathsf{E} = BP$. Classically, we can build the E-Adams spectral sequence for X using the canonical E-Adams resolution

$$(3.1) \qquad X \xleftarrow{\;g_0\;} \Sigma^{-1}\overline{\mathsf{E}} \wedge X \xleftarrow{\;g_1\;} \Sigma^{-2}\overline{\mathsf{E}}^{\wedge 2} \wedge X \xleftarrow{\;g_2\;} \cdots$$

with maps f_0, h_0, f_1, h_1, f_2, h_2 to

$$\mathsf{E} \wedge X \qquad \Sigma^{-1}\mathsf{E} \wedge \overline{\mathsf{E}} \wedge X \qquad \Sigma^{-2}\mathsf{E} \wedge \overline{\mathsf{E}}^{\wedge 2} \wedge X$$

from [36, Lem. 2.2.9] where $\overline{\mathsf{E}}$ is the cofiber of the unit map $\eta_{\mathsf{E}} \colon S \to \mathsf{E}$, $f_i = \eta_{\mathsf{E}} \wedge \Sigma^{-i}\overline{\mathsf{E}}^{\wedge i} \wedge X$ and $g_i = (\Sigma^{-1}\overline{\mathsf{E}} \to S) \wedge \Sigma^{-i}\overline{\mathsf{E}}^{\wedge i} \wedge X$ and $h_i \colon \mathsf{E} \wedge X \to \overline{\mathsf{E}} \wedge X$ induces the boundary map in the long exact sequence in homotopy groups. Applying π_* produces an exact couple with E_1-page given by the differential graded algebra

$$\mathsf{E}_* X \longrightarrow \mathsf{E}_* \overline{\mathsf{E}} \otimes_{\mathsf{E}_*} \mathsf{E}_* X \longrightarrow \mathsf{E}_* \overline{\mathsf{E}}^{\otimes_{\mathsf{E}_*} 2} \otimes_{\mathsf{E}_*} \mathsf{E}_* X \longrightarrow \cdots$$

called the (normalized) *cobar complex* with boundary maps induced by the maps $\Sigma^{-i} h_i \circ \Sigma^i f_i$. Specifically, the differential

$$d_1^{s,t} \colon [\mathsf{E}_* \overline{\mathsf{E}}^{\otimes_{\mathsf{E}_*} s} \otimes_{\mathsf{E}_*} \mathsf{E}_* X]_t \to [\mathsf{E}_* \overline{\mathsf{E}}^{\otimes_{\mathsf{E}_*} s+1} \otimes_{\mathsf{E}_*} \mathsf{E}_* X]_t$$

in the cobar complex is given by the formula

$$d_1^{s,t}(e_1 \otimes \cdots \otimes e_s \otimes m) = \sum_{i=1}^{s} (-1)^i e_1 \otimes \cdots \otimes e_{i-1} \otimes \overline{\Delta}(e_i) \otimes e_{i+1} \otimes \cdots \otimes e_s \otimes m$$
$$+ (-1)^{s+1} e_1 \otimes \cdots \otimes e_s \otimes \overline{\psi}(m)$$

when $e_1 \otimes \cdots \otimes e_s \otimes m \in [\mathsf{E}_* \overline{\mathsf{E}}^{\otimes_{\mathsf{E}_*} s} \otimes_{\mathsf{E}_*} \mathsf{E}_* X]_t$, where $\overline{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$ and $\overline{\psi}(m) = \psi(m) - 1 \otimes m$. We note that under our hypotheses on E and X the E-Adams spectral sequence has signature

$$E_2^{s,t} = [H^s(\mathsf{E}_* \overline{\mathsf{E}}^{\otimes_{\mathsf{E}_*} \bullet} \otimes_{\mathsf{E}_*} \mathsf{E}_* X; d_1)]_t \implies \pi_{t-s} X_p$$

and it strongly converges.

Remark 3.1. If we consider the E-Adams spectral sequence when X is the sphere spectrum we drop the last tensor factor from the cobar complex.

3.2. **Detecting v_2 and β_1.** The mod p Moore spectrum S/p and the Smith-Toda complex $V(1)$ are defined so that they fit into exact triangles

$$(3.2) \qquad\qquad S \xrightarrow{\ p\ } S \xrightarrow{\ i_0\ } S/p \xrightarrow{\ j_0\ } \Sigma S$$

and

$$(3.3) \qquad\qquad \Sigma^{2p-2} S/p \xrightarrow{\ v_1\ } S/p \xrightarrow{\ i_1\ } V(1) \xrightarrow{\ j_1\ } \Sigma^{2p-1} S/p$$

in the stable homotopy category of spectra. We will abuse notation and write $i_0 \colon \pi_* S \to S/p$ and $i_1 \colon \pi_* S/p \to \pi_* V(1)$ for the maps induced by i_0 and i_1 respectively. Similarly, we write $j_0 \colon \pi_* S/p \to \pi_* \Sigma S$ and $j_1 \colon \pi_* V(1) \to \pi_* \Sigma^{2p-1} S/p$ for the maps induced by j_0 and j_1. We similarly abuse notation for composites of these maps. Throughout this section we write $\overline{\mathcal{A}} = \pi_*(H\mathbb{F}_p \wedge \overline{H\mathbb{F}_p})$ for the kernel of the augmentation $\mathcal{A} \longrightarrow \mathbb{F}_p$.

Definition 3.2. Let

$$v_2 \in \pi_{2p^2-2} V(1),$$
$$\beta_1' \in \pi_{2p^2-2p-1} V(1),$$
$$\beta_1 \in \pi_{2p^2-2p-2} V(1)$$

be the classes represented in the respective cobar complexes by the permanent cycles

$$\overline{v}_2 := \tau_2 \otimes 1 + \xi_2 \otimes \tau_0 + \xi_1^p \otimes \tau_1 \in \overline{\mathcal{A}}_* \otimes H_*(V(1)),$$
$$h_1 := \bar{\xi}_1^p \otimes 1 \in \overline{\mathcal{A}}_* \otimes H_*(V(1)), \text{ and}$$
$$b_0 := \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} \bar{\xi}_1^{p-j} \otimes \bar{\xi}_1^j \otimes 1 \in \overline{\mathcal{A}}_*^{\otimes 2} \otimes H_*(V(1)).$$

The bidegrees of these elements are

$$|\overline{v}_2| = (1, 2p^2 - 1),$$
$$|h_1| = (1, 2p^2 - 2p), \text{ and}$$
$$|b_0| = (2, 2p^2 - 2p),$$

where $|x| = (s, t)$ corresponds to stem $t - s$ and Adams filtration s.

Lemma 3.3 is immediate from Proposition 2.13:

Lemma 3.3. *There is an isomorphism*

$$H_*(T_1(\mathrm{K}(\mathbb{F}_q)_p)) \cong H_*(H_*(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p) \otimes E(t); d),$$

where the right-hand side is the homology of the differential graded algebra

$$H_*(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \otimes E(t)$$

with $|t| = (0, -2)$ *and* $|x| = (s, 0)$ *if* $x \in H_s(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p))$. *The differential is* $d(x) = t\sigma x$ *for all* $x \in H_*(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)$.

In the proof of Proposition 3.4, we will make use of differentials in both the homotopy fixed point spectral sequence and the Adams spectral sequence. To differentiate between the two, we use notation d^r for differentials in the homotopy fixed point spectral sequence and we use the notation d_r for differentials in the Adams spectral sequence. The following argument is inspired by an argument of Ausoni–Rognes [5, Prop. 4.8].

Proposition 3.4. *The classes* v_2, $i_1 i_0 \beta_1$, *and* $i_1 \beta_1'$ *in* $V(1)_*$ *map nontrivially to the classes* $t\mu_2$, $t\sigma b$, *and* $t\lambda_1'$ *respectively in* $V(1)_* \mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)$ *up to multiplication by a unit in* $\mathbb{F}_p$.

Proof. We first sketch the strategy. First of all, it will suffice to prove that each of the classes v_2, $i_1 i_0 \beta_1$, and $i_1 \beta_1'$ maps to non-trivial elements $t\mu_2$, $t\sigma b$, and $t\lambda_1'$ in the homotopy groups of the skeleton $V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p)$ since the unit map $V(1) \to V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p)$ factors through the unit map $V(1) \to V(1) \wedge \mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)$. We compute $V(1)_* T_1(\mathrm{K}(\mathbb{F}_q)_p)$ using two different spectral sequences: the $H\mathbb{F}_p$-Adams spectral sequence from Section 3.1 and the truncated homotopy fixed point spectral sequence from Remark 2.15. By representing each of the classes v_2, $i_1 i_0 \beta_1$, and $i_1 \beta_1'$ by explicit classes in the E_1-page of the Adams spectral sequence, we compute the image in the E_1-page of the Adams spectral sequence for $V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p$ and we compute directly that these classes are permanent cycles. Using the truncated homotopy fixed point spectral sequence from Remark 2.15, we compute that there is only one class in the correct degree in $V(1)_* T_1(\mathrm{K}(\mathbb{F}_q)_p$ for each of the classes v_2, $i_1 i_0 \beta_1$, and $i_1 \beta_1'$. This determines that the explicit names for the images of each of the classes v_2, $i_1 i_0 \beta_1$, and $i_1 \beta_1'$ must be $t\mu_2$, $t\sigma b$, and $t\lambda_1'$ up to multiplication by a unit in $\mathbb{F}_p$.

First, v_2 is represented by $\bar{v}_2$, β_1' is represented by h_1 and β_1 is represented by b_0 in the E_1-page of the Adams spectral sequence that converges to $\pi_* V(1)$ as in Definition 3.2. We consider the map of Adams spectral sequences

$$\mathrm{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_p, H_*(V(1))) \longrightarrow \mathrm{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_p, H_*(V(1)) \otimes H_*(T_1(\mathrm{K}(\mathbb{F}_q)_p)))$$

induced by the unit map

$$V(1) \wedge S \xrightarrow{\; 1_{V(1)} \wedge \eta \;} V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p) \; .$$

We see that $\bar{v}_2$, h_1, and b_0 are permanent cycles in the source, which map to classes of the same name in the target. Since the elements in the source are infinite cycles, this implies that the elements that they map to are infinite cycles as well. We therefore just have to check that these classes are not boundaries.

We can eliminate the possibility of a d_1 differential with $\bar{v}_2$ as a boundary by computing the differential in the cobar complex for $H_*(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p))$ on each

class of the correct degree. There is a truncated homotopy fixed point spectral sequence by Remark 2.15 with signature

$$E^2_{*,*} = E(t) \otimes H_*(V(1) \wedge THH(\mathrm{K}(\mathbb{F}_q)_p)) \implies H_*(V(1)) \otimes H_*(T_1(\mathrm{K}(\mathbb{F}_q)_p)),$$

where

$$E^2_{*,*} = E(t) \otimes E(\tau_0, \tau_1) \otimes P(\tilde{\xi}_1^p, \tilde{\xi}_2, \overline{\xi}_3, \dots)$$
$$\otimes E(b, \tilde{\tau}_2, \overline{\tau}_3, \dots) \otimes E(\sigma\tilde{\xi}_1^p, \sigma\tilde{\xi}_2) \otimes P(\sigma\tilde{\tau}_2) \otimes \Gamma(\sigma b)$$

by Corollary 2.9. There is an isomorphism

$$E^4_{*,*} = H_*(V(1)) \otimes H_*(H_*(\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p) \otimes E(t); d^2),$$

where $d^2(x) = t\sigma x$ and the spectral sequence clearly collapses at the E^4-page for bi-degree reasons as described in Lemma 3.3. Consequently,

$$H_{2p^2-1}(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p)) \cong \mathbb{F}_p\{\sigma\tilde{\xi}_2, \tau_1\sigma b\}.$$

Since σb is primitive, $d_1(\sigma b) = 0$. The coaction on τ_1 is given by the co-product

$$\psi(\tau_1) = \tau_1 \otimes 1 + \xi_1 \otimes \tau_0 + 1 \otimes \tau_1$$

and the co-action on $\sigma\tilde{\xi}_2$ is described in 2.8. Consequently, $d_1(\tau_1) = \tau_1 \otimes 1 + \xi_1 \otimes \tau_0$ and we compute

$$d_1(\sigma\tilde{\xi}_2) = \overline{\xi}_1 \otimes \sigma\tilde{\xi}_1^p + \tau_1 \otimes \sigma b, \text{ and}$$
$$d_1(\tau_1\sigma b) = \tau_1 \otimes \sigma b + \xi_1 \otimes \tau_0\sigma b$$

and then observe that

$$d_1(\alpha\tau_1\sigma b + \beta\sigma\tilde{\xi}_2) = \alpha(\tau_1 \otimes \sigma b + \xi_1 \otimes \tau_0\sigma b) + \beta(\overline{\xi}_1 \otimes \sigma\tilde{\xi}_1^p + \overline{\tau}_1 \otimes \sigma b)$$
$$\neq \tau_2 \otimes 1 + \xi_2 \otimes \tau_0 + \xi_1^p \otimes \tau_1$$

for any $\alpha, \beta \in \mathbb{F}_p$. Therefore, $\overline{v}_2$ survives to the E_2-page. There are no possible longer differentials hitting $\overline{v}_2$ because $\overline{v}_2$ is in Adams filtration one; hence, it is a permanent cycle.

We eliminate the possibility that the class h_1 is a boundary of a d_1 by the same method. The source of a d_1-differential hitting h_1 must be in Adams filtration zero and degree $2p^2 - 2p$. By Proposition 2.13, we compute $H_{2p^2-2p}(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p)) \cong \mathbb{F}_p\{\sigma b\}$ we just need to compute $d_1(\sigma b)$. However, we already observed that $d_1(\sigma b) = 0$. The class h_1 is in Adams filtration one so it cannot be the target of a longer differential, therefore it is a permanent cycle.

For b_0, we need to check that it is not the boundary of a d_1 or a d_2, because it is in Adams filtration two. We first need to check that b_0 is not a d_1-boundary for any element in $\overline{\mathcal{A}}_* \otimes H_*(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p))$ in degree $2p^2 - 2p$. We check the differential in the cobar complex on all the elements here in the right degree. These classes are

$$\left\{ \begin{matrix} \overline{\tau}_0 \otimes \tau_0 t\tilde{\xi}_1^p, & \overline{\xi}_1^{p-1} \otimes \tau_0\tau_1 t, & \overline{\xi}_1^{p-2}\overline{\tau}_1 \otimes \tau_1 t, & \overline{\xi}_1^{p-1}\overline{\tau}_0 \otimes \tau_1 t, \\ \overline{\xi}_1^{p-1}\overline{\tau}_1 \otimes \tau_0 t, & \overline{\xi}_1^{p-1}\overline{\tau}_0\overline{\tau}_1 \otimes t, & \overline{\xi}_1^p \otimes 1 \end{matrix} \right\}.$$

Recall that Milnor computed the co-action of $\mathcal{A}_*$ on

$$H^*(\mathbb{C}P^\infty, \mathbb{F}_p) = H^*(B\mathbb{T}; \mathbb{F}_p) = H^*_{\mathrm{gp}}(\mathbb{T}; \mathbb{F}_p),$$

and the co-action on the class t is

$$\psi(t) = \sum_{i \geq 0} \bar{\xi}_i \otimes t^{p^i},$$

see [29]. Therefore, in the input of the truncated homotopy fixed point spectral sequence computing $V(1)_* T_1(\mathrm{K}(\mathbb{F}_q)_p)$, the $\mathcal{A}_*$ co-action on t is primitive.

We compute the differential in the cobar complex on each of the elements that could possibly have the class representing β_1 as a target. However, since all of the classes except $\bar{\xi}_1^p \otimes 1$ have a t as a factor of the rightmost tensor factor and t is a comodule primitive, it suffices to compute the d_1-differential on the class $\bar{\xi}_1^p \otimes 1$. We then compute directly that

$$d_1(\bar{\xi}_1^p \otimes 1) = 0.$$

We can conclude that b_0 is not a boundary of a d_1.

Since b_0 is in Adams filtration two, we still have to check that there is no d_2 differential hitting it in the Adams spectral sequence,

$$\mathrm{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{F}_p, H_*(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p))) \implies V(1)_* T_1(\mathrm{K}(\mathbb{F}_q)_p).$$

Since a d_2 would have to have its source on the 0-line in degree $2p^2 - 2p - 1$, it would have to be a class in $H_{2p^2-2p-1}(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p))$.

Again, we compute

$$H_{2p^2-2p-1}(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p)) \cong \mathbb{F}_p\{\tau_0 t\tilde{\xi}_1^p\}$$

by Proposition 2.13. Since $d_1(\tau_0) = -\tau_0 \otimes 1$, the Leibniz rule implies

$$d_1(\tau_0 t\tilde{\xi}_1^p) = -\tau_0 \otimes t\tilde{\xi}_1^p - \tau_0 d_1(t\tilde{\xi}_1^p),$$

so since $d_1(t\tilde{\xi}_1^p) = -\bar{\xi}_1^p \otimes t - \bar{\tau}_0 \otimes tb$, we observe that

$$d_1(\tau_0 t\tilde{\xi}_1^p) = -\tau_0 \otimes t\tilde{\xi}_1^p + \tau_0 \bar{\xi}_1^p \otimes t.$$

So $\tau_0(t\tilde{\xi}_1^p)$ does not survive to the E_2-page and therefore it cannot support a d_2 differential hitting b_0. Therefore, since b_0 is an infinite cycle and we have proven it isn't a boundary it must be a permanent cycle.

We conclude the elements v_2, β_1 and β_1' map nontrivially from $V(1)_*$ to $V(1)_* T_1(\mathrm{K}(\mathbb{F}_q)_p)$ via the map induced by the unit map $S \to T_1(\mathrm{K}(\mathbb{F}_q)_p)$. We claim that in $V(1)_* T_1(\mathrm{K}(\mathbb{F}_q)_p)$, the only possible classes in the correct degree to be v_2, β_1 and β_1' respectively are $t\mu_2$, $t\sigma b$ and $t\lambda_1'$. To see this consider the truncated homotopy fixed point spectral sequence

$$E(t) \otimes V(1)_* THH(\mathrm{K}(\mathbb{F}_q)_p) \implies V(1)_* T_1(\mathrm{K}(\mathbb{F}_q)_p),$$

the only possible d^2-differential in this spectral sequence that could affect degree $2p^2 - 2p - 1$, $2p^2 - 2p - 2$, or $2p^2 - 2$ is $d^2(\lambda_1') = at\sigma b$ for some $a \in \mathbb{F}_p$. We can show that $a = 0$ by considering the map of truncated homotopy fixed point spectral sequences with codomain

$$E(t) \otimes H_*(V(1) \wedge THH(\mathrm{K}(\mathbb{F}_q)_p) \implies H_*(V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p)$$

which sends λ_1' and $t\sigma b$ to classes with the same name. We then observe that $d^2(\lambda_1') = 0$ in the codomain spectral sequence. Consequently, we have isomorphisms

$$V(1)_{2p^2-2} T_1(\mathrm{K}(\mathbb{F}_q)_p) \cong \mathbb{F}_p\{t\mu_2\},$$
$$V(1)_{2p^2-2p-2} T_1(\mathrm{K}(\mathbb{F}_q)_p) \cong \mathbb{F}_p\{t\sigma b\},$$
$$V(1)_{2p^2-2p-1} T_1(\mathrm{K}(\mathbb{F}_q)_p) \cong \mathbb{F}_p\{t\lambda_1'\},$$

proving the claim. The unit map $S \to T_1(\mathrm{K}(\mathbb{F}_q)_p)$ factors through the unit map $S \to \mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)$, so these classes pull back to classes in $V(1)_*\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)$. $\square$

Corollary 3.5. *The classes $t\mu_2$, $t\sigma b$, and $t\lambda_1'$ are permanent cycles in the homotopy fixed point spectral sequence*

$$H^*(\mathbb{T}, V(1)_*\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \implies V(1)_*\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)).$$

Lemma 3.6. *There is a differential $d_{2p-2}(t) = t^p\alpha_1$ in the homotopy fixed point spectral sequence*

$$H^*(\mathbb{T}, V(1)_*\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \implies V(1)_*\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)).$$

Proof. First, we can show that there is a differential $d_{2p-2}(t) = t^p\alpha_1$ in the homotopy fixed point spectral sequence

$$H^*(\mathbb{T}, V(1)_*\mathrm{K}(\mathbb{F}_q)_p)) \implies V(1)_*\mathrm{K}(\mathbb{F}_q)_p^{h\mathbb{T}},$$

where $\mathrm{K}(\mathbb{F}_q)_p$ has trivial $\mathbb{T}$-action, because α_1 is an attaching map in $B\mathbb{T}$. This has already been proven in [15, Theorem 3.5], so we omit the details. There is a $\mathbb{T}$-equivariant map of commutative ring spectra

$$\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p) \to \mathrm{K}(\mathbb{F}_q)_p$$

which induces a map of homotopy fixed point spectral sequences and since this map sends t to t and α_1 to α_1, the differential $d_{2p-2}(t) = t^p\alpha_1$ also occurs in the homotopy fixed point spectral sequence

$$H^*(\mathbb{T}, V(1)_*\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \implies V(1)_*\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)).$$

Note that we could have also proven this directly by examining $\mathbb{T}$-equivariant attaching maps in $E\mathbb{T}$, but for the sake of brevity we give the simpler proof. $\square$

Corollary 3.7. *There are differentials $d_{2p-2}(\mu_2) = -t^{p-1}\alpha_1\mu_2$, $d_{2p-2}(\sigma b) = -t^{p-1}\alpha_1\sigma b$, and $d_{2p-2}(\lambda_1') = 0$.*

Proof. This is immediate from Lemma 3.6 and Corollary 3.5. Note that $\alpha_1\lambda_1' = 0$ and there are no other classes in the same bidegree as $t^{p-1}\alpha_1\lambda_1'$ so $d_{2p-2}(\lambda_1')=0$. $\square$

Now, the classes $\beta_i \in \pi_{(2p^2-2)i-2p}S$ have the property that in the BP-Adams spectral sequence for S they are represented by the classes

$$\binom{i}{2}v_2^{i-2}k_0 + iv_2^{i-1}b_{1,0} \mod (p, v_1),$$

where $\binom{i}{2} = 0$ when $i = 1$, which are in BP-Adams filtration 2 (see Definition 3.13). We will therefore give a similar argument to the one in the proof of Proposition 3.4, except that we will work in the BP-Adams spectral sequence in order to use the fact that the classes representing β_k are in low BP-Adams filtration. To do this we must compute $BP \wedge V(1)_*(T_k(\mathrm{K}(\mathbb{F}_q)_p))$ up to possible differentials of length 4 or higher.

3.3. **The $BP \wedge V(1)$-Hochschild–May spectral sequence.** In this section, we begin by computing the input of the $BP \wedge V(1)$-Hochschild–May spectral sequence.

Lemma 3.8. *There is an isomorphism of $(BP \wedge V(1))_* H\pi_* \mathrm{K}(\mathbb{F}_q)_p$-algebras*

(3.4) $(BP \wedge V(1))_* \mathrm{THH}(H\pi_* \mathrm{K}(\mathbb{F}_q)_p) \cong P(t_1, t_2, \dots) \otimes E(\epsilon_1, \lambda_1) \otimes P(\mu_1) \otimes N,$

where $N = \mathrm{HH}_*(S/p_*(H\pi_* \mathrm{K}(\mathbb{F}_q)_p))$ and the Hurewicz map

$(BP \wedge V(1))_* \mathrm{THH}(H\pi_* \mathrm{K}(\mathbb{F}_q)_p) \to (H\mathbb{F}_p \wedge BP \wedge V(1))_* \mathrm{THH}(H\pi_* \mathrm{K}(\mathbb{F}_q)_p)$

sends t_1 to $\bar{\xi}_1 - \hat{\xi}_1$, where $\bar{\xi}_1$ is the generator in degree $2p - 2$ of $H_* BP$ and $\hat{\xi}_1$ is the generator in degree $2p - 2$ of $H_*(V(1) \wedge \mathrm{THH}(H\pi_* \mathrm{K}(\mathbb{F}_q)_p))$.

Proof. Recall that $V(1) \wedge \mathrm{THH}(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p))$ is a $V(1) \wedge H\pi_*(\mathrm{K}(\mathbb{F}_q)_p)$-algebra, and hence an $H\mathbb{F}_p$ algebra, since $V(1) \wedge H\pi_*(\mathrm{K}(\mathbb{F}_q)_p)$ is itself an $H\mathbb{F}_p$-algebra. Thus, there is an equivalence

$BP \wedge V(1) \wedge \mathrm{THH}(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p)) \simeq BP \wedge H\mathbb{F}_p \wedge_{H\mathbb{F}_p} V(1) \wedge \mathrm{THH}(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p))$

and by the collapse of the Künneth spectral sequence and Proposition 2.7, the isomorphism (3.4) holds.

Since $BP \wedge V(1) \wedge \mathrm{THH}(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p))$ is an $H\mathbb{F}_p$-module we can use Lemma 2.5, which states that $(BP \wedge V(1))_* \mathrm{THH}(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p))$ includes as the comodule primitives inside of

$$(H\mathbb{F}_p \wedge BP \wedge V(1))_* \mathrm{THH}(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p)).$$

We recall that by the Künneth isomorphism and Proposition 2.6 there is an isomorphism of graded rings

$$(H\mathbb{F}_p \wedge BP \wedge V(1))_* \mathrm{THH}(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p))$$
$$\cong Z \otimes (A//E(0))_* \otimes E(\sigma\bar{\xi}_1) \otimes P(\sigma\bar{\tau}_1) \otimes N,$$

where $N = \mathrm{HH}_*(S/p_*(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p)))$ and $Z = H_*(BP \wedge V(1))$. We recall from [3, Lemma 3.4] that in degrees less than or equal to $2p - 2$ then N has generators

$$\{1, x_1, \sigma x_1, x_2\}$$

as a graded $\mathbb{F}_p$-module where $|x_1| = 2p - 3$, $|\sigma x_1| = 2p - 2$, and $|x_2| = 2p - 2$, which will suffice for our purposes. In fact, the elements $1, x_1, \sigma x_1$, and x_2 are also indecomposable algebra generators in N and they are permanent cycles in the algebraic Hochschild–May spectral sequence corresponding to $1, \alpha_1, \sigma\alpha_1$, and v_1 respectively in $\mathrm{HH}_*(S/p_* j)$.

We use the notation $(A//E(0))_* \cong P(\hat{\xi}_1, \hat{\xi}_2, \dots) \otimes E(\hat{\tau}_1, \hat{\tau}_2, \dots)$ and $H_*(BP) \cong P(\bar{\xi}_1, \bar{\xi}_2, \dots)$ to distinguish the two sets of generators. We also recall that $H_*(V(1)) \cong E(\tau_0, \tau_1)$ as a sub-A_*-comodule of the dual Steenrod algebra A_*. [5] The co-actions on $\bar{\xi}_i, \bar{\tau}_i, \hat{\tau}_i$ and $\hat{\xi}_i$ are the same as the co-product in the dual Steenrod algebra, and hence for example $\bar{\xi}_1 - \hat{\xi}_1$ is a comodule primitive, since

$$\psi(\bar{\xi}_1 - \hat{\xi}_1) = 1 \otimes \bar{\xi}_1 + \bar{\xi}_1 \otimes 1 - 1 \otimes \hat{\xi}_1 - \bar{\xi}_1 \otimes 1 = 1 \otimes \bar{\xi}_1 - 1 \otimes \hat{\xi}_1.$$

---

[5] As discussed before Proposition 2.6, a complete description of $\mathrm{HH}_*(S/p_*(H\pi_*(\mathrm{K}(\mathbb{F}_q)_p)))$ can be found in [3, p. 271-272], but we will not need an explicit description here because we will use the algebraic Hochschild–May spectral sequence of [3, §3.1.1] to compute the d^1 differentials and by the E^2-page the description is much simpler.

The co-action on the remaining algebra generators in degrees less than or equal to $2p - 2$ is

$$\psi(x_1) = 1 \otimes x_1,$$
$$\psi(\sigma x_1) = 1 \otimes \sigma x_1,$$
$$\psi(x_2) = 1 \otimes x_2 + \bar{\tau}_0 \otimes x_2$$

and we may observe that there are no other $\mathcal{A}_*$ comodule primitives in degree $2p - 2$ other than $\bar{\xi}_1 - \hat{\xi}_1$ and σx_1 so t_1 must map to a linear combination of $\bar{\xi}_1 - \hat{\xi}_1$ and σx_1. However, the May filtration of $\bar{\xi}_1 - \hat{\xi}_1$ is zero and the May filtration of σx_1 is positive so, since the Hurewicz map preserves May filtration, we determine that t_1 maps to $\bar{\xi}_1 - \hat{\xi}_1$. $\square$

Proposition 3.9. *There is an isomorphism of (BP_*, BP_*BP)-comodules*

$$(BP \wedge V(1))_* \text{THH}(\text{K}(\mathbb{F}_q)_p) \cong P(t_1^p, t_2, \dots) \otimes E(b) \otimes E(\lambda_1', \lambda_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b)$$

with BP_*BP-co-action

$$\psi(t_1^p) = 1 \otimes t_1^p + t_1^p \otimes 1,$$
$$\psi(\mu_2) = 1 \otimes \mu_2,$$
$$\psi(t_n) = \sum_{k=0}^{n} t_k \otimes t_{n-k}^{p^k} \ for \ n \geq 2,$$
$$\psi(\gamma_n(\sigma b)) = 1 \otimes \gamma_n(\sigma b),$$
$$\psi(b) = 1 \otimes b,$$
$$\psi(\lambda_2) = 1 \otimes \lambda_2 + t_1 \otimes \lambda_1', \ and$$
$$\psi(\lambda_1') = 1 \otimes \lambda_1',$$

where λ_1' is detected by σt_1^p in the E^∞-page, λ_2 is detected by σt_2 in the E^∞-page, and $\gamma_n(\sigma b)$ is detected by $\gamma_{np}(\sigma \alpha_1)$ in the E^∞-page for $k \geq 0$.

Proof. We need to compute differentials in the $BP \wedge V(1)$-Hochschild–May spectral sequence

$$E^1_{*,*} = (BP \wedge V(1))_{*,*} \text{THH}(H\pi_*(\text{K}(\mathbb{F}_q)_p))) \implies BP \wedge V(1)_* \text{THH}(\text{K}(\mathbb{F}_q)_p),$$

so we examine the map of spectral sequences
(3.5)

$$
\begin{array}{ccc}
(BP \wedge V(1))_{*,*} \text{THH}(H\pi_*(\text{K}(\mathbb{F}_q)_p))) & \Longrightarrow & BP \wedge V(1)_* \text{THH}(\text{K}(\mathbb{F}_q)_p) \\
\downarrow {\scriptstyle h} & & \downarrow \\
(H\mathbb{F}_p \wedge BP \wedge V(1))_{*,*} \text{THH}(H\pi_*(\text{K}(\mathbb{F}_q)_p)) & \Longrightarrow & (H\mathbb{F}_p \wedge BP \wedge V(1))_* \text{THH}(\text{K}(\mathbb{F}_q)_p),
\end{array}
$$

induced by the Hurewicz map $BP \to H\mathbb{F}_p \wedge BP$. Recall from Lemma 3.8 that

$$(H\mathbb{F}_p \wedge BP \wedge V(1))_* \text{THH}(H\pi_*(\text{K}(\mathbb{F}_q)_p))$$
$$\cong H_*(BP \wedge V(1)) \otimes (A//E(0))_* \otimes P(\mu_1) \otimes N,$$

where $N = \text{HH}_*(S/p_*(H\pi_*(\text{K}(\mathbb{F}_q)_p)))$. We know that in the $H\mathbb{F}_p \wedge BP \wedge V(1)$-Hochschild–May spectral sequence the classes $\bar{\xi}_i$ for $i \geq 1$ and τ_j for $j = 0, 1$ survive to E^∞, since the output of the spectral sequence is known to be

$$(H\mathbb{F}_p \wedge BP \wedge V(1))_* \text{THH}(\text{K}(\mathbb{F}_q)_p) \cong P(t_1, t_2, \dots) \otimes E(\tau_0, \tau_1) \otimes H_*(\text{THH}(\text{K}(\mathbb{F}_q)_p))$$

by Theorem 2.8 and the Künneth isomorphism. This forces the same d^1-differentials that occur in the $H\mathbb{F}_p \wedge V(1)$-Hochschild–May spectral sequence and consequently there is an additive isomorphism

$$E^2_{*,*} \cong P(t_1, t_2, \dots) \otimes E(\epsilon_1, \lambda_1, \sigma v_1, \alpha_1) \otimes P(v_1, \mu_1) \otimes \Gamma(\sigma\alpha_1).$$

The map of spectral sequences is therefore again injective on E^2-pages. In the $H\mathbb{F}_p \wedge V(1)$-Hochschild–May spectral sequence there are differentials

$$
\begin{aligned}
d^{2p-3}(t_1) &= \alpha_1, \\
d^{2p-3}(\lambda_1) &= \sigma\alpha_1, \\
d^{2p-2}(\hat{\tau}_1) &= v_1, \\
d^{2p-2}(\mu_1) &= \sigma v_1
\end{aligned}
$$

and no further nontrivial differentials besides those differentials generated by these differentials using the Leibniz rule. Since the Hurewicz map h is injective and it sends t_1 to $\bar{\xi}_1 - \hat{\xi}_1$, the differential $d^{2p-3}(t_1) = \alpha_1$ in the top spectral sequence of Equation (3.5) can be computed using the formula

$$d^{2p-3}(h(t_1)) = d^{2p-3}(\bar{\xi}_1 - \hat{\xi}_1) = \alpha_1 = h(\alpha_1).$$

Similarly, ϵ_1 maps to $\bar{\tau}_1 - \hat{\tau}_1$ implying $d^{2p-2}(\epsilon_1) = v_1$. Hence, in the $BP \wedge V(1)$-Hochschild–May spectral sequence there are differentials

$$d^{2p-3}(t_1) = \alpha_1, \quad d^{2p-3}(\lambda_1) = \sigma\alpha_1, \quad d^{2p-2}(\epsilon_1) = v_1, \quad d^{2p-2}(\mu_1) = \sigma v_1.$$

On E^{2p-1}-pages the map of spectral sequences induced by the Hurewicz map is again injective. Since $E^{2p-1} \cong E^\infty$ in the target spectral sequence, the same is true in the source. This implies that the $BP \wedge V(1)$-Hochschild–May spectral sequence collapses at the E^{2p-1}-page. From the E^{2p-1}-page on, we write

$$
\begin{aligned}
\gamma_n(\sigma b) &:= \gamma_{np}(\sigma\alpha_1), \\
\lambda_1' &:= \lambda_1\gamma_{p-1}(\sigma\alpha_1), \\
\lambda_2 &:= (\sigma v_1)\mu_1^{p-1}, \\
\mu_2 &:= \mu_1^p, \text{ and} \\
b &:= \alpha_1 t_1^{p-1}.
\end{aligned}
$$

Write $\mathrm{mfilt}(x)$ for the May filtration of an element x. We note that $\mathrm{mfilt}(\gamma_n(\sigma b)) = (2p-2)np-1$, $\mathrm{mfilt}(\lambda_1') = (p-1)(2p-2)-1$, $\mathrm{mfilt}(\lambda_2) = 2p-2$ and $\mathrm{mfilt}(b) = 2p-3$ for later use. By examining the long exact sequence

$$BP_*(V(1) \wedge \mathrm{K}(\mathbb{F}_q)_p) \to BP_*V(1) \wedge \ell \to BP_*(V(1) \wedge \Sigma^{2p-2}\ell),$$

we can determine that the co-action on t_1^p and t_i for $i \geq 2$ in $BP_*\mathrm{K}(\mathbb{F}_q)_p$ is the same as the co-action on these elements in $BP_*(V(1) \wedge \ell) \cong P(t_1, t_2, \dots)$. Note that there is no hidden comultiplication on t_1^p since there are no classes in degrees $2p^2 - 2p - (2p-2)$ or lower and the lowest degree element in BP_*BP is in degree $2p-2$. The class b is the class in lowest positive degree and therefore it is primitive. This produces the co-action on b, t_1^p, t_i for $i \geq 2$ in $BP_*(V(1) \wedge \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p))$, by using the splitting of BP_*BP-comodules

$$BP_*(V(1) \wedge \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \cong BP_*(V(1) \wedge \mathrm{K}(\mathbb{F}_q)_p) \oplus BP_*(V(1) \wedge \overline{\mathrm{THH}}(\mathrm{K}(\mathbb{F}_q)_p))$$

induced by the splitting $\mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p) \simeq \mathrm{K}(\mathbb{F}_q)_p \vee \overline{\mathrm{THH}}(\mathrm{K}(\mathbb{F}_q)_p)$, which we have because $\mathrm{K}(\mathbb{F}_q)_p$ is a commutative ring spectrum.

The co-action on μ_2 is primitive because $|\mu_2| = 2p^2$ and there are no classes in degrees $2p^2 - 2p + 2$ or $2p^2 - 4p + 4$ or lower and the classes in BP_*BP are in degrees congruent to zero mod $2p - 2$. Similarly, the co-action on λ_1' is primitive because there are no classes in degree $2p^2 - 2p + 1 - (2p - 2)$ or lower. To determine the co-action on λ_2, note that there is an isomorphism

$$
\begin{aligned}
BP_*(V(1) \wedge \mathrm{K}(\mathbb{F}_q)_p) &\cong P(\bar{\xi}_1^p, \bar{\xi}_2, \dots) \otimes E(b) \\
&\cong P(t_1^p, t_2, \dots) \otimes E(b),
\end{aligned}
$$

so $\bar{\xi}_2$ and t_2 are two names for the same basis element up to multiplication by a unit. Similarly, $\bar{\xi}_1^p$ and t_1^p are two names for the same basis element up to multiplication by a unit (cf. [47] as described in [28, p. 511]). The operation σ gives $\lambda_2 = \sigma\bar{\xi}_2 \doteq \sigma t_2$ and $\lambda_1' \doteq \sigma t_1^p$ and we can therefore compute the co-action on λ_2 using the formula $\psi(\lambda_2) = (1 \otimes \sigma)(t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2)$, due to [4]. In other words, in $(BP \wedge V(1))_*\mathrm{THH}(\mathrm{K}\mathbb{F}_q)_p)$,

$$
\psi(\lambda_2) \doteq 1 \otimes \lambda_2 + t_1 \otimes \lambda_1'.
$$

This just leaves the classes $\gamma_n(\sigma b)$ for $n \geq 1$. Since $\gamma_{p^k}(\sigma b)$ are the indecomposable algebra generators of $\Gamma(\sigma b)$ over a finite field $\mathbb{F}_p$ and products of primitives are primitive, to compute that $\gamma_n(\sigma b)$ is a comodule primitive for each $n \geq 1$ it suffices to show that $\gamma_{p^k}(\sigma b)$ is a comodule primitive for each $k \geq 0$. Note that we already checked that in the input of the $BP \wedge V(1)$-Hochschild–May spectral sequence the classes $\gamma_{p^{k+1}}(\sigma\alpha_1) = \gamma_{p^k}(\sigma b)$ are primitive. Therefore, it suffices to check that there is not a hidden BP_*BP co-action in the Hochschild–May spectral sequence.

If the co-action contains terms of the form $x \otimes m$ where $|m| \leq |\gamma_{p^k}(\sigma b)|$, then the May filtration of m must be strictly greater than the May filtration of $\gamma_{p^k}(\sigma b)$ (since it would have to arise from a hidden BP_*BP-coaction in the Hochschild–May spectral sequence). Our goal will be to reach a contradiction.

Suppose that the inequality

$$
\mathrm{mfilt}(m) > \mathrm{mfilt}(\gamma_{p^k}(\sigma b)) = (2p - 2)p^{k+1} - 1
$$

is satisfied. Then, since the only classes with positive May filtration are $\gamma_{p^j}(\sigma b)$, b, λ_1', and λ_2, the class m must be of the form

$$
(\gamma_\ell(\sigma b))b^{\epsilon_1}\lambda_1'^{\epsilon_2}\lambda_2^{\epsilon_3}z,
$$

for some element z, where $0 \leq \ell < p^k$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$. Write $\mathrm{mfilt}(x)$ for the May filtration of an element, then

$$
\begin{aligned}
\mathrm{mfilt}(\gamma_\ell(\sigma b)) &= (2p - 2)p\ell - 1, \\
\mathrm{mfilt}(b) &= 2p - 3, \\
\mathrm{mfilt}(\lambda_1') &= (p - 1)(2p - 2) - 1, \text{ and} \\
\mathrm{mfilt}(\lambda_2) &= 2p - 2,
\end{aligned}
$$

so ℓ, ϵ_1, ϵ_2, and ϵ_3 must satisfy

$$
\begin{aligned}
(3.6) \quad (2p - 2)p\ell - 1 + \epsilon_1 \cdot 2p - 3 + \epsilon_2 \cdot (p - 1)(2p - 2) - 1 + \epsilon_3 \cdot 2p - 2 \\
\geq (2p - 2)p^{k+1} - 1.
\end{aligned}
$$

We can rewrite the left-hand side as

$$(2p-2)(p\ell + \epsilon_1 + (p-1)\epsilon_2 + \epsilon_3) - 3$$

and conclude that the inequality

$$p\ell + \epsilon_1 + (p-1)\epsilon_2 + \epsilon_3 > p^{k+1}$$

must be satisfied, since $p \geq 5$. Since $p^k > \ell$, we conclude that

$$p^{k+1} > p\ell > p^{k+1} - (\epsilon_1 + (p-1)\epsilon_2 + \epsilon_3)$$

and therefore

(3.7) $$\epsilon_1 + (p-1)\epsilon_2 + \epsilon_3 > 1.$$

The degree of $(\gamma_\ell(\sigma b))b^{\epsilon_1}\lambda_1'^{\epsilon_2}\lambda_2^{\epsilon_3}z$ satisfies

$$
\begin{aligned}
|(\gamma_\ell(\sigma b))b^{\epsilon_1}\lambda_1'^{\epsilon_2}\lambda_2^{\epsilon_3}z| >& (2p^2-2p)(p^{k+1} - (\epsilon_1 + (p-1)\epsilon_2 + \epsilon_3)) \\
&+ (2p^2-2p-1)\epsilon_1 + (2p^2-2p+1)\epsilon_2 + (2p^2-1)\epsilon_3 \\
=& |\gamma_{p^k}(\sigma b)| - \epsilon_1 + ((p-2)(2p^2-2p)+1)\epsilon_2 + (2p-1)\epsilon_3 \\
>& |\gamma_{p^k}(\sigma b)|
\end{aligned}
$$

whenever $p\ell > p^{k+1} - (\epsilon_1 + (p-1)\epsilon_2 + \epsilon_3)$ and either $\epsilon_2 > 0$ or $\epsilon_3 > 0$. Since (3.7) cannot be satisfied when $\epsilon_2 = \epsilon_3 = 0$, we have reached a contradiction to the assumption that $|m| < |\gamma_p(\sigma b)|$. Thus, no such m such that $|m| \leq |\gamma_{p^k}(\sigma b)|$ and $\mathrm{mfilt}(m) > \mathrm{mfilt}(\gamma_{p^k}(\sigma b))$ exists. This implies that there are no hidden co-actions and $\gamma_{p^k}(\sigma b)$ remains a comodule primitive. $\qquad\square$

Corollary 3.10. *In the homotopy fixed point spectral sequence*

$$H^*(\mathbb{T}, (BP \wedge V(1))_* \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \implies (BP \wedge V(1))_*^c \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)^{h\mathbb{T}},$$

there are differentials

$$d^2(t_1^p) \doteq t\lambda_1',$$
$$d^2(t_2) \doteq t\lambda_2,$$
$$d^2(b) \doteq t\sigma b$$

and no further d^2 differentials besides those generated from these d^2 differentials using the Leibniz rule.

Proof. This follows from Proposition 2.13 and the fact that $\lambda_2 \doteq \sigma t_2$ and $\lambda_1' \doteq \sigma t_1^p$ as discussed in Proposition 3.9. $\qquad\square$

Remark 3.11. We also need to know the co-action of BP_*BP on

$$BP_*(V(1) \wedge T_k(\mathrm{K}(\mathbb{F}_q)_p)),$$

which is isomorphic to a sub-quotient of

$$P(t_1^p, t_2, \dots) \otimes E(b) \otimes E(\lambda_1', \lambda_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b) \otimes P(t)/t^{k+1}$$

after taking into account the differentials. Modulo higher filtration, this just amounts to describing the co-action on the class t in the input of the homotopy fixed point spectral sequence

(3.8) $$H^*(\mathbb{T}, BP_*(V(1) \wedge \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p))),$$

since the co-action on a sub-quotient of (3.8) is determined by the co-action on (3.8).

Since we know that $\psi_{H\mathbb{F}_p}(t) = \sum_{i \geq 0} \bar{\xi}_i \otimes t^{p^i}$ and $t_i \doteq \bar{\xi}_i$ in $H_* BP$ (cf. [47] as described in [28, p. 511]), $BP_* BP$-comodule structure on $H^*(\mathbb{T}, \mathbb{F}_p) \otimes BP_*/(p, v_1)$ is determined modulo higher filtration by the formula

$$\psi_{BP}(t) = \sum_{i \geq 0} t_i \otimes t^{p^i}$$

in addition to the usual $BP_* BP$-co-action on $BP_*/(p, v_1)$.

Note that there is a *truncated homotopy fixed point spectral sequence* with signature

(3.9)
$$E^2_{*,*} = P(t)/t^{k+1} \otimes BP_*(V(1) \wedge \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \implies BP_*(V(1) \wedge T_k(\mathrm{K}(\mathbb{F}_q)_p))$$

as discussed in Remark 2.15.

Corollary 3.12. *In the spectral sequence (3.9) computing* $(BP \wedge V(1))_* T_k(\mathrm{K}(\mathbb{F}_q)_p)$, *there is an isomorphism of* $P_{k+1}(t)$-*modules and* $BP_* BP$ *comodules*

$$E^4_{*,*} \cong M_1 \otimes_{P_{k+1}(t)} M_2 \otimes_{P_{k+1}(t)} M_3 \otimes_{P_{k+1}(t)} (P_{k+1}(t) \otimes P(t_k : k \geq 3) \otimes P(\mu_2)),$$

where $M_1 = H_*(P(t_1^p) \otimes E(\lambda_1') \otimes P_{k+1}(t); d^2)$, $M_2 = H_*(P(t_2) \otimes E(\lambda_2) \otimes P_{k+1}(t); d^2)$, *and* $M_3 = H_*(E(b) \otimes \Gamma(\sigma b) \otimes P_{k+1}(t); d^2)$. *Explicitly, we have*

$$M_1 = \mathbb{F}_p\{\lambda_1' t_1^{p(j-1)}, t^k t_1^{pj} : \nu_p(j) = 0, j \geq 1\} \oplus \left(P_{k+1}(t) \otimes P(t_1^{p^2}) \otimes E(\lambda_1' t_1^{p^2 - p})\right),$$

$$M_2 = \mathbb{F}_p\{\lambda_2 t_2^{j-1}, t^k t_2^j : \nu_p(j) = 0, j \geq 1\} \oplus \left(P_{k+1}(t) \otimes P(t_2^p) \otimes E(\lambda_2 t_2^{p-1})\right), \text{ and}$$

$$M_3 = \mathbb{F}_p\{\gamma_j(\sigma b), t^k b\gamma_{j-1}(\sigma b) : \nu_p(j) = 0, j \geq 1\}$$
$$\oplus (P_{k+1}(t) \otimes E(b\gamma_{p-1}(\sigma b)) \otimes \Gamma(\gamma_p(\sigma b))).$$

The co-action on an element $x \in E^4_{*,*}$ *is determined modulo higher filtration by the co-action of classes in* $(BP \wedge V(1))_* \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)$ *and the co-action of* t *from Remark 3.11 modulo the differential* d^2 *determined in Corollary 3.10. The elements in* $P(t)/t^k$ *and*

$$\mathbb{F}_p\{\lambda_1' t_1^{p(j-1)}, t^k t_1^{pj_1}, \lambda_2 t_2^{j-1}, t^k t_2^j, \gamma_j(\sigma b), t^k b\gamma_{j-1}(\sigma b) : \nu_p(j) = 0, j \geq 1\}$$

are permanent cycles in the truncated homotopy fixed point spectral sequence.

Proof. All but the last statement directly follow from Corollary 3.10. For the last statement, we note that the homotopy fixed point spectral sequence for the sphere spectrum

$$(BP \wedge V(1))_* \otimes P(t) \implies (BP \wedge V(1))^c_*(\mathrm{TC}^-(S))$$

collapses at the E^2-page for bidegree reasons. Consequently, the differentials in homotopy fixed point spectral sequence

$$P(t) \otimes (BP \wedge V(1))_* \mathrm{THH}(R) \implies (BP \wedge V(1))^c_* \mathrm{TC}^-(R)$$

are always t-linear and the same is true for the truncated homotopy fixed point spectral sequence. This implies that the elements in $P(t)/t^{k+1}$ are permanent cycles and that the elements in

$$\mathbb{F}_p\{\lambda_1' t_1^{p(j-1)}, t^k t_1^{pj_1}, \lambda_2 t_2^{j-1}, t^k t_2^j, b\gamma_{j-1}(\sigma b), t^k \gamma_j(\sigma b) : \nu_p(j) = 0, j \geq 1\}$$

are permanent cycles. $\square$

3.4. **Detecting β elements in** TC$^-$. First, we define the β-family.

Definition 3.13. Let $p \geq 5$. By [36, Example 5.1.20], the elements
$$\beta_s \in \pi_{(2p^2-2)s-2p} S_p$$
are represented by the permanent cycles

$$(3.10) \qquad \overline{\beta}_s := \binom{s}{2} v_2^{s-2} k_0 + s v_2^{s-1} b_{1,0} \mod (p, v_1)$$

in $BP_* \overline{BP}_*^{\otimes_{BP_*} 2}$ in the BP-Adams spectral sequence where

$$(3.11) \qquad k_0 := 2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{p+1} - t_1^{2p} \otimes t_1 \text{ and}$$

$$(3.12) \qquad b_{1,0} := -\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} t_1^{p-i} \otimes t_1^i.$$

Recall that $T_s(R)$ is defined to be the spectrum $F(S(\mathbb{C}^{s+1})_+, \text{THH}(R))^{\mathbb{T}}$. As noted before Proposition 1.4 in [16], $T_s(R)$ is a commutative ring spectrum whenever R is a commutative ring spectrum. In particular, TC$^-(R)$ is a commutative ring spectrum. In the following theorem, let ℓ_p be a commutative ring spectrum model for the p-complete connective Adams summand as constructed in [27]. Note that there is a map of E_∞ ring spectra $L_{K(1)}S \to L_p$ given by inclusion of homotopy fixed points since
$$L_{K(1)}S \simeq L^{h\mathbb{Z}_p},$$
where $\mathbb{Z}_p$-action on L_p is by E_∞ ring maps and consequently the inclusion of fixed point $L_{K(1)}S \to L$ is a map of E_∞ ring spectra (cf. [17, Theorem 4.4, Proposition 8.1]). There is therefore a map of E_∞ ring spectra $K(\mathbb{F}_q)_p \to \ell_p$ by applying $\tau_{\geq 0}$ to the map above and using the weak equivalences of E_∞ ring spectra $K(\mathbb{F}_q)_p \simeq \tau_{\geq 0} L_{K(1)}S$ and $\tau_{\geq 0} L_p \simeq \ell_p$ where $\tau_{\geq 0}$ is the Postnikov truncation functor (see [3, p. 260] for a discussion of the first weak equivalence and [8, Proposition 2.1] for the second weak equivalence).

Proposition 3.14. *The classes v_2^s map to nonzero classes $(t\mu_2)^s$ under the unit map*
$$V(1)_* S \to V(1)_* T_k(K(\mathbb{F}_q)_p).$$

Proof. By [5, Prop. 4.8], the element $v_2 \in V(1)_{2p^2-2}$ maps to $t\mu_2 \in V(1)_{2p^2-2}(T_1(\ell_p))$. By [5, Thm. 6.7], the elements $(t\mu_2)^k$ survive the $\mathbb{T}$-homotopy fixed point spectral sequence computing $V(1)_* \text{TC}^-(\ell_p)$ and v_2^s maps to $(t\mu_2)^s$ because the map $V(1)_* \to V(1)_* \text{TC}^-(\ell_p)$ is a ring map. The map $V(1)_* \text{TC}^-(\ell_p) \to V(1)_* T_k(\ell_p)$ sends $(t\mu_2)^s$ to $(t\mu_2)^s$ by construction of the $\mathbb{T}$-homotopy fixed point spectral sequence.

Since we showed v_2 maps to $t\mu_2$ under the unit map $V(1)_* S \to V(1)_* T_1(K(\mathbb{F}_q)_p)$, the map $V(1)_* T_1(K(\mathbb{F}_q)_p) \to V(1)_* T_1(\ell_p)$ sends $t\mu_2$ to $t\mu_2$, and the maps
$$V(1)_* S \to T_s(K(\mathbb{F}_q)_p) \to V(1)_* T_s(\ell_p)$$
are ring maps for $s \geq 1$, the classes v_2^s also map to $(t\mu_2)^s$ under the unit map
$$V(1)_* S \to V(1)_* T_s(K(\mathbb{F}_q)_p).$$

We therefore know that $(t\mu_2)^s$ are permanent cycles in the BP-Adams spectral sequence and homotopy fixed point spectral sequences computing $V(1)_* T_s K(\mathbb{F}_q)_p$. $\square$

We will continue to use notation d^r for differentials in the homotopy fixed point spectral sequence and d_r for differentials in the BP-Adams spectral sequence to differentiate the two.

Theorem 3.15. *Let $p \geq 5$. The elements $\beta_s \in \pi_{(2p^2-2)s-2p}S$ for $s \not\equiv 0 \mod p$ map to*

$$(3.13) \qquad \binom{s}{2}(t\mu_2)^{s-2}t^2\lambda_1'\lambda_2 + s(t\mu_2)^{s-1}t\sigma b \in V(1)_{(2p^2-2)s-2p}TC^-(\mathrm{K}(\mathbb{F}_q)_p)$$

up to multiplication by a unit in $\mathbb{F}_p$ for certain choices of representatives of the classes $(t\mu_2)^{s-2}t^2\lambda_1'\lambda_2$ and $(t\mu_2)^{s-1}t\sigma b$.

Proof. The element β_s is represented by the class $\overline{\beta}_s$ as in Definition 3.13 by [36, Example 5.1.20]. We therefore need to check that the classes $\overline{\beta}_s$ map to permanent cycles in the BP_*BP-cobar complex for $V(1)_*T_{s'}(\mathrm{K}(\mathbb{F}_q)_p)$. From the proof, we will see that we can accomplish this by choosing $s = s'$.

Due to the length of the proof, we break it into several steps:

Step 1. First, we show that v_2^s in the BP_*BP-cobar complex for $V(1)$ maps to $(t\mu_2)^s$ in the BP_*BP-cobar complex for $V(1)_*T_s(\mathrm{K}(\mathbb{F}_q)_p))$. By Proposition 3.14, we know that v_2^s maps to $(t\mu_2)^s$ under the map

$$V(1)_* \to V(1)_*T_s(\mathrm{K}(\mathbb{F}_q)_p).$$

By examining the map of Hochschild–May spectral sequences induced by the unit map $\eta \wedge \mathrm{id}_{V(1)}\colon S \wedge V(1) \to BP \wedge V(1)$ and the subsequent map of homotopy fixed point spectral sequences induced by this same map, we see that $(t\mu_2)^s$ maps to $(t\mu_2)^s$ under the map

$$V(1)_*T_s(\mathrm{K}(\mathbb{F}_q)_p) \to (BP \wedge V(1))_*T_s(\mathrm{K}(\mathbb{F}_q)_p).$$

We also know the map

$$\pi_*(\eta \wedge \mathrm{id}_{V(1)})\colon \pi_*(S \wedge V(1)) \to \pi_*(BP \wedge V(1))$$

sends the class v_2^s to v_2^s since the edge-homomorphism in the BP-Adams spectral sequence is a ring homomorphism. We then use the commutative diagram

$$(3.14) \qquad \begin{array}{ccc} V(1)_* & \longrightarrow & V(1)_*T_s(\mathrm{K}(\mathbb{F}_q)_p) \\ \downarrow & & \downarrow \\ BP_*V(1) & \longrightarrow & (BP \wedge V(1))_*T_s(\mathrm{K}(\mathbb{F}_q)_p) \end{array}$$

to determine that $v_2^s \in BP_*V(1)$ maps to $(t\mu_2)^s \in (BP \wedge V(1))_*T_s(\mathrm{K}(\mathbb{F}_q)_p)$ and also in the map of exact couples of the respective BP-Adams spectral sequences.

Step 2. We now show that the element β_1 has nontrivial image in $V(1)_*T_1(\mathrm{K}(\mathbb{F}_q)_p)$. The element β_1 is represented by the permanent cycle $b_{1,0}$ in the E_1-page of the BP-Adams spectral sequence for $V(1)$. This class maps to a class of the same name in E_1-page of the BP-Adams spectral sequence for $V(1) \wedge T_1(\mathrm{K}(\mathbb{F}_q)_p)$. The result then follows by Proposition 3.4 and a result of Zahler [47], which implies that $b_{1,0}$ maps to b_0 via the map from the BP-Adams spectral sequence to the Adams spectral sequence (cf. [28, p. 511]).

Step 3. Next we consider the case $s \equiv 1 \mod p$. In this case, we know that $\overline{\beta}_{pk+1} = b_{1,0}v_2^{pk}$ represents β_{pk+1} in the BP-Adams spectral sequence for $V(1)$ for $k \geq 1$. It maps to $b_{1,0}(t\mu_2)^{pk}$ in the BP-Adams spectral sequence for $V(1) \wedge T_{pk+1}(\mathrm{K}(\mathbb{F}_q)_p)$ up to multiplication by a unit. This follows by Step 1 and the fact that the cobar complex for $V(1) \wedge T_{pk+1}(\mathrm{K}(\mathbb{F}_q)_p)$ is multiplicative. Since $\overline{\beta}_{pk+1}$ is a permanent cycle in the BP-Adams spectral sequence for $V(1)$ for $k \geq 1$ (this follows from [33, Lemma 5.4]), the class $b_{1,0}(t\mu_2)^{pk}$ is an infinite cycle in the BP-Adams spectral sequence for $V(1) \wedge T_{pk+1}(\mathrm{K}(\mathbb{F}_q)_p)$ for $k \geq 1$, but it could still be a boundary. It is on the two-line of the BP-Adams spectral sequence, so we just need to check that it is not the boundary of a d^1 or d^2 differential. Note that we will prove that, in fact, the element $b_{1,0}(t\mu_2)^s$ is never a boundary for any $s \geq 1$. In particular, we also show that $v_2\beta_1$ in the image of the unit map $V(1) \longrightarrow V(1)_*\mathrm{TC}^-(K(\mathbb{F}_q)_p)$ since $v_2\beta_1$ is known to be a permanent cycle in $V(1)_*$ by [42]. We will break this into two further sub-steps:

Sub-step 1. If the class $b_{1,0}(t\mu_2)^s$ is the boundary of a d^1, then there is a sum of classes

(3.15) $$\sum_i a_i \otimes m_i \in \overline{BP_*BP} \otimes_{BP_*} BP_*(V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q)_p))$$

such that

$$d_1\left(\sum_{i \in I} a_i \otimes m_i\right) = b_{1,0} \otimes (t\mu_2)^s,$$

where $s \geq 1$ and I is a finite set. Recall that the co-action on $x \in (BP \wedge V(1))_*T_s(\mathrm{K}(\mathbb{F}_q)_p)$ is of the form

$$\psi(x) = 1 \otimes x + \sum_{j \in J} a_j \otimes x_j,$$

where $|x_j| < |x|$ and J is a finite set. Observe that the only monomial classes $x \in (BP \wedge V(1))_*T_s(\mathrm{K}(\mathbb{F}_q)_p)$ such that $x = (t\mu_2)^s$ or $x_j = (t\mu_2)^s$ are classes of the form $(t\mu_2)^s y$ for some $y \in (BP \wedge V(1))_*T_s(\mathrm{K}(\mathbb{F}_q)_p)$ up to multiplication by a unit. The co-action of such a class is

$$\psi((t\mu_2)^s y) = (1 \otimes (t\mu_2)^s)\psi(y),$$

so in order to arrange that $x_{j'} = (t\mu_2)^s$ for some j' the coaction $\psi(y)$ must be of the form

$$\psi(y) = 1 \otimes y + z \otimes 1 + \sum_{k \in K} b_k \otimes y_k$$

for some finite set K. Since the only classes in $(BP \wedge V(1))_*T_s(\mathrm{K}(\mathbb{F}_q)_p)$ that have $z \otimes 1$ as a summand in their co-action for some element $z \neq 0, 1$ are the classes t_1^p and t_i for $i \geq 2$, the class y must be a product $\prod_{i \geq 2}(t_1^p)^{k_1}(t_i)^{k_i}$ where $k_i \geq 0$, $k_i = 0$ for all but finitely many $i \geq 2$, and $\sum_{i \geq 1} k_i > 0$. Since the internal degree of $a_i \otimes (t\mu_2)^s y$ must equal $(2p^2 - 2)s + 2p^2 - 2p$ in order to be a summand of (3.15), we have

$$|a_i \otimes (t\mu_2)^s y| = (2p^2 - 2)(s) + |y| + |a_i| = (2p^2 - 2)s + 2p^2 - 2p,$$

so, the degree of $|y| + |a_i|$ must be $2p^2 - 2p$. However, the class t_1^p is the element of lowest degree of the form $(t_1^p)^{k_1}\prod_{i \geq 2}(t_i)^{k_i}$, where $k_i \geq 0$, $k_i = 0$ for all but finitely many $i \geq 1$, and $\sum_{i \geq 1} k_i > 0$, and $|t_1^p| = 2p^2 - 2p$. Also, the only classes in degrees

less than or equal to $2p^2 - 2p$ in $BP_*\overline{BP}$ are powers of v_1 and t_1. Therefore, the only possibility is $a_i = t_1^{p-j}v_1^j$ for some $0 \leq j < p$ and $y = 1$. We know that

$$\Delta(t_1^{p-j}v_1^j) = v_1^j\Delta(t_1^{p-j}) = v_1^j \cdot (t_1 \otimes 1 + 1 \otimes t_1)^{p-j}$$

and we compute

$$\begin{aligned} d_1(t_1^{p-j}v_1^j \otimes (t\mu_2)^s) &= -\overline{\Delta}(t_1^{p-j}v_1^j) \otimes (t\mu_2)^s \\ &\quad + t_1^{p-j}v_1^j \otimes \overline{\psi}((t\mu_2)^s) \\ &= \overline{\Delta}(t_1^{p-j}v_1^j) \otimes (t\mu_2)^s. \end{aligned}$$

We observe that there is no linear combination of $d_1(t_1^{p-j}v_1^j \otimes (t\mu_2)^s)$ that has the element $b_{1,0}(t\mu_2)^s$ as a summand. We conclude that $m_{i'}\doteq(t\mu_2)^s$ for at least one i'.

Now, if $m_{i'}\doteq(t\mu_2)^s$ for only one i', then the element $a_{i'}$ corresponding to $m_{i'}$ must have reduced co-product $b_{1,0} + z$ for some element $z \in BP_*\overline{BP}\otimes_{BP_*} BP_*\overline{BP}$, up to multiplication by a unit; i.e., $\bar{\Delta}(a_{i'})\doteq b_{1,0} + z$. The degree of $a_{i'}$ must be $2p^2 - 2p$, so $a_{i'}\doteq t_1^j v_1^{p-j}$ for $0 \leq j < p$. However,

$$\begin{aligned} \bar{\Delta}(t_1^j v_1^{p-j}) &= v_1^{p-j}\bar{\Delta}(t_1^j) \\ &= v_1^{p-j}(t_1 \otimes 1 + 1 \otimes t_1)^j - 1 \otimes t_1^j v_1^{p-j} - t_1^j v_1^{p-j} \otimes 1 \end{aligned}$$

and this does not equal $b_{1,0}+z$ for any j, and any element $z \in BP_*\overline{BP}\otimes_{BP_*}BP_*\overline{BP}$.

Suppose that $m_j\doteq(t\mu_2)^s$ for $j \in J$ where J is a finite set of indices with more than one element. Then $\psi(\sum_{j\in J} a_j)\doteq b_{1,0} + z'$ for some element z' in $BP_*\overline{BP}\otimes_{BP_*} BP_*\overline{BP}$. However, we checked in the proof of Proposition 3.4 that no class of the form $\sum_{j\in J} a_j$ has co-action $b_{1,0} + z'$ and the same proof applies here.

Thus, there is no sum of classes $\sum_{j\in I} a_i \otimes m_i$ such that $d_1(\sum_{i\in I} a_i \otimes m_i)\doteq b_{1,0} \otimes (t\mu_2)^s$ and therefore the class $b_{1,0}(t\mu_2)^s$ survives to the E_2-page as long as it is a d^1 cycle, which is the case when $s = pk$ for $k \geq 1$ and $s = 2$ for all $p \geq 5$.

Sub-step 2. Now suppose there is a class x in bidegree $((2p^2 - 2)s + 2p^2 - 2p + 1, 0)$ that is the source of a d^2 differential hitting $b_{1,0}(t\mu_2)^s$. Then

$$x \in BP_{(2p^2-2)s+2p^2-2p+1}(V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q)_p))$$

is a d_1-cycle. Since x is in an odd degree, we can classify all the possibilities as a linear combination of elements in the six families of elements,

$$\{\lambda_1' z_1, \lambda_2 z_2, t^s b z_3, \lambda_1' t_1^{p^2-p} z_4, \lambda_2 t_2^{p-1} z_5, b\gamma_{p-1}(\sigma b)z_6\}$$

in a sub-quotient of $BP_{(2p^2-2)s+2p^2-2p+1}(V(1) \wedge \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p)) \otimes P(t)/t^{s+1}$ by Corollary 3.12 where z_4, z_5, and z_6 are non-trivial even dimensional classes and z_1, z_2, and z_3 are non-trivial even dimensional classes that are not t-divisible, since λ_1', λ_2, and $t^s b$ are simple t-torsion elements. We know that each of the elements in the set

$$\{\lambda_2, \lambda_1', t^s b\}$$

are permanent cycles in the truncated homotopy fixed point spectral sequence computing $(BP \wedge V(1))_* T_s(\mathrm{K}(\mathbb{F}_q)_p)$ by Corollary 3.12. We know that each of the elements in the set

$$\{b\gamma_{p-1}(\sigma b), \lambda_2 t_2^{p-1}, \lambda_1 t_1^{p^2-p}\}$$

are permanent cycles by mapping to a homological homotopy fixed point spectral sequence via the Hurewicz map and applying [16, Theorem 5.1 (b)]. Note that the classes $t^s b$, $\lambda_1' t_1^{p^2-p}$, $b\gamma_{p-1}(\sigma b)$, and $\lambda_2 t_2^{p-1}$ are indecomposable in

$$BP_{(2p^2-2)s+2p^2-2p+1}(V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q)_p))$$

by Corollary 3.12.

We explicitly compute $d_1(\lambda_2) = \bar{\xi}_1 \otimes \lambda_1'$. Therefore, by the Leibniz rule, $d_1(\lambda_2 z_2) = (\bar{\xi}_1 \otimes \lambda_1')z_2 + \lambda_2 d_1(z_2) \neq 0$. Therefore, the classes of the form $\lambda_2 z_2$ do not survive to the E_2 page of the BP-Adams spectral sequence and cannot be the source of a d_2 differential hitting $b_{1,0}(t\mu_2)^s$.

Next, we check elements of the form $t^s b z_3$ or $\lambda_1' z_1$ where z_3 is not divisible by $t\mu_2$ or $t\sigma b$ and z_1 is not divisible by λ_1'. Note that the Leibniz rule implies

$$d_2(t^s b z_3) = d_2(t^s b)z_3 + t^s b d_2(z_3)$$

and similarly,

$$d_2(\lambda_1' z_2) = d_2(\lambda_1')z_2 - \lambda_1' d_2(z_2),$$

so we need to check if

$$\alpha(d_2(tb)z_3 + tb d_2(z_3)) + \beta(d_2(\lambda_1')z_2 - \lambda_1' d_2(z_2)) = b_{1,0}(t\mu_2)^s$$

for some $\alpha, \beta \in \mathbb{F}_p$. Suppose $d_2(\lambda_1' z_2) \dot{=} b_{1,0}(t\mu_2)^s + z$ for some possibly trivial class z. Then by the Leibniz rule $d_2(\lambda_1' z_2) = d_2(\lambda_1')z_2 - \lambda_1' d_2(z_2)$ and so $d_2(\lambda_1')$ would need to divide the product of $b_{1,0}$ and $(t\mu_2)^s$. Since we already proved that $b_{1,0}$ and $(t\mu_2)^s$ are permanent cycles, this would be a contradiction. It therefore suffices to show that $d_2(t^s b z_3) \neq b_{1,0}(t\mu_2)^s + z'$ for some class z'. Since $t\mu_2$ does not divide z_3 it suffices to show that $d_2(z_3)t^s b \neq b_{1,0}(t\mu_2)^s + z'$ for some class z' by the Leibniz rule. This follows because $t^s b$ does not divide $b_{1,0}(t\mu_2)^s$.

Finally, we need to check the classes $\lambda_1' t_1^{p^2-p} z_4$, $\lambda_2 t_2^{p-1} z_5$, and $b\gamma_{p-1}(\sigma b) z_6$. In order to be a source of a d_2, these classes must survive to the E_2-page, so we just need to check this in the case when the classes of the form $\lambda_1' t_1^{p^2-p} z_4$, $\lambda_2 t_2^{p-1} z_5$, and $b\gamma_{p-1}(\sigma b) z_6$ are d_1 cycles. Since the d_1-cycles on the 0-line are exactly that $BP_* BP$-comodule primitives, we just need to prove that the elements

(3.16) $$\{\lambda_1' t_1^{p^2-p}, b\gamma_{p-1}, \lambda_1' t_1^{p^2-p}\}$$

are not $BP_* BP$-comodule primitives. From Corollary 3.12, we know that they are not $BP_* BP$-comodule primitives on the E^4-page of the truncated homotopy fixed point spectral sequence, which is the E^∞-page for $BP_*(V(1) \wedge T_3(\mathrm{K}\mathbb{F}_q)_p))$. We also claim that hidden $BP_* BP$-comodule extensions in the truncated homotopy fixed point spectral sequence computing $BP_* V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q)_p)$ cannot produce additional comodule primitives. To see this, note that elements in higher skeletal filtration are always divisible by higher powers of t and therefore summing with such classes cannot reduce the number of terms in the $BP_* BP$-coaction of an element in the E^∞-page in lower skeletal filtration. Therefore, the classes $\lambda_1' t_1^{p^2-p} z_4$, $\lambda_2 t_2^{p-1} z_5$, and $b\gamma_{p-1}(\sigma b) z_6$ do not survive to the E_1-page of the BP-Adams spectral sequence for $BP_* V(1) \wedge T_3(\mathrm{K}(\mathbb{F}_q)_p)$. The elements in (3.16) in the BP-Adams spectral sequence for $V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q))$ map to classes of the same name in the BP-Adams spectral sequence via the map E_1-pages of BP-Adams spectral sequences

$$BP_* \overline{BP}_*^{\otimes_{BP_*} \bullet} \otimes BP_* V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q)) \longrightarrow BP_* \overline{BP}_*^{\otimes_{BP_*} \bullet} \otimes BP_* V(1) \wedge T_3(\mathrm{K}(\mathbb{F}_q))$$

induced by the canonical map $V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q)) \to V(1) \wedge T_3(\mathrm{K}(\mathbb{F}_q))$ and therefore they cannot be comodule primitives in the source, because this would lead to a contradiction.

In fact, our proof shows that $b_{1,0}(t\mu_2)^s + z''$ is not hit by a d_2 differential for any $s \geq 0$ and any class z'', so the exact same argument implies that, as long as $b_{1,0}(t\mu_2)^s$ and $\overline{\beta}_s$ are d_1-cycles and they are not d_1 boundaries, then they are not d_2 boundaries for any $s \geq 1$. Note that by [42, Theorem 5.2], the element $v_2\beta_1$ is non-trivial in $V(1)_{(2p^2-2)+(2p^2-2p-2)}$. Our argument also shows that $v_2\beta_1$ has non-trivial image in $V(1)_*\mathrm{TC}^-(K(\mathbb{F}_q)_p)$. We also use this to reduce the work we need to do to show that $\overline{\beta}_s$ is a permanent cycle in Step 5.

Step 4. We now discuss how to detect β_2. The element β_2 is represented by the class $k_0 + 2b_{1,0}v_2 \mod (p, v_1)$ in the input of the BP-Adams spectral sequence for S. It is also a permanent cycle in the BP-Adams spectral sequence for $V(1)$ as a consequence of [33, Lemma 5.4]. It maps to the class $k_0 + 2b_{1,0}t\mu_2$ in

$$BP_*\overline{BP}_* \otimes_{BP_*} BP_*\overline{BP}_* \otimes_{BP_*} BP_*V(1) \wedge T_2(\mathrm{K}(\mathbb{F}_q)_p)$$

under the map of E_1-pages of BP-Adams spectral sequences induced by the map $V(1) \to V(1) \wedge T_2(\mathrm{K}(\mathbb{F}_q)_p)$, by Step 1 and the multiplicativity of E_1-page of the BP-Adams spectral sequence.

Recall that $BP_*(V(1) \wedge T_2(\mathrm{K}(\mathbb{F}_q)_p))$ is isomorphic to a sub-quotient of

$$H_* \left(P(t_1^p, t_2, \dots) \otimes E(b) \otimes E(\sigma t_1^p, \sigma t_2) \otimes P(\mu_2) \otimes \Gamma(\sigma b) \otimes P_3(t); d^2(x) = t\sigma x \right)$$

after taking into account d^4 differentials. We can therefore check every element in degree $4p^2 - 2p - 2$ in

$$BP_*\overline{BP}_* \otimes_{BP_*} BP_*(V(1) \wedge T_2(\mathrm{K}(\mathbb{F}_q)_p))$$

to see if it has $k_0 + 2b_{1,0}v_2$ as a boundary up to multiplication by a unit. Note that the source of a d_1 differential hitting $k_0 + 2b_{1,0}v_2$ must be in bidegree $(1, 4p^2 - 2p - 2)$ or in other words stem $4p^2 - 2p - 3$ and BP-Adams filtration 1 because the d_1 differential decreases stem by one and increases BP-Adams filtration by one. Consequently, the source of a possible differential of length 1 hitting $k_0 + 2b_{1,0}v_2$ must be of the form $x \otimes m$ where $0 \leq |x| \leq 4p^2 - 2p - 2$ and $0 \leq |m| \leq 4p^2 - 2p - 2$. The elements $x \in BP_*\overline{BP}_*$ satisfying $0 \leq |x| \leq 4p^2 - 2p - 2$ are linear combinations of t_1^k and t_2 for $1 \leq k < 2p$. We conclude that $4p^2 - 2p - 2 - |m| \equiv 0 \mod 2p - 2$. By Corollary 3.12, the potential elements $m \in BP_*V(1) \wedge T_2(\mathrm{K}(\mathbb{F}_q)_p)$ satisfying $0 \leq |m| \leq 4p^2 - 2p - 2$ are

$$\{1, \lambda_1', \lambda_2, \sigma b, \gamma_2(\sigma b), \mu, t\mu, t^2\mu, t^2t_1^p, t^2t_1^{2p}, t^2t_2, t^2b, t^2t_1^pb, t^2bt_2\},$$

so after removing all elements that don't satisfy $4p^2 - 2p - 2 - |m| \equiv 0 \mod 2p - 2$, we are left with the elements

$$\{\sigma b, \gamma_2(\sigma b), t\mu_2\}.$$

When $m = \sigma b$, the only possibilities for x are linear combinations of t_1^{p+1} and t_2, so we need to compute $d_1(t_1^{p+1} \otimes \sigma b)$ and $d_1(t_2 \otimes \sigma b)$. When $m = \gamma_2(\sigma b)$ the only option is $x = t_1$, so we need to compute $d_1(t_1 \otimes \gamma_2(\sigma b))$. When $m = t\mu_2$, x

must be t_1^p so we need to compute $d_1(t_1^p \otimes t\mu_2)$. We compute

$$d_1(t_1^{p+1} \otimes \sigma b) = -\sum_{i=1}^{p} \binom{p+1}{i} t_1^i \otimes t_1^{p+1-i} \otimes \sigma b,$$

$$d_1(t_2 \otimes \sigma b) = -t_1^p \otimes t_1 \sigma b,$$

$$d_1(t_1 \otimes \gamma_2(\sigma b)) = 0,$$

$$d_1(t_1^p \otimes t\mu_2) = -\sum_{i=1}^{p-1} \binom{p}{i} t_1^i \otimes t_1^{p-i} \otimes t\mu = 0.$$

We then note that the equation

$$a_1 d_1(t_1^{p+1} \otimes \sigma b) + a_2 d_1(t_2 \otimes \sigma b) + a_3 d_1(t_1^p \otimes t\mu_2) = k_0 + 2b_{1,0}v_2$$

is not satisfied for any $a_1, a_2, a_3 \in \mathbb{F}_p$. We conclude that $k_0 + 2b_{1,0}v_2$ survives to the E_2-page of the BP-Adams spectral sequence.

We next need to check if it is the boundary of a d_2. By Corollary 3.12, we know that $BP_{4p^2-4p-5}(V(1) \wedge T_2(\mathrm{K}(\mathbb{F}_q)_p)) = \mathbb{F}_p\{t^2 t_1 b\}$. However, we know $\psi(t_1) = t_1 \otimes 1 + 1 \otimes t_1$, so $d_1(t_1) = t_1 \otimes 1$. Since $t^2 b$ is a comodule primitive, we compute $d_1(t_1 t^2 b) = t_1 \otimes t^2 b$ and consequently, $t_1 t^2 b$ does not survive to the E_2-page of the BP-Adams spectral sequence. Therefore, the class $t_1 t^2 b$ cannot be the source of a d_2 differential hitting $k_0 + 2b_{1,0}v_2$. Consequently, the class $k_0 + 2b_{1,0}v_2$ survives to the E^3-page of the BP-Adams spectral sequence. Since it is in BP-Adams filtration 2 and it is an infinite cycle, it also survives to the E_∞-page of the BP-Adams spectral sequence.

We now identify an element in $V(1)_{4p^2-2p-4}T_2(\mathrm{K}(\mathbb{F}_q)_p)$ which projects onto $k_0 + 2b_{1,0}t\mu_2$ in the E_∞-page of the BP-Adams spectral sequence. We claim that the only classes in $V(1)_{4p^2-2p-4}T_2(\mathrm{K}(\mathbb{F}_q)_p)$ are $t^2\lambda_1'\lambda_2$ and $t^2\mu_2\sigma b$. To see this, note that the there is a truncated homotopy fixed point spectral sequence

$$P_2(t) \otimes V(1)_* \mathrm{THH}(\mathrm{K}(\mathbb{F}_q)_p) \implies V(1)_{4p^2-2p-4}T_2(\mathrm{K}(\mathbb{F}_q)_p),$$

where the input is

$$\mathbb{F}_p\{1, \alpha_1, \lambda_1', \alpha_1\lambda_2, \lambda_1'\lambda_2, \alpha_1\lambda_1'\lambda_2\} \otimes P(\mu_2) \otimes \Gamma(\sigma b) \otimes P_2(t).$$

Concentrating on elements in degree $4p^2 - 2p - 4$ allows us to remove all elements in odd degree and degree greater than $4p^2 - 2p - 4$ from consideration, leaving us with considering elements in

$$\mathbb{F}_p\{1, \alpha_1\lambda_2, \lambda_1'\lambda_2\} \otimes E(\mu_2) \otimes P_2(\sigma b) \otimes P_2(t).$$

Direct computation shows that the only possible elements in the correct degree in the E^2-page of the truncated homotopy fixed point spectral sequence are $t^2\lambda_1'\lambda_2$ and $t^2\mu\sigma b$. We already showed that $v_2\beta_1$ has nontrivial image in $V(1)_{4p^2-2p-4}\mathrm{TC}^-(K(\mathbb{F}_q)_p)$ in the previous step. We also just showed that β_2 has nontrivial image. Therefore, both $t^2\lambda_1'\lambda_2$ and $t^2\mu_2\sigma b$ must be permanent cycles or else we reach a contradiction. Since we know v_2 maps to $t\mu_2$ and β_1 maps to $t\sigma b$, multiplicativity implies that $v_2\beta_1$ is detected by $t^2\mu_2\sigma b$. Therefore, we conclude that the Hurewicz image of β_2 is detected by a linear combination $ct^2\lambda_1'\lambda_2 + c't^2\mu_2\sigma b \in V(1)_{4p^2-2p-4}T_2(\mathrm{K}(\mathbb{F}_q)_p)$ where $c \neq 0$. We may arrange that β_2 maps to $t^2\lambda_2\lambda_1' + 2\mu_2\sigma b$ by judiciously choosing our choice of representatives for the classes $t^2\mu_2\sigma b$ and $t^2\lambda_2\lambda_1'$ and our basis for the vector space $\mathbb{F}_p\{t^2\mu_2\sigma b, t^2\lambda_2\lambda_1'\}$.

In fact, we can conclude that k_0 representing $\beta_2 - 2v_2\beta_1$ and $v_2 b_{1,0}$ representing $v_2\beta$ also have non-trivial Hurewicz image.

Step 5. Finally, we discuss how to detect β_s where $s \not\equiv 0, 1 \mod p$. In this case, β_s is represented by $\overline{\beta}_s$, which maps to

$$(t\mu_2)^{s-1}\left(\binom{s}{2}k_0 + kb_{1,0}(t\mu_2)\right)$$

by Step 1. We just need to check the d_1 differentials on classes of the form $(t\mu_2)^{s-1}w$ where w is an element in $BP_*\overline{BP}_* \otimes_{BP_*} BP_*\overline{BP}_*$. If the class

$$2t_1^p \otimes t_2 \otimes (t\mu_2)^{s-1} - 2t_1^p \otimes t_1^{1+p} \otimes (t\mu_2)^{s-1} - t_1^{2p} \otimes t_1 \otimes (t\mu_2)^{s-1}$$

is the boundary of a d_1, then there is a sum of classes

$$\sum_{i\in I} a_i \otimes m_i \in \overline{BP_*BP} \otimes_{BP_*} BP_*V(1) \wedge T_k(\mathrm{K}(\mathbb{F}_q)_p)$$

for some finite set I such that

$$d_1(\sum_{i\in I} a_i \otimes m_i) \doteq 2t_1^p \otimes t_2 \otimes (t\mu_2)^{s-1} - 2t_1^p \otimes t_1^{1+p} \otimes (t\mu_2)^{s-1} - t_1^{2p} \otimes t_1 \otimes (t\mu_2)^{s-1}.$$

Sub-step 1. Recall that the co-action on x is of the form

$$\psi(m) = 1 \otimes x + \sum_{j\in J} a_j \otimes x_j$$

for some finite set J where $|x_j| < |x|$. Again, observe that the only elements in $(BP \wedge V(1))_* T_k(\mathrm{K}(\mathbb{F}_q)_p)$ whose co-action contains $(t\mu_2)^{s-1}$ as either x or x_j for some j are classes of the form $(t\mu_2)^{s-1}y$ for some $y \in (BP \wedge V(1))_* T_s(\mathrm{K}(\mathbb{F}_q)_p)$ up to multiplication by a unit. The co-action of such a class is $\psi((t\mu_2)^{s-1}y) = (1 \otimes (t\mu_2)^{s-1})\psi(y)$, and $\psi(y)$ must be of the form $\psi(y) = 1 \otimes y + z \otimes 1 + \sum b_i \otimes y_i$ since $\psi((t\mu_2)^{s-1}y)$ must have $1 \otimes (t\mu_2)^{s-1}$ as a term, up to multiplication by a unit. Since the only classes in $(BP \wedge V(1))_* T_s(\mathrm{K}(\mathbb{F}_q)_p)$ that have a term $z \otimes 1$ in their co-action are the classes t_1^p, t_i for $i \geq 2$ the class y must be a product of these. Since $|(u \cdot t\mu_2)^{s-1}y| = (2p^2-2)(s-1) + |y|$ and the degree must equal $4p^2 - 2p + s(2p^2-2)$, the degree of y must be $4p^2 - 2p$. However, the class t_1^p is the element of lowest degree of the form

$$t_1^{pi_1} \prod_{j\geq 2} t_j^{i_j},$$

where $i_j \geq 0$, $i_j = 0$ for all but finitely many $j \geq 1$, and $\sum_{j\geq 2} i_j > 0$. Also, $|t_1^p| = 2p^2 - 2p$ and the next lowest degree element is t_2 with $|t_2| = 2p^2 - 2$ so no product of classes in this set can be in degree $4p^2 - 2p$. Thus, $m_{i'} \doteq (t\mu_2)^{s-1}$ for at least one $i' \in I$.

Now, if $m_{i'} \doteq (t\mu_2)^{s-1}$ for only one i', then the element $a_{i'}$ corresponding to $m_{i'}$ must have reduced co-product $2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} \otimes 1 - t_1^{2p} \otimes t_1$; i.e.

$$\overline{\Delta}(a_{i'}) \doteq 2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} - t_1^{2p} \otimes t_1.$$

The degree of $a_{i'}$ must be $4p^2 - 2p$, so $a_{i'} \doteq t_1^j v_1^{p-j} v_2^{\epsilon_1} t_2^{\epsilon_2}$.

However,

$$\Delta(t_1^j v_1^{p-j} v_2^{\epsilon_1} t_2^{\epsilon_2}) = v_1^{p-j} v_2^{\epsilon_1}(t_1 \otimes 1 + 1 \otimes t_1)^j (t_2 \otimes 1 + 1 \otimes t_2 + t_1^p \otimes t_1)^{\epsilon_2}$$

and so $\overline{\Delta}(t_1^j v_1^{p-j} v_2^{\epsilon_1} t_2^{\epsilon_2})$ does not equal

$$t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} - t_1^{2p} \otimes t_1$$

for any $0 \le j \le p$.

Suppose that $m_k = (t\mu_2)^{s-1}$ for $k \in K \subset I$ where K contains more than one element. Then

$$\psi(\sum_{k \in K} a_k) \doteq 2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} - t_1^{2p} \otimes t_1$$

for some possibly trivial element $z' \in BP_* \overline{BP}_* \otimes_{BP_*} BP_* \overline{BP}_*$. However, we checked in Step 4 that no class of the form $\sum_{k \in K} a_k$ has co-action

$$(2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} - t_1^{2p} \otimes t_1)$$

and the same proof applies here.

Thus, there is no sum of classes $\sum_{i \in I} a_i \otimes m_i$ such that

$$d_1(\sum_{i \in I} a_i \otimes m_i) = (2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} - t_1^{2p} \otimes t_1) \otimes (t\mu_2)^{s-1}$$

and therefore the class $(2t_1^p \otimes t_2 - 2t_1^p \otimes t_1^{1+p} - t_1^{2p} \otimes t_1) \otimes (t\mu_2)^{s-1}$ survives to the E_2-page.

Sub-step 2. The argument that there are no d_2 differentials that hit

$$\binom{s}{2}(t\mu_2)^{s-1}k_0 + sb_{1,0}(t\mu_2)^s$$

is the same as that of Step 2.

Therefore, $\binom{s}{2}(t\mu_2)^{s-1}k_0 + sb_{1,0}(t\mu_2)^s$ is a permanent cycle in the BP-Adams spectral sequence for $V(1) \wedge T_s(\mathrm{K}(\mathbb{F}_q)_p)$.

We now just need to show that

$$\binom{s}{2}(t\mu_2)^{s-1}(t\lambda_1'\lambda_2) + s(t\sigma b)(t\mu_2)^s \in V(1)_{(2p^2-2)s-2p} T_s(\mathrm{K}(\mathbb{F}_q)_p)$$

projects onto the permanent cycle $\binom{s}{2}(t\mu_2)^{s-1}k_0 + sb_{1,0}(t\mu_2)^s$ in the E_∞-page of the BP-Adams spectral sequence. Since $\beta_2 - 2v_2\beta_1$, β_1, and v_2 in $V(1)_*$ map to $t^2\lambda_1'\lambda_2$, $t\sigma b$, and $t\mu_2$ by the previous steps and these project onto k_0, β_1, and v_2 respectively in the E_∞-page of the BP-Adams spectral sequence, we can conclude that

$$\binom{s}{2}(t\mu_2)^{s-1}(t\lambda_1'\lambda_2) + s(t\sigma b)(t\mu_2)^s$$

projects onto $\binom{s}{2}(t\mu_2)^{s-1}k_0 + sb_{1,0}(t\mu_2)^s$ in the E_∞-page of the BP-Adams spectral sequence because the unit map $V(1) \longrightarrow V(1) \wedge T_s K(\mathbb{F}_q)_p)$ induces a multiplicative map of BP-Adams spectral sequences. $\qquad\square$

3.5. Detecting β elements in iterated algebraic K-theory of the integers.
The goal of this section is to prove that the β-family mod (p, v_1) is detected in the iterated algebraic K-theory of finite fields. We prove this as a corollary to Theorem 3.15. The proof relies on the fact that the trace map $\mathrm{K}(R) \to \mathrm{TC}^-(R)$ is a map of commutative ring spectra when R is a commutative ring spectrum. The proof that the cyclotomic trace map $\mathrm{K}(R) \to \mathrm{TC}(R)$ is a map of commutative ring spectra when R is a commutative ring spectrum is due to Hesselholt–Geisser [18] for Eilenberg-MacLane spectra and Blumberg–Gepner–Tabuada [14] for commutative

ring spectra. The advantage of the approach of Blumberg–Gepner–Tabuada [14] is that they prove that the multiplicative cyclotomic trace map is also unique. This work builds on their proof that algebraic K-theory is the universal additive functor [13] (for a different take on universal properties of algebraic K-theory see Barwick [9]).

Let R be a connective commutative ring spectrum. We write $\mathrm{TP}(R) := \mathrm{THH}(R)^{t\mathbb{T}}$. Recall from [30, Prop. II.1.9] and [30, Rem. II.4.3] that TC is the homotopy equalizer of the maps

$$\mathrm{can}\colon \mathrm{TC}^-(R) \to \mathrm{TP}(R)^{\wedge} = \prod_p \mathrm{TP}(R)_p,$$

$$\varphi_p^{h\mathbb{T}}\colon \mathrm{TC}^-(R) \to \mathrm{TP}(R)^{\wedge} = \prod_p \mathrm{TP}(R)_p,$$

where $\mathrm{can}\colon \mathrm{TC}^-(R) \to \mathrm{TP}(R)^{\wedge} \to \mathrm{TP}(R)$ is the canonical map from $\mathrm{TC}^-(R)$ to the cofiber of the norm map $\Sigma\mathrm{THH}(R)_{h\mathbb{T}} \to \mathrm{TC}^-(R)$, and

$$\varphi_p\colon \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

is the Tate valued Frobenius map. This description relies on the identification

$$(\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}} \simeq \mathrm{TP}(R)_p$$

from [30, Lemma II.4.2.], which uses the assumption that R is bounded below. Since the maps can and $\varphi_p^{h\mathbb{T}}$ are each commutative ring spectrum maps and the forgetful functor $\mathrm{Comm}(\mathsf{Sp}) \to \mathsf{Sp}$ is a right adjoint, we can take the equalizer of can and $\varphi_p^{h\mathbb{T}}$ in $\mathrm{Comm}(\mathsf{Sp})$. Therefore, the map $\mathrm{TC}(R) \to \mathrm{TC}^-(R)$ is a map of commutative ring spectra. The following is then an easy consequence of [14, Thm. 7.4]. The author would like to thank the anonymous referee for pointing out this simpler argument.

Lemma 3.16. *Suppose R is a connective commutative ring spectrum, then the trace map*

$$\mathrm{K}(R) \to \mathrm{TC}(R) \to \mathrm{TC}^-(R)$$

is a map of commutative ring spectra.

Corollary 3.17. *Let $p \geq 5$ be a prime. Then each element of the p-primary β-family $\{\beta_k : k \not\equiv 0 \mod p\}$ maps to a nonzero element in $\pi_*\mathrm{K}(\mathrm{K}(\mathbb{F}_q))$ under the unit map. If $q = \ell^m$ for some prime ℓ, then each element of the p-primary β-family $\{\beta_k : k \not\equiv 0 \mod p\}$ maps to a nonzero element in $\pi_*\mathrm{K}(\mathrm{K}(\mathbb{F}_\ell))$ under the unit map.*
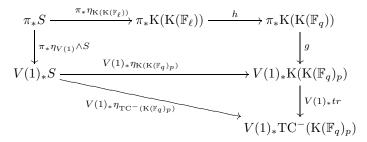
Proof. The elements $\beta_k \in \pi_{(2p^2-2)k-2p}(S)$ for $k \not\equiv 0 \mod p$ map nontrivially to

$$V(1)_{(2p^2-2)k-2p}\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)$$

under the unit map by Theorem 3.15. Since the cyclotomic trace is multiplicative, the unit map

$$\pi_*S \to V(1)_*\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)$$

factors through $\pi_*\mathrm{K}(\mathrm{K}(\mathbb{F}_q))$; i.e., there is a commutative diagram

$$
\begin{array}{ccc}
\pi_*S & \xrightarrow{\ \pi_*\eta_{\mathrm{K}(\mathrm{K}(\mathbb{F}_\ell))}\ } \pi_*\mathrm{K}(\mathrm{K}(\mathbb{F}_\ell)) \xrightarrow{\ h\ } \pi_*\mathrm{K}(\mathrm{K}(\mathbb{F}_q)) \\
\downarrow{\scriptstyle \pi_*\eta_{V(1)}\wedge S} & \Big\downarrow{\scriptstyle g} \\
V(1)_*S \xrightarrow{\qquad V(1)_*\eta_{\mathrm{K}(\mathrm{K}(\mathbb{F}_q)_p)}\qquad} V(1)_*\mathrm{K}(\mathrm{K}(\mathbb{F}_q)_p) \\
\qquad \searrow{\scriptstyle V(1)_*\eta_{\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)}} \qquad \Big\downarrow{\scriptstyle V(1)_*tr} \\
V(1)_*\mathrm{TC}^-(\mathrm{K}(\mathbb{F}_q)_p)
\end{array}
$$

where g is the composite of the unit map $\pi_*\eta_{V(1)} \wedge \mathrm{K}(\mathrm{K}(\mathbb{F}_q)): \pi_*\mathrm{K}(\mathrm{K}(\mathbb{F}_q)) \longrightarrow V(1)_*\mathrm{K}(\mathrm{K}(\mathbb{F}_q))$ with the map $V(1)_*K(f_p)$ where f_p is the p-completiffon map $f_p: \mathrm{K}(\mathbb{F}_q) \longrightarrow \mathrm{K}(\mathbb{F}_q)_p$. The map h is induced by the canonical map of fields $\mathbb{F}_\ell \longrightarrow \mathbb{F}_q$. $\qquad\square$

Corollary 3.18. *Let R be any E_2 ring spectrum equipped with a map of E_2 ring spectra to $H\mathbb{F}_q$ and let $p \geq 5$. Then the p-primary β-family $\{\beta_s : s \not\equiv 0 \mod p\}$ is detected in $\mathrm{K}(\mathrm{K}(R))$.*

Proof. Since there exists a map of E_2 ring spectra $R \to H\mathbb{F}_q$ inducing a map $\mathrm{K}(\mathrm{K}(R)) \to \mathrm{K}(\mathrm{K}(\mathbb{F}_q))$. Therefore, there is a commutative diagram

(3.17)
$$
\begin{array}{ccc}
S \longrightarrow \mathrm{K}(\mathrm{K}(S)) \longrightarrow \mathrm{K}(\mathrm{K}(R)) \\
\searrow \qquad \Big\downarrow \\
\mathrm{K}(\mathrm{K}(\mathbb{F}_q)).
\end{array}
$$

Since the β-family is nontrivial in the image of the map $\pi_*S \to \pi_*\mathrm{K}(\mathrm{K}(\mathbb{F}_q))$, it is also nontrivial in the image of the unit map $\pi_*S \to \pi_*(\mathrm{K}(\mathrm{K}(R)))$. $\qquad\square$

In particular, this implies the following result:

Corollary 3.19. *The p-primary beta family $\{\beta_s : s \not\equiv 0 \mod p\}$ is detected in $\mathrm{K}(\mathrm{K}(\mathbb{Z}))$ for all $p \geq 5$.*

Remark 3.20. Note that the α-family is detected in $\mathrm{K}(\mathbb{Z})$. Since $\mathrm{K}_0(\mathbb{Z}) \cong \mathbb{Z}$, there is a map of commutative ring spectra $\mathrm{K}(\mathbb{Z}) \to H\mathbb{Z}$. We may consider the infinite family of maps

$$S \to \ldots \to \mathrm{K}(\mathrm{K}(\mathrm{K}(\mathbb{Z}))) \to \mathrm{K}(\mathrm{K}(\mathbb{Z})) \to \mathrm{K}(\mathbb{Z})$$

and a specialization of the Greek-letter family red-shift conjecture is that the p-primary n-th Greek letter family is in the image of the unit map $S \to \mathrm{K}^{(n)}(\mathbb{Z})$, where $\mathrm{K}^{(n)}(\mathbb{Z})$ is algebraic K-theory iterated n-times, for p sufficiently large. Corollary 3.19 is therefore evidence for this conjecture when $n = 2$.

<div align="center">ACKNOWLEDGMENTS</div>

would like to thank each of these mathematics departments for their hospitality. The author would also like to thank Christian Ausoni and John Rognes for posing the red-shift conjectures, which have been immensely inspirational to the author. Also, the author would like to thank Bob Bruner for offering his insight about the homological homotopy fixed point spectral sequence. The author also thanks W. Stephen Wilson for comments which led to a correction of an error in an early draft and Gerd Laures for feedback on the results of this paper in the context of the double transfer and the f-invariant.

References

[1] J. F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71, DOI 10.1016/0040-9383(66)90004-8. MR198470

[2] Gabe Angelini-Knoll and Andrew Salch, *A May-type spectral sequence for higher topological Hochschild homology*, Algebr. Geom. Topol. **18** (2018), no. 5, 2593–2660, DOI 10.2140/agt.2018.18.2593. MR3848395

[3] Gabriel Angelini-Knoll, *On topological Hochschild homology of the $K(1)$-local sphere*, J. Topol. **14** (2021), no. 1, 258–290, DOI 10.1112/topo.12182. MR4235012

[4] Vigleik Angeltveit and John Rognes, *Hopf algebra structure on topological Hochschild homology*, Algebr. Geom. Topol. **5** (2005), 1223–1290, DOI 10.2140/agt.2005.5.1223. MR2171809

[5] Christian Ausoni and John Rognes, *Algebraic K-theory of topological K-theory*, Acta Math. **188** (2002), no. 1, 1–39, DOI 10.1007/BF02392794. MR1947457

[6] Christian Ausoni and John Rognes, *The chromatic red-shift in algebraic k-theory*, Monogr. Enseign. Math., vol. 40, L'Enseignement Mathématique, Geneva, 2008. MR2499538

[7] Christian Ausoni and John Rognes, *Algebraic K-theory of the first Morava K-theory*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 4, 1041–1079, DOI 10.4171/JEMS/326. MR2928844

[8] Andrew Baker and Birgit Richter, *Uniqueness of E_∞ structures for connective covers*, Proc. Amer. Math. Soc. **136** (2008), no. 2, 707–714. MR2358512

[9] Clark Barwick, *On the algebraic K-theory of higher categories*, J. Topol. **9** (2016), no. 1, 245–347, DOI 10.1112/jtopol/jtv042. MR3465850

[10] Mark Behrens, *Congruences between modular forms given by the divided β family in homotopy theory*, Geom. Topol. **13** (2009), no. 1, 319–357, DOI 10.2140/gt.2009.13.319. MR2469520

[11] Mark Behrens and Gerd Laures, *β-family congruences and the f-invariant*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 9–29, DOI 10.2140/gtm.2009.16.9. MR2544384

[12] Mark Behrens and Satya Pemmaraju, *On the existence of the self map v_2^9 on the Smith-Toda complex $V(1)$ at the prime 3*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 9–49, DOI 10.1090/conm/346/06284. MR2066495

[13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838, DOI 10.2140/gt.2013.17.733. MR3070515

[14] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *Uniqueness of the multiplicative cyclotomic trace*, Adv. Math. **260** (2014), 191–232. MR3209352

[15] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Astérisque **226** (1994), 7–8, 57–143. K-theory (Strasbourg, 1992). MR1317117

[16] Robert R. Bruner and John Rognes, *Differentials in the homological homotopy fixed point spectral sequence*, Algebr. Geom. Topol. **5** (2005), 653–690, DOI 10.2140/agt.2005.5.653. MR2153113

[17] Ethan S. Devinatz and Michael J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), no. 1, 1–47, DOI 10.1016/S0040-9383(03)00029-6. MR2030586

[18] Thomas Geisser and Lars Hesselholt, *Topological cyclic homology of schemes*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 41–87, DOI 10.1090/pspum/067/1743237. MR1743237

[19] Christian Haesemeyer and Charles A. Weibel, *The norm residue theorem in motivic cohomology*, Annals of Mathematics Studies, vol. 200, Princeton University Press, Princeton, NJ, 2019. MR3931681

[20] L. Hesselholt, *Topological Hochschild homology and the Hasse-Weil zeta function*, E-prints (February 2016), available at 1602.01980.

[21] Lars Hesselholt, *On the p-typical curves in Quillen's K-theory*, Acta Math. **177** (1996), no. 1, 1–53, DOI 10.1007/BF02392597. MR1417085

[22] Eva Höning, *The topological Hochschild homology of algebraic K-theory of finite fields*, Ann. K-Theory **6** (2021), no. 1, 29–96, DOI 10.2140/akt.2021.6.29. MR4283094

[23] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49. MR1652975

[24] Donald M. Larson, *Explicit modular forms from the divided beta family*, Topology Appl. **273** (2020), 107094, 19, DOI 10.1016/j.topol.2020.107094. MR4062995

[25] Gerd Laures, *The topological q-expansion principle*, Topology **38** (1999), no. 2, 387–425, DOI 10.1016/S0040-9383(98)00019-6. MR1660325

[26] Stephen Lichtenbaum, *Values of zeta-functions, étale cohomology, and algebraic K-theory*, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973, pp. 489–501. MR0406981

[27] J. E. McClure and R. E. Staffeldt, *The chromatic convergence theorem and a tower in algebraic K-theory*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 1005–1012, DOI 10.2307/2160154. MR1164148

[28] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. (2) **106** (1977), no. 3, 469–516, DOI 10.2307/1971064. MR458423

[29] John Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171, DOI 10.2307/1969932. MR99653

[30] Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), no. 2, 203–409, DOI 10.4310/ACTA.2018.v221.n2.a1. MR3904731

[31] Shichirô Oka, *A new family in the stable homotopy groups of spheres*, Hiroshima Math. J. **5** (1975), no. 1, 87–114.

[32] Shichirô Oka, *A new family in the stable homotopy groups of spheres. II*, Hiroshima Math. J. **6** (1976), no. 2, 331–342.

[33] Shichirô Oka and Katsumi Shimomura, *On products of the β-elements in the stable homotopy of spheres*, Hiroshima Math. J. **12** (1982), no. 3, 611–626. MR676562

[34] Daniel Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2) **96** (1972), 552–586. MR0315016

[35] Daniel Quillen, *Higher algebraic K-theory*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Canad. Math. Congress, Montreal, Que., 1975, pp. 171–176. MR0422392

[36] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR860042

[37] Katsumi Shimomura, *A beta family in the homotopy of spheres*, Proc. Amer. Math. Soc. **142** (2014), no. 8, 2921–2928, DOI 10.1090/S0002-9939-2014-12009-0. MR3209345

[38] Larry Smith, *On realizing complex bordism modules. Applications to the stable homotopy of spheres*, Amer. J. Math. **92** (1970), 793–856. MR0275429

[39] V. P. Snaith, *Towards the Lichtenbaum-Quillen conjecture concerning the algebraic K-theory of schemes*, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math., vol. 1046, Springer, Berlin, 1984, pp. 349–356, DOI 10.1007/BFb0072030. MR750689

[40] R. W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 437–552. MR826102 (87k:14016)

[41] Hirosi Toda, *p-primary components of homotopy groups. IV. Compositions and toric constructions*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **32** (1959), 297–332, DOI 10.1215/kjm/1250776579. MR111041

[42] Hirosi Toda, *On spectra realizing exterior parts of the Steenrod algebra*, Topology **10** (1971), 53–65, DOI 10.1016/0040-9383(71)90017-6. MR271933

[43] Vladimir Voevodsky, *On motivic cohomology with $\mathbf{Z}/l$-coefficients*, Ann. of Math. (2) **174** (2011), no. 1, 401–438, DOI 10.4007/annals.2011.174.1.11. MR2811603

[44] Friedhelm Waldhausen, *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy*, Algebraic topology, Aarhus 1982 (Aarhus, 1982), Lecture Notes in Math., vol. 1051, Springer, Berlin, 1984, pp. 173–195, DOI 10.1007/BFb0075567. MR764579

[45] Charles A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory, DOI 10.1090/gsm/145. MR3076731

[46] A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. of Math. (2) **131** (1990), no. 3, 493–540. MR1053488

[47] Raphael Zahler, *The Adams-Novikov spectral sequence for the spheres*, Ann. of Math. (2) **96** (1972), 480–504. MR319197

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824; and Institut fur Mathematik, Freie Universität Berlin, Arnimallee 7, 14195 Berlin, Germany

Current address: Department of Mathematics, Institut Galilée, Université Sorbonne Paris Nord, 99 Av. JB Clément, FR-93430 Villetaneuse, France

Email address: `angelini-knoll@math.univ-paris13.fr`

URL: `https://gangeliniknoll.com`