COMMUTING UNBOUNDED HOMOTOPY LIMITS WITH MORAVA K-THEORY

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ABSTRACT. This paper provides conditions for Morava K-theory to commute with certain homotopy limits. These conditions extend previous work on this question by allowing for homotopy limits of sequences of spectra that are not uniformly bounded below. As an application, we prove the $K(n)$-local triviality (for sufficiently large n) of the algebraic K-theory of algebras over truncated Brown-Peterson spectra, building on work of Bruner and Rognes and extending a classical theorem of Mitchell on $K(n)$-local triviality of the algebraic K-theory spectrum of the integers for large enough n.

CONTENTS

1. INTRODUCTION

Given a generalized homology theory E_* and a sequence

$$\cdots \to X_2 \to X_1 \to X_0$$

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of spectra, one often needs to know, for computations, whether there is an isomorphism

$$\lim_i E_*(X_i) \cong E_*(\text{holim}_i X_i).$$

This cannot be true in full generality. For example, the limit of the sequence

$$\cdots \to S/p^2 \to S/p$$

is the p-complete sphere $\hat{S}_p$ and therefore

$$H_*(\text{holim}_i S/p^i; \mathbb{Q}) \cong H_*(\hat{S}_p; \mathbb{Q}) \cong \mathbb{Q}_p$$

whereas $H_*(S/p^i; \mathbb{Q}) \cong 0$ for $i \geqslant 1$ and therefore $\lim_i H_*(S/p^i; \mathbb{Q}) \cong 0$, and $\lim_i^1 H_*(S/p^i; \mathbb{Q})$ also vanishes, so we do not even have hope of recovering

$$H_*(\text{holim}_i S/p^i; \mathbb{Q})$$

from a "Milnor sequence."

This motivates the question: what conditions on E_* and the sequence

$$\cdots \to X_2 \to X_1 \to X_0$$

allow us to commute the homotopy limit with E_*? There are known results along these lines, most famously a commonly-used result of Adams from [1], but the usual hypothesis is that the spectra X_i are uniformly bounded below and the homology theory E_* is connective. In this paper, we remove each of these assumptions, under some reasonable additional hypotheses. Our particular focus is on the case where E_* is a Morava K-theory $K(n)_*$.[1]

This paper is written with a view towards filtered spectra that arise when studying topological periodic cyclic homology, in particular, the Greenlees filtration (28) on topological periodic cyclic homology

$$TP(R) := THH(R)^{t\mathbb{T}},$$

is not uniformly bounded below. Nevertheless, these filtered spectra often have nice enough homological properties to apply the main result of this paper.

Following the red-shift program of Ausoni and Rognes [6], we are most interested in the chromatic complexity of topological periodic cyclic homology. Therefore, a generalized homology theory of primary interest is Morava K-theory $K(n)_*$. Calculating Morava K-theory of topological periodic cyclic homology using the Greenlees filtration requires that one be able to commute a non-bounded-below generalized homology theory (Morava K-theory) with a non-uniformly-bounded-below homotopy limit, so existing results on generalized homology of limits, like Adams' theorem from [1] reproduced as Theorem 3.2 below, do not suffice.

Classically, the vanishing of the Margolis homology $H(H_*(X; \mathbb{F}_p), Q_n)$ of a bounded-below spectrum X with finite-type mod p homology implies that X has is $K(n)_*$-acyclic. However, even if each spectrum X_i is bounded below, has finite-type homology, and $H(H_*(X; \mathbb{F}_p), Q_n)$ vanishes, it is not always the case that $\text{holim}_i X_i$ has trivial $K(n)$-homology. Our main theorem establishes sufficient conditions for this homotopy limit to indeed have trivial $K(n)$-homology. Given a spectrum X, we adopt the notation that $X^{<N}$ is its Postnikov truncation constructed by attaching

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[1]The easiest case of a Morava K-theory—in particular, the only case which is bounded below—is $K(0)_*$, which coincides with rational homology. So the above example with the rational homology of the p-complete sphere shows that some hypotheses are needed in order to commute Morava K-theory with a sequential limit.

cells to kill all the homotopy groups of X in degrees $\geqslant N$. Our main result may then be summarized as follows.

Theorem 1.1. *(Theorem 3.7) Suppose M is an integer and*

$$\cdots \to Y_2 \to Y_1 \to Y_0$$

is a sequence of bounded below finite-type spectra which are $H\mathbb{F}_p$-nilpotently complete, such that the largest grading degree of a comodule primitive in $H_*(Y_i; \mathbb{F}_p)$ is strictly less than M, the homology groups $H_*(Y_i; \mathbb{F}_p)$ and $H_*(Y_i^{<M}; \mathbb{F}_p)$ are finitely generated for each i, and the limit

$$\lim_{i \to \infty} H(H_*(Y_i; \mathbb{F}_p), Q_n)$$

of the Margolis homologies vanishes. Then the n-th Morava K-theory of $\mathrm{holim}_i Y_i$ is trivial. That is, we have an isomorphism

$$K(n)_*(\mathrm{holim}_i Y_i) \cong 0.$$

This result relies on another result which resolves the question of when Postnikov truncations $X \to X^{<N}$ induce injections in homology.

Theorem 1.2. *(Theorem 2.6) Let M, N be integers with $N \geqslant M$, and let X be a bounded below $H\mathbb{F}_p$-nilpotent spectrum with the grading degrees of the comodule primitives in $H_*(X; \mathbb{F}_p)$ bounded above by M. Then the map*

$$H_*(X; \mathbb{F}_p) \to H_*(X^{<N}; \mathbb{F}_p),$$

induced by the canonical map $X \to X^{<N}$, is injective.

As the main application in the present paper, we prove a higher chromatic height analogue of Mitchell's theorem for algebraic K-theory of truncated Brown-Peterson spectra, building on work of Bruner and Rognes [12]. In particular, Mitchell proves in [28] that

$$K(m)_*(K(\mathbb{Z})) = 0$$

for $m \geqslant 2$, and consquently the same vanishing of Morava K-theory occurs for any $H\mathbb{Z}$-algebra. Let $BP\langle n \rangle$ denote the p-completion of the truncated Brown-Peterson spectrum. As an application of the main result of this paper, we prove the following result, building on [12, Prop. 6.1].

Theorem 1.3. *(Theorem 4.10) There are isomorphisms*

$$K(m)_*(K(BP\langle n \rangle)) \cong 0$$

for $m \geqslant n + 2$, which holds at all primes when $n = 0, 1$ and holds at $p = 2, 3$ when $n = 2$. Consequently, the same vanishing of Morava K-theory holds for any $BP\langle n \rangle$-algebra.[2]

In particular, we note that this recovers the vanishing of $K(m)_*(K(BP\langle 0 \rangle))$ for $m \geqslant 2$ of [28] and the vanishing of $K(m)_*(K(BP\langle 1 \rangle))$ for $m \geqslant 3$ shown by Ausoni-Rognes [5], although our proof uses entirely different methods. After work of Mitchell in 1990 [28] and work of Ausoni-Rognes in 2002 [5], there has been renewed interest and progress on questions of this nature in the last five years. In particular, building on joint work with Naumann and Noel in [25], Mathew (unpublished) has shown that $K(i)_*K(E_n)$ vanishes for $i \geqslant n + 2$, where E_n is

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[2]This result also holds for arbitrary p and n if certain plausible-looking conditions on E_3-ring spectra can be shown to hold. See Remark 4.3 for explanation.

the n-th Lubin-Tate E-theory. Also, Land, Meier, and Tamme [20] recently proved several results of this flavor, for example, they show that if R is a $K(1)$-acyclic E_∞-algebra then $K(n)_*(K(R)) = 0$ for $n \geqslant 2$ (as a consequence of [20, Cor. 2.29]). We regard our Theorem 4.10 as complementary to these recent results of Mathew or of Land-Meier-Tamme: our results are not special cases of theirs, and theirs are not special cases of ours.

Our work in this paper was motivated initially by a conjecture in a first draft of a paper by the first author and Quigley [3]. We restate the version for topological periodic cyclic homology. Here we write $TP(y(n))[k]$ for the Greenlees filtration of topological periodic cyclic homology of $y(n)$ (see (28)).

Conjecture 1.4. There are isomorphisms

$$\lim_{k \to -\infty} K(m)_*(TP(y(n))[k]) \cong K(m)_*(TP(y(n))$$

for $1 \leqslant m \leqslant n$ and all $1 \leqslant n < \infty$.

This conjecture is resolved by the main theorem in the present paper, given above as Theorem 1.1, together with calculations of the first author and Quigley which verify that the hypotheses (on comodule primitives and finite-typeness of homology) of that main theorem are satisfied. The first author and Quigley plan to update [3] with these stronger results.

1.1. **Conventions.** When κ is a cardinal number and A an object in some category, we will write A^κ for the κ-fold categorical product and $A^{\coprod \kappa}$ for the κ-fold categorical coproduct. If our category is additive, then we write $A^{\oplus \kappa}$ for the κ-fold categorical biproduct of A with itself. Let R be a commutative ring. In this paper, the term "finite type" is used in the sense common in algebraic topology: a graded R-module V is *finite type* if, for each integer n, the grading degree n summand V^n is a finitely generated R-module. In particular, if R is a field, then we say V is finite type if, for each integer n, the grading degree n summand V^n is a finite dimensional vector space.

We will write $P(x_1, \ldots, x_n)$ for a polynomial algebra with generators $x_1, \ldots, x_n$ over $\mathbb{F}_p$, $E(x_1, \ldots, x_n)$ for an exterior algebra over $\mathbb{F}_p$ with generators $x_1, \ldots, x_n$ and we write $P_k(x_1, \ldots, x_n)$ for the truncated polynomial algebra with generators $x_1, \ldots, x_n$, where $P_k(x)$ is often also denoted $\mathbb{F}_p[x]/x^k$ in the literature. Given categories $\mathcal{C}$ and $\mathcal{D}$, we will write $\mathcal{D}^\mathcal{C}$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$.

Given a ring R, a non-zero-divisor $r \in R$, and a left R-module M, we say that M is *r-power-torsion*[3] if, for each $m \in M$, there exists some integer n such that $r^n m = 0$. We say that M is *simple r-torsion* if $rm = 0$ for all $m \in M$.

1.2. **Organization.** In Section 2, we give sufficient conditions for the canonical map $X \to X^{<N}$ to induce an injection on homology. This is a key result used to prove the main theorem in Section 3 and we believe it is a useful contribution to the literature in its own right. In Section 3, we prove the main theorem about when we may commute Morava K-theory with a non-uniformly bounded below

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[3]In other words, M is r-power-torsion if the zeroth local cohomology $H^0_{(r)}(M) \cong \mathrm{colim}_n \hom_R(R/(r^n), M)$ vanishes. Some references say that M is "r-torsion" instead of "r-power-torsion," but then again, some references say that M is "r-torsion" when M is what we call simple r-torsion; for the sake of clarity, we prefer to only use the unambiguous terms "r-power-torsion" and "simple r-torsion.".

sequential limits. In Section 4, we give our main application, which is a proof of a higher chromatic height analogue of Mitchell's theorem about vanishing of Morava K-theory of algebraic K-theory of the integers. Finally, we give some supplementary results and background on Margolis homology in Appendix A.

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2. When does killing homotopy induce an injection in homology?

This section proves a general result, Theorem 2.6, which will be used to prove Theorem 3.7. However, it takes some work to prove and is useful in its own right. The question is when, given a spectrum X and an integer N, the map $X \to X^{<N}$ given by attaching cells to kill all the homotopy groups of X in degrees $\geqslant N$ induces an injection in mod p homology groups. Theorem 2.6 gives some practical sufficient conditions for $H_*(X; \mathbb{F}_p) \to H_*(X^{<N}; \mathbb{F}_p)$ to be injective. A simple example where the hypotheses, and therefore the result, hold can be constructed by letting $X = S^0$, $M = N = 1$ so that $\mathbb{F}_p \hookrightarrow (A//E(0))_*$ is an inclusion. A simple example where the hypotheses are *not* satisfied is the case where $X = S^0$ and $M = N = 0$, where of course $X^{<N}$ is the 0 spectrum and the result cannot hold.

The notation A_* in the statement of Lemma 2.1 refers to the dual mod p Steenrod algebra, as usual, but the lemma and its proof both work for any graded Hopf algebra over a field.

Lemma 2.1. *Let p be a prime number, let I be a set, and for each $i \in I$, let M_i be a graded A_*-comodule. Then, for each integer t and nonnegative integer s, the natural map of $\mathbb{F}_p$-vector spaces*

$$\coprod_{i \in I} \mathrm{Cotor}_{A_*}^{s,t}(\mathbb{F}_p, M_i) \to \mathrm{Cotor}_{A_*}^{s,t}\left(\mathbb{F}_p, \coprod_{i \in I} M_i\right)$$

is an isomorphism.

Proof. This result follows by observing that each step in the construction of the cobar complex of A_*, with coefficients in an A_*-comodule M, commutes with co-products in the variable M. □

Definition 2.2. *Let p be a prime number and let M be an integer. We will say X satisfies condition $\mathrm{H}(M)$ if it is bounded below and $H\mathbb{F}_p$-nilpotently complete[4] and the A_*-comodule primitives of $H_*(X; \mathbb{F}_p)$ are trivial in grading degrees $\geqslant M$.*

We suppress the prime p from the notation $\mathrm{H}(M)$ because it will always be clear from the context.

Remark 2.3. Given a bounded below $H\mathbb{F}_p$-nilpotent complete spectrum X, we note that the 0-line of the E_2-page of the Adams spectral sequence

$$\mathrm{Ext}^{*,*}_{\mathrm{Comod}(A_*)}(\mathbb{F}_p, H_*(X)) \to \pi_* X$$

is isomorphic to

$$\mathrm{Hom}_{\mathrm{Comod}(A_*)}(\mathbb{F}_p, H_* X),$$

which is isomorphic to the sub-A_*-comodule of A_*-comodule primitives in $H_* X$. So, if X is bounded below and $H\mathbb{F}_p$-nilpotently complete, then condition $\mathrm{H}(M)$ is equivalent to the following statements being true for all integers $j \geqslant M$:

(1) the Hurewicz map

$$\pi_j(X) \to H_j(X; \mathbb{F}_p)$$

is zero, and

(2) for all $r \geqslant 2$, the $H\mathbb{F}_p$-Adams spectral sequence for X does not have any nonzero d_r-differentials supported in bidegree $(0, j)$ (i.e., $s = 0$ and $t = j$).

It is the above two conditions that actually play a role in the proofs in this section, but the form of condition $H(M)$ given in Definition 2.2 is a relatively familiar and easily-checked condition which is equivalent (for bounded below $H\mathbb{F}_p$-nilpotently complete spectra) to the two properties given above.

In Theorem 6.6 of [10], Bousfield proved that, if E is a connective ring spectrum with $\pi_0(E) \cong \mathbb{F}_p$ and X is a connective spectrum, then the E-nilpotent completion $\hat{X}_E$ of X is weakly equivalent to the Bousfield localization $L_E X$. As a special case, we have:

Lemma 2.4. *Let X be a connective spectrum. Then the $H\mathbb{F}_p$-nilpotent completion map $X \to \hat{X}_{H\mathbb{F}_p}$ is an $H\mathbb{F}_p$-local equivalence. That is, the induced map of spectra $X \wedge H\mathbb{F}_p \to \hat{X}_{H\mathbb{F}_p} \wedge H\mathbb{F}_p$ is a weak equivalence.*

Lemma 2.5. *Let p be a prime number and let I be a set. Let X satisfy condition $H(M)$ for some integer M. For each $i \in I$, let $f_i \colon S^n \to X$ be a map of spectra which induces the zero map in mod p homology, and let $f \colon (\coprod_{i \in I} S^n)\hat{}_{H\mathbb{F}_p} \to \hat{X}_{H\mathbb{F}_p} \xrightarrow{\cong} X$ denote the map given by $H\mathbb{F}_p$-nilpotent completion and the universal property of the coproduct applied to the set of maps $\{f_i : i \in I\}$. Suppose that the map*

$$(1) \qquad\qquad \pi_n\left(\coprod_{i \in I} S^n\right) \otimes_{\mathbb{Z}} \mathbb{F}_p \to \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{F}_p$$

is injective. Finally, suppose that $n < M$. Then the following statements are true:

(1) *the map $H_*(X; \mathbb{F}_p) \to H_*(\mathrm{cof}\, f; \mathbb{F}_p)$ is injective, and*

(2) *$\mathrm{cof}\, f$ satisfies condition $H(M)$.*

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[4]Readers who are not used to thinking about nilpotent completion may be relieved to know that a bounded-below spectrum is $H\mathbb{F}_p$-nilpotently complete if and only if its homotopy groups are Ext-p-complete, by [10, Prop. 2.5]. In particular, if all the homotopy groups of a bounded-below spectrum are p-adically complete, then that spectrum is also $H\mathbb{F}_p$-nilpotently complete.

Proof. The $H\mathbb{F}_p$-nilpotent completion map $\coprod_{I_k} S^n \to \left(\coprod_{I_k} S^n\right)^{\widehat{}}_{H\mathbb{F}_p}$ is an $H\mathbb{F}_p$-local equivalence, by Lemma 2.4, so we have a natural isomorphism

$$\coprod_{I_k} H_*(S^n; \mathbb{F}_p) \xrightarrow{\cong} H_*\left(\left(\coprod_{I_k} S^n\right)^{\widehat{}}_{H\mathbb{F}_p}; \mathbb{F}_p\right).$$

Since each f_i induces the zero map in mod p homology, so does the map $\coprod_{i \in I_k} f_i$, and consequently so does the composite

$$(2) \qquad \coprod_{I_k} S^n \xrightarrow{\coprod_{i \in I_k} f_i} \coprod_{I_k} X \xrightarrow{\nabla} X,$$

i.e., the map $\coprod_{I_k} S \to X$ given by the universal property of the coproduct. (The symbol ∇ in (2) stands for the fold map, given again by the universal property of the coproduct.) We have the commutative square

$$\begin{array}{ccc}
\left(\coprod_{I_k} S^n\right) \wedge H\mathbb{F}_p & \longrightarrow & X \wedge H\mathbb{F}_p \\
\downarrow{\cong} & & \downarrow{\cong} \\
\left(\coprod_{I_k} S^n\right)^{\widehat{}}_{H\mathbb{F}_p} \wedge H\mathbb{F}_p & \xrightarrow{f \wedge H\mathbb{F}_p} & \hat{X}_{H\mathbb{F}_p} \wedge H\mathbb{F}_p
\end{array}$$

in which the vertical maps are isomorphisms in SHC by Lemma 2.4. The top horizontal map was already shown to be zero, so the bottom horizontal map is as well. So f induces the zero map in mod p homology. So we have the short exact sequence of A_*-comodules

$$(3) \qquad 0 \to H_*(X; \mathbb{F}_p) \to H_*(\text{cof } f; \mathbb{F}_p) \to \coprod_{i \in I} H_*(\Sigma S^n; \mathbb{F}_p) \to 0$$

and hence the claimed injectivity of $H_*(X; \mathbb{F}_p) \to H_*(\text{cof } f; \mathbb{F}_p)$, as well as an induced long exact sequence

(4)

$$
\begin{array}{ccc}
0 \longrightarrow \text{Cotor}_{A_*}^{0,j}\left(\mathbb{F}_p, H_*(X; \mathbb{F}_p)\right) \xrightarrow{a} \text{Cotor}_{A_*}^{0,j}\left(\mathbb{F}_p, H_*(\text{cof } f; \mathbb{F}_p)\right) \\
\xrightarrow{b} \\
\text{Cotor}_{A_*}^{0,j}\left(\mathbb{F}_p, \coprod_{i \in I} H_*(\Sigma S^n; \mathbb{F}_p)\right) \longrightarrow \cdots.
\end{array}
$$

Condition H(M) gives us that the left-hand term in (4) vanishes for $j \geqslant M$, while the right-hand term in (4) vanishes for $j \geqslant n + 1$, so cof f satisfies condition H($\max\{M, n+1\}$). When $n < M$, then clearly $\max\{M, n+1\} = M$. $\qquad \square$

Theorem 2.6. *Let p be a prime number, let M, N be integers with $N \geqslant M$, and let X satisfy condition H(M). Let $X^{<N}$ be X with cells attached to kill all the homotopy groups of X in degrees $\geqslant N$. Then the map $H_*(X; \mathbb{F}_p) \to H_*(X^{<N}; \mathbb{F}_p)$, induced by the canonical map $X \to X^{<N}$, is injective.*

Proof. This is a proof by induction. Suppose k is an integer, $k \geqslant N$, and that we have already constructed a sequence of maps

$$(5) \qquad X = X(N) \to X(N+1) \to \cdots \to X(k)$$

satisfying the properties:

(1) for each $j > N$, the map $X(j-1) \to X(j)$ induces an isomorphism in π_i for all $i < j - 1$, and
(2) for each $j > N$, $\pi_{j-1}(X(j))/p \cong 0$, and
(3) for each $j > N$, the induced map $H_*(X(j-1); \mathbb{F}_p) \to H_*(X(j); \mathbb{F}_p)$ is injective, and
(4) for each $j \geqslant N$, $X(j)$ satisfies condition H(j).

Note that if $X(j)$ satisfies condition H(j), then it is $H\mathbb{F}_p$-nilpotently complete. Consequently, the group $\pi_j(X(j))$ is Ext-p-complete by [10, Thm. 2.5,6.6]. The vanishing mod p of an Ext-p-complete abelian group implies the vanishing of that abelian group, as a special case of [19, Thm. A.6(d)]; so $\pi_{j-1}(X(j)) \cong 0$ for all $j > N$. We want to construct a map $X(k) \to X(k+1)$ which extends the sequence (5) one step to the right and satisfies the same four properties listed above. Clearly, if this can be done, then the third property implies that the map

$$ H_*(X; \mathbb{F}_p) \to H_*(\operatorname{hocolim}_k X(k); \mathbb{F}_p) $$

is injective, as desired. The first two of the four properties listed above also imply that the map $\pi_j(X) \to \pi_j(\operatorname{hocolim}_k X(k))$ is an isomorphism for $j < N$. Properties two and four then imply that $\pi_j(\operatorname{hocolim}_k X(k))$ vanishes for all $j \geqslant N$.

Those observations are enough to show that $\operatorname{hocolim}_k X(k)$ agrees with any of the usual "attach cells to kill homotopy in degrees $> N$" constructions applied to X, for the following reason: if $A_N \colon \mathrm{SHC} \to \mathrm{SHC}$ is any functor and $\eta_N \colon \mathrm{id}_{\mathrm{SHC}} \to A_j$ any natural transformation such that, for all spectra X, $A_N(X)$ has homotopy concentrated in degrees $< N$ and $\eta_N(X) \colon X \to A_j(X)$ induces an isomorphism in homotopy in degrees $< j$, then applying A_N to the maps $\eta_N(X)$ and $X \to \operatorname{hocolim}_k X(k)$ yields a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \operatorname{hocolim}_k X(k) \\
\downarrow{\scriptstyle \eta_N(X)} & \cong \downarrow{\scriptstyle \eta_N(\operatorname{hocolim}_k X(k))} \\
A_N(X) & \longrightarrow & A_N(\operatorname{hocolim}_k X(k))
\end{array}
$$

by naturality of η_N. The bottom horizontal map induces an isomorphism

$$ \pi_j(A_N(X)) \to \pi_j(A_N(\operatorname{hocolim}_k X(k))) $$

for $j < N$, and both sides vanish if $j \geqslant N$, so the bottom horizontal map is an isomorphism in SHC. So $A_n(X) \cong \operatorname{hocolim}_k X(k)$. So although we will not construct a spectrum $X^{<N}$ by simply attaching cells as one does classically, our spectrum $\operatorname{hocolim}_k X(k)$ will indeed be isomorphic to the result of that classical construction.

We construct $X(k+1)$, and the map $X(k) \to X(k+1)$, as follows: choose a set I_k of elements of $\pi_k(X(k))$ with the property that the image of I_k in $\pi_k(X(k))/p$ is a *minimal* set of generators for the group $\pi_k(X(k))/p$. Then we apply Lemma 2.5, letting the I, X, n, and M in the statement of Lemma 2.5 be $I_k, X(k), k$, and $k+1$, respectively. We check that the hypotheses of Lemma 2.5 are satisfied:

- for each $i \in I_k$, the map $f_i \colon S^k \to X(k)$ is zero in mod p homology by the $j = k$ case of the fourth inductive hypothesis (see Remark 2.3 Item 1), above,

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[4]This is because each $X(i)$ is $H\mathbb{F}_p$-nilpotently complete and has the property that $\pi_{i-1}X(i)/p$ is trivial, and so $\pi_{i-1}X(i)$ is trivial; this is a typical argument about Ext-p-completeness.

- the map (1) is an isomorphism, by the minimality hypothesis on I_k as a set of generators for $\pi_k(X(k)) \otimes_{\mathbb{Z}} \mathbb{F}_p$,
- and $X(k)$ satisfies condition $H(k)$ by the fourth inductive hypothesis.

Now we define $X(k+1)$ to be the cofiber of the map

$$f \colon \left(\coprod_{i \in I_k} S^k \right)^{\hat{}}_{H\mathbb{F}_p} \longrightarrow (X(k))^{\hat{}}_{H\mathbb{F}_p} \xrightarrow{\cong} X(k)$$

given by $H\mathbb{F}_p$-nilpotent completion and the universal property of the coproduct applied to the maps $\{f_i : i \in I_k\}$, just as in the statement of Lemma 2.5. By construction, the first and second inductive hypotheses then hold for $j = k + 1$. Lemma 2.5 then gives us that the third inductive hypothesis also holds in the case $j = k + 1$, and that $X(k+1)$ satisfies condition $\mathrm{H}(k+1)$ so the fourth hypothesis is also true when k is replaced by $k + 1$. This completes the induction.

The resulting map $X \to \operatorname{hocolim}_k X(k)$ is an isomorphism in homotopy in degrees $< N$, and injective in mod p homology, as desired. We still need to show that $\pi_j(\operatorname{hocolim}_k X(k))$ vanishes for all $j \geqslant N$. By construction, $\pi_j(\operatorname{hocolim}_k X(k))/p \cong \pi_j(X(j+1))/p$ is trivial for all $j \geqslant N$. Since $X(j+1)$ is also $H\mathbb{F}_p$-nilpotently complete, the group $\pi_j(X(j+1))$ is Ext-p-complete by Theorems 2.5 and 6.6 of [10]. The vanishing mod p of an Ext-p-complete abelian group implies the vanishing of that abelian group, as a special case of Theorem A.6(d) of [19]; so $\pi_j(X(j+1)) \cong \pi_j(\operatorname{hocolim}_k X(k)) \cong 0$ for all $j \geqslant N$, as desired. $\qquad\square$

3. Morava K-theory of homotopy limits

Recall that, for each prime number p and positive integer n, we have the homotopy fiber sequence

$$\Sigma^{2(p^n-1)} k(n) \to k(n) \to H\mathbb{F}_p,$$

and the composite map

(6) $$H\mathbb{F}_p \to \Sigma^{2p^n-1} k(n) \to \Sigma^{2p^n-1} H\mathbb{F}_p$$

is the cohomology operation Q_n, which satisfies $Q_n^2 = 0$. This implies some useful relationships between Morava K-theories and Margolis homology of $E(Q_n)$-modules, which we summarize in an appendix to this paper, appendix A, containing various results which are basically well-known but which are not well-documented in the literature. That appendix does not logically depend on anything earlier in the paper. The reader who is not already familiar with Margolis homology and its topological applications can consult appendix A for a crash course.

Definition 3.1. *We say that an $E(Q_n)$-module is Q_n-acyclic if the Margolis Q_n-homology $H(M; Q_n)$ vanishes. We say that a morphism of $E(Q_n)$-modules is a Q_n-equivalence if it induces an isomorphism in Q_n-Margolis homology.*

Recall the following useful result of Adams, Theorem III.15.2 of [1]:

Theorem 3.2. *Suppose that R is a subring of $\mathbb{Q}$, E is a bounded-below spectrum such that $H_r(E)$ is a finitely generated R-module for all r, and $\{Y_i\}_{i \in I}$ is a set of spectra such that $\pi_r(Y_i)$ is an R-module for all r. Suppose that there exists a*

uniform lower bound for $\pi_*(Y_i)$, *i.e., there exists an integer N such that $\pi_n(Y_i) \cong 0$ for all $n < N$. Then the canonical map of spectra*

$$E \wedge \prod_{i \in I} X_i \to \prod_{i \in I} (E \wedge X_i)$$

is a weak equivalence.

As an easy corollary of Theorem 3.2:

Corollary 3.3. *Let*

$$\cdots \to Y_2 \to Y_1 \to Y_0$$

be a sequence of morphisms of spectra. Suppose that there exists a uniform lower bound on $\pi_*(Y_i)$. Then the canonical map of spectra

$$H\mathbb{F}_p \wedge \mathrm{holim}_i X_i \to \mathrm{holim}_i (H\mathbb{F}_p \wedge X_i)$$

is a weak equivalence.

Consequently, for each nonnegative integer n, we have a short exact sequence

$$0 \to R^1 \lim_i H_{n+1}(X_i; \mathbb{F}_p) \to H_n(\mathrm{holim}_i X_i; \mathbb{F}_p) \to \lim_i H_n(X_i; \mathbb{F}_p) \to 0.$$

The dual action of Q_n on mod p homology is compatible with the comparison map (3.2), so that the direct sum of the sequences (3.3),

$$0 \to \Sigma^{-1} R^1 \lim_i H_*(X_i; \mathbb{F}_p) \to H_*(\mathrm{holim}_i X_i; \mathbb{F}_p) \to \lim_i H_*(X_i; \mathbb{F}_p) \to 0,$$

is a short exact sequence of graded $E(Q_n)$-modules.

Proof. We have the commutative diagram in the stable homotopy category

$$
\begin{array}{ccccc}
E \wedge \mathrm{holim}_i X_i & \longrightarrow & E \wedge \prod_i X_i & \xrightarrow{\mathrm{id}\,-T} & E \wedge \prod_i X_i \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{holim}_i(E \wedge X_i) & \longrightarrow & \prod_i(E \wedge X_i) & \xrightarrow{\mathrm{id}\,-T} & \prod_i(E \wedge X_i)
\end{array}
$$

whose rows are homotopy fiber sequences. The middle and right-hand vertical maps in (3) are weak equivalence, by the special case $R = \mathbb{Z}$, $E = H\mathbb{F}_p$ of Theorem 3.2. The map of long exact sequences induced in π_* by (3), together with the Five Lemma, now imply that the left-hand vertical map in (3) is also a weak equivalence.

The sequence (3.3) is simply the Milnor exact sequence for $\pi_* \mathrm{holim}_i (H\mathbb{F}_p \wedge X_i)$. $\square$

Lemma 3.4. *Let $n > 0$, let p be a prime, and let X be a bounded spectrum (i.e., X is bounded above and bounded below) such that $H_*(X; \mathbb{F}_p)$ is finite type. Then the E_∞-page of the Adams spectral sequence*

$$(7) \qquad E_2^{s,t} \cong \mathrm{Ext}_{\mathrm{gr\,Comod}(A_*)}^{s,t} (\mathbb{F}_p, H_*(k(n) \wedge X; \mathbb{F}_p)) \Rightarrow \pi_{t-s} \left((k(n) \wedge X)_{H\mathbb{F}_p}^{\hat{}} \right)$$

$$d_r \colon E_r^{s,t} \to E_r^{s+r,t+r-1}$$

is v_n-power-torsion. That is, every bihomogeneous element x in that E_∞-page satisfies $v_m x = 0$ for some m.

Proof. Since X is bounded, it is in particular bounded below, so $k(n) \wedge X$ is also bounded below. Since $k(n)$ is a ring spectrum with $\pi_0(k(n)) \cong \mathbb{F}_p$, $\pi_*(k(n) \wedge X)$ is a $\mathbb{F}_p$-vector space, hence is automatically p-adically complete. So $k(n) \wedge X$ is $H\mathbb{F}_p$-nilpotently complete, so the E_∞-page of spectral sequence (7) is the associated graded of a filtration on $k(n)_*(X)$. Since X is bounded, it is $K(n)$-acyclic, so $v_n^{-1}(k(n)_*(X))$ must vanish. The associated graded of a filtered $\mathbb{F}_p[v_n]$-module M can have fewer v_n-torsion-free elements than M, but it cannot have more, so inverting v_n on the E_∞-page of spectral sequence (7) must yield zero. $\square$

In fact, it will be useful to know more information about the bound on the Adams filtration of the v_n-torsion in the E_∞-page of the Adams spectral sequence for $k(n)$ homology of a bounded spectrum. First we prove a lemma that we expect to be well known, but we could not find in the literature.

Lemma 3.5. *Let M be a finite type $\hat{\mathbb{Z}}_p$-module. Then the graded $k(n)_*$-module $k(n)_*(HM)$ is simple v_n-torsion.*

Proof. First, note that a finite type $\hat{\mathbb{Z}}_p$-module M is of the form

$$M = \hat{\mathbb{Z}}_p^{n_0} \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus n_1} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{\oplus n_2} \oplus \cdots \oplus (\mathbb{Z}/p^k\mathbb{Z})^{\oplus n_k}$$

so it suffices to show that $k(n)_*(H\mathbb{Z}_p)$ is simple v_n torsion and that $k(n)_**(H\mathbb{Z}/p^j\mathbb{Z})$ is simple v_n-torsion for all positive integers j and n and primes p.

First, note that there is an isomorphism of modules over the Steenrod algebra $H_*(H\mathbb{Z}_p) \cong (A//E(Q_0))_*$, which is free over $E(Q_n)$ for any positive integer n. Consequently, the Margolis homology $H(H_*(H\mathbb{Z}_p), Q_n)$ vanishes. The E_2-page of the Adams spectral sequence converging to $k(n)_*(H\hat{\mathbb{Z}}_p)$ is isomorphic to

$$\text{Ext}_{E(Q_n)}^{*,*}(\mathbb{F}_p, H_*(H\hat{\mathbb{Z}}_p))$$

by Proposition A.7, from the appendix on Margolis homology, so above the zero line these groups can be identified with Margolis homology, which vanishes as we just concluded. Therefore, the Adams spectral sequence collapses to the zero line, which is simple v_n-torsion. [5] Therefore $k(n)_*(\hat{\mathbb{Z}}_p)$ is simple v_n-torsion.

Now we know that the multiplication-by-p^j map $S^0 \to S^0$ on the sphere spectrum is nulhomotopic after applying the function spectrum functor $F(-, H\mathbb{F}_p)$. So, if we smash $p^j \colon S^0 \longrightarrow S^0$ with $H\mathbb{Z}$, the resulting map $p^j \colon H\mathbb{Z} \longrightarrow H\mathbb{Z}$ is nulhomotopic after applying $F(-, H\mathbb{F}_p)$, by closedness of SHC as a monoidal category. Consequently the induced map $H^*(H\mathbb{Z}; \mathbb{F}_p) \to H^*(H\mathbb{Z}; \mathbb{F}_p)$ is zero, and the long exact sequence induced in mod p cohomology by the fiber sequence

$$H\mathbb{Z} \xrightarrow{p^j} H\mathbb{Z} \to H\mathbb{Z}/p^j\mathbb{Z}$$

splits into short exact sequences:

(8)
$$
\begin{array}{ccccccccc}
0 & \longleftarrow & H^*(H\mathbb{Z}; \mathbb{F}_p) & \longleftarrow & H^*(H\mathbb{Z}/p^j\mathbb{Z}; \mathbb{F}_p) & \longleftarrow & H^*(\Sigma H\mathbb{Z}; \mathbb{F}_p) & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle\cong} & & \downarrow \\
0 & \longleftarrow & A//E(0) & \longleftarrow & H^*(H\mathbb{Z}/p^j\mathbb{Z}; \mathbb{F}_p) & \longleftarrow & \Sigma A//E(0) & \longleftarrow & 0
\end{array}
$$

---

[5]Note that $k(n) \wedge H\hat{\mathbb{Z}}_p$ is bounded below and has Ext-p-complete homotopy groups so it is $H\mathbb{F}_p$-nilpotently complete and the Adams spectral sequence indeed converges to $k(n)_* H\hat{\mathbb{Z}}_p$.

Since $A//E(0)$ is Q_n-acyclic for positive n the long exact sequence induced in Q_n-Margolis homology by the bottom row of (8) gives us that $H^*(H\mathbb{Z}/p^j\mathbb{Z}; \mathbb{F}_p)$ is also Q_n-acyclic. Proposition A.7, from the appendix on Margolis homology, then gives us $\mathrm{Ext}^s_{E(Q_n)}\left(H^*(H\mathbb{Z}/p^j\mathbb{Z}; \mathbb{F}_p), \mathbb{F}_p\right) \cong H\left(H^*(H\mathbb{Z}/p^j\mathbb{Z}; \mathbb{F}_p); Q_n\right)^* \cong 0$. So the E_2-page of the Adams spectral sequence

$$
\begin{aligned}
E_2^{s,t} &\cong \mathrm{Ext}_A^{s,t}\left(H^*(k(n) \wedge H\mathbb{Z}/p^j\mathbb{Z}; \mathbb{F}_p), \mathbb{F}_p\right) \\
&\cong \mathrm{Ext}_{E(Q_n)}^{s,t}\left(H^*(H\mathbb{Z}/p^j\mathbb{Z}; \mathbb{F}_p), \mathbb{F}_p\right) \\
&\Rightarrow \pi_{t-s}\left((k(n) \wedge H\mathbb{Z}/p^j\mathbb{Z})\right)
\end{aligned}
$$

is concentrated on the $s = 0$-line. Since v_n-multiplication increases Adams degree in this spectral sequence, $\pi_*\left(k(n) \wedge H\mathbb{Z}/p^j\mathbb{Z}\right)$ must be simple v_n-torsion. $\square$

Lemma 3.6. *Let n be a positive integer, let p be a prime, and let $\{X_i\}$ be a sequence of bounded spectra, each with maximal nontrivial homotopy group in degree $M - 1$, such that c is a finite-type graded $\hat{\mathbb{Z}}_p$-module for each i. Then*

$$
\lim_i k(n)_*(X_i)
$$

is v_n-power-torsion, and consequently the inverse limit Adams spectral sequence

$$
\lim_i \mathrm{Ext}_{E(Q_n)_*}^{*,*}(\mathbb{F}_p, H_*(X_i)) \Rightarrow \lim_i k(n)_* X_i
$$

has v_n-power-torsion E_∞-page.

Proof. We will compare two different exact couples in order to produce a spectral sequence

(1) whose E_2-page is simple v_n-torsion,
(2) that strongly converges to

$$
\lim_i k(n)_*(X_i),
$$

(3) and that has a vanishing line that implies that there cannot be any infinite v_n-towers produced by passing from the E_∞-page to the abutment $\lim_i k(n)_*(X_i)$.

Applying $k(n)_*$ to the Postnikov tower of X_i yields the spectral sequence

$$
\begin{aligned}
(9) \qquad\qquad E_{s,t}^1 &\cong k(n)_s(\Sigma^t H\pi_t X_i) \Rightarrow k(n)_s X_i \\
d^r &: E_{s,t}^r \to E_{s-1,t+r}^r,
\end{aligned}
$$

with E^2-page isomorphic to $H_*(k(n); \pi_*(X_i))$. The spectral sequence is functorial, and its E^∞-page is the associated graded of the filtration on $k(n)_*(X_i)$ in which an element $x \in k(n)_*(X_i)$ has filtration $\geqslant j$ if and only if the projection map from X_i to its jth Postnikov truncation $X_i^{\leqslant j}$ sends x to zero in $k(n)$-homology. In particular, this is a decreasing filtration, so the map $k(n)_*(X_{i+1}) \to k(n)_*(X_i)$ may raise filtration, but cannot decrease filtration.

Beginning with the E^2-page, [32, Thm. 4.1] establishes[6] that this spectral sequence is isomorphic to the usual Atiyah-Hirzebruch spectral sequence

$$H_*(k(n); \pi_*(X_i)) \Rightarrow k(n)_* X_i$$

constructed by applying X_i-homology to a CW-decomposition of $k(n)$. But spectral sequence (9) has an important property which the usual Atiyah-Hirzebruch spectral sequence (AHSS), arising from a CW-decomposition of $k(n)$, doesn't obviously have: the E^1-page of (9) consists of simple v_n-torsion, by Lemma 3.5. Consequently all later pages in (9) are, as $k(n)_*$-modules, simple v_n-torsion. Consequently, for each $r \geqslant 2$, the E^r-page in the homological Atiyah-Hirzebruch spectral sequence $H_*(k(n); \pi_*(X_i)) \Rightarrow k(n)_* X_i$ consists of simple v_n-torsion, by the following argument: for each pair i, t, the graded $k(n)_*$-module $k(n)_*(\Sigma^t H \pi_t X_i)$ consists of *simple v_n-torsion in grading degrees* $\geqslant t$, by Lemma 3.5.

That observation is our first reason to consider spectral sequence (9) built using the Postnikov system of X_i. For both spectral sequences, we will show that the hypothesis that $\pi_* X_i$ is a finite type graded $\hat{\mathbb{Z}}_p$-module is sufficient to produce a limit of exact couples, however it is more clear from the AHSS that this spectral sequence strongly converges to the desired abutment.

We now give an explicit construction of the AHSS by choosing a finite-type CW-decomposition[7] of $k(n)$, and letting

(10) $$k(n)^{(0)} \to k(n)^{(1)} \to k(n)^{(2)} \to \dots$$

be the associated skeletal filtration. Applying X_i-homology to (10) yields the unrolled (in the sense of [9]) exact couple

(11)
$$\dots \longrightarrow (X_i)_*(k(n)^{(1)}) \longrightarrow (X_i)_*(k(n)^{(2)}) \longrightarrow \dots$$
$$(X_i)_* \left(k(n)^{(1)}/k(n)^{(0)}\right) \qquad (X_i)_* \left(k(n)^{(2)}/k(n)^{(1)}\right)$$

which is natural in the variable i. Here we are writing $k(n)^{(j)}/k(n)^{(j-1)}$ for the homotopy cofiber of the inclusion map $k(n)^{(j-1)} \to k(n)^{(j)}$.

Since $\pi_*(X_i)$ is a finite-type graded $\hat{\mathbb{Z}}_p$-module, so is each of the graded abelian groups in (11). So $R^1 \lim_i$ of each of those groups vanishes, so we get an unrolled exact couple

(12)
$$\dots \longrightarrow \lim_i (X_i)_*(k(n)^{(1)}) \longrightarrow \lim_i (X_i)_*(k(n)^{(2)}) \longrightarrow \dots$$
$$\lim_i (X_i)_* \left(k(n)^{(1)}/k(n)^{(0)}\right) \qquad \lim_i (X_i)_* \left(k(n)^{(2)}/k(n)^{(1)}\right)$$

---

[6]It is a classical theorem of Maunder [26] that the *cohomological* Atiyah-Hirzebruch spectral sequence $H^*(X; \pi_*(E)) \Rightarrow [\Sigma^* X, E]$, arising from a CW-decomposition of X, is isomorphic (beginning with the E_2-pages) to the spectral sequence $H^*(X; \pi_*(E)) \Rightarrow [\Sigma^* X, E]$ arising from the Postnikov system of E. As far as we know, the analogous results for the *homological* Atiyah-Hirzebruch spectral sequence did not appear in the literature until Tene's paper.

[7]I.e., a CW-decomposition with finitely many cells in each dimension. This is possible because the homology of $k(n)$ is finite-type.

The spectral sequence of (11) is the classical homological Atiyah-Hirzebruch spectral sequence of the bounded spectrum X_i, so it is strongly convergent to the colimit, by section 12 of [9]. In particular, the limit $\lim_j (X_i)_* \left(k(n)^{(j)} \right)$ vanishes. So we have

$$\lim_j \lim_i (X_i)_* \left(k(n)^{(j)} \right) \cong \lim_i \lim_j (X_i)_* \left(k(n)^{(j)} \right) \cong 0,$$

i.e., the spectral sequence of (12) is also strongly convergent to the colimit, and that colimit is

$$\mathrm{colim}_j \lim_i (X_i)_* \left(k(n)^{(j)} \right).$$

Finally, since (10) is a *finite-type* CW-decomposition, each skeleton $k(n)^{(j)}$ is a finite spectrum, so we have a Milnor sequence

$$0 \to R^1 \lim_i \Sigma k(n)_*^{(j)}(X_i) \to k(n)_*^{(j)} \left(\mathrm{holim}_i X_i \right) \to \lim_i k(n)_*^j(X_i) \to 0$$

whose $R^1 \lim_i$ term vanishes due to our finiteness hypotheses. Consequently we have isomorphisms

$$\begin{aligned}
\mathrm{colim}_j \lim_i (X_i)_* \left(k(n)^{(j)} \right) &\cong \mathrm{colim}_j \pi_* \left(k(n)^j \wedge \mathrm{holim}_i X_i \right) \\
&\cong \pi_* \left(\mathrm{hocolim}_j \left(k(n)^{(j)} \wedge \mathrm{holim}_i X_i \right) \right) \\
&\cong \pi_* \left(k(n) \wedge (\mathrm{holim}_i X_i) \right) \\
&\cong k(n)_* \left(\mathrm{holim}_i X_i \right),
\end{aligned}$$

i.e., the spectral sequence of the exact couple (12) converges strongly to

$$k(n)_* \left(\mathrm{holim}_i X_i \right).$$

We want to understand the action of v_n-multiplication on $k(n)_* \left(\mathrm{holim}_i X_i \right)$, but the exact couples (11) and (12) aren't exact couples of $k(n)_*$-modules, and in particular, v_n-multiplication isn't defined on these exact couples! However, by Tene's homological version of Maunder's theorem, the spectral sequence of (11) is isomorphic (starting with the E^2-page) to the spectral sequence (9), whose exact couple *is* an exact couple of $k(n)_*$-modules, and whose functoriality gives us v_n-multiplication as an endomorphism of the whole exact couple and consequently of its spectral sequence. Let $\mathcal{E}_i$ denote the unrolled exact couple

(13)
$$\ldots \longrightarrow k(n)_*(X_i^{\leqslant 2}) \longrightarrow k(n)_*(X_i^{\leqslant 1}) \longrightarrow \ldots$$
$$k(n)_* \left(\Sigma^2 H\pi_2 X_i \right) \qquad k(n)_* \left(\Sigma^1 H\pi_1 X_i \right)$$

Our finite-type assumptions on $\pi_*(X_i)$ give us that each of the groups in $\mathcal{E}_i$ is finite, and hence the relevant $R^1 \lim_i$ groups all vanish, so that $\lim_i \mathcal{E}_i$ is an exact couple. The spectral sequence of the exact couple $\lim_i \mathcal{E}_i$ is isomorphic to that of (12), hence converges to $k(n)_*(\mathrm{holim}_i X_i)$. We showed already that the E^1-page of the spectral sequence (9) associated to $\mathcal{E}_i$ is simple v_n-torsion, and since a limit of *simple* v_n-torsion $k(n)_*$-modules remains simple v_n-torsion, we have that the E^1-page of the spectral sequence of $\lim_i \mathcal{E}_i$ is simple v_n-torsion. So its later pages, and its E^∞-page (whose associated graded is $k(n)_*(\mathrm{holim}_i X_i)$) must also be simple v_n-torsion.

We have therefore produced a spectral sequence whose E_∞-page is simple v_n-torsion and which strongly converges to $\lim_i k(n)_* X_i$ so it suffices to produce a vanishing line. This vanishing line is clear from the description of the E_2-page as

$$\lim_i H_*(k(n), \pi_*(X_i))$$

and the assumption that $\pi_j X_i \cong 0$ for $j \geqslant M$ for each i. $\square$

Finally, we come to the main technical tool obtained in this paper.

Theorem 3.7. *Let n, N, M be integers such that $n > 0$ and $N \geqslant M$, and let*

$$\cdots \to Y_2 \to Y_1 \to Y_0$$

be a sequence of morphisms of bounded-below $H\mathbb{F}_p$-nilpotently complete spectra. Make the following assumptions:

- *the graded $\mathbb{Z}_p$-modules $\pi_* Y_i$ are finite type*
- *there is an isomorphism $\lim_i H(H_*(Y_i; \mathbb{F}_p); Q_n) \cong 0$, and*
- *each spectrum Y_i satisfies condition $H(M)$.*

Then $\operatorname{holim}_i Y_i$ is $K(n)$-acyclic.

Proof. We prove this in several steps:

Step 1: We consider the homotopy fiber sequence

$$K(n) \wedge \operatorname{holim}_i Y_i^{\geqslant N} \to K(n) \wedge \operatorname{holim}_i Y_i \to K(n) \wedge \operatorname{holim}_i Y_i^{<N}$$

in which the right-hand term is contractible since bounded-above spectra are $K(n)$-acyclic for $n > 0$. So $\operatorname{holim}_i Y_i$ is $K(n)$-acyclic if and only if $\operatorname{holim}_i Y_i^{\geqslant N}$ is $K(n)$-acyclic. So we only need to show that $\operatorname{holim}_i Y_i^{\geqslant N}$ is $K(n)$-acyclic. We therefore need to show that $k(n)_*(\operatorname{holim}_i Y_i^{\geqslant N})$ is v_n-power-torsion. Here we note that since the spectra $Y_i^{\geqslant N}$ are uniformly bounded below there is a weak equivalence

$$k(n) \wedge (\operatorname{holim}_i Y_i^{\geqslant N}) \simeq \operatorname{holim}_i \left(k(n) \wedge Y_i^{\geqslant N} \right)$$

by Corollary 3.3. It therefore suffices to show that $\pi_*(\operatorname{holim}_i k(n)_* Y_i^{\geqslant N})$ is v_n-power-torsion. Since $\pi_* Y_i$ is a finite type $\hat{\mathbb{Z}}_p$-module and consequently $\pi_* Y_i^{\geqslant N}$ is a finite type $\mathbb{Z}_p$-module, we know that $H_*(Y_i)$ and $H_*(Y_i^{\geqslant N})$ are finite type graded $\mathbb{F}_p$-modules. Since $H_*(Y_i^{\geqslant N}; \mathbb{F}_p)$ is finite type and $k(n) \wedge Y_i^{\geqslant N}$ is bounded below and $H\mathbb{F}_p$-nilpotent, we can apply [23, Prop. 2.2] to produce a strongly convergent inverse-limit Adams spectral sequence of the form

$$(14) \qquad \lim_i \operatorname{Ext}^{*,*}_{\operatorname{gr} \operatorname{Mod}(E(Q_n))}(H^*(Y_i^{\geqslant N}; \mathbb{F}_p), \mathbb{F}_p) \Rightarrow \pi_*(\operatorname{holim}_i k(n) \wedge Y_i^{\geqslant N}).$$

We therefore need to prove:

(1) that the E_∞-page of (14) is v_n-power-torsion, and
(2) there cannot exist an infinite sequence of nontrivial extensions in the E_∞-page of (14) which could yield a non-v_n-power-torsion element in

$$\pi_*(\operatorname{holim}_i k(n) \wedge Y_i^{\geqslant N}).$$

Step 2: The second task, to show that we cannot have an infinite sequence of nontrivial v_n-extensions, is easy as we will now show explicitly. The E_2-page in

question is isomorphic to

$$\lim_i \operatorname{Ext}^{*,*}_{\operatorname{gr} \operatorname{Comod}(A_*)}(\mathbb{F}_p, H_*(k(n); \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y_i^{\geq N}; \mathbb{F}_p))$$

$$(15) \qquad \cong \lim_i \operatorname{Ext}^{*,*}_{\operatorname{gr} \operatorname{Comod}(A_*)}\left(\mathbb{F}_p, (A_* \square_{E(Q_n)_*} \mathbb{F}_p) \otimes_{\mathbb{F}_p} H_*(Y_i^{\geq N}; \mathbb{F}_p)\right)$$

$$(16) \qquad \cong \lim_i \operatorname{Ext}^{*,*}_{\operatorname{gr} \operatorname{Comod}(E(Q_n)_*)}\left(\mathbb{F}_p, H_*(Y_i^{\geq N}; \mathbb{F}_p)\right)$$

$$(17) \qquad \cong \lim_i \operatorname{Ext}^{*,*}_{\operatorname{grMod}(E(Q_n))}\left(H^*(Y_i^{\geq N}; \mathbb{F}_p), \mathbb{F}_p\right)$$

with isomorphism (15) by the well-known description of $H_*(k(n); \mathbb{F}_p)$ given by [7], isomorphism (16) by a standard change-of-rings isomorphism as in Corollary A1.3.13 of [30], and isomorphism (17) by the assumption that $H_*(Y_i^{\geq N}; \mathbb{F}_p)$ is finite type and $Y_i^{\geq N}$ is bounded below.

Since every graded $E(Q_n)$-module is a coproduct of cyclic $E(Q_n)$-modules (a special case of the Cohen-Kaplansky theorem, from [13]), we know

$$\operatorname{Ext}^{*,*}_{\operatorname{grMod}(E(Q_n))}\left(H^*(Y_i^{\geq N}; \mathbb{F}_p), \mathbb{F}_p\right)$$

is a direct product of copies of

$$(18) \qquad \operatorname{Ext}^{*,*}_{\operatorname{gr} \operatorname{Mod}(E(Q_n))}(\Sigma^m \mathbb{F}_p, \mathbb{F}_p) \quad \text{and}$$

$$(19) \qquad \operatorname{Ext}^{*,*}_{\operatorname{gr} \operatorname{Mod}(E(Q_n))}(\Sigma^{m'} E(Q_n), \mathbb{F}_p)$$

for various integers m and m'. The Ext-module (18) is isomorphic to $\mathbb{F}_p[v_n]$, while the Ext-module (19) is isomorphic to $\mathbb{F}_p[v_n]/v_n$, concentrated on the cohomological degree 0 line.

Since $Y_i^{\geq N}$ is N-connective, $H^*(k(n) \wedge Y_i^{\geq N}; \mathbb{F}_p)$ vanishes below grading degree N. So, while we may have a copy of $\mathbb{F}_p[v_n]$ generated in bidegree $(0, N)$ in the E_2-page of the Adams spectral sequence for $Y_i^{\geq N}$, the E_2-page of that spectral sequence vanishes above the line formed by such a potential v_n-tower, i.e., $E_2^{s,t} \cong 0$ if $t < (2p^n - 1)s + N$. Of course the E_∞-page must then vanish above the same line. So, given an element $x \in E_\infty^{s,t}$, although we may have that $v_n x = 0$ in E_∞ but $v_n x \neq 0$ in $k(n)_*(Y_i^{\geq N})$ due to a filtration jump in the E_∞-page, a fixed choice of element x cannot support an infinite sequence of such filtration jumps, since after finitely many such filtration jumps the resulting element in E_∞ would necessarily be above the vanishing line, and hence would be zero. Write $E_r^{*,*}(Y_i^{\geq N})$ for the E_r-page of the Adams spectral sequence for $Y_i^{\geq N}$ and simply $E_r^{*,*}$ for the E_r-page of (14). Then there are isomorphisms

$$\lim_i E_r^{*,*}(Y_i^{\geq N}) \cong E_r^{*,*},$$

by the construction of the inverse limit Adams spectral sequence (see remark above Proposition 2.2 in [23]). Since there is a uniform vanishing line for each Adams spectral sequence for $Y_i^{\geq N}$, we have the same vanishing line in (22) and the same argument implies that there cannot be any infinite v_n-towers after resolving multiplicative extensions.

Step 3: Therefore, it suffices to prove that the E_∞-page of (14) is v_n-power torsion. By Theorem 2.6, the sequence

$$(20) \qquad 0 \to H_*(Y_i; \mathbb{F}_p) \to H_*(Y_i^{<N}; \mathbb{F}_p) \to H_*(\Sigma Y_i^{\geq N}; \mathbb{F}_p) \to 0$$

is exact for each i. Since $H_*(Y_i; \mathbb{F}_p)$ and $H_*(Y_i^{<N}; \mathbb{F}_p)$, and consequently $H_*(Y_i^{\geq N}; \mathbb{F}_p)$, are finite type, and since $H(H_*(X; \mathbb{F}_p); Q_n)$ is a subquotient of $H_*(X)$ for any spectrum X, we know that the we know that $H(H_*(Y_i; \mathbb{F}_p); Q_n)$, $H(H_*(Y_i^{<N}; \mathbb{F}_p); Q_n)$, and $H(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n)$ are each finite type. Since $R^1 \lim$ vanishes on sequences of finite-dimensional vector spaces, we now have

$$R^1 \lim H(H_*(Y_i; \mathbb{F}_p); Q_n) \cong R^1 \lim H(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n)$$
$$\cong R^1 \lim_i H(H_*(Y_i^{<N}; \mathbb{F}_p); Q_n) \cong 0.$$

We apply Margolis homology to the sequence (20) to get a long exact sequence for each i, and consequently an inverse sequence of long exact sequences. In general, applying $\lim$ to an inverse sequence of long exact sequences yields a chain complex with no guarantee of exactness anywhere at all. But in our case, we have just shown that $R^1 \lim_i H(-; Q_n)$ vanishes on each term in (20), so applying $\lim_i$ *does* produce a long exact sequence. In this long exact sequence, we know that every third term is some suspension of

$$\lim_i H(H_*(Y_i; \mathbb{F}_p); Q_n).$$

Since this vanishes by assumption, we get an isomorphism

$$(21) \qquad \lim_i H(H_*(\Sigma^{-1} Y_i^{<N}; \mathbb{F}_p); Q_n) \to \lim_i H(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n).$$

induced by the right-hand map in (20).

We note that $Y_i^{<N}$ is a bounded spectrum and $k(n) \wedge Y_i^{<N}$ is therefore bounded below and $H\mathbb{F}_p$-nilpotent complete so we may use functoriality of [23, Prop. 2.2] to produce a map of inverse limit Adams spectral sequences from

$$(22) \quad \lim_i \mathrm{Ext}^{*,*}_{\mathrm{gr}\,\mathrm{Mod}(E(Q_n))}(H^*(\Sigma^{-1} Y_i^{<N}; \mathbb{F}_p), \mathbb{F}_p) \Rightarrow \pi_*(\mathrm{holim}_i k(n) \wedge (\Sigma^{-1} Y_i^{<N}))$$

to the spectral sequence (14).

We then observe that this map of spectral sequences induces the composite isomorphism

$$\lim_i \mathrm{Ext}^{s+1,t}_{\mathrm{gr}\,\mathrm{Comod}(E(Q_n)*)} \left(H^*(\Sigma^{-1} Y_i^{<N}; \mathbb{F}_p), \mathbb{F}_p \right)$$

$$(23) \qquad \cong \lim_i \left(\Sigma^{-s(2(p^n-1))} H \left(H_*(\Sigma^{-1} Y_i^{<N}; \mathbb{F}_p); Q_n \right) \right)^{-(t+2p^n-1)}$$

$$(24) \qquad \cong \lim_i \left(\Sigma^{-s(2(p^n-1))} H \left(H_*(Y_i^{\geq N}; \mathbb{F}_p); Q_n \right) \right)^{-(t+2p^n-1)}$$

$$(25) \qquad \cong \lim_i \mathrm{Ext}^{s+1,t}_{\mathrm{gr}\,\mathrm{Mod}(E(Q_n))} \left(H_*(Y_i^{\geq N}; \mathbb{F}_p), \mathbb{F}_p \right)$$

with isomorphisms (23) and (25) due to Proposition A.4 from the appendix on Margolis homology, and with isomorphism (24) given by (21). To see that this isomorphism is the same as the the map of E_2 pages induced by the map

$$\Sigma^{-1} Y_i^{<N} \to Y_i^{\geq N},$$

as we have claimed, we must simply observe that the isomorphism (23) is a natural isomorphism. Also, to understand the grading shift of Margolis homology in isomorphism (23) and isomorphism (25) we point the reader to our grading convention in Conventions A.2, though this grading shift plays no significant role in the argument. Note that we are not claiming that these two spectral sequences have isomorphic E_2 pages, but rather that their E_2 pages are isomorphic above the $s = 0$-line.

By Lemma 3.4 the E_∞-page of the $H\mathbb{F}_p$-Adams spectral sequence for $k(n) \wedge Y_i^{<N}$ is v_n-power-torsion for each i. To draw the same conclusion for the spectral sequence (14), we need more information. In fact, spectral sequence (22) has a vanishing line of slope $1/|v_n|$ that crosses the x-axis at N, as we argued earlier. Consequently, the E_2-page of the inverse-limit Adams spectral sequence (14) vanishes above the $s = 0$-line in the region to the left of this vanishing line, by the sequence of isomorphisms above. Since the E_∞-page of (14) is a limit of v_n-power-torsion E_∞-pages, the only possible problem would be if a sequence of v_n-power-torsion elements became v_n-power-torsion free in the limit, but the vanishing region and Lemma 3.6 makes this impossible. Therefore, the E_∞-page of (14) and consequently the E_∞-page of (22) is v_n-power-torsion. □

4. A higher chromatic height analogue of Mitchell's theorem

In [28], Mitchell proved that $K(m)_*(K(\mathbb{Z})) \cong 0$ for $m \geqslant 2$ and consequently $K(m)_*(K(R)) \cong 0$ for any $H\mathbb{Z}$-algebra R. Throughout this section we will write $BP\langle n \rangle$ for the truncated Brown-Peterson spectrum with coefficients $\pi_*(BP\langle n \rangle) \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$. We also use the convention that $BP\langle -1 \rangle$ is $H\mathbb{F}_p$. Since $BP\langle 0 \rangle = H\mathbb{Z}_{(p)}$, we may consider the following higher chromatic height analogue of Mitchell's result.

Question 4.1. Suppose n is some integer, $n \in [-1, \infty)$. If R is a $BP\langle n \rangle$-algebra spectrum, then does $K(m)_*(K(BP\langle n \rangle))$ vanish for all $m \geqslant n + 2$?

This gives an upper bound on on the chromatic complexity of $K(BP\langle n \rangle)$ so that if there is a "red-shift" in algebraic K-theory of a $BP\langle n \rangle$-algebra spectra, then this shift is a shift of at most one. The main goal of this section is to answer this question for all (n, p) such that $BP\langle n \rangle$ can be modeled by an E_∞ ring spectrum, which is known to be $n = -1, 0, 1$ at all primes and $n = 2$ at $p = 2, 3$. The cases $n = -1$ and $n = 0$ are clear since $H\mathbb{F}_p$ and $H\mathbb{Z}_p$ are E_∞ ring spectra. There is an E_∞ ring spectrum model for $BP\langle 1 \rangle$ by McClure-Staffeldt [27] and there is an E_∞ ring spectrum model for $BP\langle 2 \rangle$ at the prime $p = 2$ by Lawson-Naumann [22] and at $p = 3$ by Hill-Lawson [17].

Conventions 4.2. When $n = -1, 0, 1$ at all primes and $n = 2$ at the primes $p = 2$ and $p = 3$, we write $BP\langle n \rangle$ for an E_∞ ring spectrum model for the truncated Brown-Peterson spectrum, which exist by the discussion above.

Remark 4.3. One can ask whether our positive answer to Question 4.1, Theorem 4.10, in this section can be extended to those values of (n, p) for which $BP\langle n \rangle$ is *not* known to admit an E_∞ ring structure—or even those values of (n, p) for which $BP\langle n \rangle$ is known *not* to admit an E_∞ ring structure, when $n \geqslant 4$ by Lawson [21] at $p = 2$ and Senger [31] for $p > 2$.

We suspect that the answer is "yes." Our arguments in this section are far from using the full strength of an E_∞ multiplication on $BP\langle n \rangle$: if the results of Bruner-Rognes [12] can be extended to the setting of E_3 ring spectra leading to the same computations of $H_*^c(TC^-(BP\langle n \rangle); \mathbb{F}_p)$ and $H_*^c(TP(BP\langle n \rangle); \mathbb{F}_p)$ as appear there, then our main theorem in this section, Theorem 4.10, will apply for any (n, p) such that $BP\langle n \rangle$ has a model as an E_3 ring spectrum. The spectrum BP was shown to admit an E_4 ring spectrum model by Basterra-Mandell [8]. We have heard other mathematicians claim that the spectra $BP\langle n \rangle$ also admit E_3 multiplications, but

as far as we have been able to tell, this is folklore and does not appear in the literature. There is currently work in progress of Hahn-Wilson on showing that $BP\langle n \rangle$ is indeed E_3 as a BP-algebra.

If $BP\langle n \rangle$ admits an E_3 multiplication, and if the calculations of [12] can be made to work using only the E_3 Dyer-Lashof-Kudo-Araki operations rather than the classical (E_∞) Dyer-Lashof-Kudo-Araki operations, then our Theorem 4.10 would give a positive answer to Question 4.1 for all primes p and heights n.

We first briefly recall the setup for topological periodic cyclic homology and topological negative cyclic homology. Let R be an E_1 ring spectrum and write $\mathbb{T}$ for the circle group. Recall that the topological Hochschild homology of R, denoted $THH(R)$ has a canonical $\mathbb{T}$-action. We define *topological negative cyclic homology* as the homotopy fixed-point spectrum

$$TC^-(R) := THH(R)^{h\mathbb{T}}$$

and *topological periodic cyclic homology* as the Tate spectrum

$$TP(R) := THH(R)^{t\mathbb{T}}.$$

There is a homological $\mathbb{T}$-homotopy fixed point spectral sequence

$$(26) \qquad E_2^{*,*}(R) = H^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*^c(TC^-(R); \mathbb{F}_p)$$

with abutment

$$H_*^c(TC^-(R); \mathbb{F}_p) := \lim_{k \to -\infty} H_*(TC^-(R)[k]; \mathbb{F}_p)$$

where

$$TC^-(R)[k] := F(E\mathbb{T}^{(k)}, THH(R))^{\mathbb{T}}$$

and $E\mathbb{T}^{(k)}$ is the k-skeleton of $E\mathbb{T}$. There is also a homological $\mathbb{T}$-Tate spectral sequence

$$(27) \qquad \hat{E}_2^{*,*}(R) = \hat{H}^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*^c(TP(R); \mathbb{F}_p)$$

with abutment

$$H_*^c(TP(R); \mathbb{F}_p) := \lim_{k \to -\infty} H_*(TP(R)[k]; \mathbb{F}_p)$$

where

$$(28) \qquad TP(R)[k] := \left(\widetilde{E\mathbb{T}}/\widetilde{E\mathbb{T}}_k \wedge F(E\mathbb{T}, THH(R)) \right)^{\mathbb{T}}$$

is the Greenlees filtration [16] where $\widetilde{E\mathbb{T}}_k$ is the cofiber of the map $E\mathbb{T}_+^{(k)} \to S^0$ for $k \geqslant 0$ and the Spanier-Whitehead dual of $\widetilde{E\mathbb{T}}_{-k-1}$ if $k < 0$.

By considering the filtrations on $E\mathbb{T}$ and on $\widetilde{E\mathbb{T}}$ only in a range of dimensions, there is also a spectral sequence

$$(29) \qquad E_2^{*,*}(R) = H_{\geqslant -2k}^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*(TC^-(R)[k]; \mathbb{F}_p)$$

whose input is $P_{k+1}(t) \otimes H_*(THH(R); \mathbb{F}_p)$ with $|t| = -2$ and a spectral sequence

$$(30) \qquad E_2^{*,*}(R) = \hat{H}_{\geqslant -2k}^*(\mathbb{T}, H_*(THH(R); \mathbb{F}_p)) \Rightarrow H_*(TP(R)[-k]; \mathbb{F}_p)$$

whose input is $P(t^{-1})\{t^k\} \otimes H_*(THH(R); \mathbb{F}_p)$ for $k \geqslant 0$.

More generally, for a homology theory E_* we write

$$E_*^c(TP(R)) := \lim_{k \to -\infty} E_*(TP(R)[k]) \text{ and}$$

$$E_*^c(TC^-(R)) := \lim_{k \to -\infty} E_*(TC^-(R)[k]).$$

We now recall that by Angeltveit-Rognes [4, Thm. 5.12], there is an isomorphism

$$H_*(THH(BP\langle n\rangle); \mathbb{F}_p) \cong H_*(BP\langle n\rangle; \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \ldots, \sigma\bar{\xi}_{n+1}) \otimes P(\sigma\bar{\tau}_{n+1})$$

of A_*-comodules with A_*-coaction

$$\nu_n : H_*(THH(BP\langle n\rangle); \mathbb{F}_p) \to A_* \otimes H_*(THH(BP\langle n\rangle); \mathbb{F}_p)$$

given by the restriction of the coproduct of A_* to $H_*(BP\langle n\rangle; \mathbb{F}_p) \subset A_*$ on elements in $H_*(BP\langle n\rangle; \mathbb{F}_p)$, and by the formula

$$\nu_n(\sigma x) = (1 \otimes \sigma)(\nu_n(x))$$

in [4, Eq. 511] for elements of the form $\sigma\bar{\xi}_i$ for $1 \leqslant i \leqslant n$ and $\sigma\bar{\tau}_{n+1}$, and then for the remaining elements by the formula

$$\nu_n(xy) = \nu_n(x)\nu_n(y).$$

Bruner-Rognes [12] then compute $H_*^c(TC^-(BP\langle n\rangle); \mathbb{F}_p)$ and $H_*^c(TP(BP\langle n\rangle); \mathbb{F}_p)$, as we recall below. We will focus on odd primes for simplicity, but essentially the same results will hold at the prime 2. In [12], they compute

$$\begin{aligned} E_\infty^{*,*}(BP\langle n\rangle) \quad \cong \quad & P(t) \otimes P(\bar{\xi}_k^p | 1 \leqslant k \leqslant n+1) \otimes P(\bar{\xi}_{k+1} | k \geqslant n+1) \otimes \\ & E(\bar{\tau}_{k+1}' | k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1}\sigma\bar{\xi}_k | 1 \leqslant k \leqslant n) \oplus T \end{aligned}$$

where T consists of classes x in filtration $s = 0$ with $tx = 0$ and

$$\bar{\tau}_{k+1}' = \bar{\tau}_{k+1} - \bar{\tau}_k(\sigma\bar{\tau}_k)^{p-1}$$

for $k \geqslant m$. One can easily deduce the computation

$$\begin{aligned} \hat{E}_\infty^{*,*}(BP\langle n\rangle) \quad \cong \quad & P(t, t^{-1}) \otimes P(\bar{\xi}_k^p | 1 \leqslant k \leqslant n+1) \otimes P(\bar{\xi}_{k+1} | k \geqslant n+1) \otimes \\ & E(\bar{\tau}_{k+1}' | k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1}\sigma\bar{\xi}_k | 1 \leqslant k \leqslant n) \end{aligned}$$

for topological periodic cyclic homology from their computation. There are no possible additive extensions because the abutment is an $\mathbb{F}_p$-vector space.

It will also be useful to record the computations for each spectrum in the filtration:

$$\begin{aligned} \hat{E}_\infty^{*,*}(BP\langle n\rangle)[-k] = & [P(t^{-1})\{t^{k-1}\} \otimes P(\bar{\xi}_k^p | 1 \leqslant k \leqslant n+1) \otimes P(\bar{\xi}_{k+1} | k \geqslant n+1) \otimes \\ & E(\bar{\tau}_{k+1}' | k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1}\sigma\bar{\xi}_k | 1 \leqslant k \leqslant n)] \oplus V(n, k) \end{aligned}$$

where

(31) $$V(n, k) := [\cap\{\ker d_{s,t} | s = -2k, t \geqslant 0\}]$$

and we write $d_{s,t}$ for the differential in the spectral sequence computing

$$H_*^c(TP(BP\langle n\rangle); \mathbb{F}_p)$$

rather than the truncated one. We now compute the "continuous Margolis homology," with respect to the Greenlees filtration, of the homology groups of the topological periodic homology of $BP\langle n\rangle$; i.e., the limit of the Margolis homology of $H_*(TP(BP\langle n\rangle)[k]; \mathbb{F}_p)$ as k goes to negative infinity.

Proposition 4.4. *There are isomorphisms*

$$\lim_{k \to -\infty} H(H_*(TP(BP\langle n\rangle)[k]; \mathbb{F}_p), Q_m) \cong 0$$

for all $m \geqslant n + 2$.

Proof. First, we note that Q_m acts trivially on any $P(t^{-1})$ module contained in $P(t^{\pm 1})$ for all $m \geqslant 0$ because t is in an even degree. We also claim that the tensor product

$$P(t^{-1})\{t^{k-1}\} \otimes P(\bar{\xi}_{k+1}|k \geqslant n + 1) \otimes E(\bar{\tau}'_{k+1}|k \geqslant n + 1) \otimes$$
$$P(\bar{\xi}^p_k|1 \leqslant k \leqslant n + 1) \otimes E(\bar{\xi}^{p-1}_k \sigma\bar{\xi}_k|1 \leqslant k \leqslant n) \otimes V(n, k)$$

in the abutment of the truncated homological $\mathbb{T}$-Tate spectral sequence is still a tensor product of Q_m-modules, in other words, we will argue that there are no possible hidden $E(Q_n)$-module extensions.

To see this we note that there is no room for hidden Q_m-module extension in the truncated homological $\mathbb{T}$-Tate spectral sequence for the sphere spectrum

$$P(t)\{t^{k-1}\} \Rightarrow H_*(TP(S)[-k])$$

which maps to the truncated homological $\mathbb{T}$-Tate spectral sequence for $BP\langle n\rangle$. The truncated $\mathbb{T}$-Tate spectral sequence for the sphere spectrum collapses and has abutment $P(t^{-1})\{t^{k-1}\}$ with trivial Q_m-action. The map to the abutment $H_*(TP(BP\langle n\rangle)[k]; \mathbb{F}_p)$ is an injection and it is a map of Q_n-modules so the elements t^k must have trivial Q_m-action. Here we use the fact that the abutment is a graded $\mathbb{F}_p$-module so there are no hidden additive extensions involving t^j. Therefore, all elements of the form xt^j in the abutment must have Q_m-action $Q_m(xt^j) = Q_m(x)t^j + xQ_m(t^j) = Q_m(x)t^j$. So the Q_m-action preserves the filtration in the truncated homological $\mathbb{T}$-Tate spectral sequence. Since hidden comodule extensions would shift filtration, we know that the abutment is a tensor product of $E(Q_m)$-modules and it is isomorphic to the E_∞-page as $E(Q_m)$-modules.

Now that we have proven the claim, we can apply the Künneth isomorphism for Margolis homology [24, Prop. 18.1.2(a)] and the fact that Margolis homology commutes with coproducts of $E(Q_m)$-modules to produce the isomorphism

$$H(H_*(TP(BP\langle n\rangle)[k]; \mathbb{F}_p), Q_m) \cong \big(P(t^{-1})\{t^{k-1}\} \otimes H(M_1, Q_m) \otimes H(M_2, Q_m)\big)$$
$$\oplus H(V(n, k), Q_m)\{t^k\}$$

where

$$M_1 := P(\bar{\xi}_{k+1}|k \geqslant n + 1) \otimes E(\bar{\tau}'_{k+1}|k \geqslant n + 1),$$
$$M_2 := P(\bar{\xi}^p_k|1 \leqslant k \leqslant n + 1) \otimes E(\bar{\xi}^{p-1}_k \sigma\bar{\xi}_k|1 \leqslant k \leqslant n),$$

and $V(n, k)$ is defined as in (31), above. We also recall that $Q_m(\bar{\tau}_{k+1}) = \bar{\xi}^{p^{m+1}}_{k-m}$ for $k \geqslant m + 1$. This holds by the coaction

$$\nu(\bar{\tau}_{k+1}) = \bar{\tau}_m \otimes \bar{\xi}^{p^{m+1}}_{k-m} + \bar{\nu}(\bar{\tau}_{k+1})$$

where the $\bar{\nu}(\bar{\tau}_{k+1})$ consists of terms such that $\bar{\tau}_m \otimes x$ does not appear for any nontrivial x. It is easy to observe that the same will be true of τ'_{k+1} for $k \geqslant n + 1$. Therefore, we can compute Q_{k+1}-Margolis homology of $M_1 \otimes M_2$ as the tensor product of the chain complex

$$(c_{k+1}) := \big(\mathbb{F}_p\{1\} \leftarrow \mathbb{F}_p\{\tau'_{k+1}\}\big)$$

with the chain complex whose Margolis homology is $H(\overline{M}_1 \otimes M_2, Q_{k+1})$ where $M_1 \cong \overline{M}_1 \otimes E(\bar{\tau}_{k+1})$. Since the chain complex (c_{k+1}) is acyclic, we observe that the Margolis homology of $P(t^{-1})\{t^{k-1}\} \otimes M_1 \otimes M_2$ is trivial for $k \geqslant n+1$. We also observe that the map

$$H_*(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m) \to H_*(H_*(TP(BP\langle n \rangle)[k+1]; \mathbb{F}_p), Q_m)$$

maps the summand $H_*(V(n,k), Q_m)$ to zero and maps zero to $H_*(V(n,k+1), Q_m)$ by examination of the map of truncated Tate spectral sequences and consequently the map is the zero map. Therefore,

$$\{H(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m)\}$$

is pro-equivalent to the constant pro-object on the zero object and in particular

$$\lim_{k \to -\infty} H(H_*(TP(BP\langle n \rangle)[k]; \mathbb{F}_p), Q_m) = 0.$$

$\square$

Lemma 4.5. *Let R be a connective p-local E_1 ring spectrum such that $\pi_0(R) \cong \mathbb{Z}_{(p)}$ and such that $\pi_*(R)$ is finite type as a graded $\mathbb{Z}_{(p)}$-module. Then the homotopy groups of the enveloping algebra*

$$\pi_*(R \wedge R^{\mathrm{op}})$$

are also a finite type graded $\mathbb{Z}_{(p)}$-module.

Proof. We first show that $\pi_*(R \wedge_{S_{(p)}} R^{\mathrm{op}})$ is finite type as a $\mathbb{Z}_{(p)}$-module. We apply the Künneth spectral sequence

$$\mathrm{Tor}_{*,*}^{\pi_*(S_{(p)})}(\pi_*(R), \pi_*(R^{\mathrm{op}})) \Rightarrow \pi_*(R \wedge_{\hat{S}_{(p)}} R^{op}).$$

Since $\pi_*(R)$ is a finite type $\mathbb{Z}_{(p)}$-module and $\pi_*(S)$ is a finite type $\mathbb{Z}_{(p)}$-module and the unit map $S_{(p)} \to R$ is an isomorphism on π_0, we may see by an easy resolution argument that each bidegree in the spectral sequence is a finite type $\mathbb{Z}_{(p)}$-module. Since this is a first quadrant spectral sequence graded with the Serre convention, we know that only finitely many bidegrees contribute to each homotopy degree in the abutment.

Note that $R \wedge_{S_{(p)}} R^{\mathrm{op}}$ is defined as the equalizer

$$(32) \qquad R \wedge S_{(p)} \wedge R^{\mathrm{op}} \underset{1 \wedge (\psi_L)_{(p)}}{\overset{(\psi_R)_{(p)} \wedge 1}{\rightrightarrows}} R \wedge R^{\mathrm{op}}$$

where $(\psi_R)_{(p)}$ and $(\psi_L)_{(p)}$ are the usual right and left action of $S_{(p)}$ on the p-local spectrum R respectively. We may then simply observe that Bousfield localization at the Moore spectrum $S\mathbb{Z}_{(p)}$ is smashing so the equalizer (32) is equivalent to the equalizer

$$S_{(p)} \wedge (R \wedge S \wedge R^{\mathrm{op}}) \underset{1 \wedge 1 \wedge \psi_L}{\overset{1 \wedge \psi_R \wedge 1}{\rightrightarrows}} S_{(p)} \wedge (R \wedge R^{\mathrm{op}}),$$

which in turn is equivalent to $(R \wedge R^{\mathrm{op}})_{(p)}$ where we write $(-)_{(p)}$ for Bousfield localization at $S\mathbb{Z}_{(p)}$. Since $R \wedge R^{\mathrm{op}}$ is already p-local, there is an equivalence to $R \wedge R^{\mathrm{op}} \simeq (R \wedge R^{\mathrm{op}})_{(p)}$ and consequently,

$$\pi_*(R \wedge R^{\mathrm{op}}) \cong \pi_*(R \wedge_{S_{(p)}} R^{\mathrm{op}})$$

as graded $\mathbb{Z}_{(p)}$-modules. $\square$

We now show that the filtration $\{TP(B\hat{P}\langle n\rangle)[i]_p\}$ satisfies the remaining hypotheses of Theorem 3.7.

Lemma 4.6. *The graded $\mathbb{Z}_{(p)}$-modules $\pi_*(TP(BP\langle n\rangle)[i])$ are finite type for all i and all pairs (n,p) such that $BP\langle n\rangle$ is an E_∞ ring spectrum. Consequently, $\pi_*(TP(B\hat{P}\langle n\rangle)[i]_p)$ is a finite type graded $\mathbb{Z}_p$-modules for all i under the same conditions on the pairs (n,p).*

Proof. Since the truncated Tate spectral sequence
$$\hat{H}^*_{\geqslant -2k}(\mathbb{T}; \pi_*(THH(BP\langle n\rangle))) \Rightarrow \pi_*(TP(BP\langle n\rangle)[-k])$$
has a vanishing line of vertical slope with the Serre convention, which implies only finitely many bidegrees contribute to each homotopy degree in the abutment, it suffices to show that $\pi_*(THH(BP\langle n\rangle))$ is a finite type graded $\mathbb{Z}_{(p)}$-module for all n. To show this, we apply [2, Lem. 5.2.5], which states that if R is a connective E_∞ ring spectrum whose homotopy groups $\pi_*(R \otimes R)$ are finite type $\pi_0(R)$-modules then, in particular, $THH_*(R)$ is finite type. Note that
$$\pi_*(BP\langle n\rangle) \cong \mathbb{Z}_{(p)}[v_1, \ldots v_n]$$
is a connective commutative graded $\pi_0(BP\langle n\rangle) \cong \mathbb{Z}_{(p)}$-module and by assumption $BP\langle n\rangle$ is E_∞, so to apply [2, Lem. 5.2.5] it suffices to show that
$$\pi_*(BP\langle n\rangle \wedge BP\langle n\rangle)$$
is finite type, but this follows by Lemma 4.5. $\square$

Remark 4.7. The only place where the E_∞ ring spectrum structure is used in the result above is in the hypothesis of [2, Lem. 5.2.5]. However, [2, Lem. 5.2.5] can be easily generalized to E_1 ring spectra when we are only applying it to topological Hochschild homology, and not to the "higher THH" constructions obtained by tensoring with simplicial sets other than the standard simplicial circle, which were under consideration in the paper [2] and which motivated the assumption of an E_∞ ring structure in that paper.

Lemma 4.8. *The spectra $(TP(B\hat{P}\langle n\rangle)[-k])_p$ satisfy condition $H(M)$ for all $k \geqslant 2$ and a fixed M depending on n, but not on k.*

Proof. Note that $H_*((TP(B\hat{P}\langle n\rangle)[-k])_p) \cong H_*(TP(BP\langle n\rangle)[-k])$. It suffices to check that the sub-A_*-comodule
$$\mathrm{Hom}_{\mathcal{A}}(\mathbb{F}_p, H_*(TP(BP\langle n\rangle[k]); \mathbb{F}_p)) \subset H_*(TP(BP\langle n\rangle)[k]; \mathbb{F}_p)$$
of A_*-comodule primitives is bounded above. We first choose a cofinal sequence that always contains elements in homotopy that are not t-divisible to make the argument easier. Recall that
$$
\begin{aligned}
\hat{E}^{*,*}_\infty(BP\langle n\rangle)[-k] \quad = \quad & (P(t^{-1})\{t^{k-1}\} \otimes P(\bar{\xi}^p_k | 1 \leqslant k \leqslant n+1) \otimes P(\bar{\xi}_{k+1} | k \geqslant n+1) \otimes \\
& E(\bar{\tau}'_{k+1} | k \geqslant n+1) \otimes E(\bar{\xi}^{p-1}_k \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n)) \oplus \\
& (H_*(THH(BP\langle n\rangle); \mathbb{F}_p)/\mathrm{im}\,(d^2_{-2k+2,*}))
\end{aligned}
$$
where we assume $k > 1$.

We first need to rule out certain hidden A_*-comodule extensions. We claim that after resolving hidden comodule extensions there cannot be additional comodule primitives. Suppose
$$x \in H_*(TP(BP\langle n\rangle)[-k]; \mathbb{F}_p)$$

is not a comodule primitive, and has coaction
$$\nu_n(x) = \sum x_{(1)} \otimes x_{(2)},$$
adapting Sweedler's notation used for coproducts in coalgebras to this context. Then after resolving hidden A_*-comodule extensions, we will have
$$\nu_n(x) = \sum x_{(1)} \otimes x_{(2)} + \sum y_{(1)} \otimes y_{(2)},$$
and the terms in the sum $\sum y_{(1)} \otimes y_{(2)}$ must be in higher filtration in the spectral sequence's E_∞-page, and therefore cannot cancel with terms in the sum $\sum x_{(1)} \otimes x_{(2)}$. Thus, we cannot have an element that is not a comodule primitive in the E_∞-page become a comodule primitive after resolving hidden comodule extensions. It therefore suffices to consider the comodule primitives in $\hat{E}_\infty^{*,*}(BP\langle n\rangle)[k]$, which by the argument above, are the same as the comodule primitives in the abutment.

Now, we know that the only A_*-comodule primitive in
$$P(\bar{\xi}_k^p | 1 \leqslant k \leqslant n+1) \otimes P(\bar{\xi}_{k+1} | k \geqslant n+1) \subset A_*$$
is the element 1 because it is a sub-Hopf algebra of the dual Steenrod algebra and its coaction is the restriction of the coalgebra structure on A_*. Also,
$$E(\bar{\tau}'_{k+1} | k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n)$$
is clearly bounded above by the same m independent of k. Also, t^j for $j \neq 0$ is not a comodule primitive except for t^j such that $2j > k$, which is in degree $-k+2$ or lower. To see this, note that the coaction
$$\psi \colon \hat{H}^*(S^1, \mathbb{F}_p) \to \hat{H}^*(\mathbb{T}, \mathbb{F}_p) \otimes A_*$$
of the dual Steenrod algebra on $\hat{H}^*(\mathbb{T}, \mathbb{F})_p$ is
$$\psi(t) = 1 \otimes t + \bar{\xi}_1 \otimes t^2 + \bar{\xi}_2 \otimes t^4 + \dots$$
when $j > 0$ and can be computed by the formula
$$\psi(t) \cdot \psi(t^{-1}) = 1 \otimes 1$$
when $j < 0$. The coaction
$$\psi_k \colon \hat{H}^*_{\geqslant -2k}(\mathbb{T}; \mathbb{F}_p) \to \hat{H}^*_{\geqslant -2k}(\mathbb{T}; \mathbb{F}_p) \otimes A_*$$
can then be computed as
$$\psi_k(t) = \psi(t) \mod t^{k+1}.$$
So the only way a power t^k could be a comodule primitive is when $k = 0$ or $2j > k+1$ Consequently, the coaction of $t^j x$ for
$$x \in (P(t^{-1})\{t^{k-1}\} \otimes P(\bar{\xi}_k^p | 1 \leqslant k \leqslant n+1) \otimes P(\bar{\xi}_{k+1} | k \geqslant n+1) \otimes$$
$$E(\bar{\tau}'_{k+1} | k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n))$$
will always have a summand
$$\bar{\xi}_1^j \otimes t^{2j} x$$
when k is sufficiently large so $t^j x$ cannot be a comodule primitive unless $2j > k$. Therefore the only comodule primitives are elements in
$$E(\bar{\tau}'_{k+1} | k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n),$$
possibly plus some correcting terms, or elements in
$$E(\bar{\tau}'_{k+1} | k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n)\{t^j\}$$

such that $2j > k$ plus some correcting terms. We claim that these are both bounded above by the maximal degree of

$$E(\bar{\tau}'_{k+1}|k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n).$$

This follows since t^j is in a negative degree and if

$$y \in E(\bar{\tau}'_{k+1}|k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n)$$

is not a comodule primitive then it will be of the form

$$y \mapsto 1 \otimes y + \sum y_1^{(1)} \otimes y_2^{(2)},$$

but then $y_1^{(1)} \otimes t^j y_2^{(2)}$ will be nonzero, since t^j is not a zero divisor. Therefore, $t^j y$ will not be a comodule primitive unless y is also a comodule primitive, which proves the claim.

We still need to show that the comodule primitives in

$$H_*(THH(BP\langle n\rangle); \mathbb{F}_p)/\operatorname{im}(d^2_{-2k+2,*}),$$

are bounded above, which is not immediately obvious because the comodule primitives in

$$H_*(THH(BP\langle n\rangle))$$

are not bounded above. However, the elements $(\sigma(\bar{\tau}_n))^k$ for $k \geqslant 0$ are in $\operatorname{im}(d^2)$ and all remaining potential comodule primitives in $H_*(THH(BP\langle n\rangle); \mathbb{F}_p)$ are bounded above by the same bound as the highest degree element in the A_*-comodule

$$E(\bar{\tau}'_{k+1}|k \geqslant n+1) \otimes E(\bar{\xi}_k^{p-1} \sigma \bar{\xi}_k | 1 \leqslant k \leqslant n).$$

Therefore, it remains to check that sums of homogeneous elements in the abutment coming from different filtrations in the truncated $\mathbb{T}$-tate spectral sequence are not comodule primitives. We divide into two cases. First, consider $t^j x$ where x is a not comodule primitive. If x is not a comodule primitive, then the coaction on x is

$$1 \otimes x + \sum_{i \geqslant 1} x_i^{(1)} \otimes x_i^{(2)}$$

and there cannot be a term in the coaction on $t^{j+k} y$ that cancels out the term $x_1^{(1)} \otimes x_1^{(2)} t^j$ in the coaction on $t^j x$. Second, consider the case $t^j x$ where x is a comodule primitive. The only comodule primitives are products of $\sigma \bar{\xi}_i$ or possibly τ'_{k+1} plus some additional classes. In each case, there will be a term of the form $\bar{\xi}_1^{2j} \otimes t^j x$ when $j > 0$ or $\bar{\xi}_1^{2j} \otimes 1$ if $j > 0$ that cannot be canceled by adding on terms in higher filtration. □

Remark 4.9. Note that the comodule primitives are not necessarily bounded above in $H_*(THH(BP\langle n\rangle); \mathbb{F}_p)$, but they are bounded above in $H_*(TP(BP\langle n\rangle)[k]; \mathbb{F}_p)$ nonetheless.

Theorem 4.10. *When $BP\langle n\rangle$ is an E_∞ ring spectrum, there are isomorphisms*

$$K(m)_*(TP(BP\langle n\rangle)) \cong 0 \text{ and}$$
$$K(m)_*(TC^-(BP\langle n\rangle)) \cong 0$$

for $m \geqslant n+2$ for all $n \geqslant 0$.

Proof. The statement

$$K(m)_*(TP(BP\langle n\rangle)) \cong 0$$

follows by Theorem 3.7 together with Proposition 4.4, Lemma 4.6, and Lemma 4.8. The fact that $K(m)_*(TC^-(BP\langle n\rangle)) \cong 0$ for $m \geqslant n + 2$ then follows by the fiber sequence

$$\Sigma THH(BP\langle n\rangle)_{h\mathbb{T}} \to THH(BP\langle n\rangle)^{h\mathbb{T}} \to THH(BP\langle n\rangle)^{t\mathbb{T}}$$

and the claim that

$$K(m)_*(\Sigma THH(BP\langle n\rangle)_{h\mathbb{T}}) \cong 0$$

for $m \geqslant n + 2$, which we now prove. Since $K(m)_*$ has a Künneth isomorphism, there is a Bökstedt spectral sequence

$$HH_*^{K(m)*}(K(m)_*(BP\langle n\rangle)) \Rightarrow K(m)_* THH(BP\langle n\rangle).$$

When $m \geqslant n+2$ we know that $K(m)_*(BP\langle n\rangle) \cong 0$, so clearly the spectral sequence converges and is trivial so that

$$K(m)_*(THH(BP\langle n\rangle)) \cong 0$$

for $m \geqslant 2$. We then simply use the fact that smashing with $K(m)$ commutes with homotopy colimits to show that

$$K(m)_*(\Sigma THH(BP\langle n\rangle)_{h\mathbb{T}}) \cong 0$$

for $m \geqslant n + 2$. $\qquad\qquad\square$

Corollary 4.11. *When $BP\langle n\rangle$ is an E_∞ ring spectrum, there are isomorphisms*

$$K(m)_*(K(BP\langle n\rangle)) \cong 0$$

for $m \geqslant n + 2 \geqslant 1$.

Proof. The result for topological cyclic homology follows by Theorem 4.10 together with the long exact sequence in Morava K-theory associated to fiber sequence

$$TC(BP\langle n\rangle)_p \to TC^-(BP\langle n\rangle)_p \to TP(BP\langle n\rangle)_p$$

of [29, Cor. 1.5]. For algebraic K-theory, the result follows by [14, Thm. 7.3.1.8], which produces a fiber sequence

$$K(BP\langle n\rangle)_p \to TC(BP\langle n\rangle)_p \to \Sigma^{-1}H\mathbb{Z}_p$$

and the associated long exact sequence in Morava K-theory, since

$$K(m)_*(\Sigma^{-1}H\mathbb{Z}_p) = 0 \text{ for } m \geqslant 1.$$

Note that our result does not give a new proof in the case $n = -1$ since [14, Thm. 7.3.1.8] depends on the computation of algebraic K-theory of $\mathbb{F}_p$. $\qquad\square$

Appendix A. Review of Margolis homology.

This appendix, which does not logically rely on anything earlier in the paper, consists of material that is well-known to users of Margolis homology. We think it is useful to provide these results, which in a few cases we do not know written references for (but are nevertheless not difficult, and certainly not new).

A.1. **Basics of Margolis homology.** Given a graded ring R, we write $\operatorname{gr}\operatorname{Mod}(R)$ for the category of graded R-modules and grading-preserving R-module homomorphisms.

Definition A.1. *Let k be a field and let $E(Q)$ be the exterior k-algebra on a single homogeneous generator Q in an odd grading degree $|Q|$. By Q-Margolis homology we mean the functor $H(-;Q)\colon \operatorname{gr}\operatorname{Mod}(E(Q)) \to \operatorname{gr}\operatorname{Mod}(E(Q))$ given on a graded $E(Q)$-module M by the quotient*

$$H(M;Q) = \left(\ker(M \xrightarrow{Q} \Sigma^{-|Q|}M)\right) \Big/ \left(\operatorname{im}\left(\Sigma^{|Q|}M \xrightarrow{Q} M\right)\right).$$

Of course we shouldn't use the word "homology" in the phrase "Margolis homology" unless $H(-;Q_n)$ is a functor which turns short exact sequences of $E(Q_n)$-modules to long exact sequences, but happily it does have this property: given a short exact sequence of graded $E(Q)$-modules

$$0 \to M' \to M \to M'' \to 0$$

the associated short exact sequence of chain complexes produces a long exact sequence

$$\cdots \longrightarrow \Sigma^{|Q|}H(M'';Q)$$
$$H(M';Q) \longrightarrow H(M;Q) \longrightarrow H(M'';Q)$$
$$\Sigma^{-|Q|}H(M';Q) \longrightarrow \cdots.$$

Here is a quick note on gradings; it is extremely elementary, but not taking a moment to "fix notations" on this point tends to lead to sign errors in the gradings.

Conventions A.2. Given a graded ring R and graded R-modules M and N, we write $\hom_R(M,N)$ for the degree-preserving R-linear morphisms $M \to N$, and we write $\underline{\hom}_R(M,N)$ for the graded abelian group whose degree n summand is $\hom_R(\Sigma^n M, N)$.[8] We write $\operatorname{Ext}_R^{s,t}(M,N)$ for the sth right-derived functor of $\hom_R(-,\Sigma^t N)$ on the opposite category of graded R-modules applied to M, i.e. (up to isomorphism), the sth right-derived functor of $\hom_R(\Sigma^{-t}M,-)$ on the category of graded R-modules applied to N. We write $\operatorname{Ext}_R^{s,*}(M,N)$ for the graded abelian group whose degree t summand is $\operatorname{Ext}_R^{s,t}(M,N)$, and we refer to this grading as the *internal* or *topological* grading, to distinguish it from the *cohomological* degree given by s.

In particular, the k-linear dual of a graded k-vector space has the signs of the gradings reversed, i.e.,

$$(\Sigma^n V)^* = \underline{\hom}_k(\Sigma^n V, k) \cong \Sigma^{-n}(V^*).$$

Now given a spectrum X, the action of Q_n on $H^*(X;\mathbb{F}_p)$ is the one induced in homotopy by the map of function spectra $F(X, H\mathbb{F}_p) \to F(X, \Sigma^{2p^n-1}H\mathbb{F}_p)$ induced by the composite (6). Somewhat less famous than the action of Steenrod

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[8]Some references, e.g. [11], use the opposite grading on $\underline{\hom}_R$—hence the need to give Convention A.2 explicitly. The argument for our choice of gradings is that it is the one that yields the hom-tensor adjunction $\hom_R(L, \underline{\hom}_R(M,N)) \cong \hom_R(L \otimes_R M, N)$.

operations on mod p cohomology, we have also the dual action of Steenrod operations on mod p homology: the action of Q_n on $H_*(X; \mathbb{F}_p)$ is the one induced in homotopy by the map of spectra $X \wedge H\mathbb{F}_p \to X \wedge \Sigma^{2p^n - 1} H\mathbb{F}_p$ induced by the composite (6). These operations are $\mathbb{F}_p$-linearly dual under the isomorphism $H^i(X; \mathbb{F}_p) \cong \hom_{\mathbb{F}_p}(H_i(X; \mathbb{F}_p), \mathbb{F}_p)$, which holds for all spectra X; see Proposition III.13.5 of [1] or Theorem IV.4.5 of [15].

Lemma A.3. *Let $k, E(Q), |Q|$ be as in Definition A.1. Let M be a graded $E(Q)$-module. Then we have an isomorphism of graded $E(Q)$-modules*

$$\underline{\hom}_{E(Q)}(M, E(Q)) \cong \underline{\hom}_k(M, \Sigma^{|Q|} k)$$

natural in the choice of M.

Proof. Let $\mathrm{pr}\colon E(Q) \to \Sigma^{|Q|} k$ denote the morphism of k-vector spaces given by $\mathrm{pr}(a + bQ) = b$ for all $a, b \in k$, and let $g(M)$ denote the morphism of graded k-vector spaces given by

$$\begin{aligned}
g\colon \underline{\hom}_{E(Q)}(M, E(Q)) &\to \underline{\hom}_k(M, \Sigma^{|Q|} k) \\
(g(f))(a + bQ) &= \mathrm{pr}(f(a + bQ)) \\
&= a\beta + b\alpha, \quad \text{where } f(1) = \alpha + \beta Q.
\end{aligned}$$

We claim that $g(M)$ is actually a morphism of $E(Q)$-modules, where the $E(Q)$-action on $\underline{\hom}_k(M, \Sigma^{|Q|} k)$ is by precomposition, i.e., $(Qf)(m) = f(Qm)$ for $f \in \underline{\hom}_k(M, \Sigma^{|Q|} k)$. Clearly $g(M)$ is k-linear, so we only need to show that $g(M)$ commutes with multiplication by Q, which is easily verified:

$$\begin{aligned}
(g(M)(Qf))(a + bQ) &= a\alpha \\
&= (g(M)(f))(Q(a + bQ)),
\end{aligned}$$

where $f(1) = \alpha + \beta Q$. The function $g(M)$ is natural in M, i.e., g is a natural transformation $g\colon \underline{\hom}_{E(Q)}(-, E(Q)) \to \underline{\hom}_k(-, \Sigma^{|Q|} k)$ of the functors

$$(33) \quad \underline{\hom}_{E(Q)}(-, E(Q)), \underline{\hom}_k(-, \Sigma^{|Q|} k)\colon \ \mathrm{gr}\,\mathrm{Mod}(E(Q)) \to \mathrm{gr}\,\mathrm{Mod}(E(Q))^{\mathrm{op}}.$$

The domain and codomain of g are each coproduct-preserving functors (because of the op in (33)), so if we can show that the two claims

(1) every object of $\mathrm{gr}\,\mathrm{Mod}(E(Q))$ is a coproduct of indecomposable objects, and

(2) g is an isomorphism when evaluated on each indecomposable object in $\mathrm{gr}\,\mathrm{Mod}(E(Q))$,

are true, then we will know that g is a natural isomorphism.

The first claim is true, since every $E(Q)$-module decomposes as a direct sum of cyclic $E(Q)$-modules, e.g. by the Cohen-Kaplansky theorem, [13]. In particular, every object of $\mathrm{gr}\,\mathrm{Mod}(E(Q))$ is isomorphic to a coproduct of suspensions of $E(Q)$ and suspensions of k.

We have that $(g(\Sigma^n E(Q)))(f)(a + bQ) = 0$ if and only if $\alpha b = -a\beta$, where $f(1) = \alpha + \beta Q$, so $f \in \ker g(\Sigma^n E(Q))$ if and only if $\alpha = 0 = \beta$, i.e., if and only if $f = 0$. So $g(\Sigma^n E(Q))$ is injective. Since $g(\Sigma^n E(Q))$ is an injective homomorphism between finite-dimensional k-vector spaces, it is bijective. The same argument applies with $\Sigma^n k$ in place of $\Sigma^n E(Q)$, so the second claim is also true. So g is a natural isomorphism. $\qquad\square$

Proposition A.4. *Let $k, E(Q), |Q|$ be as in Definition A.1. Then we have isomorphisms of graded $E(Q)$-modules:*

$$(34) \qquad H(M^*; Q) \cong H(M; Q)^*,$$

$$(35) \qquad \operatorname{Ext}^{s,-*}_{E(Q)}(k, M) \cong \Sigma^{-s|Q|} H(M; Q) \quad \text{if } s > 0,$$

$$(36) \qquad \operatorname{Ext}^{s,-*}_{E(Q)}(M, k) \cong \Sigma^{-(s+1)|Q|} H\left(\underline{\hom}_{E(Q)}(M, E(Q)); Q\right) \quad \text{if } s > 0,$$

$$(37) \qquad \cong \Sigma^{-s|Q|} H(M; Q)^*.$$

natural in the choice of graded $E(Q)$-module M. (The notation M^* is for the graded k-linear dual of M, i.e., $M^* = \underline{\hom}_k(M, k)$.)

Proof. We handle each of the isomorphisms (34) through (37) in turn:

The Q-Margolis homology of M is the homology of the chain complex

$$(38) \qquad \cdots \xrightarrow{Q} \Sigma^{|Q|} M \xrightarrow{Q} M \xrightarrow{Q} \Sigma^{-|Q|} M \xrightarrow{Q} \cdots$$

and so, since the k-linear dual of the multiplication-by-Q map on a $E(Q)$-module is the multiplication-by-Q map on the k-linear dual of that module, the cohomology of the k-linear dual of the chain complex (38) is $H(M^*; Q)$. So the classical universal coefficient sequence for chain complexes (e.g. as in 3.6.5 of [33]) yields the isomorphism (34).

Applying $\underline{\hom}_{E(Q)}(-, M)$ to the projective graded $E(Q)$-module resolution of k

$$(39) \qquad 0 \leftarrow E(Q) \xleftarrow{Q} \Sigma^{|Q|} E(Q) \xleftarrow{Q} \Sigma^{2|Q|} E(Q) \xleftarrow{Q} \cdots$$

yields the cochain complex

$$0 \rightarrow M \xrightarrow{Q} \Sigma^{-|Q|} M \xrightarrow{Q} \Sigma^{-2|Q|} M \xrightarrow{Q} \cdots$$

whose homology is $\Sigma^{-s|Q|} H(M; Q)$ in each cohomological degree $s > 0$. This gives us isomorphism (35). (See Convention A.2 for the sign change in grading degrees.)

We take advantage of the fact that $E(Q)$ is self-injective, so that

$$(40) \qquad 0 \rightarrow \Sigma^{-|Q|} E(Q) \xrightarrow{Q} \Sigma^{-2|Q|} E(Q) \xrightarrow{Q} \Sigma^{-3|Q|} E(Q) \xrightarrow{Q} \cdots$$

is an injective graded $E(Q)$-module resolution of k. [9] Applying $\underline{\hom}_{E(Q)}(M, -)$ to (40) yields the cochain complex

$$(41) \qquad 0 \rightarrow \underline{\hom}_{E(Q)}(M, \Sigma^{-|Q|} E(Q)) \xrightarrow{Q} \underline{\hom}_{E(Q)}(M, \Sigma^{-2|Q|} E(Q)) \xrightarrow{Q} \cdots,$$

hence isomorphism (36).

Isomorphism (37) then follows from the chain of isomorphisms

$$\Sigma^{-(s+1)|Q|} H\left(\underline{\hom}_{E(Q)}(M, E(Q)); Q\right) \cong \Sigma^{-(s+1)|Q|} H\left(\underline{\hom}_k(M, \Sigma^{|Q|} k); Q\right)$$
$$\cong \Sigma^{-(s+1)|Q|} H\left(\Sigma^{|Q|} M^*; Q\right)$$
$$\cong \Sigma^{-s|Q|} H(M; Q)^*,$$

due to Lemma A.3. $\qquad\qquad\square$

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[9] To be absolutely clear: it is not a typo that (40) has $\Sigma^{-|Q|} E(Q)$, not $E(Q)$, in cohomological degree 0. This is because the kernel of multiplication by Q on $E(Q)$ is $\Sigma^{|Q|} k$, so we need to desuspend to get an injective resolution of k and not $\Sigma^{|Q|} k$.

A.2. **The relationships between Margolis homology and Morava K-theory.**
Recall that, for each prime number p and each positive integer n, we have the ring
spectrum $K(n)$, the *p-primary height n Morava K-theory spectrum,* whose ring of
homotopy groups is given by $\pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}]$ with $|v_n| = 2(p^n - 1)$. The
connective cover of $K(n)$, written $k(n)$ and called *p-primary connective height n
Morava K-theory,* is also a ring spectrum, and of course has its ring of coefficients
given by $\pi_*(k(n)) \cong \mathbb{F}_p[v_n]$. The ring spectra $K(n)$ and $k(n)$ each depend on a
choice of prime number p, but the prime number p is traditionally suppressed from
the notations for $K(n)$ and $k(n)$.

Proposition A.5. *The mod p cohomology of the connective p-primary height n
Morava K-theory is given, as a graded module over the Steenrod algebra A, by*

$$H^*(k(n); \mathbb{F}_p) \cong A \otimes_{E(Q_n)} \mathbb{F}_p,$$

where Q_n is the nth Milnor primitive in A. Meanwhile, $H^*(K(n); \mathbb{F}_p) \cong 0$.

Proof. See [7]. $\qquad\qquad\square$

Theorem A.6, like everything else in this appendix, is basically well-known; for
example, that X finite and $H(H^*(X; \mathbb{F}_p); Q_n)$ is trivial implies that $K(n) \wedge X$ is
contractible appears as Corollary 4.9 in [18]. We do not know any written reference,
however, which is stated or worked out in the level of generality of Theorem A.6.

Theorem A.6. *Let p be a prime number and let n be a positive integer. Let X be a
spectrum with finite-type mod p homology groups, and such that $H(H^*(X; \mathbb{F}_p); Q_n)$
vanishes.*

- *Then the homotopy colimit of the sequence*

(42) $$\cdots \xrightarrow{v_n} \left(k(n) \wedge \Sigma^{2p^n - 2} X \right)^{\wedge}_{H\mathbb{F}_p} \xrightarrow{v_n} (k(n) \wedge X)^{\wedge}_{H\mathbb{F}_p} \xrightarrow{v_n} \cdots$$

  is contractible.
- *If X is furthermore assumed to be bounded below, then $K(n) \wedge X$ is also
  contractible, and the mapping spectrum $F(X, K(n))$ is also contractible.*

Proof. Consider the spectral sequence

(43) $$E_2^{s,t} \cong \operatorname{Ext}^{s,t}_{\operatorname{gr Comod}(A_*)} (\mathbb{F}_p, H_*(k(n) \wedge X; \mathbb{F}_p)),$$

$$\Rightarrow \pi_{t-s} \left((k(n) \wedge X)^{\wedge}_{H\mathbb{F}_p} \right)$$

$$d_r \colon E_r^{s,t} \to E_r^{s+r, t+r-1}.$$

We have the change-of-rings isomorphism

$$\operatorname{Ext}^{s,t}_{\operatorname{gr Comod}(A_*)} (\mathbb{F}_p, H_*(k(n) \wedge X; \mathbb{F}_p))$$

$$\cong \operatorname{Ext}^{s,t}_{\operatorname{gr Comod}(A_*)} \left(\mathbb{F}_p, E(Q_n)^* \square_{A_*} \mathbb{F}_p \otimes_{\mathbb{F}_p} H_*(X; \mathbb{F}_p) \right)$$

$$\cong \operatorname{Ext}^{s,t}_{\operatorname{gr Comod}(E(Q_n)^*)} (\mathbb{F}_p, H_*(X; \mathbb{F}_p))$$

$$\cong \operatorname{Ext}^{s,t}_{\operatorname{gr Mod}(E(Q_n))} (H^*(X; \mathbb{F}_p), \mathbb{F}_p)$$

so isomorphism (37) in Proposition A.4 gives us that, if $H(H^*(X; \mathbb{F}_p); Q)$ vanishes,
then the E_2-page of spectral sequence (43) collapses on to the $s = 0$-line. The same
argument as above then gives us that the colimit of the sequence

(44) $$\cdots \xrightarrow{v_n} \pi_* \left((k(n) \wedge \Sigma^{2p^n - 2} X)^{\wedge}_{H\mathbb{F}_p} \right) \xrightarrow{v_n} \pi_* \left((k(n) \wedge X)^{\wedge}_{H\mathbb{F}_p} \right) \xrightarrow{v_n} \cdots$$

is zero. Compactness of the sphere spectrum gives us that (44) computes the homotopy groups of the homotopy colimit of (42).

If X is assumed to be bounded below, then the nilpotent completion map $k(n) \wedge X \to (k(n) \wedge X)^{\wedge}_{H\mathbb{F}_p}$ is a weak equivalence, since $k(n)$ is a ring spectrum and so $\pi_*(k(n) \wedge X)$ is a $\pi_0(k(n))$-module, i.e., an $\mathbb{F}_p$-module, hence $\pi_*(k(n) \wedge X)$ is p-adically complete, hence $k(n) \wedge X$ is $H\mathbb{F}_p$-nilpotently complete by Theorem 6.6 of [10]. So the homotopy colimit of (44) is weakly equivalent to the telescope of v_n on $k(n) \wedge X$, i.e., $K(n) \wedge X$, since smashing with X commutes with homotopy colimits. Finally, since $K(n)$ is a field spectrum, we have $K(n)^*(X) \cong \underline{\hom}_{K(n)_*}(K(n)_*(X), K(n)_*)$ (a nice general way to prove this duality isomorphism is by using the universal coefficient theorem IV.4.5 of [15], but the result was certainly known earlier). $\square$

The boundedness hypothesis in the statement of Theorem A.6 cannot be done away with: if $X = K(n)$, for example, then neither $K(n) \wedge X$ nor $F(K(n), K(n))$ are contractible. The sequence (42), of course, still has contractible homotopy colimit in that case since each term in that sequence is contractible.

Proposition A.7 is a simple cohomological duality. For clarity, we drop the gradings:

Proposition A.7. *Let $k, E(Q), |Q|$ be as in Definition A.1. For each $E(Q)$-module M, we have an isomorphism of $E(Q)$-modules*

$$\text{(45)} \qquad \text{Ext}^s_{E(Q)}(M, k) \cong \text{Ext}^{s,*}_{E(Q)}(k, M^*)$$

for all integers s. If $s > 0$, then each side of (45) is furthermore isomorphic to $\text{Ext}^{s,*}_{E(Q)}(k, M)^*$.

Proof. The $s = 0$ case of isomorphism (45) is simply the claim that the tensor product over k is a *closed* monoidal product on the category of $E(Q)$-modules. Consequently, for the rest of this proof we assume $s > 0$, and consequently the hypotheses of Proposition A.4 are fulfilled. Stringing together isomorphisms from Proposition A.4:

$$\text{(46)} \qquad \text{Ext}^s_{E(Q)}(M, k) \cong H(M; Q)^*$$
$$\cong H(M^*; Q)$$
$$\cong \text{Ext}^s_{E(Q)}(k, M^*).$$

The right-hand side of (46) is also isomorphic to $\text{Ext}^s_{E(Q)}(k, M)^*$, by isomorphism (35). $\square$

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