# THE SPECTRUM OF EXCISIVE FUNCTORS 

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#### Abstract

We prove a thick subcategory theorem for the category of $d$-excisive functors from finite spectra to spectra. This generalizes the Hopkins-Smith thick subcategory theorem (the $d=1$ case) and the $C_{2}$-equivariant thick subcategory theorem (the $d=2$ case). We obtain our classification theorem by completely computing the Balmer spectrum of compact $d$-excisive functors. A key ingredient is a non-abelian blueshift theorem for the generalized Tate construction associated to the family of non-transitive subgroups of products of symmetric groups. Also important are the techniques of tensor triangular geometry and striking analogies between functor calculus and equivariant homotopy theory. In particular, we introduce a functor calculus analogue of the Burnside ring and describe its Zariski spectrum à la Dress. The analogy with equivariant homotopy theory is strengthened further through two applications: We explain the effect of changing coefficients from spectra to HZZ-modules and we establish a functor calculus analogue of transchromatic Smith-Floyd theory as developed by Kuhn-Lloyd. Our work offers a new perspective on functor calculus which builds upon the previous approaches of Arone-Ching and Glasman


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## 1. Introduction

Goodwillie's calculus of functors [Goo90, Goo92, Goo03] provides a powerful tool for studying functors between categories of homotopical origin. It approximates a homotopical functor $F$ via a tower of simpler functors

$$
F \rightarrow \ldots \rightarrow P_{d} F \rightarrow P_{d-1} F \rightarrow \ldots P_{2} F \rightarrow P_{1} F
$$

analogous to the Taylor tower in ordinary calculus which approximates a smooth function $f: \mathbb{R} \rightarrow \mathbb{R}$ by polynomials of increasing degree $d$. In the calculus of functors, polynomials of degree $d$ are replaced by so-called $d$-excisive functors which are homotopical functors satisfying a certain weak form of excision. The functor $P_{d} F$ is the best approximation of $F$ by a $d$-excisive functor.

A prominent domain for this theory is the study of (homotopical) functors $\mathcal{S} p \rightarrow \mathcal{S} p$ from the category of spectra to itself. For technical reasons, we restrict attention to functors that preserve filtered colimits, which can be equivalently regarded as functors $\mathcal{S} p^{c} \rightarrow \mathcal{S} p$ from finite spectra to spectra. A fundamental problem then is to understand the structure of the category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ of $d$-excisive functors $\mathcal{S} p^{c} \rightarrow \mathcal{S} p$; in other words, to understand what "polynomials" are in this context.

For example, a linear functor $S p^{c} \rightarrow \mathcal{S} p$ is essentially the same thing as a homology theory and we have an equivalence $\operatorname{Exc}_{1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq \mathcal{S} p$ but the structure of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is considerably more complicated for $d>1$. As in ordinary calculus, one can define the derivatives $\partial_{1} F, \ldots, \partial_{d} F \in \mathcal{S} p$ of a $d$-excisive functor $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ but, in contrast to ordinary polynomials, a $d$-excisive functor is far from being determined by its derivatives when $d>1$. Previous attempts to understand the category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ have focused on describing the extra structure possessed by the derivatives [AC15, AC16] or by the cross-effects [Gla18] of a $d$-excisive functor.

Here we take a different approach to analyzing $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ by regarding it as a tensor triangulated ("tt") category under Day convolution. Our main result is a computation of the Balmer spectrum of the subcategory of compact objects in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ for any $d \geq 1$. This leads to a complete classification of compact $d$-excisive functors up to tt-equivalence. Two compact functors $F$ and $G$ are tt-equivalent if they can be built from each other using triangles, retracts and Day convolution with arbitrary compact functors; that is, $F$ and $G$ are tt-equivalent if they generate the same thick tensor-ideal.

We proceed to describe our classification theorem. For simplicity, let us restrict attention to $p$-local spectra $\mathcal{S} p_{(p)}$ for a given prime $p$. The type $(x) \in \mathbb{N}_{\infty}=$ $\{0,1,2, \ldots\} \cup\{\infty\}$ of a $p$-local finite spectrum $x$ is the minimal $h$ with $K(p, h)_{*}(x) \neq 0$ if $x$ is nonzero and $\infty$ otherwise; here $K(p, h)$ is height $h$ Morava $K$-theory at $p$. This invariant extends to functors $F: \mathcal{S} p \rightarrow \mathcal{S} p_{(p)}$ by setting

$$
\operatorname{type}(F): \mathbb{N} \rightarrow \mathbb{N}_{\infty}, \quad l \mapsto \operatorname{type}\left(\partial_{l} F\right)
$$

If $F$ is $d$-excisive, then we may restrict the domain to $\{1, \ldots, d\}$. We now can state the $p$-local version of our main result (Theorem 12.5):

Main Theorem. The assignment $F \mapsto$ type $(F)$ induces a bijection

$$
\left\{\begin{array}{c}
\text { tt-equivalence classes }\langle F\rangle: \\
F \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p_{(p)}\right) \text { compact }
\end{array}\right\} \xrightarrow{\sim}\left\{\begin{array}{c}
\text { functions } f:\{1, \ldots, d\} \rightarrow \mathbb{N}_{\infty} \text { such that } \\
f(k) \leq \delta_{p}(k, l)+f(l) \text { if } p-1 \mid k-l \geq 0
\end{array}\right\}
$$

where $\delta_{p}(k, l)$ is an explicit function depending on $l$ and the base $p$ expansion of $k$.

To put this result into perspective, let us consider two special cases. If $d=1$, then $\operatorname{Exc}_{1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq \mathcal{S} p$ and the above bijection is the content of the thick subcategory theorem of Devinatz, Hopkins and Smith [DHS88, HS98] which we however take as an input to our theorem. When $d=2$, there is an equivalence $\operatorname{Exc}_{2}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq \mathcal{S} p_{C_{2}}$ with the category of genuine $C_{2}$-equivariant spectra. In this case, our theorem recovers the classification of compact $C_{2}$-spectra due to Balmer and Sanders [BS17]. Generalizing from cyclic groups of order $p$ to arbitrary finite abelian groups $G$, the analogous classification problem for compact $G$-spectra has subsequently been settled by Barthel, Hausmann, Naumann, Nikolaus, Noel and Stapleton in [BHN+ 19], but the general case of a non-abelian finite group $G$ remains elusive.

The theory we develop here in our study of $d$-excisive functors will demonstrate that the categories $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ behave a great deal like categories of $\Sigma_{d}$-equivariant spectra indexed on the family of non-transitive subgroups of $\Sigma_{d}$. In light of the ambiguity left in the tt-classification of compact $G$-spectra for general $G$, we view the complete classification furnished by our main theorem as somewhat surprising. It requires precise control over the chromatic behaviour of generalized Tate constructions with respect to families of non-transitive subgroups of products of symmetric groups, i.e., non-abelian analogs of the blueshift theorems of [Kuh04] and $\left[\mathrm{BHN}^{+} 19\right]$.

Further details. We will now say more about the techniques that go into proving the main theorem. As mentioned above, our starting point is the observation that $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a tensor triangulated category via (localized) Day convolution, and our first task is to show that it has excellent structural properties, making it amenable to the ample toolkit of tt-geometry. The following is a summary of results obtained in Corollaries 5.25 and 5.34 and Theorem 5.33.

Theorem A. The category of d-excisive functors $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a compactly generated presentably symmetric monoidal stable $\infty$-category whose compact and dualizable objects coincide. A set of compact generators is given by the functors

$$
P_{d} h_{\mathbb{S}}(i)(-), \quad X \mapsto P_{d}\left(\Sigma^{\infty} \operatorname{Map}_{S p}(\mathbb{S}, X) \otimes \cdots \otimes \Sigma^{\infty} \operatorname{Map}_{S_{p}}(\mathbb{S}, X)\right)
$$

for $1 \leq i \leq d$, where $P_{d} h_{\mathbb{S}}=P_{d} h_{\mathbb{S}}(1)$ is the monoidal unit. Moreover, these generators are self-dual separable commutative algebras.

We deduce most of this result from global statements about Fun $\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ by applying the (smashing) localization functor $P_{d}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. This leads in particular to a strong compatibility among the given sets of compact generators for varying $d$. Indeed, we observe that for $1 \leq i \leq d-1$ the generators $P_{d} h_{\mathbb{S}}(i)$ of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ are mapped by $P_{d-1}$ to the generators $P_{d-1} h_{\mathbb{S}}(i)$ of $\operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. One is then led to wonder if $P_{d-1}$ is the finite localization that kills $P_{d} h_{\mathbb{S}}(d)$. We verify in Theorem 6.19 that this is indeed the case:

Theorem B. The induced functor $P_{d-1}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a finite localization whose ideal of acyclics is generated by $P_{d} h_{\mathbb{S}}(d)$. In particular, the category $\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ of d-homogeneous functors is the localizing tensor-ideal generated by $P_{d} h_{\mathbb{S}}(d)$.

The combination of Theorem A and Theorem B puts us in the position to study $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ via tt-geometry and, in particular, to reason inductively over $d$. Recall that tt -geometry as initiated by Balmer [Bal05, Bal10c] views a rigidly-compactly
generated tt-category $\mathcal{T}$ as a sheaf of local tt-categories over a naturally associated spectral space $\operatorname{Spc}\left(\mathcal{T}^{c}\right)$, the Balmer spectrum of the full subcategory $\mathcal{T}^{c}$ of compact objects in $\mathcal{T}$. The spectrum (whose points are the prime ideals of $\mathcal{T}^{c}$ ) plays a dual role. On the one hand, it reveals the geometric structure of $\mathcal{T}$ and brings a geometric set of methods to its analysis. On the other hand, its topology encodes the classification of thick tensor-ideals of $\mathcal{T}^{c}$. This leads us to:

Goal. Compute the spectrum of prime ideals $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ for all $d \geq 1$.
The first step is to determine its underlying set. We show that the Goodwillie derivatives $\partial_{i}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ for $1 \leq i \leq d$ form a collection of symmetric monoidal and jointly conservative tt-functors (Lemma 6.32 and Lemma 6.34). Therefore, one can find prime ideals of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ by pulling back prime ideals of $\mathcal{S} p^{c}$ along $\partial_{i}$. The prime ideals of $\mathcal{S} p^{c}$ are well known by the tt-geometric form of the thick subcategory theorem [HS98] as recalled in Figure 2. In this way we obtain in Theorem 7.14 a description of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ as a set:

Theorem C. Every prime ideal in $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ is of the form

$$
\mathcal{P}_{d}([i], p, h)=\left\{F \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c} \mid K(p, h-1)_{*} \partial_{i}(F)=0\right\}
$$

for some triple $(i, p, h)$ consisting of an integer $1 \leq i \leq d$, a prime number $p$ or $p=0$, and a chromatic height $1 \leq h \leq \infty$. Moreover, we have $\mathcal{P}_{d}([i], p, h)=\mathcal{P}_{d}\left([j], p^{\prime}, h^{\prime}\right)$ if and only if $[i]=[j]$ and $h=h^{\prime}$ and if $h=h^{\prime}>1$, then also $p=p^{\prime}$.

It then remains to determine the topology of the space $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$, which is a significantly more difficult problem. It turns out that it is enough to determine all the inclusions $\mathcal{P}_{d}([i], p, h) \subseteq \mathcal{P}_{d}\left([j], p^{\prime}, h^{\prime}\right)$ among the primes (Proposition 11.1). One general technique to get information about these inclusions is the comparison map introduced by Balmer [Bal10b]. This is an inclusion-reversing continuous map

$$
\begin{equation*}
\rho: \operatorname{Spc}(\mathcal{K}) \rightarrow \operatorname{Spec}\left(\operatorname{End}_{\mathcal{K}}(\mathbb{1})\right) \tag{1.1}
\end{equation*}
$$

from the Balmer spectrum of any tt-category $\mathcal{K}$ to the Zariski spectrum of the endomorphism ring of the unit in $\mathcal{K}$. In order to apply this in our setting, we introduce in Section 8 a commutative ring $A(d)$ for each $d \geq 1$ that we call the Goodwillie-Burnside ring. It is a combinatorially defined ring constructed from the category of finite sets of cardinality at most $d$ and surjections. More specifically, as an abelian group, it is free on $d$ generators $x_{1}, \ldots, x_{d}$ and its multiplication is determined by the formula

$$
x_{i} x_{j}=\sum_{1 \leq l \leq d} \mu(i, j, l) x_{l},
$$

where $\mu(i, j, l)$ counts the number of good subsets (Definition 4.9) of $[i] \times[j]$ of cardinality $l$, where $[i]$ is the set with $i$ elements. Its relation to the category of $d$-excisive functors is expressed by the following result (Theorem 9.23):
Theorem D. The endomorphism ring $\pi_{0} \operatorname{End}\left(P_{d} h_{\mathbb{S}}\right)$ of the unit in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is isomorphic to the Goodwillie-Burnside ring $A(d)$ introduced above.

In other words, Theorem D supplies a functor calculus analogue of the computation of the Burnside ring in equivariant homotopy theory due to Segal [Seg71] and tom Dieck [tD79]. We are then able to give (in Theorem 8.28 and Proposition 9.25) a complete description of the Zariski spectrum of $A(d)$ and the comparison map

$$
\begin{equation*}
\rho: \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \rightarrow \operatorname{Spec}(A(d)) \tag{1.2}
\end{equation*}
$$

This severely restricts which inclusions can possible occur, and altogether, we show that to determine the topology, it suffices to compute the minimal $\beth \geq 0$, the so-called geometric blueshift, such that

$$
\begin{equation*}
\mathcal{P}_{d}([k], p, h+\beth) \subseteq \mathcal{P}_{d}([l], p, h) \tag{1.3}
\end{equation*}
$$

whenever $p-1 \mid k-l \geq 0$ and for any $1 \leq h \leq \infty$.
To facilitate this computation, we proceed inductively on the ambient degree $d$ with the base $d=1$ case reducing to the classical thick subcategory theorem. At the level of spectra, Theorem B induces an open embedding

$$
\begin{equation*}
\operatorname{Spc}\left(P_{d-1}\right): \operatorname{Spc}\left(\operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \hookrightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \tag{1.4}
\end{equation*}
$$

which in particular implies that it suffices to consider the case $k=d$ in (1.3). The values of the corresponding geometric blueshift numbers are stated in the following theorem, which forms the heart of the description of the topology of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$.

Theorem E. Let $p, q$ be prime numbers, $1 \leq k, l \leq d$ integers, and suppose $1 \leq h, h^{\prime} \leq \infty$. There is an inclusion $\mathcal{P}_{d}\left([k], p, h^{\prime}\right) \subseteq \mathcal{P}_{d}([l], q, h)$ if and only if the following three conditions hold:
(a) $p-1 \mid k-l \geq 0$;
where $\delta_{p}(k, l):= \begin{cases}0 & \text { if } k=l ; \\ 1 & \text { if } p-1 \mid k-l>0 \text { and } l \geq s_{p}(k) ; \\ 2 & \text { if } p-1 \mid k-l>0 \text { and } l<s_{p}(k) ; \\ \infty & \text { otherwise. }\end{cases}$
Here $s_{p}(k)$ denotes the sum of the coefficients of the base $p$ expansion of $k$.
In particular, this result shows that, somewhat surprisingly, the geometric blueshift $\beth$ is never greater than 2. Its proof occupies the bulk of Part III and is summarized in the following subsection. It completes our computation of the Balmer spectrum of compact $d$-excisive functor for all $d \geq 1$. The $d=3$ case is depicted in Figure 1 below.


Figure 1. The Balmer spectrum of 3-excisive functors, along with the comparison map to the Zariski spectrum of $A(3)$.

Outline of the proof of Theorem E. The tt-geometric classification of compact $d$-excisive functors stated at the beginning of the introduction is equivalent to Theorem E; the translation between the two formulations follows from general tt-geometric principles. In particular, the tt-classification theorem is extracted from Theorem E in Theorem 12.5; see also Corollary 12.10.

The proof of Theorem E takes a number of steps, whose overall strategy is modelled on the work of Balmer and Sanders [BS17] in equivariant homotopy theory. First of all, the condition that $p-1 \mid k-l$ already appears in the Zariski spectrum of the Goodwillie-Burnside ring while the requirement that $k-l \geq 0$ follows from a straightforward computation; thus it remains to determine the values of $\delta_{p}(k, l)$, as explained above. This geometric blueshift function measures the chromatic shift that occurs in inclusions among prime ideals, and it is essentially controlled by two interacting pieces of structure, to be explained in more detail below:

- the chromatic blueshift behaviour of the Tate-derivatives; and
- the combinatorics of $p$-power partitions.

In order to explain our strategy, recall from (1.4) that $P_{d-1}$ induces an open embedding on spectra. Reinterpreted tt-geometrically, this results in a categorical open-closed decomposition of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$, which on objects decategorifies to the Kuhn-McCarthy pullback square [McC01, Kuh04] for $d$-excisive functors $F$ :


Here, the Tate construction appearing in the lower right corner governs the gluing data between $(d-1)$-excisive and $d$-homogeneous functors needed to reconstruct a $d$-excisive functor. The idea is then to argue inductively on $d$, where the induction step requires us to glue $\operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ and $\operatorname{Spc}\left(\operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ via the Tate construction.

In order to understand the gluing, we first reduce to the $p$-local situation for a given $p$ and then consider the ( $p$-local) Tate-derivatives defined as

$$
\partial_{l} t_{d} i_{d}: \mathcal{S} p_{(p)} \rightarrow \mathcal{S} p_{(p)}, \quad A \mapsto \partial_{l}\left(X \mapsto\left(A \otimes X^{\otimes d}\right)^{t \Sigma_{d}}\right)
$$

for any $l<d$. These functors exhibit a chromatic height-shifting behaviour-a phenomenon generally known as blueshift-which determines what we call elementary inclusions among prime ideals.

Theorem F. The p-local Tate-derivatives $\partial_{l} t_{d} i_{d}$ are contractible unless $d>l$ and there exists a partition of $d$ by powers of $p$ of length $l$. If such a partition exists, then the p-local Tate-derivatives exhibit a blueshift by 1, i.e., the kernel of the functor

$$
\partial_{l}\left(\left(L_{p, h}^{f} \mathbb{S} \otimes(-)^{\otimes d}\right)^{t \Sigma_{d}}\right): \mathcal{S} p_{(p)}^{c} \longrightarrow \mathcal{S} p
$$

consists precisely of the type $h$ finite p-local spectra.
The proof of Theorem F proceeds by using work of Arone-Ching [AC15] to translate the problem to stable equivariant homotopy theory. More precisely, we express the Tate-derivatives in terms of generalized Tate constructions with respect
to families $\mathcal{F}_{\text {nt }}$ of non-transitive subgroups of symmetric groups:

$$
\partial_{l}\left(\left(L_{p, h}^{f} \mathbb{S} \otimes(-)^{\otimes d}\right)^{t \Sigma_{d}}\right) \simeq\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\mathrm{nt}}(\lambda)}:=\Phi^{\mathcal{F}_{\mathrm{nt}}(\lambda)}\left(\underline{\operatorname{infl}_{e}^{\Sigma_{\lambda}}} L_{p, h}^{f} \mathbb{S}\right) .
$$

We are then able to determine the corresponding blueshift behaviour using the main theorem of previous work $\left[\mathrm{BHN}^{+} 19\right]$ on the topology of Balmer spectrum of finite abelian groups, see Theorem 10.33, which in turn extended Kuhn's blueshift theorem [Kuh04]. Since it involves generalized Tate constructions based on symmetric groups, we may view Theorem F as an instance of non-abelian blueshift. We note that, in general, determining the blueshift behaviour for arbitrary families of subgroups of symmetric groups would suffice to compute the Balmer spectrum of compact $G$-spectra for all finite groups $G$, a problem that remains open for every $\Sigma_{d}$ with $d \geq 4$.

Returning to the problem of computing the blueshift function $\delta_{p}(k, l)$, Theorem F accounts only for part of the possible relations; indeed, in general there can (and will!) be chains of inclusions

$$
\mathcal{P}_{d}([k], p, h)=\mathcal{P}_{d}\left(\left[k_{1}\right], p, h_{1}\right) \subseteq \mathcal{P}_{d}\left(\left[k_{2}\right], p, h_{2}\right) \subseteq \ldots \subseteq \mathcal{P}_{d}\left(\left[k_{j}\right], p, h_{j}\right)=\mathcal{P}_{d}\left([l], p, h^{\prime}\right)
$$

in which only the adjacent inclusions are captured by the Tate-derivatives. One outcome of our analysis is that all inclusions among prime ideals are formed by such chains, so it remains to find the minimal shift among all such chains, which is precisely the value of $\delta_{p}(k, l)$. This is then translated into a combinatorial problem on $p$-power partitions, which we are able to solve using some elementary number theory (Proposition 11.38).

Parerga and paralipomena. We conclude our summary of the paper by mentioning some applications and further directions.

Change of categories. In this paper we focus on functors from the category of spectra to itself. But the methods of functor calculus apply much more broadly. Indeed, one can apply functor calculus to study the $\infty$-category of functors Fun $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ and $\mathcal{D}$ are general $\infty$-categories, perhaps satisfying some mild technical assumptions. It is natural to wonder how our methods work for other categories. In order to use tt-geometric techniques, we should require our categories to be symmetric monoidal and stable; in addition, the theory is particularly effective and well-developed in the rigidly-compactly generated case.

It is relatively straightforward to extend our methods from $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ to the study of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right)$, where $\mathcal{D}$ is a rigidly-compactly generated symmetric monoidal stable $\infty$-category. Indeed, we show in Proposition 12.21 that, as a set, the primes are always obtained by pulling back primes along the derivatives as in the case $\mathcal{D}=\mathcal{S} p$. In contrast, it is likely that new ideas are needed to replace $\mathcal{S} p^{c}$ in the source with a more general category. In particular Lemma 7.5 and Lemma 7.7 seem to depend on the source being $\mathcal{S} p^{c}$.

As a proof of concept, when the target is $\operatorname{Mod}_{H Z}$, we completely determine the topology of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)^{c}\right)$ in Theorem 12.27, a computation analogous to the work of Patchkoria-Sanders-Wimmer in equivariant homotopy theory [PSW22]. In this case, all primes are of the form $\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p})$ for $1 \leq k \leq d$ and $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z})$ and the topology is given by:

Theorem G. Let $d \geq k, l \geq 1$ be integers and consider two prime ideals $\mathfrak{p}, \mathfrak{q} \in$ $\operatorname{Spec}(\mathbb{Z})$. Then there is an inclusion $\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p}) \subseteq \mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{q})$ if and only if one of the following two conditions is satisfied:
(a) $\mathfrak{p}=(p)$ for some prime $p, \mathfrak{q}=(p)$ or $\mathfrak{q}=(0)$, and $p-1 \mid k-l \geq 0$;
(b) $\mathfrak{p}=(0)=\mathfrak{q}$ and $k=l$.

Moreover, $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S p}^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)^{c}\right)$ is noetherian, so that the topology is determined by these inclusions. Base-change $\operatorname{S} p \rightarrow \operatorname{Mod}_{\mathrm{HZ}}$ induces a map

$$
\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)^{c}\right) \rightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)
$$

which is a homeomorphism onto its image. It maps $\mathcal{P}_{d}^{\mathbb{Z}}([k],(p))$ to $\mathcal{P}_{d}([k], p, \infty)$ and maps $\mathcal{P}_{d}^{\mathbb{Z}}([k],(0))$ to $\mathcal{P}_{d}([k], 0,1)$. Finally, $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{H} Z}\right)$ is stratified and costratified over its spectrum in the sense of [BHS23b] and satisfies the telescope conjecture.

We refer the reader to Figure 5 for an illustration of the case $d=3$. More generally, it is possible to deduce an abstract description of the resulting spectrum of the category $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right)^{c}\right)$ for any rigidly-compactly generated symmetric monoidal stable $\infty$-category $\mathcal{D}$ from Theorem E, but we will not pursue this here.

Another important example of change of coefficients is that of functors from spectra to telescopically localized spectra, studied in depth by Kuhn in [Kuh04]. Using the aforementioned telescopic blueshift theorem proven in the same paper, Kuhn deduced from (1.5) that the Taylor tower of any $F \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ splits completely after telescopic localization. From this perspective, our main result is a far-reaching delocalization of Kuhn's splitting theorem.

Smith-Floyd theory in functor calculus. The geometric blueshift occurring between inclusions of prime ideals in (1.3), as determined in Theorem E, is equivalent to the implication:

$$
\begin{equation*}
K(p, h-1+\beth)_{*} \partial_{k} F=0 \Longrightarrow K(p, h-1)_{*} \partial_{l} F=0 \tag{1.6}
\end{equation*}
$$

whenever $F \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$. Instead of a conditional vanishing result, it is desirable to have quantitative control over the transchromatic relation between different derivatives.

In equivariant homotopy theory, the analogous problem is to find relations between the dimensions of Morava $K$-theories of geometric fixed points $\Phi^{H}(x), \Phi^{K}(x)$ of a given finite $G$-spectrum $x$, for various subgroups $H, K$ of $G$. On the one hand, since the geometric $H$-fixed points of a $G$-suspension spectrum of a $G$-space $X$ coincide with the $G$-suspension spectrum of the $H$-fixed points of $X$, at height $\infty$ this is the topic of classical Smith [Smi41] and Floyd [Flo52] theory for mod $p$ homology. On the other hand, non-equivariantly, Ravenel [Rav84] showed that the dimension of $K(h)_{*}(x)$ provides an upper bound for the dimension of $K(n)_{*}(x)$ for all $n \leq h$ and all finite spectra $x$; this is a transchromatic version of Floyd theory for the trivial action. Simultaneously generalizing these theorems, Hausmann and Kuhn-Lloyd [KL20] reinterpreted the results of [BS17, BHN ${ }^{+}$19] as transchromatic Smith-Floyd theory.

In light of the strong analogy between excisive functors $\mathcal{S} p^{c} \rightarrow \mathcal{S} p$ and stable equivariant homotopy theory, we can view the sought-after quantitative version of (1.6) as a functor calculus analogue of transchromatic Smith-Floyd theory. As one application of our main theorem, we show that this story indeed carries over to the calculus context, see Corollary 12.19:

Theorem H. For any compact $F \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$, the following inequality holds:

$$
\begin{equation*}
\operatorname{dim}_{K(p, n)_{*}} K(p, n)_{*}\left(\partial_{k} F\right) \geq \operatorname{dim}_{K(p, h)_{*}} K(p, h)_{*}\left(\partial_{l} F\right) \tag{1.7}
\end{equation*}
$$

whenever $p-1 \mid k-l \geq 0$ and $n \geq h+\delta_{p}(k, l)$.
We refer to Proposition 12.16 for the precise equivalence between transchromatic Smith theory (1.6) and transchromatic Floyd theory (1.7) in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$.

Spectral Mackey functors. As witnessed repeatedly above, the abstract tt-geometry of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ closely resembles that of $\mathcal{S} p_{G}$ for $G$ a finite group. The underlying reason for why the two stories are parallel is given in the setting of spectral Mackey functors on epiorbital categories developed by Barwick and Glasman [Bar17, Gla15]. Specifically, $G$-spectra can be modelled as spectral Mackey functors on the category of finite $G$-sets [Bar17], while $d$-excisive functors can be modelled as spectral Mackey functors on the free coproduct completion of the category of finite sets of cardinality at most $d$ and surjections [Gla18]. In fact, when $d=2$, this induces the aforementioned equivalence between 2-excisive functors and $C_{2}$-spectra, as the indexing categories are equivalent; however, the stories bifurcate as soon as $d>2$. For the benefit of the reader conversant in equivariant homotopy theory, we will often indicate when a construction in Goodwillie calculus is the analog of a well-known construction in equivariant homotopy theory; for example, the Goodwillie derivatives correspond to geometric fixed points. We have also tabulated the most prominent instances of this correspondence in Appendix A.

Spaces vs spectra. One aspect of this work that we find intriguing is that it is particularly adapted to the study of functors from spectra to spectra - and, more generally, to functors between rigidly-compactly generated symmetric monoidal stable $\infty$-categories - but it does not readily apply to functors from spaces to spectra, because the category of such functors is not rigidly-compactly generated. This is in contrast to previous approaches to classification of excisive functors, where functors from spaces to spectra tended to be easier to understand than functors from spectra to spectra.

Frequently used notations. Here is some notation that will be used throughout:

- For any $\infty$-category $\mathcal{C}, \mathfrak{C}^{c}$ denotes the category of compact objects in $\mathcal{C}$.
- Coproducts and smash product of spectra are denoted by $\oplus$ and $\otimes$ respectively. Smash product of pointed spaces is denoted by $\wedge$. Day convolution is denoted by $\circledast$. Internal homs are denoted hom $(-,-)$.
- In a tensor triangulated category $\mathcal{T}$, the localizing ideal (resp. localizing subcategory) generated by a subcategory $\mathcal{E} \subset \mathcal{T}$ is denoted Locid $\langle\mathcal{E}\rangle$ (resp. $\operatorname{Loc}\langle\mathcal{E}\rangle$ ) while the thick ideal generated by $\mathcal{E}$ is denoted thickid $\langle\mathcal{E}\rangle$.
- If $\mathcal{C}$ and $\mathcal{D}$ are presentably symmetric monoidal stable $\infty$-categories, then a geometric functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric monoidal functor which preserves all colimits. A geometric equivalence is a geometric functor which is an equivalence.
- For an integer $n \geq 0$, the standard set with $n$ elements will be denoted by $[n]:=\{1, \ldots, n\}$
- $\operatorname{surj}(m, n)$ will denote the set of surjective functions from $[m]$ to $[n]$. Accordingly, $|\operatorname{surj}(m, n)|$ denotes the number of surjections from $[m]$ to $[n]$.
- We write $\mathbb{N}_{\infty}=\{0,1,2, \ldots\} \cup\{\infty\}$.

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## Part I. Categorical foundations

## 2. The tensor triangulated category of $d$-excisive functors

We begin with a treatment of Goodwillie calculus for functors from spectra to spectra and introduce the tensor triangulated category of $d$-excisive functors. Most, if not all, of the definitions are due to Goodwillie [Goo03]. For an $\infty$-categorical treatment of Goodwillie's work we refer the reader to [Lur17, Chapter 6]. A survey of Goodwillie calculus can be found in [AC20].

## $d$-cubes and $d$-excisive functors.

Definition 2.1. Let $\mathcal{C}$ be an $\infty$-category with finite limits and finite colimits. The category of $d$-cubes in $\mathcal{C}$ is the functor category $\operatorname{Fun}(\mathcal{P}([d]), \mathcal{C})$ where $\mathcal{P}([d])$ denotes the poset of subsets of a finite set of cardinality $d$. A $d$-cube $X$ is said to be
(i) cartesian, if the canonical map

$$
X(\varnothing) \rightarrow \lim _{\varnothing \neq S \subseteq[d]} x(S)
$$

is an equivalence, and
(ii) cocartesian, if the canonical map

$$
\operatorname{colim}_{S \subsetneq[d]} X(S) \rightarrow X([d])
$$

is an equivalence.
Finally, a d-cube $X$ is strongly cocartesian if it is left Kan extended from subsets of cardinality at most 1 (equivalently, any 2 -face in the cube is a pushout).

Example 2.2. When $d=2$ the conditions above reduce to the usual notions of pushout and pullback squares. A 3-cube is of the form


It is strongly cocartesian if every face is a pushout or, equivalently, if it is left Kan extended from the diagram


Remark 2.3. When $\mathcal{C}$ is a stable $\infty$-category, a cubical diagram in $\mathcal{C}$ is cocartesian if and only if it is cartesian. For $d=2$ this is essentially the definition of a stable
$\infty$-category, see [Lur17, Proposition 1.1.3.4], and for $d>2$ it is proved by induction on $d$, see [Lur17, Proposition 1.2.4.13].

Definition 2.4. A functor $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ is reduced if it preserves the zero object. We let $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ denote the stable $\infty$-category of reduced functors from $\mathcal{S} p^{c}$ to $\mathcal{S} p$.

Remark 2.5. The category $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a full subcategory of the category of all functors from $\mathcal{S} p^{c}$ to $\mathcal{S} p$. In fact, $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is strongly reflexive, i.e., it is presentable, stable under equivalence in the category of all functors, and the inclusion admits a left adjoint; see [Lur17, Remark 1.4.2.4]. Furthermore, for any functor $F$ there is a natural equivalence $F \simeq \bar{F} \times F(0)$ where $\bar{F}(X):=\mathrm{fib}(F(X) \rightarrow F(0))$ is the reduced part of $F$. It follows that there is no real loss of information in focusing on reduced functors.

Definition 2.6. Let $d \geq 1$ be an integer and let $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ be a functor from finite spectra to spectra. We say that $F$ is $d$-excisive if it takes strongly cocartesian $(d+1)$-cubes in $\mathcal{S} p^{c}$ to cartesian (or, equivalently, cocartesian by Remark 2.3) $(d+1)$-cubes in $\mathcal{S} p$. We let $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ denote the full stable subcategory of reduced $d$-excisive functors from $\mathcal{S} p^{c}$ to $\mathcal{S} p$.

Example 2.7. A functor $F: S p^{c} \rightarrow \mathcal{S} p$ is 1-excisive precisely when it carries pushout squares in $\mathcal{S} p^{c}$ to pushout squares in $\mathcal{S} p$. Reduced 1-excisive functors are called linear. Evaluation at the sphere spectrum gives an equivalence $\operatorname{Exc}_{1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{\sim} \mathcal{S} p$ between the category of linear functors and the category of spectra, whose inverse sends a spectrum $A$ to the functor $X \mapsto A \otimes X$.

Remark 2.8. By taking Ind-completions, there is an equivalence $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq$ $\operatorname{Exc}_{d}^{c}(\mathcal{S} p, \mathcal{S} p)$ between $d$-excisive functors $\mathcal{S} p^{c} \rightarrow \mathcal{S} p$ and $d$-excisive functors $\mathcal{S} p \rightarrow \mathcal{S} p$ that preserve filtered colimits; see [Lur17, Proposition 6.1.5.4].

Proposition 2.9. There are inclusions

$$
\operatorname{Exc}_{1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \cdots \subseteq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \operatorname{Exc}_{d+1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \cdots \subseteq \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)
$$

Proof. Any $d$-excisive functor is $m$-excisive for each $m \geq d$ by [Goo03, Corollary 1.11] or [Lur17, Corollary 6.1.1.14].

Theorem 2.10. The inclusion $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \hookrightarrow \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ admits an exact left adjoint $P_{d}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Proof. This is one of the main theorems of Goodwillie [Goo03, Section 1] and is established for $\infty$-categories in [Lur17, Theorem 6.1.1.10].

Remark 2.11. By slight abuse of notation we will also write $P_{d}$ for the composition of $P_{d}$ with the inclusion functor:

$$
\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{P_{d}} \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \hookrightarrow \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) .
$$

This agrees with the notation used in [Goo03]. Note that since $\mathcal{S} p$ is stable, the category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is closed under all colimits; see [Lur17, Remark 6.1.5.10]. This means that the inclusion functor $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \hookrightarrow \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ commutes with colimits. It follows that $P_{d}$ commutes with colimits both when regarded as a functor $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ and as an endofunctor on $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Definition 2.12. For all $k \geq 0$, we have $P_{d} P_{d+k} \simeq P_{d}$ as functors $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow$ $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. Indeed, just observe that they are both left adjoint to the inclusion. We therefore obtain natural transformations $P_{d} \rightarrow P_{d-1}$. The Taylor tower (or Goodwillie tower) of $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ is the following sequence of natural transformations of functors $S p^{c} \rightarrow \mathcal{S} p$ :

$$
F \rightarrow \cdots \rightarrow P_{d+1} F \rightarrow P_{d} F \rightarrow P_{d-1} F \rightarrow \cdots \rightarrow P_{1} F \rightarrow P_{0} F \simeq 0
$$

Definition 2.13. The $d$-th layer in the Taylor tower is $D_{d} F:=\operatorname{fib}\left(P_{d} F \rightarrow P_{d-1} F\right)$. The functor $D_{d} F$ is both $d$-excisive $\left(P_{d} F \simeq F\right)$ and $d$-reduced $\left(P_{d-1} F \simeq 0\right)$. We call such functors $d$-homogeneous. These functors form a full stable subcategory $\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$; see [Lur17, Corollary 6.1.2.8].
Remark 2.14 (Multi-variable calculus). There is also a multi-variable version of Goodwillie calculus. In particular, we say that a functor $F:\left(\mathcal{S} p^{c}\right)^{\times n} \rightarrow \mathcal{S} p$ is $\vec{d}$-excisive for $\vec{d}=\left(d_{1}, \ldots, d_{n}\right)$ if, for all $1 \leq i \leq n$ and every sequence of objects $\left\{X_{j} \in \mathcal{S} p^{c}\right\}_{j \neq i}$, the composite

$$
\mathcal{S} p^{c} \hookrightarrow \mathcal{S} p^{c} \times \prod_{j \neq i}\left\{X_{j}\right\} \hookrightarrow\left(\mathcal{S} p^{c}\right)^{\times n} \xrightarrow{F} \mathcal{S} p^{c}
$$

is $d_{i}$-excisive. The inclusion of $\vec{d}$-excisive functors into the category of all functors also has a left adjoint (see [AK98, Section 1] or [Lur17, Proposition 6.1.3.6]). We will only need the following generalization of Example 2.7: There is an equivalence

$$
\begin{equation*}
\operatorname{Exc}_{(1, \ldots, 1)}\left(\left(\mathcal{S} p^{c}\right)^{\times d}, \mathcal{S} p\right) \xrightarrow{\sim} S p \tag{2.15}
\end{equation*}
$$

between multi-linear functors (that is, functors of $d$ variables that are reduced and 1 -excisive in each variable) and spectra, given by evaluating at ( $\mathbb{S}, \ldots, \mathbb{S}$ ). This is shown in [Goo03, Section 5.2] at the level of homotopy categories and in [Lur17, Remark 6.1.3.3] at the level of $\infty$-categories.

The symmetric monoidal structure on $d$-excisive functors. We now explain how to construct a symmetric monoidal structure on the category of $d$-excisive functors. We begin with all functors from finite spectra to spectra.
Construction 2.16. Recall that the category of functors from finite spectra to spectra obtains a symmetric monoidal structure via Day convolution (see [Gla16] or [Lur17, Remark 4.8.1.13]): Given such functors $F$ and $G$ we define $F \circledast G$ as the left Kan extension in the following diagram:


Informally, we have

$$
\begin{equation*}
(F \circledast G)(C)=\operatorname{colim}_{C_{0} \otimes C_{1} \rightarrow C} F\left(C_{0}\right) \otimes F\left(C_{1}\right) \tag{2.17}
\end{equation*}
$$

The unit is the functor $\Sigma_{+}^{\infty} \operatorname{Map}_{\mathcal{S}_{p}}^{\circ}(\mathbb{S},-)$ where $\operatorname{Map}_{S_{p}}^{\circ}(-,-)$ denotes the (unpointed) mapping space.

The category of reduced functors $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ inherits a symmetric monoidal structure from the category of all functors via the localization of Remark 2.5. This follows, for example, by applying $\left[\mathrm{CDH}^{+} 23\right.$, Lemma 5.3 .4$]$ to the constant diagram on 0 . In fact, (2.17) implies that the Day convolution of two reduced functors is
reduced. It follows that the localized Day convolution on $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ actually coincides with the ordinary Day convolution - all that changes is that the unit of Fun $\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is the localization of the unit for ordinary Day convolution. This is the reduced functor $\Sigma^{\infty} \operatorname{Map}_{S p}(\mathbb{S},-)$ corepresented by the sphere spectrum, where $\operatorname{Map}_{S_{p}}(-,-)$ denotes the pointed mapping space; see [Chi21, Lemma 2.10]. That is, it is the functor

$$
\Sigma^{\infty} \Omega^{\infty}: \mathcal{S} p^{c} \rightarrow \mathcal{S} p, \quad M \mapsto \Sigma^{\infty} \Omega^{\infty} M
$$

Remark 2.18. The symmetric monoidal structure on $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is closed, since the category Fun $\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is presentably symmetric monoidal by [Nik16, Proposition 3.3], that is, it is a presentable symmetric monoidal $\infty$-category whose tensor product commutes with colimits in both variables. In particular, it has an internal hom object which can be computed via

$$
\begin{aligned}
\operatorname{hom}_{\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)}(F, G)(x) & \simeq \int_{d \in \mathcal{S} p^{c}} \operatorname{hom}_{\mathcal{S} p}(F(d), G(x \otimes d)) \\
& \simeq \operatorname{Hom}_{\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)}(F, G(x \otimes-))
\end{aligned}
$$

See [Nik16, Proposition 3.11] and [GHN17, Proposition 5.1].
Notation 2.19. For any $x \in \mathcal{S} p^{c}$ let

$$
h_{x}:=\Sigma^{\infty} \operatorname{Map}_{\mathcal{S} p}(x,-) \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)
$$

be the corresponding corepresentable functor (where again $\operatorname{Map}_{\mathcal{S}_{p}}(-,-)$ denotes the pointed mapping space). Note that

$$
\begin{equation*}
\operatorname{hom}_{\operatorname{Fun}\left(\delta p^{c}, \delta p\right)}\left(h_{x}, F\right) \simeq F(x \otimes-) \tag{2.20}
\end{equation*}
$$

Remark 2.21. There is a formula for Day convolution with a corepresentable functor that we now describe. For any reduced functor $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$, there is a natural assembly map

$$
h_{x}\left(y_{1}\right) \otimes F\left(y_{2}\right) \rightarrow F\left(\operatorname{hom}_{s_{p}}\left(x, y_{1} \otimes y_{2}\right)\right)
$$

which, by the universal property of Day convolution, induces a natural transformation

$$
\begin{equation*}
\left(h_{x} \circledast F\right)(-) \rightarrow F\left(\operatorname{hom}_{s p}(x,-)\right) \tag{2.22}
\end{equation*}
$$

of reduced functors $\mathcal{S} p^{c} \rightarrow \mathcal{S} p$.
Lemma 2.23. The map (2.22) is an equivalence.
Proof. It is enough to show that for any reduced functor $G: S p^{c} \rightarrow \mathcal{S} p$, the map (2.22) induces an equivalence

$$
\operatorname{Hom}_{\operatorname{Fun}\left(\delta p^{c}, \delta p\right)}\left(F\left(\operatorname{hom}_{s p}(x,-)\right), G\right) \rightarrow \operatorname{Hom}_{\operatorname{Fun}\left(\delta p^{c}, \delta p\right)}\left(h_{x} \circledast F, G\right)
$$

This map factors as a composition of equivalences

$$
\begin{aligned}
\operatorname{Hom}_{\operatorname{Fun}\left(\delta p^{c}, S p\right)}\left(F\left(\operatorname{hom}_{\mathcal{S} p}(x,-)\right), G\right) & \xrightarrow{\simeq} \operatorname{Hom}_{\operatorname{Fun}\left(\mathcal{S} p^{c}, \delta p\right)}(F, G(x \otimes-)) \\
& \xrightarrow{\simeq} \operatorname{Hom}_{\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)}\left(F, \operatorname{hom}_{\operatorname{Fun}\left(\mathcal{S} p^{c}, \delta p\right)}\left(h_{x}, G\right)\right) \\
& \xrightarrow{\hookrightarrow} \operatorname{Hom}_{\operatorname{Fun}\left(\mathcal{S} p^{c}, \delta p\right)}\left(F \circledast h_{x}, G\right)
\end{aligned}
$$

and so is itself an equivalence. The first equivalence follows from the following observation: Given an adjunction $L: \mathcal{C} \leftrightarrows \mathcal{D}: R$ and functors $F: \mathcal{C} \rightarrow \mathcal{Z}$ and $G: \mathcal{D} \rightarrow \mathcal{Z}$, there is a natural equivalence

$$
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{D}, z)}(F R, G) \simeq \operatorname{Hom}_{\operatorname{Fun}(\mathcal{e}, z)}(F, G L)
$$

The second equivalence uses (2.20) while the third equivalence is the closed monoidal adjunction.

Remark 2.24. It follows from Lemma 2.23 that $h_{\mathcal{S}}$ is indeed the unit of $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ under the Day convolution monoidal structure of Construction 2.16.

The following corollary is a variant of [Chi21, Lemma 2.18].
Corollary 2.25. The canonical natural transformation

$$
h_{x_{1}}\left(y_{1}\right) \otimes h_{x_{2}}\left(y_{2}\right) \rightarrow h_{x_{1} \otimes x_{2}}\left(y_{1} \otimes y_{2}\right)
$$

induces an equivalence

$$
h_{x_{1}} \circledast h_{x_{1}} \xrightarrow{\simeq} h_{x_{1} \otimes x_{2}} .
$$

Remark 2.26. This corollary can also be proved by observing that $x \mapsto h_{x}$ under the stable Yoneda embedding

$$
\begin{equation*}
\mathcal{S} p^{c} \rightarrow \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{\text {op }} \tag{2.27}
\end{equation*}
$$

which is the composite of the usual space-valued Yoneda embedding, followed by composition with $\Sigma_{+}^{\infty}$ and then followed by the localization functor of Remark 2.5. The usual Yoneda embedding is symmetric monoidal by [Gla16, Section 3], as is post-composition with the symmetric monoidal functor $\Sigma_{+}^{\infty}$. The localization functor is symmetric monoidal by construction. This is another way of appreciating that $h_{\mathbb{S}}$ is the monoidal unit of $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Day convolution of excisive functors. We now return to $d$-excisive functors. Our goal is to show that $P_{d}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a smashing localization which is compatible with the Day convolution monoidal structure on $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Definition 2.28. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between $\infty$-categories is a localization if $F$ has a fully faithful right adjoint $G$.

Remark 2.29. It follows that there is an equivalence between $\mathcal{D}$ and the full subcategory of $\mathcal{C}$ given by the essential image of $G$. We will sometimes abuse terminology and refer to the endofunctor $L:=G \circ F$ as the localization. A map $f: X \rightarrow Y$ in $\mathcal{C}$ is said to be an $L$-local equivalence if $L f: L X \rightarrow L Y$ is an equivalence. For example, the unit $X \rightarrow L X$ of the adjunction is an $L$-local equivalence for each $X \in \mathcal{C}$.

Example 2.30. The $d$-excisive approximation $P_{d}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a localization by Theorem 2.10; cf. Remark 2.11.

Definition 2.31. Suppose that $\mathcal{C}$ is a symmetric monoidal $\infty$-category. We say that a localization $L: \mathcal{C} \rightarrow \mathcal{C}$ is compatible with the symmetric monoidal structure if, whenever $f: X \rightarrow Y$ is a $L$-local equivalence and $Z \in \mathcal{C}$ is any object, the map $f \otimes \mathrm{id}: X \otimes Z \rightarrow Y \otimes Z$ is also an $L$-local equivalence.

The following follows from [Lur17, Proposition 2.2.1.9]:
Proposition 2.32. Let $\left(\mathcal{C}, \otimes, \mathbb{1}_{\mathfrak{C}}\right)$ be a symmetric monoidal $\infty$-category. Suppose that $L: \mathcal{C} \rightarrow \mathcal{C}$ is a localization which is compatible with the symmetric monoidal structure. Then LE inherits the structure of a symmetric monoidal $\infty$-category with unit $L\left(\mathbb{1}_{\mathcal{C}}\right)$ and monoidal product $L(-\otimes-)$. In particular, the localization $\mathcal{C} \rightarrow L \mathcal{C}$ is a symmetric monoidal functor.
Definition 2.33. Let $\left(\mathcal{C}, \otimes, \mathbb{1}_{\mathfrak{C}}\right)$ be a presentably symmetric monoidal stable $\infty$ category. A localization $L: \mathcal{C} \rightarrow \mathcal{C}$ is smashing if it preserves colimits.

Remark 2.34. If a localization $L: \mathcal{C} \rightarrow \mathcal{C}$ is compatible with the symmetric monoidal structure, then the map $X \rightarrow L\left(\mathbb{1}_{\mathcal{C}}\right) \otimes X$ induced by the canonical map $\mathbb{1}_{\mathfrak{C}} \rightarrow L\left(\mathbb{1}_{\mathcal{C}}\right)$ is a natural $L$-local equivalence. Hence we obtain a natural map $\alpha_{X}: L\left(\mathbb{1}_{\mathcal{C}}\right) \otimes X \rightarrow$ $L(X)$ as the composite

$$
L\left(\mathbb{1}_{\mathbb{C}}\right) \otimes X \rightarrow L\left(L\left(\mathbb{1}_{\mathbb{C}}\right) \otimes X\right) \simeq L(X)
$$

Proposition 2.35. Let $\left(\mathcal{C}, \otimes, \mathbb{1}_{\mathfrak{C}}\right)$ be a presentably symmetric monoidal stable $\infty$-category. Suppose that $L: \mathcal{C} \rightarrow \mathcal{C}$ is a smashing localization which is compatible with the symmetric monoidal structure. Then:
(a) The natural map $\alpha_{X}: L\left(\mathbb{1}_{\mathfrak{C}}\right) \otimes X \rightarrow L(X)$ is an equivalence for all $X \in \mathcal{C}$.
(b) LЄ inherits a symmetric monoidal structure with unit $L\left(\mathbb{1}_{\mathbb{C}}\right)$ and monoidal product $-\otimes-$.
(c) There is a symmetric monoidal equivalence of stable $\infty$-categories

$$
\operatorname{Mod}_{\mathcal{C}}\left(L\left(\mathbb{1}_{\mathcal{C}}\right)\right) \simeq L \mathcal{C}
$$

that exhibits LC as base change along the map $\mathbb{1} \rightarrow L\left(\mathbb{1}_{\mathbb{C}}\right)$.
Proof. The argument for part (a) is the same as the argument in [HPS97] which establishes the equivalence of the three conditions in their Definition 3.3.2. ${ }^{1}$ Part (b) then follows from part (a) and Proposition 2.32. Part (c) is a well-known consequence: it follows, for example, by applying [MNN17, Proposition 5.29] to the adjunction $L: \mathcal{C} \leftrightarrows L \mathcal{C}: \iota$ where $\iota: L \mathcal{C} \hookrightarrow \mathcal{C}$ is the inclusion.

Remark 2.36. We now turn our attention back to $d$-excisive functors. In fact, we work a little more generally, following $\left[\mathrm{CDH}^{+} 23\right]$ and [HR21]. To that end, let $\mathcal{J}=\left\{\bar{p}_{\alpha}: K_{\alpha}^{\triangleright} \rightarrow \mathcal{S} p^{c}\right\}$ be a small collection of diagrams in $\mathcal{S} p^{c}$. Let Fun $\left.\mathcal{J}^{(\mathcal{S}} p^{c}, \mathcal{S} p\right) \subseteq$ Fun $\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ denote the full subcategory spanned by those functors which send every diagram in $\mathcal{J}$ to a limit diagram. As explained in [CDH ${ }^{+} 23$, p. 159] or [HR21, Theorem 4.2], the subcategory $\operatorname{Fun}_{\mathcal{J}}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a localization of $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. In fact, it is the collection of $S$-local objects for some set of maps in $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ which implies that $\operatorname{Fun}_{\mathcal{J}}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is presentable and that the inclusion has a left adjoint $L_{\mathcal{J}}$ [Lur09a, Proposition 5.5.4.15].

Proposition 2.37. Suppose that the small set of diagrams $\mathcal{J}$ is closed under post composition with $x \otimes(-): \mathcal{S} p^{c} \rightarrow \mathcal{S} p^{c}$ for all $x \in \mathcal{S} p^{c}$. Then the localization

$$
L_{\mathcal{J}}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Fun}_{\mathcal{J}}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)
$$

is a smashing localization which is compatible with Day convolution. In particular, $\operatorname{Fun}_{\mathcal{J}}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a presentably symmetric monoidal $\infty$-category with symmetric monoidal structure given by Day convolution $F \circledast G$ and with tensor unit $L_{\mathcal{J}} h_{\mathcal{S}}$.

Proof. This is all contained in [CDH ${ }^{+}$23, Lemma 5.3.4] and [HR21, Theorem 4.5] except for the fact that the localization is smashing, so that Proposition 2.35 applies. However, the inclusion $\operatorname{Fun}_{\mathcal{J}}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \hookrightarrow \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ preserves filtered colimits (and hence all colimits), since filtered colimits in $\mathcal{S} p$ are left exact.

Theorem 2.38. The d-excisive approximation $P_{d}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a smashing localization on $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ which is compatible with the Day convolution. It follows that the category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a presentably symmetric monoidal stable

[^0]$\infty$-category with the tensor product and internal hom both computed in $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. The monoidal unit is the functor $P_{d} h_{\mathbb{S}}: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$. Moreover, $P_{d} h_{\mathbb{S}}=P_{d} \Sigma^{\infty} \Omega^{\infty}$.

Proof. This follows from Proposition 2.37, since $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is of the form $\operatorname{Fun}_{\mathcal{J}}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ for an appropriate choice of $\mathcal{J}$, namely the collection corresponding to all strongly cocartesian cubes. The condition in Proposition 2.37 is satisfied because the functor $x \otimes(-)$ preserves strongly cocartesian cubes (which follows, for example, from the characterization in [Lur17, Proposition 6.1.1.15]).

Remark 2.39. In Corollary 6.21 we show that $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a finite localization of $\operatorname{Exc}_{d+k}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ for all $k \geq 0$ and provide an explicit set of compact generators for the kernel of the localization functor. However, we suspect that the smashing localization $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is not a finite localization, i.e., its kernel is not generated by compact objects. In Example 6.22 we show that the "obvious" set of compact generators does not work.

Corollary 2.40. For any $F \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$, there is a natural equivalence

$$
\begin{equation*}
P_{d} F \xrightarrow{\simeq} F \circledast P_{d} \Sigma^{\infty} \Omega^{\infty} \tag{2.41}
\end{equation*}
$$

Corollary 2.42. Suppose $F$ and $G$ are $m$-excisive and $d$-excisive, respectively. Then $F \circledast G$ is $\min (m, d)$-excisive.

Proof. It follows from Corollary 2.40 that

$$
P_{m} \Sigma^{\infty} \Omega^{\infty} \circledast P_{d} \Sigma^{\infty} \Omega^{\infty} \simeq P_{\min (m, d)} \Sigma^{\infty} \Omega^{\infty}
$$

and hence

$$
\begin{aligned}
F \circledast G \simeq P_{m} F \circledast P_{d} G & \simeq F \circledast P_{m} \Sigma^{\infty} \Omega^{\infty} \circledast G \circledast P_{d} \Sigma^{\infty} \Omega^{\infty} \\
& \simeq F \circledast G \circledast P_{\min (m, d)} \Sigma^{\infty} \Omega^{\infty} \\
& \simeq P_{\min (m, d)}(F \circledast G) .
\end{aligned}
$$

Remark 2.43. Recall that if $\mathcal{C}$ is a symmetric monoidal stable $\infty$-category then there is an essentially unique symmetric monoidal colimit-preserving functor $\mathcal{S} p \rightarrow \mathcal{C}$ given by $A \mapsto A \otimes \mathbb{1}_{\mathcal{C}}$; see [Lur17, Corollary 4.8.2.19]. For example, if $\mathcal{C}=\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$, then this is the functor $A \mapsto F_{A}$, where $F_{A} \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ sends $X \in \mathcal{S} p^{c}$ to $A \otimes \Sigma^{\infty} \Omega^{\infty} X$. Applying $P_{d}$ (or appealing to the universal property directly) we can make the following definition:

Definition 2.44. The inflation functor $i_{d}: \mathcal{S} p \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is the (essentially unique) symmetric monoidal colimit-preserving functor. It is given by $A \mapsto P_{d}\left(F_{A}\right)$.

Example 2.45. Suppose $d=1$. In this case, it is well-known that $P_{1}\left(F_{A}\right)$ is equivalent to the functor $X \mapsto A \otimes X$. For example, see [Lur17, Example 6.1.1.28] for a proof in the language of $\infty$-categories (or see Remark 5.35). In particular, the equivalence $\operatorname{Exc}_{1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq \mathcal{S} p$ of Example 2.7 is an equivalence of symmetric monoidal stable $\infty$-categories. In fact, the same statement is true for the equivalence of (2.15) which is given by the canonical symmetric monoidal functor $\mathcal{S} p \rightarrow \operatorname{Exc}_{(1, \ldots, 1)}\left(\left(\mathcal{S} p^{c}\right)^{\times n}, \mathcal{S} p\right)$ that sends $A \in \mathcal{S} p$ to the functor $\left(X_{1}, \ldots, X_{n}\right) \mapsto A \otimes X_{1} \otimes \ldots \otimes X_{n}$.

Remark 2.46 (Change of target category). If $\mathcal{D}$ is a presentable symmetric monoidal stable $\infty$-category, then we could also consider the category of reduced $d$-excisive functors $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right)$. Much of what we have said thus far can be repeated for this
category. Our present goal is to explain how this construction can be viewed in terms of the Lurie tensor product of presentable $\infty$-categories.

Indeed, let $\operatorname{Pr}^{L}$ denote the $\infty$-category of presentable $\infty$-categories with the colimit-preserving functors as morphisms. This category has a symmetric monoidal structure constructed in [Lur17, Section 4.8.1] with the $\infty$-category of spaces $\mathcal{S}$ serving as the unit. The full subcategory $\operatorname{Pr}_{s t}^{L}$ consisting of the presentable stable $\infty$-categories inherits the monoidal structure from $\operatorname{Pr}^{L}$ with the category of spectra $\mathcal{S} p$ now serving as the unit. Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal, then the Lurie tensor product $\mathcal{C} \otimes \mathcal{D}$ also inherits a symmetric monoidal structure.

We claim there is an equivalence of symmetric monoidal stable $\infty$-categories

$$
\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right) \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \otimes \mathcal{D}
$$

where the left-hand side has the localized Day convolution monoidal structure, and the right-hand side has the symmetric monoidal structure coming from the Lurie tensor product. Indeed, because $\mathcal{D}$ is stable, there is an equivalence $\mathcal{D} \simeq \mathcal{D} \otimes \mathcal{S} p \otimes \mathcal{S}$. Then we have an equivalence of presentable stable $\infty$-categories
$\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right) \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D} \otimes \mathcal{S} p \otimes \mathcal{S}\right) \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S}\right) \otimes \mathcal{D} \otimes \mathcal{S} p \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \otimes \mathcal{D}$
by applying [HR21, Theorem 4.2(2)] twice. Moreover, it follows from [HR21, Lemma 4.8] that this is an equivalence of symmetric monoidal $\infty$-categories.

Example 2.47. Taking $\mathcal{D}=\operatorname{Mod}_{\mathrm{H} \mathbb{Z}}$ in the previous example, we see that

$$
\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right) \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \otimes \operatorname{Mod}_{\mathrm{HZ}} .
$$

This is the $d$-excisive version of Kaledin's derived Mackey functors; cf. [PSW22].

## 3. Cross-effects and idempotents

The cross-effects play an important role in the study of polynomial and excisive functors. In this section we will review several notions of cross-effect and show that for functors with values in a stable $\infty$-category, the definition of the crosseffect via idempotents, in the style of Eilenberg-Mac Lane [EML54], is equivalent to the definition of the cross-effect via total homotopy (co)fibers introduced by Goodwillie [Goo03]. The idempotent model is convenient for analyzing certain operations with cross-effects that we will need. In particular, it is useful for calculating natural transformations and Day convolutions of cross-effects.

Notation 3.1. Throughout this section $\mathcal{D}$ denotes a stable $\infty$-category and $F$ denotes a (not necessarily reduced) functor from $\mathcal{S} p^{c}$ to $\mathcal{D}$.

Remark 3.2. If $\mathcal{D}$ is a stable $\infty$-category, then so is $\mathcal{D}^{\mathrm{op}}$. This means that the material in this section applies both to covariant and contravariant functors from $\mathcal{S} p^{c}$ to $\mathcal{D}$. In subsequent sections we will apply the cross-effect construction to a contravariant functor (a version of the Yoneda embedding) from $\mathcal{S} p^{c}$ to $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Definition 3.3 (Goodwillie). The $d$-th cross-effect functor of $F$ is the functor $\mathrm{cr}_{d} F:\left(\mathcal{S} p^{c}\right)^{\times d} \rightarrow \mathcal{D}$ defined by

$$
\operatorname{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right):=\operatorname{tofib}_{S \subseteq[d]}\left\{F\left(\bigoplus_{i \notin S} x_{i}\right)\right\}
$$

the iterated (or total) fiber of a $d$-cube formed by applying $F$ to the direct sums of subsets of $x_{1} \ldots, x_{d}$. The morphisms in the cube are given by the relevant collapse maps $x_{i} \rightarrow 0$. See [Lur17, Construction 6.1.3.20].

Dually, we define the $d$-th co-cross-effect $\mathrm{cr}^{d} F:\left(\mathcal{S} p^{c}\right)^{\times d} \rightarrow \mathcal{D}$ by

$$
\operatorname{cr}^{d} F\left(x_{1}, \ldots, x_{d}\right):=\operatorname{tcof}_{S \subseteq[d]}\left\{F\left(\bigoplus_{i \notin S} x_{i}\right)\right\}
$$

the total cofiber of the cubical diagram with the same objects as before, but where the morphisms are induced by the inclusions $0 \rightarrow x_{i}$.

Remark 3.4. By permutation of variables, the (co-)cross-effect has a (naive) action of $\Sigma_{d}$. At the $\infty$-categorical level, this corresponds to the fact that the $d$-th crosseffect is a symmetric $d$-ary functor; see [Lur17, Section 6.1.4]. Using the coreduction functor defined in [Lur17, Construction 6.2.3.6] one can also check that the $d$-th co-cross-effect is a symmetric $d$-ary functor.

Example 3.5. The first cross-effect of $F$ is the fiber

$$
\operatorname{cr}_{1} F(x)=\operatorname{fib}(F(x) \rightarrow F(0))
$$

and the first co-cross-effect is the cofiber

$$
\operatorname{cr}^{1} F(x)=\operatorname{cofib}(F(0) \rightarrow F(x))
$$

The second cross-effect is the total fiber

$$
\operatorname{cr}_{2} F\left(x_{1}, x_{2}\right)=\text { tofib }\left(\begin{array}{cc}
F\left(x_{1} \oplus x_{2}\right) \longrightarrow & F\left(x_{2}\right) \\
\downarrow & \downarrow \\
F\left(x_{1}\right) \longrightarrow & F(0)
\end{array}\right)
$$

and the second co-cross-effect is the total cofiber

$$
\operatorname{cr}^{2} F\left(x_{1}, x_{2}\right)=\operatorname{tcof}\left(\begin{array}{cc}
F\left(x_{1} \oplus x_{2}\right) \longleftarrow & F\left(x_{2}\right) \\
\uparrow & \uparrow \\
F\left(x_{1}\right) \longleftarrow & F(0)
\end{array}\right)
$$

Lemma 3.6. For any functor $F: \mathcal{S} p^{c} \rightarrow \mathcal{D}$, there are natural transformations of functors of d variables

$$
\begin{equation*}
\operatorname{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right) \rightarrow F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \rightarrow \operatorname{cr}^{d} F\left(x_{1}, \ldots, x_{d}\right) \tag{3.7}
\end{equation*}
$$

whose composition is an equivalence.
Proof. This is shown in [Heu21, Lemma B.1].
Remark 3.8. It follows from Lemma 3.6 that there is an idempotent endomorphism in the homotopy category of functors of $d$ variables:

Remark 3.10. The notion of an idempotent endomorphism in an $\infty$-category is somewhat involved, since it has to incorporate higher coherences; see [Lur09b, Definition 4.4.5.4]. However, for a stable $\infty$-category, every idempotent in the homotopy category lifts to an idempotent in the $\infty$-category [Lur17, Proof of Lemma
1.2.4.6 and Warning 1.2.4.8]; in other words, the existence of higher coherences comes for free. In the sequel, when we say "idempotent", we will mean "idempotent endomorphism" in the relevant $\infty$-category, but when we say that idempotents "commute" or are "orthogonal", we will mean that they do so in the homotopy category.
Remark 3.11. Suppose $e: X \rightarrow X$ is an idempotent in a stable $\infty$-category which admits sequential colimits. We define

$$
e X:=\operatorname{colim}(X \xrightarrow{e} X \xrightarrow{e} \cdots) .
$$

The map $(1-e)$ is an idempotent as well, and there is a direct sum decomposition

$$
X \simeq e X \oplus(1-e) X
$$

which is natural with respect to maps that commute with $e$; see [Nee01, Proposition 1.6.8], for example. More generally, if $e_{1}, \ldots, e_{d}: X \rightarrow X$ are idempotents that are pairwise orthogonal in the homotopy category and such that $e_{1}+\cdots+e_{d} \simeq 1_{X}$ then there is an equivalence

$$
X \simeq e_{1} X \oplus \cdots \oplus e_{d} X
$$

Remark 3.12. It follows from Remark 3.10 that the idempotent map (3.9) can be thought of as an idempotent in the $\infty$-category of functors $\left(S p^{c}\right)^{\times d} \rightarrow \mathcal{D}$. We denote this idempotent by $\operatorname{cr}(d)$. It follows that the $d$-th cross-effect $\operatorname{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right)$ is a direct summand of $F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$, split off by the idempotent $\operatorname{cr}(d)$. In other words, $\operatorname{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right) \simeq \operatorname{cr}(d) F\left(x_{1}, \ldots, x_{d}\right)$ using the notation of Remark 3.11.
Remark 3.13. Our next goal is to give a "formula" for the idempotent $\operatorname{cr}(d)$ which expresses it in terms of more elementary idempotents. This formula was first given in [EML54] for functors between abelian categories. We will adapt their constructions to the setting of stable $\infty$-categories.

Definition 3.14. Let $x_{1}, \ldots, x_{d} \in \mathcal{S} p^{c}$ and $U \subseteq[d]$. Define

$$
\psi_{U}: x_{1} \oplus \cdots \oplus x_{d} \rightarrow x_{1} \oplus \cdots \oplus x_{d}
$$

to be the sum of the identity morphisms $x_{i} \xrightarrow{1} x_{i}$ for all $i \in U$ and the zero morphisms $x_{i} \xrightarrow{0} x_{i}$ for $i \notin U$. When $U=\{i\}$ is a singleton, we denote $\psi_{U}$ by $\psi_{i}$.
Remark 3.15. We regard $\psi_{U}$ as a natural endomorphism of the functor

$$
\bigoplus_{d}:\left(\mathcal{S} p^{c}\right)^{\times d} \rightarrow \mathcal{S} p^{c}
$$

which sends $\left(x_{1}, \ldots, x_{d}\right)$ to $x_{1} \oplus \cdots \oplus x_{d}$. For any functor $F: \mathcal{S} p^{c} \rightarrow \mathcal{D}$, we then obtain a natural endomorphism $F * \psi_{U}$ of $F \circ \bigoplus_{d}$.
Lemma 3.16. The morphisms $\left\{\psi_{U} \mid U \subseteq[d]\right\}$ are pairwise commuting idempotents. For every $U, V \subseteq[d], \psi_{U} \circ \psi_{V}=\psi_{U \cap V}$. For every $U \subseteq[d]$, there is an equivalence

$$
\psi_{U} \simeq \sum_{i \in U} \psi_{i}
$$

Proof. These are routine verifications which we leave to the reader.
Remark 3.17. Given a functor $F: \mathcal{S} p^{c} \rightarrow \mathcal{D}$, it follows from Lemma 3.16 that

$$
\left\{F * \psi_{U} \mid U \subseteq[d]\right\}
$$

are pairwise commuting idempotents of the functor $\left(x_{1}, \ldots, x_{d}\right) \mapsto F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$. Furthermore, $\left(F * \psi_{U}\right)\left(F * \psi_{V}\right)=F * \psi_{U \cap V}$ for all $U, V \subseteq[d]$.

Definition 3.18. For each subset $U \subseteq[d]$, define the natural transformation

$$
\operatorname{cr}(U): F \circ \bigoplus_{d} \rightarrow F \circ \bigoplus_{d}
$$

as follows:

$$
\begin{aligned}
\operatorname{cr}(U) & :=F * \psi_{U} \circ\left(\bigcirc_{i \in U}\left(1-F * \psi_{U \backslash\{i\}}\right)\right) \\
& \simeq \bigcirc_{i \in U}\left(F * \psi_{U}-F * \psi_{U \backslash\{i\}}\right) \\
& \simeq \sum_{V \subseteq U}(-1)^{|U|-|V|} F * \psi_{V} .
\end{aligned}
$$

Here $\circ$ and $\bigcirc$ denote composition. All maps of the form $F * \psi_{U}$ and $1-F * \psi_{U \backslash\{i\}}$ commute with each other, so the order of composition is not important. That the different formulas for $\operatorname{cr}(U)$ are equivalent is an exercise.

Example 3.19. When $U=[d]$, we denote $\operatorname{cr}(U)$ by $\operatorname{cr}[d]$. Thus we have

$$
\operatorname{cr}[d]=\bigcirc_{i \in[d]}\left(1-F * \psi_{[d] \backslash\{i\}}\right) \simeq \sum_{V \subseteq[d]}(-1)^{d-|V|} F * \psi_{V}
$$

Remark 3.20. The $\operatorname{cr}(U)$ are analogous to the maps $D_{\sigma}$ in [EML54, (9.7), page 77].
Lemma 3.21 ([EML54], Proof of Theorem 9.1). The maps $\{\operatorname{cr}(U) \mid U \subseteq[d]\}$ are pairwise orthogonal idempotents whose sum is the identity.

Proof. The fact that the $\psi_{U}$ are pairwise commuting homotopy idempotents implies that all maps of the form $F * \psi_{U}$ commute with each other and also with maps of the form $1-F * \psi_{V}$. Therefore, the maps $\operatorname{cr}(U)$ commute with each other. Each map $\operatorname{cr}(U)$ is a composition of commuting idempotents, so it is itself an idempotent.

To prove that the idempotents $\operatorname{cr}(U)$ are pairwise orthogonal, let $U, U^{\prime}$ be distinct subsets of $[d]$. Without loss of generality we may assume that there exists an element $i \in U \backslash U^{\prime}$. Then $U^{\prime} \subseteq[d] \backslash\{i\}$ and thus $\operatorname{cr}\left(U^{\prime}\right)=F * \psi_{[d] \backslash\{i\}} \operatorname{cr}\left(U^{\prime}\right)$. On the other hand, since $i \in U, F * \psi_{U}-F * \psi_{U \backslash\{i\}}$ is a factor of $\operatorname{cr}(U)$. So we may write $\operatorname{cr}(U)=\overline{\operatorname{cr}}(U) \circ\left(F * \psi_{U}-F * \psi_{U \backslash\{i\}}\right)$ for some map $\overline{\operatorname{cr}}(U)$. It follows that there are equivalences

$$
\begin{aligned}
\operatorname{cr}(U) \circ \operatorname{cr}\left(U^{\prime}\right) & \simeq \overline{\operatorname{cr}}(U) \circ\left(F * \psi_{U}-F * \psi_{U \backslash\{i\}}\right) F * \psi_{[d] \backslash\{i\}} \circ \operatorname{cr}\left(U^{\prime}\right) \\
& \simeq \overline{\operatorname{cr}}(U) \circ\left(F * \psi_{U \backslash\{i\}}-F * \psi_{U \backslash\{i\}}\right) \circ \operatorname{cr}\left(U^{\prime}\right) \\
& \simeq 0 .
\end{aligned}
$$

Finally, we want to prove that $\sum_{U \subseteq[d]} \operatorname{cr}(U) \simeq 1$. If we adopt the formula $\operatorname{cr}(U)=$ $\sum_{V \subseteq U}(-1)^{|U|-|V|} F * \psi_{V}$, then it is clear that

$$
\sum_{U \subseteq[d]} \operatorname{cr}(U) \simeq \sum_{V \subseteq[d]} \sum_{V \subseteq U \subseteq[d]}(-1)^{|U|-|V|} F * \psi_{V}
$$

For all $V \subsetneq[d], \sum_{V \subseteq U \subseteq[d]}(-1)^{|U|-|V|}=0$. So $\sum_{U} \operatorname{cr}(U) \simeq F * \psi_{[d]}$, which is the identity.

Notation 3.22. We will write $\operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$ for the direct summand of $F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$ split off by the idempotent $\operatorname{cr}(U): F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \rightarrow F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$.

Remark 3.23. If $U=\left\{i_{1}, \ldots, i_{s}\right\} \subseteq[d]$, then $\operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$ is denoted by $F\left(x_{i_{1}}|\cdots| x_{i_{s}}\right)$ in [EML54]. We will see in Lemma 3.26 below that indeed $\operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$ really only depends on $x_{i_{1}}, \ldots, x_{i_{s}}$.
Corollary 3.24. Let $F \mathcal{S} p^{c} \rightarrow \mathcal{D}$ be a functor and let $x_{1}, \ldots, x_{d} \in \mathcal{S} p^{c}$. There is an equivalence

$$
F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \simeq \prod_{U \subseteq[d]} \operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right)
$$

which is natural in $x_{1}, \ldots, x_{d}$.
Proof. This is an immediate consequence of Lemma 3.21. It also is [EML54, Theorem 9.1] lifted to functors between stable $\infty$-categories.

The next two lemmas record some elementary properties of the idempotents $\operatorname{cr}(U)$.
Lemma 3.25. Let $x_{1}, \ldots, x_{d} \in \mathcal{S} p^{c}$ and suppose that $x_{i} \simeq 0$ for some $1 \leq i \leq d$. Then for all subsets $\left\{i_{1}, \ldots, i_{s}\right\} \subseteq[d]$ that contain $i, \operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \simeq 0$.
Proof. Suppose $x_{i} \simeq 0$ and $i \in U$. Then $\psi_{U}=\psi_{U \backslash\{i\}}$ and therefore

$$
F * \psi_{U}-F * \psi_{U \backslash\{i\}}: F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \rightarrow F\left(x_{1} \oplus \cdots \oplus x_{d}\right)
$$

is (homotopic to) the trivial map. But $F * \psi_{U}-F * \psi_{U \backslash\{i\}}$ is a factor of $\operatorname{cr}(U)$. It follows that $\operatorname{cr}(U)$ acts trivially on $F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$ in this case, and therefore $\operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \simeq 0$.

Lemma 3.26. Let $y_{1}, \ldots y_{d} \in \mathcal{S} p^{c}$ and suppose we are given maps $x_{i} \rightarrow y_{i}$ for $1 \leq i \leq d$. Suppose there is a set $U \subseteq[d]$ such that the map $x_{i} \rightarrow y_{i}$ is an equivalence for all $i \in U$. Then the following induced map is an equivalence:

$$
\operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \xrightarrow{\simeq} \operatorname{cr}(U) F\left(y_{1} \oplus \cdots \oplus y_{d}\right) .
$$

Proof. Note that our assumptions imply that the map

$$
x_{1} \oplus \cdots \oplus x_{d} \rightarrow y_{1} \oplus \cdots \oplus y_{d}
$$

induces an equivalence

$$
\psi_{V}\left(x_{1} \oplus \cdots \oplus x_{d}\right) \xrightarrow{\simeq} \psi_{V}\left(y_{1} \oplus \cdots \oplus y_{d}\right)
$$

for all $V \subseteq U$. It follows that $\operatorname{cr}(U)=\sum_{V \subseteq U}(-1)^{|U|-|V|} F\left(\psi_{V}\right)$ also induces an equivalence

$$
\operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \xrightarrow{\simeq} \operatorname{cr}(U) F\left(y_{1} \oplus \cdots \oplus y_{d}\right) .
$$

Remark 3.27. Now we are ready to express the cross-effect of a functor $F$ in terms of the idempotents $\operatorname{cr}(U)$.
Proposition 3.28. Let $x_{1}, \ldots, x_{d} \in \mathcal{S} p^{c}$ and let $F$ be a functor from $\mathcal{S} p^{c}$ to $\mathcal{D}$. Consider the two d-dimensional cubical diagrams that both send a subset $S \subseteq[d]$ to $F\left(\bigoplus_{i \notin S} x_{i}\right)$ and whose morphisms are induced by the relevant collapse maps $x_{i} \rightarrow 0$ for the first diagram and the relevant inclusion maps $0 \rightarrow x_{i}$ for the second diagram. These diagrams are equivalent to the diagrams that send $S$ to the product

$$
\prod_{U \subseteq[d] \backslash S} \operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right)
$$

and whose maps are the obvious projections (for the first diagram) and the obvious inclusions (for the second diagram).

Proof. We can identify $\bigoplus_{i \notin S} x_{i}$ with $\bigoplus_{i=1}^{d} y_{i}$ where $y_{i}=0$ for $i \in S$ and $y_{i}=x_{i}$ for $i \notin S$. It follows that we may identify either one of the two versions of the cubical diagram $S \mapsto F\left(\bigoplus_{i \notin S} x_{i}\right)$ with a cubical diagram where $F$ is evaluated at a wedge sum of $d$ spectra, and where the maps are induced by a wedge sum of $d$ maps of spectra.

By Corollary 3.24, $F\left(y_{1} \oplus \cdots \oplus y_{d}\right)$ splits as a product of $\operatorname{cr}(U) F\left(y_{1} \oplus \cdots \oplus y_{d}\right)$, where $U$ ranges over subsets of $[d]$, and the maps in either version of the cubical diagram are compatible with the splitting. By Lemma 3.25, for each $U$ that is not contained in $[d] \backslash S, \operatorname{cr}(U) F\left(y_{1} \oplus \cdots \oplus y_{d}\right) \simeq 0$, and so can be dropped from the product. By Lemma 3.26, whenever $S \subseteq S^{\prime} \subseteq[d]$, for any $U \subseteq[d] \backslash S^{\prime}$ the factors $\operatorname{cr}(U) F\left(\bigoplus_{i \notin S} x_{i}\right)$ and $\operatorname{cr}(U) F\left(\bigoplus_{i \notin S^{\prime}} x_{i}\right)$ are equivalent, and can be identified with $\operatorname{cr}(U) F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$.

We now can state the precise relationship between $\mathrm{cr}_{d}, \mathrm{cr}^{d}, \mathrm{cr}[d]$ and $\mathrm{cr}(d)$.
Lemma 3.29. The idempotent $\operatorname{cr}[d]: F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \rightarrow F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$ of Example 3.19 is equivalent to the idempotent $\operatorname{cr}(d): F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \rightarrow F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$ of Remark 3.8. For any functor $F$ there are natural equivalences

$$
\operatorname{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right) \simeq \operatorname{cr}^{d} F\left(x_{1}, \ldots, x_{d}\right) \simeq \operatorname{cr}[d] F\left(x_{1} \oplus \cdots \oplus x_{d}\right)
$$

Proof. We already saw in Lemma 3.6 that $\operatorname{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right) \simeq \operatorname{cr}^{d} F\left(x_{1}, \ldots, x_{d}\right)$, but this also follows from Proposition 3.28. The proposition says that $\mathrm{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right)$ (respectively, $\operatorname{cr}^{d} F\left(x_{1}, \ldots, x_{d}\right)$ ) is equivalent to the total fiber (respectively, cofiber) of the cubical diagram that sends a subset $S \subseteq[d]$ to $\prod_{U \subseteq[d] \backslash S} \operatorname{cr}(U) F\left(\bigoplus_{i \notin S} x_{i}\right)$ where the maps are given by projections (respectively, inclusions). A straightforward calculation shows that the factors corresponding to $\operatorname{cr}(U) F$ with $U$ a proper subset of [d] "cancel out", and the total fiber/cofiber is equivalent to $\operatorname{cr}[d] F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$. It also follows that the map $F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \rightarrow \operatorname{cr}^{d} F\left(x_{1}, \ldots, x_{d}\right)$ is equivalent to the projection $F\left(x_{1} \oplus \cdots \oplus x_{d}\right) \rightarrow \operatorname{cr}[d] F\left(x_{1} \oplus \cdots \oplus x_{d}\right)$, and the map $\mathrm{cr}_{d} F\left(x_{1}, \ldots, x_{d}\right) \rightarrow$ $F\left(x_{1}, \ldots, x_{d}\right)$ is equivalent to the section of the projection map. It follows that $\operatorname{cr}[d]$ is equivalent to the idempotent $\mathrm{cr}(d)$ of line (3.9) in Remark 3.8.

We have one final lemma, which gives a kind of generalized converse of the formula for $\operatorname{cr}(U)$ in Definition 3.18.

Lemma 3.30. For any $U \subseteq[d]$, there are equivalences

$$
\begin{equation*}
F * \psi_{U} \simeq \sum_{V \subseteq U} \operatorname{cr}(V) \tag{3.31}
\end{equation*}
$$

and

$$
\begin{equation*}
1-F * \psi_{U} \simeq \sum_{V \not \subset U} \operatorname{cr}(V) \tag{3.32}
\end{equation*}
$$

Moreover, for any collection of subsets $U_{1}, \ldots, U_{d} \subseteq[d]$, there is an equivalence

$$
\begin{equation*}
\bigcirc_{i=1}^{d}\left(1-F * \psi_{U_{i}}\right) \simeq \sum_{V \nsubseteq U_{1}, \ldots, U_{d}} \operatorname{cr}(V) \tag{3.33}
\end{equation*}
$$

where the sum on the right-hand side is over all subsets $V \subseteq[d]$ that are not contained in any of the sets $U_{1}, \ldots, U_{d}$.

Proof. Observe that we have equivalences

$$
\begin{aligned}
1 & \simeq \bigcirc_{i \in U}\left(F * \psi_{[d] \backslash\{i\}}+\left(1-F * \psi_{[d] \backslash\{i\}}\right)\right) \\
& \simeq \sum_{V \subseteq U} F * \psi_{[d] \backslash V} \circ\left(\bigcirc_{i \in U \backslash V}\left(1-F * \psi_{[d] \backslash\{i\}}\right)\right) .
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
F * \psi_{U} & \simeq F * \psi_{U} \sum_{V \subseteq U} F * \psi_{[d] \backslash V} \circ\left(\bigcirc_{i \in U \backslash V}\left(1-F * \psi_{[d] \backslash\{i\}}\right)\right) \\
& \left.\simeq \sum_{V \subseteq U} \bigcirc_{i \in U \backslash V}\left(F * \psi_{U \backslash V}-F * \psi_{(U \backslash V) \backslash\{i\}}\right)\right) \\
& \simeq \sum_{V \subseteq U} \operatorname{cr}(U \backslash V)=\sum_{V \subseteq U} \operatorname{cr}(V)
\end{aligned}
$$

which proves (3.31). The equality (3.32) follows from (3.31) and the fact that $\sum_{U \subseteq[d]} \operatorname{cr}(U) \simeq 1$. Finally (3.33) follows from (3.32) and the fact that the $\operatorname{cr}(U)$ are pairwise orthogonal idempotents.

## 4. The cross-effects of Representable functors

We now investigate the cross-effects of representable functors and show that they provide a convenient set of generators for $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. These generators will play an important role in this work. The material in the previous section will help us analyze the Day convolutions and natural transformations between these generators.

Remark 4.1. Recall from Notation 2.19 that $h_{x} \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is the reduced functor corepresented by $x \in \mathcal{S} p^{c}$, i.e., $h_{x}(y)=\Sigma^{\infty} \operatorname{Map}_{\mathcal{S}_{p}}(x, y)$. In this section we will view the assignment $x \mapsto h_{x}$ as a contravariant functor from $\mathcal{S} p^{c}$ to $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$, or equivalently as a functor from $\mathcal{S} p^{c}$ to $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{\text {op }}$. Let us formalize this in a definition:

Definition 4.2. Let $F: \mathcal{S} p^{c} \rightarrow \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{\text {op }}$ be the functor defined by the formula

$$
F(x):=h_{x} .
$$

One can view $F$ as a stable and reduced variant of the Yoneda embedding.
Definition 4.3. Fix a finite spectrum $x \in \mathcal{S} p^{c}$. For each integer $i \geq 1$, we define $h_{x}(i) \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ by

$$
h_{x}(i):=\operatorname{cr}_{i} F(x, \ldots, x) .
$$

More generally, given a finite set $U$, we define

$$
h_{x}(U):=\operatorname{cr}(U) F\left(\bigoplus_{U} x\right)
$$

where the right-hand side uses Notation 3.22.
Remark 4.4. For each $i \geq 1$, we thus have a bi-functor $(x, y) \mapsto h_{x}(i)(y)$ which is contravariant in $x$ and covariant in $y$.

Remark 4.5. By Lemma 3.29, there are three equivalent models for $h_{x}(i)$ : as the cross-effect, the co-cross-effect, and as the image of the idempotent $\operatorname{cr}(i)$ acting on

$$
h_{\underbrace{}_{i} \oplus \cdots \oplus x}
$$

The co-cross-effect leads to a particularly simple description of $h_{x}(i)$. Recall that the co-cross-effect of $h_{x}$ in the variable $x$ is the total cofiber of the cubical diagram

$$
S \mapsto h\left(\underset{j \in[i] \backslash S}{\bigvee} x_{j}\right)(-)=\Sigma^{\infty} \prod_{j \in[i] \backslash S} \operatorname{Map}_{S_{p}}\left(x_{j},-\right)
$$

where the maps are inclusions of factors. The total cofiber of this diagram is $h_{x_{1}} \otimes \cdots \otimes h_{x_{i}}$. This is just saying that $n$-fold smash product of spaces is equivalent to the total cofiber of a cube of cartesian products of spaces, indexed by all subsets of $n$. After substituting $x$ for $x_{1}, \ldots, x_{i}$, we conclude that

$$
\begin{equation*}
h_{x}(i)(y) \simeq h_{x}(y) \otimes \cdots \otimes h_{x}(y) \tag{4.6}
\end{equation*}
$$

In other words, $h_{x}(i)$ is given by the pointwise tensor product of $i$ copies of $h_{x}$.
Remark 4.7. There is a natural equivalence $h_{x}(y \oplus y) \xrightarrow{\sim} h_{x \oplus x}(y)$. It follows that we could equivalently define $h_{x}(i)$ to be the $i$-th cross-effect in the covariant variable. However, it suits our goals better to take the cross-effects in the contravariant variable, as in Definition 4.3.

Remark 4.8. From now on we will mostly take $x=\mathbb{S}$ and consider the functors $h_{\mathbb{S}}(i)$ given by

$$
h_{\mathbb{S}}(i)(y) \simeq h_{\mathbb{S}}(i)(y)^{\otimes i} \simeq\left(\Sigma^{\infty} \Omega^{\infty} y\right)^{\otimes i} \simeq \Sigma^{\infty}\left(\Omega^{\infty} y\right)^{\wedge i}
$$

Our next task is to calculate the Day convolutions of the functors $h_{\mathbb{S}}(i)$ for various $i$. We will use the notation

$$
[i]_{+} \wedge x:=\underbrace{x \oplus \ldots \oplus x}_{i} .
$$

We also need the following definition, which will show up again later in the paper.
Definition 4.9. Let $i$ and $j$ be positive integers, and write $[i]=\{1, \ldots, i\}$ for the finite set with $i$ elements. A subset of $[i] \times[j]$ is said to be good if its projections onto $[i]$ and $[j]$ are both surjective.
Remark 4.10. Note that in order to be good, a subset must have cardinality between $\max (i, j)$ and $i j$.

Proposition 4.11. There is an equivalence

$$
h_{\mathbb{S}}(i) \circledast h_{\mathbb{S}}(j) \simeq \bigoplus_{\substack{U \subseteq[i] \times[j] \\ \bar{U} \text { good }}} h_{\mathbb{S}}(|U|)
$$

in $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Proof. To prove the proposition, we will use the idempotent model of the cross-effect. By Lemma 3.29, $h_{\mathbb{S}}(i)$ is equivalent to $\operatorname{cr}(i) h_{[i]_{+} \wedge \mathbb{S}}$ where

$$
\begin{equation*}
\operatorname{cr}(i)=\bigcirc_{1 \leq s \leq i}\left(1-h_{\psi_{[i] \backslash\{s\}}}\right) \tag{4.12}
\end{equation*}
$$

Here $\psi_{[i] \backslash\{s\}}:[i]_{+} \wedge \mathbb{S} \rightarrow[i]_{+} \wedge \mathbb{S}$ is the idempotent map obtained by collapsing copy number $s$ of $\mathbb{S}$ to a point. Thus

$$
h_{\psi_{[i] \backslash\{s\}}}: h_{[i]_{+} \wedge \mathcal{S}} \rightarrow h_{[i]_{+} \wedge \mathcal{S}}
$$

is the idempotent map induced by $\psi_{[i \backslash \backslash\{s\}}$.
Day convolution commutes with colimits in each variable, so there are equivalences

$$
h_{\mathbb{S}}(i) \circledast h_{\mathbb{S}}(j) \simeq \operatorname{cr}(i) h_{[i]_{+} \wedge \mathbb{S}} \circledast \operatorname{cr}(j) h_{[j]_{+} \wedge \mathbb{S}} \simeq \operatorname{cr}(i) \operatorname{cr}(j)\left(h_{[i]_{+} \wedge \mathbb{S}} \circledast h_{[j]_{+} \wedge \mathbb{S}}\right)
$$

where $\operatorname{cr}(i)$ and $\operatorname{cr}(j)$ act on $[i]_{+} \wedge \mathbb{S}$ and $[j]_{+} \wedge \mathbb{S}$ respectively. Next we need to figure out the effect of $\operatorname{cr}(i)$ and $\operatorname{cr}(j)$ on $h_{[i]_{+} \wedge \mathcal{S}} \circledast h_{[j]_{+} \wedge \mathcal{S}}$. Recall that $h_{x}=$ $\Sigma^{\infty} \operatorname{Map}_{\mathcal{S} p}(x,-)$. By Corollary 2.25 there is a natural equivalence

$$
h_{[i]_{+} \wedge \mathbb{S}} \circledast h_{[j]_{+} \wedge \mathbb{S}} \xrightarrow{\simeq} h_{[i] \times[j]_{+} \wedge \mathcal{S}} .
$$

It follows that given, say, an $s \in[i]$, the action of $h_{\psi_{[i] \backslash\{s\}}}$ on $h_{[i]_{+} \wedge \mathcal{S}} \circledast h_{[j]_{+} \wedge \mathcal{S}}$ can be identified with the action of $h_{\psi_{([i] \backslash\{s\}) \times[j]}}$ on $h_{[i] \times[j]_{+} \wedge \mathcal{S}}$. Here

$$
\psi_{([i] \backslash\{s\}) \times[j]}:[i] \times[j]_{+} \wedge \mathbb{S} \rightarrow[i] \times[j]_{+} \wedge \mathbb{S}
$$

is the idempotent that collapses $\{s\} \times[j]$ to the basepoint.
It follows that $h_{\mathbb{S}}(i) \circledast h_{\mathbb{S}}(j)$ is equivalent to

$$
\bigcirc_{s \in[i]}\left(1-h_{\left.\psi_{([i] \backslash\{s\}) \times[j]}\right)}\right) \bigcirc_{t \in[j]}\left(1-h_{\psi_{[i] \times([j] \backslash\{t\})}}\right) h_{[i] \times[j]++} .
$$

By formula (3.33) of Lemma 3.30, this is equivalent to

$$
\sum_{U} \operatorname{cr}(U) h_{[i] \times[j]_{+} \wedge \mathcal{S}}
$$

where $U$ ranges over subsets $U \subseteq[i] \times[j]$ that are not contained in a subset of the form $([i] \backslash\{s\}) \times[j]$ or $[i] \times([j] \backslash\{t\})$. But these are precisely the good subsets of $[i] \times[j]$, and for each good subset $U, \operatorname{cr}(U) h_{[i] \times[j]_{+} \wedge \mathbb{S}}$ is precisely $h_{\mathbb{S}}(|U|)$.
Example 4.13. We have that

$$
h_{\mathbb{S}}(2) \circledast h_{\mathbb{S}}(2) \simeq h_{\mathbb{S}}(2)^{\oplus 2} \oplus h_{\mathbb{S}}(3)^{\oplus 4} \oplus h_{\mathbb{S}}(4)
$$

As a consequence of (4.6), we obtain the following.
Lemma 4.14. For each $i \geq 1$ the functor $h_{\mathbb{S}}(i)$ is a commutative algebra in $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Proof. We recall that commutative algebras with respect to Day convolution are exactly the lax symmetric monoidal functors, as proven in the $\infty$-categorical setting in [Lur17, Example 2.2.6.9] or [Gla16, Proposition 2.12]. If $x$ is a cocommutative coalgebra spectrum, then the functor $h_{x}$ is lax symmetric monoidal via the composition

$$
\begin{gathered}
h_{x}\left(y_{1}\right) \otimes h_{x}\left(y_{2}\right)=\quad \Sigma^{\infty} \operatorname{Map}_{s p}\left(x, y_{1}\right) \otimes \Sigma^{\infty} \operatorname{Map}_{s p}\left(x, y_{2}\right) \\
\Sigma^{\infty} \operatorname{Map}_{\mathcal{S}_{p}}\left(x \otimes x, y_{1} \otimes y_{2}\right) \\
\sum^{\downarrow}{ }^{\downarrow} \operatorname{Map}_{s p}\left(x, y_{1} \otimes y_{2}\right)=h_{x}\left(y_{1} \otimes y_{2}\right)
\end{gathered}
$$

In particular, it follows that the functor $h_{[i]_{+} \wedge \mathcal{S}}$ is lax symmetric monoidal, and thus a commutative algebra with respect to Day convolution. Furthermore, if we have an inclusion of finite sets $U \hookrightarrow V$, the induced map $U_{+} \wedge \mathbb{S} \rightarrow V_{+} \wedge \mathbb{S}$ is a map of cocommutative coalgebra spectra. It follows that the induced natural transformation $h_{V_{+} \wedge \mathscr{S}} \rightarrow h_{U_{+} \wedge \mathcal{S}}$ is symmetric monoidal. This implies that the total fiber of the contravariant cubical diagram

$$
U \mapsto h_{U_{+} \wedge \mathcal{S}}
$$

as $U$ ranges over the subsets of $[i]$, being a limit of a diagram of lax symmetric monoidal functors and symmetric monoidal natural transformations, is lax symmetric monoidal. This total fiber is equivalent to $h_{\mathbb{S}}(i)$ and thus we conclude that $h_{\mathbb{S}}(i)$ is a commutative algebra with respect to Day convolution.
Corollary 4.15. Each $P_{d} h_{\mathbb{S}}(i)$ is a commutative algebra in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Proof. This follows from Lemma 4.14 since $P_{d}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is symmetric monoidal by Theorem 2.38.

The following lemma describes the multiplication on $h_{\mathbb{S}}(i)$ in terms of the splitting of Proposition 4.11.

Lemma 4.16. The multiplication map $h_{\mathbb{S}}(i) \circledast h_{\mathbb{S}}(i) \rightarrow h_{\mathbb{S}}(i)$ corresponds, under the splitting of Proposition 4.11, to projection onto the summand corresponding to the diagonal $[i] \subseteq[i] \times[i]$.
Proof. The map $h_{\mathbb{S}}(i) \circledast h_{\mathbb{S}}(i) \rightarrow h_{\mathbb{S}}(i)$ is a retract of the restriction map

$$
\begin{equation*}
h_{[i] \times[i]_{+} \wedge \mathbb{S}} \rightarrow h_{[i]_{+} \wedge \mathbb{S}} \tag{4.17}
\end{equation*}
$$

induced by the diagonal inclusion $[i] \hookrightarrow[i] \times[i]$. By Corollary 3.24, there are equivalences

$$
h_{[i] \times[i]_{+} \wedge \mathbb{S}} \simeq \prod_{U \subset[i] \times[i]} h_{\mathbb{S}}(U)
$$

and

$$
h_{[i]_{+} \wedge \mathbb{S}} \simeq \prod_{U \subset[i]} h_{\mathbb{S}}(U)
$$

It follows from the proof of Corollary 3.24 that under these splittings the restriction map (4.17) sends a summand $h_{\mathbb{S}}(U)$ of $h_{[i] \times[i]_{+} \wedge \mathcal{S}}$ to itself if $U \subseteq[i] \times[i]$ is a subset of the diagonal, and sends $h_{\mathbb{S}}(U)$ to zero otherwise. By Proposition 4.11, $h_{\mathbb{S}}(i) \circledast h_{\mathbb{S}}(i)$ is the wedge of those summands of $h_{[i] \times[i]+\wedge \mathcal{S}}$ which correspond to good subsets of $[i] \times[i]$. The only good subset of $[i] \times[i]$ that is contained in the diagonal is the diagonal itself, and it gets mapped to $h_{\mathbb{S}}(i)$.

Remark 4.18. An algebra $A$ in a symmetric monoidal category $\mathcal{C}$ is separable if the multiplication map $A \otimes A \rightarrow A$ has an $A$ - $A$-bilinear section. Separable algebras have been studied by Balmer [Bal11] in the context of additive and triangulated categories motivated by the close connection between commutative separable algebras and the notion of étale morphism in tensor triangular geometry; see [Bal16, BDS15, San22]. Recent work of Ramzi [Ram23, Theorem B] shows that a commutative separable algebra in the homotopy category of an additive symmetric monoidal $\infty$-category $\mathcal{C}$ admits an essentially unique lift to a commutative algebra object in $\mathcal{C}$ itself.
Lemma 4.19. The commutative algebra structure on $h_{\mathscr{S}}(i)$ is separable.
Proof. We begin by noting that the algebra $h_{[i]_{+} \wedge \mathcal{S}}$ is separable. Indeed, the algebra structure is induced by the coalgebra structure on the spectrum $[i]_{+} \wedge S$ which is determined by the diagonal map $[i]_{+} \wedge \mathbb{S} \rightarrow[i] \times[i]_{+} \wedge \mathbb{S}$. This map has a retraction $[i] \times[i]_{+} \wedge \mathbb{S} \rightarrow[i]_{+} \wedge \mathbb{S}$. It is induced by the map of pointed sets $[i] \times[i]_{+} \rightarrow[i]_{+}$ which sends $(x, y)$ to $x$ when $x=y \in[i]$ and sends $(x, y)$ to the basepoint if $x \neq y$. It is straightforward to check that this retraction is a map of bi-comodules over $[i]_{+} \wedge \mathbb{S}$. The retraction induces a map $h_{[i]_{+} \wedge \mathbb{S}} \rightarrow h_{[i] \times[i]_{+} \wedge \mathbb{S}}$ which is the desired section of the multiplication map.

Next, consider the multiplication map $h_{\mathbb{S}}(i) \circledast h_{\mathbb{S}}(i) \rightarrow h_{\mathbb{S}}(i)$. By the proof of Lemma 4.14, this multiplication can be identified with a map between homotopy fibers of the following form

$$
\text { tofib } h_{U \times V_{+} \wedge \mathbb{S}} \rightarrow \text { tofib } h_{W_{+} \wedge \mathscr{S}}
$$

where $U, V, W$ range over subsets of $[i]$, and the map is induced by the diagonal functor $W \mapsto W \times W$. This map has a section

$$
\text { tofib } h_{W_{+} \wedge \mathscr{S}} \rightarrow \text { tofib } h_{U \times V_{+} \wedge \mathscr{S}}
$$

The section is induced by the functor from the poset of subsets of $[i] \times[i]$ to the poset of subsets of $[i]$ that sends $(U, V)$ to $U \cap V$, and the collapse maps $U \times V_{+} \rightarrow(U \cap V)_{+}$. It is straightforward to check that the section is well-defined and that it is indeed a map of bimodules as required.

## 5. Compact generation of the category of d-EXCISIVE Functors

In this section we prove that the functors $P_{d} h_{\mathbb{S}}(i), 1 \leq i \leq d$, form a set of compact generators for $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. To this end, we first record some features of cross-effects of excisive functors.

Remark 5.1. Recall that the $m$-th cross-effect $\mathrm{cr}_{m}(F)$ of a functor $F$ is a functor of $m$ variables (see Section 3). If $F$ is $d$-excisive then for each $1 \leq m \leq d+1$, the cross-effect $\mathrm{cr}_{m}(F)$ is $(d-m+1)$-excisive in each variable; see [Goo03, Proposition 3.3] or [Lur17, Proposition 6.1.3.22]. In particular, the $d$-th cross-effect of a $(d-1)$-excisive functor is trivial. Therefore, by applying $\mathrm{cr}_{d}$ to the fiber sequence $D_{d} F \rightarrow P_{d} F \rightarrow P_{d-1} F$, we obtain

$$
\operatorname{cr}_{d}\left(D_{d} F\right) \simeq \operatorname{cr}_{d}\left(P_{d} F\right)
$$

for any functor $F$.
Remark 5.2. Recall that $h_{x}$ is the corepresentable functor $h_{x}(y)=\Sigma^{\infty} \operatorname{Map}_{s p}(x, y)$ and that the $i$-th cross-effect of the contravariant functor $x \mapsto h_{x}$ is equivalent to $h_{x_{1}} \otimes \cdots \otimes h_{x_{i}}$ (see Section 4). The following lemma is well-known; for example, see [Chi21, Lemma 2.11]:

Lemma 5.3. Let $F \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ and $x_{1}, \ldots, x_{i} \in \mathcal{S} p^{c}$. There is an equivalence

$$
\operatorname{Hom}_{\operatorname{Fun}\left(\delta p^{c}, s p\right)}\left(h_{x_{1}}(-) \otimes \cdots \otimes h_{x_{i}}(-), F(-)\right) \simeq \operatorname{cr}_{i} F\left(x_{1}, \ldots, x_{i}\right)
$$

In $d$-excisive functors, this implies:
Lemma 5.4. Let $F \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right), 1 \leq i \leq d$ and $x_{1}, \ldots, x_{i} \in \mathcal{S} p^{c}$. There is an equivalence

$$
\operatorname{Hom}_{\operatorname{Exc}_{d}\left(s p^{c}, s p\right)}\left(P_{d}\left(h_{x_{1}}(-) \otimes \cdots \otimes h_{x_{i}}(-)\right), F(-)\right) \simeq \operatorname{cr}_{i} F\left(x_{1}, \ldots, x_{i}\right)
$$

Proof. Since $F$ is $d$-excisive, there is an equivalence

$$
\begin{aligned}
\operatorname{Hom}_{\operatorname{Exc}_{d}\left(\delta p^{c}, s p\right)}\left(P _ { d } \left(h_{x_{1}}(-)\right.\right. & \left.\left.\otimes \cdots \otimes h_{x_{i}}(-)\right), F(-)\right) \\
& \simeq \operatorname{Hom}_{\operatorname{Fun}\left(S p^{c}, s p\right)}\left(h_{x_{1}}(-) \otimes \cdots \otimes h_{x_{i}}(-), F(-)\right)
\end{aligned}
$$

and the lemma then follows from Lemma 5.3.

Remark 5.5. In terms of the analogy with equivariant stable homotopy theory, we may regard the functors $\operatorname{cr}_{i}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ as functor calculus analogs of the categorical fixed point functors $(-)^{H}: \mathcal{S} p_{G} \rightarrow \mathcal{S} p$ and we think of the functors $P_{d} h_{\mathbb{S}}(i) \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ as analogous to the $G$-spectra $\Sigma_{G}^{\infty} G / H_{+} \in \mathcal{S} p_{G}$. From this perspective, Lemma 5.3 is a functor calculus version of the fact that the $G$-spectrum $\Sigma_{G}^{\infty} G / H_{+}$represents $H$-fixed points in $G$-equivariant stable homotopy theory. See Appendix A for a dictionary between functor calculus and equivariant stable homotopy theory.

Example 5.6 (The first cross-effect). Let $\mathrm{cr}_{1}(-)(\mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ be the functor that takes the first cross-effect and evaluates it at $\mathbb{S}$. Since our $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ consists of reduced functors, this is the same as just evaluating a functor at $\mathbb{S}$. We observe that this functor is also right adjoint to the inflation functor $i_{d}: \mathcal{S} p \rightarrow$ $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ of Definition 2.44:

Lemma 5.7. The functor $\mathrm{cr}_{1}(-)(\mathbb{S})$ is right adjoint to $i_{d}: \mathcal{S} p \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. In particular, the functor $\mathrm{cr}_{1}(-)(\mathbb{S})$ is lax symmetric monoidal.

Proof. Let $\alpha: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ be the right adjoint to $i_{d}$, which exists by the adjoint functor theorem. Note that we have

$$
\operatorname{Hom}_{E x c_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)}\left(i_{d} \mathbb{S}, F\right) \simeq \operatorname{Hom}_{\mathcal{S} p}(\mathbb{S}, \alpha(F)) \simeq \alpha(F)
$$

by adjunction and the Yoneda lemma. Since $i_{d} \mathbb{S} \simeq P_{d} h_{\mathbb{S}}$ (as $i_{d}$ is symmetric monoidal), this shows that $\alpha$ is corepresented by $P_{d} h_{\mathfrak{S}}$. By Lemma 5.3 this identifies $\alpha$ with $\mathrm{cr}_{1}(-)(\mathbb{S})$. Finally, as the right adjoint of a symmetric monoidal functor, $\mathrm{cr}_{1}(-)(\mathbb{S})$ is lax symmetric monoidal [Lur17, Corollary 7.3.2.7].

Remark 5.8. This matches the perspective of Remark 5.5 in which the categorical fixed point functor $(-)^{G}$ is right adjoint to the canonical geometric functor $\mathcal{S} p \rightarrow \mathcal{S} p_{G}$ which equips a spectrum with the "trivial" $G$-action.

Definition 5.9. For any $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ and $i \geq 1$, let $c_{i} F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ denote the composite

$$
\mathcal{S} p^{c} \xrightarrow{\Delta}\left(\mathcal{S} p^{c}\right)^{\times i} \xrightarrow{\mathrm{cr}_{i} F} \mathcal{S} p
$$

where $\Delta$ is the diagonal. That is, $c_{i} F$ is the functor $X \mapsto \operatorname{cr}_{i} F(X, \ldots, X)$. In particular, note that there is an equivalence of functors $c_{i} h_{x}=h_{x}(i)$ by Remark 5.2.

Lemma 5.10. If $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ is $d$-excisive, then $c_{d} F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ is $d$-homogeneous.
Proof. This follows from Propositions 3.1 and 3.3 of [Goo03].
Lemma 5.11. For each $1 \leq i \leq d$, we have $P_{d} c_{i} F \simeq c_{i} P_{d} F$.
Proof. In the case $i=d$ this is shown in the proof of [Goo03, Theorem 6.1] (in Goodwillie's notation $\lambda \mathrm{cr}_{d}$ is what we write as $c_{d}$ ); the same proof works for arbitrary $i$. Indeed, $c_{i} F(x)$ is the total fiber of the $i$-dimensional cubical diagram that sends a subset $U \subseteq[i]$ to $F\left(\bigoplus_{[i] \backslash U} x\right)$. The functor $P_{d}$ commutes with finite limits and in particular commutes with total fibers of cubical diagrams [Goo03, Proposition 1.7].

As a consequence of Lemma 5.11 we obtain the following.

Lemma 5.12. For each $1 \leq i \leq d$, there are natural equivalences

$$
c_{i} P_{d} h_{\mathbb{S}} \simeq P_{d} c_{i} h_{\mathbb{S}} \simeq P_{d} h_{\mathbb{S}}(i)
$$

of functors $\mathcal{S} p^{c} \rightarrow \mathcal{S} p$.
Proposition 5.13. The functor $P_{d} h_{\mathbb{S}}(d)$ is d-homogeneous.
Proof. Since $P_{d} h_{\mathcal{S}}$ is $d$-excisive, Lemma 5.10 implies that $c_{d} P_{d} h_{\mathcal{S}}$ is $d$-homogeneous. The result then follows from Lemma 5.12.

Corollary 5.14. We have $P_{d} h_{\mathbb{S}}(d)(X) \simeq X^{\otimes d}$ where $\Sigma_{d}$ acts on $X^{\otimes d}$ by permuting the factors.

Proof. We have equivalences

$$
P_{d} h_{\mathbb{S}}(d)(X) \simeq c_{d} P_{d} h_{\mathbb{S}}(X) \simeq \operatorname{cr}_{d}\left(P_{d} h_{\mathbb{S}}\right)(X, \ldots, X) \simeq \operatorname{cr}_{d}\left(D_{d} h_{\mathbb{S}}\right)(X, \ldots, X)
$$

The functor $h_{\mathbb{S}}$ is the functor $\Sigma^{\infty} \Omega^{\infty}$ whose Goodwillie tower is well-known. In particular, we have $D_{d} h_{\mathbb{S}}(X) \simeq\left(X^{\otimes d}\right)_{h \Sigma_{d}}$. Computing $d$-th cross-effects (for example, by a minor modification of the method used in the proof of [AC15, Proposition 5.2]), we obtain the desired result.

Corollary 5.15. If $u>d$, then $P_{d} h_{\mathbb{S}}(u)=0$.
Proof. Note that $P_{d} h_{\mathbb{S}}(u) \simeq P_{d} P_{u} h_{\mathbb{S}}(u)$, but $P_{u} h_{\mathbb{S}}(u)$ is $u$-homogeneous (Proposition 5.13).

For the next statement, recall the notion of a "good" subset from Definition 4.9.
Proposition 5.16. For any $i, j \geq 1$, there is an equivalence

$$
\begin{equation*}
P_{d} h_{\mathbb{S}}(i) \circledast P_{d} h_{\mathbb{S}}(j) \simeq \bigoplus_{\substack{\mathcal{U} \subseteq[i] \times[j] \\ \text { good } \\|\mathcal{U}| \leq d}} P_{d} h_{\mathbb{S}}(|\mathcal{U}|) \tag{5.17}
\end{equation*}
$$

in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Proof. This follows from Proposition 4.11 and Corollary 5.15.
Remark 5.18. We view this proposition as a functor calculus version of the Mackey decomposition

$$
\Sigma_{G}^{\infty} G / H_{+} \otimes \Sigma_{G}^{\infty} G / K_{+} \simeq \bigoplus_{g \in H \backslash G / K} \Sigma_{G}^{\infty} G /\left(H^{g} \cap K\right)_{+}
$$

that holds in the category of $G$-spectra.
Example 5.19. We have

$$
P_{d} h_{\mathbb{S}}(i) \circledast P_{d} h_{\mathbb{S}}(d) \simeq \bigoplus_{|\operatorname{surj}(d, i)|} P_{d} h_{\mathbb{S}}(d)
$$

where $|\operatorname{surj}(d, i)|$ denotes the number of surjections from a set of cardinality $d$ to a set of cardinality $i$. This follows from the observation that the only summands in (5.17) correspond to those good subsets of $[i] \times[d]$ of cardinality exactly $d$, and there are exactly $|\operatorname{surj}(d, i)|$ of them.

Remark 5.20. It is useful to recall, for example using inclusion-exclusion (or alternatively using the table on page 73 along with Equation 1.94(a) of [Sta12]), that

$$
\begin{equation*}
|\operatorname{surj}(i, j)| \cong j^{i}+\sum_{s=1}^{j-1}(-1)^{s}\binom{j}{s}(j-s)^{i} \tag{5.21}
\end{equation*}
$$

Remark 5.22. The $G$-spectra $\Sigma_{G}^{\infty} G / H_{+}$are dualizable and in fact (for finite $G$ ) self-dual in the category of $G$-spectra. Our next goal is to show that the same holds for their functor calculus analogs $P_{d} h_{\mathbb{S}}(i)$; see Remark 5.5. In fact, this holds before applying $P_{d}$.

Proposition 5.23. For each $i \geq 1$ the functor $h_{\mathbb{S}}(i)$ is a self-dual dualizable object of $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Proof. We first establish that $h_{\mathbb{S}}(i)$ is dualizable in $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ for each $i \geq 1$. Indeed, by Remark 2.26 if $x \in \mathcal{S} p^{c}$ is dualizable, then $h_{x} \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is dualizable, and the dual of $h_{x}$ is $h_{\mathbb{D} x}$, where $\mathbb{D} x$ is the Spanier-Whitehead dual of $x$. In particular, if $x \in \mathcal{S} p^{c}$ is self-dual, then $h_{x}$ is self-dual. This implies that $h_{\oplus_{i} \mathscr{S}}$ is self-dual, and in particular $h_{\mathbb{S}}(i)$ is dualizable, as it is a summand of a dualizable object.

Moreover, we will show that $h_{\mathbb{S}}(i)$ is itself self-dual. Indeed, suppose $F$ is a dualizable object of a symmetric monoidal stable $\infty$-category and $e: F \rightarrow F$ is an idempotent map. Recall that $e F$ is the summand of $F$ split off by $e$. Then the object $e F$ is dualizable, and furthermore the dual of $e F$ is equivalent to $\mathbb{D} e(\mathbb{D} F)$, i.e., the summand of $\mathbb{D} F$ split off by the dual idempotent $\mathbb{D} e$. In particular, if $F$ and $e$ are both self-dual, then $e F$ is self-dual. Now recall from (4.12) that $h_{\mathbb{S}}(i)$ is split off the self-dual object $h_{\oplus_{i} S}$ by the idempotent

$$
\operatorname{cr}[i]=\bigcirc_{1 \leq s \leq i}\left(1-h_{\psi_{[i] \backslash\{s\}}}\right) .
$$

Here $h_{\psi_{[i] \backslash s\}}}: h_{\oplus_{i} \mathbb{S}} \rightarrow h_{\oplus_{i} \mathbb{S}}$ is the idempotent induced by the map $\oplus_{i} \mathbb{S} \rightarrow \oplus_{i} \mathbb{S}$ that collapses summand $s$ to a point. This map is self-dual and therefore $h_{\psi_{[i] \backslash\{s\}}}$ is self-dual. It follows that $\operatorname{cr}[i]$ is self-dual and thus $h_{\mathbb{S}}(i)$ is self-dual.

Remark 5.24. In fact, since the commutative algebra $h_{\mathbb{S}}(i)$ is separable (Lemma 4.19), dualizability of $h_{\mathbb{S}}(i)$ actually forces $h_{\mathbb{S}}(i)$ to be self-dual; see [San22, Section 2].

Corollary 5.25. Each $P_{d} h_{\mathbb{S}}(i)$ is a self-dual dualizable object of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Proof. This follows from Proposition 5.23 and Theorem 2.38 since symmetric monoidal functors preserve dualizable objects and their duals.

Remark 5.26. In equivariant homotopy theory, restricting to the trivial subgroup $\operatorname{res}_{1}^{G}: \mathcal{S} p_{G} \rightarrow \mathcal{S} p$ is strong symmetric monoidal, whereas $(-)^{H}: \mathcal{S} p_{G} \rightarrow \mathcal{S} p$ is only lax symmetric monoidal for $e \neq H \leq G$. The corresponding result in functor calculus is the following:

Proposition 5.27. The functor

$$
\operatorname{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p
$$

admits a symmetric monoidal refinement.

Proof. The following is essentially the proof given for the $d=2$ case on page 108 of $\left[\mathrm{CDH}^{+} 23\right]$. Let $\Delta: S p^{c} \rightarrow\left(S p^{c}\right)^{\times d}$ denote the diagonal functor, and consider the induced functor $\Delta^{*}: \operatorname{Fun}\left(\left(\mathcal{S} p^{c}\right)^{\times d}, \mathcal{S} p\right) \rightarrow \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. This functor admits a left adjoint, via the left Kan extension along $\Delta$. Since the diagonal is symmetric monoidal, so is the left Kan extension. Moreover, since the diagonal has a two sided adjoint given by the direct sum functor $q:\left(\mathcal{S} p^{c}\right)^{\times d} \rightarrow \mathcal{S} p$, this left Kan extension is just given by restriction along $q^{*}$.

We note that by [Lur17, Corollary 6.1.3.5] the functor $\Delta^{*}$ restricts to a functor $\operatorname{Exc}_{(1, \ldots, 1)}\left(\left(\mathcal{S} p^{c}\right)^{\times d}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. Moreover, as in Examples 2.7 and 2.45 we can identify

$$
\begin{equation*}
\operatorname{Exc}_{(1, \ldots, 1)}\left(\left(\mathcal{S} p^{c}\right)^{\times d}, \mathcal{S} p\right) \simeq \mathcal{S} p \tag{5.28}
\end{equation*}
$$

as symmetric monoidal $\infty$-categories, given by evaluation at $(\mathbb{S}, \ldots, \mathbb{S})$. Using the universal property of the localization $P_{d}$ (see [Lur09b, Proposition 5.5.4.20]), there then exists a symmetric monoidal functor $\bar{q}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{(1, \ldots, 1)}\left(\left(\mathcal{S} p^{c}\right)^{\times d}, \mathcal{S} p\right)$ which fits in the commutative diagram:


The composite $p_{1, \ldots, 1} \circ q^{*}$ is the $d$-th cross-effect by [Lur17, Remark 6.1.3.23]. Unwinding the definitions, we see that the dashed arrow $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ is a symmetric monoidal functor whose underlying functor is exactly $\mathrm{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S})$.

Remark 5.29. For $i<d$ the functor

$$
\operatorname{cr}_{i}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p
$$

is lax symmetric monoidal (for example, because it is corepresented by the commutative algebra object $\left.P_{d} h_{\mathbb{S}}(i)\right)$ but not symmetric monoidal. For example, one has $\operatorname{cr}_{1} P_{d} h_{\mathbb{S}}(d) \simeq \mathbb{S}$ but $\mathrm{cr}_{1}\left(P_{d} h_{\mathbb{S}}(d) \circledast P_{d} h_{\mathbb{S}}(d)\right) \simeq \mathbb{S}^{\oplus d!}$ as follows from Example 5.19.

Remark 5.30. In equivariant homotopy theory, the categorical fixed point functors $\left\{(-)^{H}\right\}_{H \leq G}$ are jointly conservative on $G$-spectra. Our next goal is to show that the same is true for the collection $\left\{\operatorname{cr}_{i}(-)(\mathbb{S}, \ldots, \mathbb{S})\right\}_{1 \leq i \leq d}$ on the category of $d$-excisive functors. We begin by working with $d$-homogeneous functors (Definition 2.13).

Lemma 5.31. Let $F$ be a d-homogeneous functor. Then

$$
F=0 \Longleftrightarrow \mathrm{cr}_{d} F(\mathbb{S}, \ldots, \mathbb{S})=0
$$

Proof. By [Goo03, Proposition 3.4] (see also [Lur17, Corollary 6.1.4.11]) we have that $F=0 \Longleftrightarrow \operatorname{cr}_{d} F\left(X_{1}, \ldots, X_{d}\right)=0$ for all $X_{1}, \ldots, X_{d} \in \mathcal{S} p^{c}$. Thus, it suffices to show that

$$
\operatorname{cr}_{d} F(\mathbb{S}, \ldots, \mathbb{S})=0 \Longrightarrow \operatorname{cr}_{d} F\left(X_{1}, \ldots, X_{n}\right)=0
$$

We first show that

$$
\operatorname{cr}_{d} F(\mathbb{S}, \ldots, \mathbb{S})=0 \Longrightarrow \operatorname{cr}_{d} F\left(X_{1}, \mathbb{S}, \ldots, \mathbb{S}\right)=0
$$

Indeed, the collection $\mathcal{C}:=\left\{X \in \mathcal{S} p^{c} \mid \operatorname{cr}_{d} F(X, \mathbb{S}, \ldots, \mathbb{S})=0\right\}$ is a thick subcategory (this uses that $F$ is reduced to show that it is closed under extensions) which contains $\mathbb{S}$ by assumption. Therefore, $\mathcal{C}=\mathcal{S} p^{c}$. Repeating the argument in each variable then gives the desired result.

Proposition 5.32. The functor

$$
\prod_{1 \leq i \leq d} \operatorname{cr}_{i}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \prod_{1 \leq i \leq d} \mathcal{S} p
$$

is conservative. In other words, a functor $F \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is trivial if and only if $\operatorname{cr}_{i}(F)(\mathbb{S}, \ldots, \mathbb{S})=0$ for $1 \leq i \leq d$.

Proof. We prove this by induction on $d$. For $d=1$, under the identification $\operatorname{Exc}_{1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq \mathcal{S} p$ of Example 2.7, $\mathrm{cr}_{1}(-)(\mathbb{S})$ is the identity functor, and so the claim becomes a tautology. Next assume inductively that the statement is true for $(d-1)$-excisive functors, and let $F$ be an $d$-excisive functor whose cross-effects vanish. Recall that we have a fiber sequence

$$
D_{d} F \rightarrow F \rightarrow P_{d-1} F
$$

where $D_{d} F$ is $d$-homogeneous and $P_{d-1} F$ is $(d-1)$-excisive. By Remark 5.1 and our hypothesis we have $\mathrm{cr}_{d}\left(D_{d} F\right) \simeq \mathrm{cr}_{d}(F)=0$, so that $D_{d} F=0$ by Lemma 5.31. In particular, $F \simeq P_{d-1} F$ and $\operatorname{cr}_{i} P_{d-1} F(\mathbb{S}, \ldots, \mathbb{S})=0$ for $1 \leq i \leq d-1$. By induction, $F \simeq P_{d-1} F=0$, and we are done.
Theorem 5.33. The category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is compactly generated. Moreover, the functors $P_{d}\left(h_{\mathbb{S}}(i)\right)$ for $1 \leq i \leq d$ form a set of compact generators of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Proof. Let $\mathcal{G} \subseteq \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ denote the full subcategory of corepresentable functors. Ching [Chi21, Lemma 4.14] establishes that $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is compactly generated, that its compact objects are precisely the retracts of finite colimits of diagrams in $\mathcal{G}$, and that these compact objects are closed under the objectwise smash product of functors. By Remark 2.11, $P_{d}$ is a smashing localization, so [Lur09b, Corollary 5.5.7.3] implies that the category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is also compactly generated. In fact, the same corollary shows that the functor $P_{d}$ preserves compact objects and that a functor $G \in \operatorname{Exc}_{n}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is compact if and only if there exists $F \in \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ such that $G$ is a retract of $P_{d} F$.

We now show that $\left\{P_{d}\left(h_{\mathbb{S}}(i)\right) \mid 1 \leq i \leq d\right\}$ is a set of compact generators. Since the objectwise smash product of compact functors in $\operatorname{Fun}(\mathcal{S} p, \mathcal{S} p)$ is still compact, each $h_{\mathbb{S}}(i)$ is compact in $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ and hence is compact in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ after applying $P_{d}$, as explained above. To show that this set of compact objects is a set of generators, it suffices by [SS03, Lemma 2.2.1] to show that if

$$
\operatorname{Hom}_{\operatorname{Exc}_{d}\left(s_{\left.p^{c}, S p\right)}\right.}\left(P_{d}\left(h_{\mathscr{S}}(i)\right), F\right)=0 \quad \text { for all } 1 \leq i \leq d
$$

then $F=0$. Applying Lemma 5.4, we see that this is equivalent to the statement that if $\operatorname{cr}_{i} F(\mathbb{S}, \ldots, \mathbb{S})=0$ for $1 \leq i \leq d$, then $F=0$, which is precisely the content of Proposition 5.32.

Corollary 5.34. The category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is rigidly-compactly generated; that is, it is compactly generated and the compact and dualizable objects coincide.

Proof. We have already seen in Theorem 5.33 that $\operatorname{Exc}_{d}\left(S p^{c}, \mathcal{S} p\right)$ is compactly generated by $\left\{P_{d}\left(h_{\mathbb{S}}(i)\right) \mid 1 \leq i \leq d\right\}$. Moreover, each of these compact generators
is dualizable by Corollary 5.25. In the language of [HPS97], we have shown that $\operatorname{Exc}_{d}\left(S p^{c}, \mathcal{S} p\right)$ is a unital algebraic stable homotopy category. It then follows from [HPS97, Theorem 2.1.3(d)] that the compact and dualizable objects coincide.
Remark 5.35. We can also deduce from the proof fo Proposition 5.27 that

$$
\operatorname{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p
$$

has equivalent left and right adjoints, given by the composite $\mathcal{S} p \xrightarrow{i} \operatorname{Exc}_{(1, \ldots, 1)} \xrightarrow{\Delta^{*}}$ $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$, i.e., sending a spectrum $A$ to the $d$-excisive functor $X \mapsto A \otimes X^{\otimes d}$. (This follows since the diagonal $\Delta: S p^{c} \rightarrow\left(\mathcal{S} p^{c}\right)^{\times d}$ has equivalent left and right adjoints.) Let us denote this adjoint by $\kappa_{d}$. We then obtain the following, which is the functor calculus analog of [BDS15, Theorem 1.1]:

Theorem 5.36. There is an equivalence of symmetric monoidal $\infty$-categories

$$
\operatorname{Mod}_{\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)}\left(P_{d} h_{\mathbb{S}}(d)\right) \simeq \mathfrak{S} p
$$

under which $\operatorname{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S})$ is identified with extension of scalars along the commutative algebra $P_{d} h_{\mathbb{S}}(d)$. That is, we have a commutative diagram

where $F$ and $U$ denote extension and restriction of scalars along $P_{d} h_{\mathbb{S}} \rightarrow P_{d} h_{\mathbb{S}}(d)$.
Proof. Note that by Corollary 5.14 the functor $\kappa_{d}$ sends the unit $\mathbb{S}$ to $P_{d} h_{\mathbb{S}}(d)$. Moreover, $\mathrm{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S})$ is symmetric monoidal by Proposition 5.27. The theorem will then be a consequence of [MNN17, Proposition 5.29] if we can show the following:
(a) The adjunction $\operatorname{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S}) \dashv \kappa_{d}$ satisfies the projection formula;
(b) $\kappa_{d}$ preserves colimits; and
(c) $\kappa_{d}$ is conservative.

The first follows formally from [BDS16, Proposition 2.16] using Corollary 5.34; the second follows from the fact that $\kappa_{d}$ is both a left and right adjoint; and the third follows from the observation that $\mathrm{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S})$ is essentially surjective (as the $d$-th cross-effect of the functor $X \mapsto P_{d} h_{\mathbb{S}}(A \otimes X)$ evaluated at $(\mathbb{S}, \ldots, \mathbb{S})$ is $\left.A\right)$. Here we are using the observation that $\kappa_{d}$ is conservative if and only if the essential image of $\mathrm{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S})$ contains a family of generators.

Remark 5.37. The commutative algebra $P_{d} h_{\mathbb{S}}(d)$ is separable, as a consequence of Lemma 4.19. In other words, up to equivalence, the functor

$$
\operatorname{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p
$$

is given by extension of scalars along a compact commutative separable algebra. Such extensions are called finite étale; see [Bal16, San22].

## 6. Tensor idempotents and Goodwillie derivatives

We now recall some facts about tensor idempotents and their relation with completeness, torsion and the Tate construction. We will utilize the geometric perspective afforded by the Balmer spectrum. In the context of $d$-excisive functors, these constructions will also lead naturally to the Goodwillie derivatives.

## Tensor idempotents and the Tate construction.

Definition 6.1 (The Balmer spectrum). Let $\mathcal{T}$ be a rigidly-compactly generated tensor triangulated category such as (the homotopy category of) $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. We write $\mathcal{T}^{c}$ for the tensor triangulated subcategory of compact (=dualizable) objects. The Balmer spectrum of $\mathcal{T}^{c}$ is the topological space $\operatorname{Spc}\left(\mathcal{T}^{c}\right)$ whose points are the prime tt-ideals of $\mathcal{T}^{c}$ and whose topology has $\{\operatorname{supp}(a)\}_{a \in \mathcal{T}^{c}}$ as a basis of closed sets, where

$$
\operatorname{supp}(a):=\left\{\mathcal{P} \in \operatorname{Spc}\left(\mathcal{T}^{c}\right) \mid a \notin \mathcal{P}\right\}
$$

This topology is spectral in the sense of [DST19] and the closure of a prime tt-ideal $\mathcal{P} \in \operatorname{Spc}\left(\mathcal{T}^{c}\right)$ is given by $\overline{\{\mathcal{P}\}}=\{\mathcal{Q} \mid \mathcal{Q} \subseteq \mathcal{P}\}$. See [Bal05] for more details.
Remark 6.2. The spectrum is contravariantly functorial. In particular, if $F: \mathcal{T} \rightarrow \mathcal{U}$ is a tt-functor of rigidly-compactly generated tt-categories, then there is an induced continuous map $\operatorname{Spc}(F): \operatorname{Spc}\left(\mathcal{U}^{c}\right) \rightarrow \operatorname{Spc}\left(\mathcal{T}^{c}\right)$ given by $\mathcal{P} \mapsto\left\{x \in \mathcal{T}^{c} \mid F(x) \in \mathcal{P}\right\}$. Note that $\operatorname{Spc}(F)$ preserves inclusions $Q \subseteq \mathcal{P}$ of prime tt-ideals by construction.

Definition 6.3. A subset $Y \subseteq \operatorname{Spc}\left(\mathcal{T}^{c}\right)$ is a Thomason subset if it can be written as a union $\cup_{\alpha} Y_{\alpha}$ in which each $Y_{\alpha}$ is a closed set whose complement is quasi-compact.

Remark 6.4. By [Bal05, Theorem 4.10] there is a bijection

$$
\left\{\begin{array}{c}
\text { thick ideals } \\
\text { of } \mathcal{T}^{c}
\end{array}\right\} \stackrel{\sim}{\longleftrightarrow}\left\{\begin{array}{c}
\text { Thomason subsets } \\
\text { of } \operatorname{Spc}\left(\mathcal{T}^{c}\right)
\end{array}\right\}
$$

which sends a thick ideal $\mathcal{J}$ to $\operatorname{supp}(\mathcal{J}):=\bigcup_{a \in \mathcal{J}} \operatorname{supp}(a)$ and with inverse given by $Y \mapsto \mathcal{T}_{Y}^{c}:=\left\{a \in \mathcal{T}^{c} \mid \operatorname{supp}(a) \subseteq Y\right\}$.

Example 6.5. The Balmer spectrum of $\mathcal{S} p^{c}$ is described in [Bal10b, Section 9] by reinterpreting the work of Hopkins-Smith [HS98]. For a prime number $p$ and chromatic height $1 \leq h \leq \infty$, we have a prime ideal $\mathcal{C}_{p, h} \in \operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ which consists of those finite spectra annihilated by the $p$-local Morava $K$-theory $K(p, h-1)$ :

$$
\mathcal{C}_{p, h}:=\left\{x \in \mathcal{S} p^{c} \mid K(p, h-1)_{*}(x)=0\right\} .
$$

In particular, $\mathcal{C}_{p, \infty}$ is the kernel of $\mathbb{F}_{p}$-homology $\mathrm{HF}_{p}$, while $\mathcal{C}_{p, 1}$ is the kernel of rational homology HQ (that is, the finite torsion spectra), independently of $p$. Consequently, we also write $\mathcal{C}_{0,1}:=\mathcal{C}_{p, 1}$ (for all $p$ ). With these definitions in hand, $\operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ is the topological space depicted in Figure 2 below. In this figure, inclusion goes downwards, while closure goes upwards; in particular, the points $\mathcal{C}_{p, \infty}$ are closed points, while $\mathcal{C}_{0,1}$ is a generic point. A more detailed explanation of the topology may be found in [Bal10b, Corollary 9.5]. For later use, we also introduce the notation $\mathcal{C}_{p, 0}:=\mathcal{S} p_{(p)}^{c}$ for the category of finite $p$-local spectra.
Remark 6.6. We now recall some constructions from [BF11] and [BS17, Section 5]. Let $Y \subseteq \operatorname{Spc}\left(\mathcal{T}^{c}\right)$ be a Thomason subset and let $V:=\operatorname{Spc}\left(\mathcal{T}^{c}\right) \backslash Y$ be its complement. Then there is an associated idempotent triangle

$$
\begin{equation*}
e_{Y} \rightarrow \mathbb{1} \rightarrow f_{Y} \rightarrow \Sigma e_{Y} \tag{6.7}
\end{equation*}
$$

in $\mathcal{T}$, i.e., an exact triangle with $e_{Y} \otimes f_{Y}=0$. Moreover, we have

$$
\mathcal{T}_{Y}:=e_{Y} \otimes \mathcal{T}=\operatorname{ker}\left(f_{Y} \otimes-\right)=\operatorname{Locid}\left\langle e_{Y}\right\rangle=\operatorname{Loc}\left\langle\mathcal{T}_{Y}^{c}\right\rangle
$$

and

$$
\mathcal{T}(V):=f_{Y} \otimes \mathcal{T} \simeq \mathcal{T} / \operatorname{Locid}\left\langle e_{Y}\right\rangle=\mathcal{T} / \operatorname{Loc}\left\langle\mathcal{T}_{Y}^{c}\right\rangle
$$



Figure 2. The Balmer spectrum $\operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ of the category of finite spectra
and

$$
\mathcal{T}^{Y}:=\operatorname{hom}\left(e_{Y}, \mathcal{T}\right)=\operatorname{Coloc}\left\langle\mathcal{T}_{Y}^{c} \otimes \mathcal{T}\right\rangle
$$

The three categories $\mathcal{T}_{Y}, \mathcal{T}(V)$ and $\mathcal{T}^{Y}$ are related by taking right orthogonals:

$$
\mathcal{T}(V)=\left(\mathcal{T}_{Y}\right)^{\perp}=\left\{t \in \mathcal{T} \mid \operatorname{hom}(s, t)=0 \text { for all } s \in \mathcal{T}_{Y}\right\}
$$

and

$$
\mathcal{T}^{Y}=(\mathcal{T}(V))^{\perp}=\{t \in \mathcal{T} \mid \operatorname{hom}(s, t)=0 \text { for all } s \in \mathcal{T}(V)\}
$$

Moreover, the categories $\mathcal{T}_{Y}$ and $\mathcal{T}^{Y}$ are equivalent via the functors hom $\left(e_{Y},-\right)$ and $e_{Y} \otimes-$. Note in particular that $\mathcal{T} \rightarrow \mathcal{T}(V)$ is a finite localization whose compactly generated subcategory of acyclic objects is $\mathcal{T}_{Y}=\operatorname{Loc}\left\langle\mathcal{T}_{Y}^{c}\right\rangle \subseteq \mathcal{T}$. On Balmer spectra, this finite localization induces the embedding

$$
\operatorname{Spc}\left(\mathcal{T}(V)^{c}\right) \cong V \hookrightarrow \operatorname{Spc}\left(\mathcal{T}^{c}\right)
$$

which explains the notation $\mathcal{T}(V)$; see [BHS23b, Remark 1.23].
Remark 6.8. We can represent the situation in the previous remark as follows:

where the dashed arrows indicate orthogonality. In other words, we have a recollement; see [BS17, Section 5].
Example 6.10. For any compact object $A \in \mathcal{T}^{c}$, we can consider the Thomason closed subset $Y:=\operatorname{supp}(A)$. Then the categories $\mathcal{T}_{\operatorname{supp}(A)}, \mathcal{T}\left(\operatorname{supp}(A)^{\complement}\right)$ and $\mathcal{T}^{\operatorname{supp}(A)}$ are precisely the categories of $A$-torsion, $A$-local, and $A$-complete objects respectively, in the sense of [HPS97, Theorem 3.35], [BHV18, Theorem 2.21], or [MNN17,

Theorem 3.9]. In particular, we have that $\mathcal{T}_{\operatorname{supp}(A)}=\operatorname{Locid}\langle A\rangle \subseteq \mathcal{T}$. However, these categories only depend on the thick ideal in $\mathcal{T}^{c}$ generated by $A$ or, equivalently by Remark 6.4, the Thomason subset $\operatorname{supp}(A)$. Our approach emphasizes the geometric perspective by regarding these constructions as associated to a Thomason subset of the Balmer spectrum rather than to a collection of objects of the category. Finally, note that $\mathcal{T} \rightarrow \mathcal{T}\left(\operatorname{supp}(A)^{\complement}\right)$ is the finite localization away from the object $A$. Geometrically it restricts to the open complement of $\operatorname{supp}(A)$.

The following definition is due to Greenlees [Gre01] in the axiomatic setting.
Definition 6.11. Let $Y \subseteq \operatorname{Spc}\left(\mathcal{T}^{c}\right)$ be a Thomason subset. We define the Tate functor with respect to $Y$ to be

$$
t_{Y}:=f_{Y} \otimes \operatorname{hom}\left(e_{Y},-\right): \mathcal{T} \rightarrow \mathcal{T}
$$

Remark 6.12. Greenlees' Warwick duality [Gre01, Corollary 2.5] shows that the Tate construction satisfies $t_{Y} \simeq \operatorname{hom}\left(f_{Y}, \Sigma e_{Y} \otimes-\right)$. Moreover, there is a pullback square

for any $X \in \mathcal{T}$.
The following result is due to Balmer and Sanders [BS17, Proposition 5.11].
Proposition 6.14 (Balmer-Sanders). Let $F: \mathcal{T} \rightarrow \mathcal{U}$ be a coproduct-preserving tensor triangulated functor and let $\phi: \operatorname{Spc}\left(\mathcal{U}^{c}\right) \rightarrow \operatorname{Spc}\left(\mathcal{T}^{c}\right)$ be the induced map. Let $Y \subseteq \operatorname{Spc}\left(\mathcal{T}^{c}\right)$ be a Thomason subset and set $Y^{\prime}:=\phi^{-1}(Y) \subseteq \operatorname{Spc}\left(\mathcal{U}^{c}\right)$. Then $\mathcal{U}_{Y^{\prime}}^{c}=\operatorname{thickid}\left\langle F\left(\mathcal{T}_{Y}^{c}\right)\right\rangle$ and there is a unique isomorphism of idempotent triangles

$$
F\left(e_{Y} \rightarrow \mathbb{1} \rightarrow f_{Y} \rightarrow \Sigma e_{Y}\right) \simeq\left(e_{Y^{\prime}} \rightarrow \mathbb{1} \rightarrow f_{Y^{\prime}} \rightarrow \Sigma e_{Y^{\prime}}\right)
$$

in $\mathcal{U}$. If moreover $F$ is closed monoidal, then $F \circ t_{Y} \simeq t_{Y^{\prime}} \circ F$.
Example 6.15 (Chromatic truncation). Suppose that $\mathcal{C}$ is a symmetric monoidal stable $\infty$-category, so that it admits a unique symmetric monoidal colimit preserving functor $i: \mathcal{S} p \rightarrow \mathcal{C}$ as in Remark 2.43. Let $1 \leq h \leq \infty$ and consider the Thomason subset

$$
Y_{p, h}:=\left\{\mathcal{C}_{p, m} \mid m>h\right\} \cup\left\{\mathcal{C}_{q, m} \mid q \neq p, m>1\right\} \subseteq \operatorname{Spc}\left(\mathcal{S} p^{c}\right)
$$

with associated idempotent triangle $e_{p, h} \rightarrow \mathbb{1} \rightarrow f_{p, h} \rightarrow \Sigma e_{p, h}$ in $\mathcal{S} p$, as in [BS17, Example 5.12]. The right idempotent $f_{p, h}$ is the finite localization $L_{h-1}^{f} \mathbb{S}$ of the sphere spectrum. By Proposition 6.14, the Thomason subset

$$
Y_{p, h}^{\prime}:=\operatorname{Spc}(i)^{-1}\left(Y_{p, h}\right) \subseteq \operatorname{Spc}\left(\mathcal{C}^{c}\right)
$$

has associated idempotent triangle

$$
i\left(e_{p, h}\right) \rightarrow \mathbb{1} \rightarrow i\left(f_{p, h}\right) \rightarrow \Sigma i\left(e_{p, h}\right)
$$

in $\mathcal{C}$. The chromatic truncation of $\mathcal{C}$ below height $h$ (at the prime $p$ ) is the finite localization

$$
\mathcal{C}_{p, \leq h}:=i\left(f_{p, h}\right) \otimes \mathcal{C} .
$$

It has the property that $\operatorname{Spc}\left(\left(\mathcal{C}_{p, \leq h}\right)^{c}\right) \cong \operatorname{Spc}\left(\mathcal{C}^{c}\right) \backslash Y_{p, h}^{\prime}$. Note that $\mathcal{C}_{p, \leq \infty}$ is simply the $p$-localization of $\mathcal{C}$.

Warning 6.16. Because of the way the primes $\mathcal{C}_{p, h}$ of $\operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ are indexed (as the kernel of $K(p, h-1)$ ), there can be an off-by-one discrepancy in different usages of the term "height" which can lead to confusion. For example, note that the prime $\mathcal{C}_{p, h}$ has height $h-1$ in the sense of Krull dimension. Geometrically, truncation below height $h$ is restriction to the open piece


This explains the notation $\mathcal{S} p \rightarrow \mathcal{S} p_{p, \leq h}$. However, note that this is the finite localization associated to Morava $E$-theory $E(h-1)$ which is a ring spectrum of height $h-1$ in the sense of chromatic homotopy theory; see Definition 10.27 below. Nevertheless, it is convenient to use language like "consider a chromatic height $1 \leq h \leq \infty$ " even if the construction associated to this choice of $h$ results in something that really has "height" $h-1$.

Remark 6.17. In the situation of Proposition 6.14, if $G$ denotes the right adjoint of $F: \mathcal{T} \rightarrow \mathcal{U}$ then one readily checks that

$$
G t_{\phi^{-1}(Y)}(F t) \simeq t_{Y}(t \otimes G \mathbb{1})
$$

for all $t \in \mathcal{T}$ by using the standard isomorphisms of [BDS16].
Completeness and torsion in $d$-excisive functors. We now specialize the discussion to the category of $d$-excisive functors. Although we have not yet determined its spectrum, we can proceed as in Example 6.10 with respect to the compact $d$-excisive functor $A=P_{d} h_{\mathbb{S}}(d)$, that is, with respect to the Thomason closed subset $Y=\operatorname{supp}\left(P_{d} h_{\Im}(d)\right)$. Our goal is to understand the recollement (6.9) and the associated Tate construction in this example.

Notation 6.18. We will write $t_{d}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ for the Tate functor associated to the Thomason closed subset $Y:=\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right)$. That is, $t_{d}:=$ $t_{\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right)}$ in the notation of Definition 6.11.
Theorem 6.19. Let $Y:=\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right) \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$. Then

$$
e_{Y} \otimes \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)=\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)
$$

and the associated idempotent triangle in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is

$$
D_{d} h_{\mathbb{S}} \rightarrow P_{d} h_{\mathbb{S}} \rightarrow P_{d-1} h_{\mathbb{S}} \rightarrow \Sigma D_{d} h_{\mathbb{S}}
$$

In particular, $P_{d-1}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is the finite localization away from $P_{d} h_{\mathbb{S}}(d)$.
Proof. We need to show that $\operatorname{ker}\left(P_{d-1}\right)=\operatorname{Locid}\left(P_{d} h_{\mathbb{S}}(d)\right)$. To see this, observe first that $P_{d-1} P_{d} h_{\mathbb{S}}(d) \simeq P_{d-1} h_{\mathbb{S}}(d)=0$ because $P_{d} h_{\mathbb{S}}(d)$ is $d$-homogeneous (Proposition 5.13). It follows that $P_{d} h_{\mathscr{S}}(d) \in \operatorname{ker}\left(P_{d-1}\right)$, so that $\operatorname{Locid}\left(P_{d} h_{\mathbb{S}}(d)\right) \subseteq \operatorname{ker}\left(P_{d-1}\right)$.

On the other hand, let $H \in \operatorname{ker}\left(P_{d-1}\right)$ and note that $H$ is $d$-homogeneous. Suppose then that

$$
\operatorname{Hom}_{\operatorname{Exc}_{d}\left(s p^{c}, s p\right)}\left(P_{d} h_{\mathbb{S}}(d), H\right)=0
$$

By Lemma 5.4 we have that $\mathrm{cr}_{d}(H)(\mathbb{S}, \ldots, \mathbb{S})=0$ and hence $H=0$ by Lemma 5.31. We deduce that $\operatorname{ker}\left(P_{d-1}\right)$ coincides with the localizing ideal generated by the compact object $P_{d} h_{\mathbb{S}}(d)$. Hence the localization $P_{d-1}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ coincides with the finite localization away from $P_{d} h_{\mathbb{S}}(d)$.

Remark 6.20. This result shows that the category of $d$-homogeneous functors in Goodwillie calculus is analogous to the category of free $G$-spectra in equivariant homotopy theory.

Corollary 6.21. The functor $P_{d-i}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d-i}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a finite localization whose ideal of acyclics is generated by $\left\{P_{d}\left(h_{\mathbb{S}}(j)\right) \mid d-i+1 \leq j \leq d\right\}$.

Proof. We prove this by induction on $i$. One can either start the induction at $i=0$, where the statement is vacuous, or at $i=1$, where it is the content of Theorem 6.19. In either case suppose the statement of the corollary holds for the functor $P_{d-i}$. The localization $P_{d-i-1}: \operatorname{Exc}_{d-i}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d-i-1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is the finite localization that kills $P_{d-i}\left(h_{\mathbb{S}}(d-i)\right)$. Noting that this is the image of the generator $P_{d}\left(h_{\mathbb{S}}(d-i)\right)$ under the localization $P_{d-i}$, the inductive hypothesis and the evident 'third isomorphism theorem' for localizations gives the result.

Example 6.22. The obvious extension of Corollary 6.21 obtained by letting $d$ go to $\infty$ is false. By this we mean that the kernel of the functor

$$
P_{k}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{k}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)
$$

is not generated by $\left\{h_{\mathbb{S}}(j) \mid k<j\right\}$. To see this, let us consider the case $k=0$. In this case $P_{0}$ is the trivial functor and the kernel of $P_{0}$ is all of $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. This category is not generated by $\left\{h_{\mathbb{S}}(j) \mid 0<j\right\}$. Indeed, consider the functor $F(X)=\Sigma^{\infty} \Omega^{\infty+1}(H \mathbb{Q} \wedge X)$. This functor has the property that $F(X) \simeq 0$ whenever $X$ is a wedge of finitely many copies of the sphere spectrum. It follows that all the cross-effects of $F$ are trivial, i.e., $\operatorname{cr}_{j} F(\mathbb{S}, \ldots, \mathbb{S}) \simeq 0$ for all $j>0$. By Lemma 5.3, this means that for all $j>0$ we have

$$
\operatorname{Hom}_{\operatorname{Fun}\left(S p^{c}, S p\right)}\left(h_{\mathbb{S}}(j), F\right) \simeq 0
$$

but $F$ is clearly not a trivial functor.
Going further, we suspect that $\operatorname{Exc}_{k}\left(S p^{c}, S p\right)$ is not a finite localization of Fun $\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$, i.e., that the kernel of the localization is not generated by compact objects.

With the torsion categories now identified, let us turn to the complete category.
Proposition 6.23. In the situation of Theorem 6.19, we have an equivalence of symmetric monoidal $\infty$-categories

$$
\operatorname{hom}\left(e_{Y}, \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)\right) \simeq \operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)
$$

In particular, there is an equivalence of symmetric monoidal stable $\infty$-categories $\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq \operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)$.

Proof. Our proof is modeled on [MNN17, Proposition 6.17], which is the corresponding statement in equivariant homotopy theory. For brevity, let us write $\mathcal{C}$ for $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. Because $P_{d} h_{\mathbb{S}}(d)$ is dualizable, we can apply [MNN17, Theorem 2.30] to deduce that
$\operatorname{hom}\left(e_{Y}, \mathcal{C}\right) \simeq \operatorname{Tot}\left(\operatorname{Mod}_{\mathcal{C}}\left(P_{d} h_{\mathbb{S}}(d)\right) \Longrightarrow \operatorname{Mod}_{\mathcal{C}}\left(P_{d} h_{\mathbb{S}}(d) \circledast P_{d} h_{\mathbb{S}}(d)\right) \Longrightarrow \cdots\right)$.
We have seen in Theorem 5.36 that $\operatorname{Mod}_{\mathcal{C}}\left(P_{d} h_{\mathscr{S}}(d)\right) \simeq \mathcal{S} p$. Using Example 5.19 we deduce moreover that

$$
\operatorname{Mod}_{\mathcal{C}}\left(P_{d} h_{\mathbb{S}}(d)^{\circledast k}\right) \simeq \prod_{\left(\Sigma_{d}\right) \times(k-1)} \mathcal{S} p
$$

Unwinding the definitions, we find that hom $\left(e_{Y}, \mathcal{C}\right)$ is identified with a totalization

$$
\operatorname{hom}\left(e_{Y}, \mathcal{C}\right) \simeq \operatorname{Tot}\left(\mathcal{S} p \Longrightarrow \prod_{\Sigma_{d}} \mathcal{S} p \Longrightarrow \cdots\right)
$$

which recovers the functor category $\operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)$ for the standard simplicial decomposition of $B \Sigma_{d}$. The final claim then follows from the symmetric monoidal equivalence of $\infty$-categories

$$
e_{Y} \otimes \mathcal{C} \simeq \operatorname{hom}\left(e_{Y}, \mathcal{C}\right)
$$

as noted in Remark 6.6.
Remark 6.24. The local duality diagram can then be written in the following form:


The equivalence $\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{\sim} \operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)$ is Goodwillie's classification of $d$-homogeneous functors [Goo03, Theorem 3.5 and $\S 5$ ]:
Lemma 6.26 (Goodwillie). The equivalence $\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{\sim} \operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)$ sends a d-homogeneous functor to $\operatorname{cr}_{d}(\mathbb{S}, \ldots, \mathbb{S})$. Its inverse sends $A \in \operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)$ to the functor $G_{A}$ with $G_{A}(X)=\left(A \otimes X^{\otimes d}\right)_{h \Sigma_{d}}$.
Definition 6.27. Let $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ be a reduced functor. The $d$-th Goodwillie derivative of $F$ is defined by

$$
\partial_{d} F:=\operatorname{cr}_{d}\left(D_{d} F\right)(\mathbb{S}, \ldots, \mathbb{S})
$$

Remark 6.28. Note that Remark 5.1 implies that $\mathrm{cr}_{d}\left(D_{d} F\right) \simeq \mathrm{cr}_{d}\left(P_{d} F\right)$, so we could have alternatively defined $\partial_{d}$ in terms of its $d$-excisive approximation. In particular, if $F$ is $d$-excisive, then $\partial_{d}\left(D_{d} F\right) \simeq \partial_{d}(F)$. More generally, for any $d$-excisive functor $F$, we have a natural equivalence

$$
\begin{equation*}
\partial_{d}\left(D_{d} F(X \otimes-)\right) \simeq \partial_{d}(F) \otimes X^{\otimes d} \tag{6.29}
\end{equation*}
$$

Indeed there are equivalences

$$
\begin{aligned}
\partial_{d}\left(D_{d} F(X \otimes-)\right) & \simeq \operatorname{cr}_{d} D_{d} F(X \otimes-)(\mathbb{S}, \ldots, \mathbb{S}) \\
& \simeq \operatorname{cr}_{d} D_{d} F(X, \ldots, X) \\
& \simeq \partial_{d}(F) \otimes X^{\otimes d}
\end{aligned}
$$

Remark 6.30. As a special case of the previous definition, we have $d$-derivatives defined on the category of $d$-excisive functors:

Definition 6.31. For a $d$-excisive functor $F$, we define the $k$-th Goodwillie derivative $\partial_{k} F$ for each $1 \leq k \leq d$ by

$$
\partial_{k} F:=\operatorname{cr}_{k}\left(P_{k} F\right)(\mathbb{S}, \ldots, \mathbb{S}) \simeq \operatorname{cr}_{k}\left(D_{k} F\right)(\mathbb{S}, \ldots, \mathbb{S})
$$

Lemma 6.32. For each $1 \leq k \leq d$, the functor $\partial_{k}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ is symmetric monoidal and preserves colimits.

Proof. By definition, $\partial_{k}$ is the composite of $P_{k}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{k}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ and $\operatorname{cr}_{k}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{k}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$, which are symmetric monoidal by Corollary 6.21 and Proposition 5.27, respectively. That $\partial_{k}$ preserves colimits is already clear from Goodwillie's original constructions; alternatively, it also follows from the fact that $P_{k}$ and $\mathrm{cr}_{k}(-)(\mathbb{S}, \ldots, \mathbb{S})$ both have right adjoints. For the latter, recall Remark 5.35.

Remark 6.33. The top derivative $\partial_{d}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ given by $\mathrm{cr}_{d}(-)(\mathbb{S}, \ldots, \mathbb{S})$ is finite étale (Remark 5.37).

Lemma 6.34. The collection of functors $\left\{\partial_{k}\right\}_{1 \leq k \leq d}$ are jointly conservative on $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Proof. The equivalence of categories $\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \simeq \operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)$ of Proposition 6.23 implies that $D_{d} F=0 \Longleftrightarrow \partial_{d} F=0$. Therefore, if $\partial_{d} F=0$, then $F$ is $(d-1)$-excisive, and the result follows by induction.

Remark 6.35. It is a consequence of Lemma 6.34 that every compact separable algebra in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ has finite degree in the sense of [Bal14]. This follows from the corresponding statement for $\mathcal{S} p$ established in [Bal14, Corollary 4.8]. Indeed, $d_{k}:=\operatorname{deg}\left(\partial_{k}(A)\right)<\infty$ for each $1 \leq k \leq d$ implies that for $d:=\max _{1 \leq k \leq d} d_{k}$ we have $\partial_{k}\left(A^{[d+1]}\right)=\partial_{k}(A)^{[d+1]}=0$ for all $1 \leq k \leq d$ by [Bal14, Theorem 3.7(a)]. Hence $A^{[d+1]}=0$ by Lemma 6.34 so that $\operatorname{deg}(A) \leq d$.

Example 6.36. The finite étale map $\partial_{d}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ of Remark 6.33 has finite degree.

Remark 6.37. We now give an explicit description of the completion functor $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{hom}\left(e_{Y}, \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)\right)$ associated to $Y=\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right)$. In other words, bearing in mind Theorem 6.19, we compute the internal hom hom $\left(D_{d} h_{\mathbb{S}}, F\right)$ for a $d$-excisive functor $F$.

Proposition 6.38. Suppose that $F$ is d-excisive. Then

$$
\operatorname{hom}_{\operatorname{Exc}_{d}\left(\delta p^{c}, \delta p\right)}\left(D_{d} h_{\mathbb{S}}, F\right)(X) \simeq\left(\partial_{d} F \otimes X^{\otimes d}\right)^{h \Sigma_{d}}
$$

Proof. Suppose that $F$ and $G$ are $d$-excisive functors. We recall from Theorem 2.38 that the internal hom in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is computed in $\operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ where it is given by

$$
\begin{aligned}
\operatorname{hom}_{\operatorname{Fun}\left(\delta p^{c}, s p\right)}(G, F)(X) & \simeq \operatorname{Hom}_{\operatorname{Fun}^{\left(s p^{c}, s p\right)}}(G, F(X \otimes-)) \\
& \simeq \operatorname{Hom}_{\operatorname{Exc}_{d}\left(\delta p^{c}, s p\right)}(G, F(X \otimes-))
\end{aligned}
$$

where the last step follows because $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \subseteq \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is a full subcategory. Therefore,

$$
\begin{aligned}
\operatorname{hom}_{\operatorname{Exc}_{d}\left(s p^{c}, s p\right)}\left(D_{d} h_{\mathbb{S}}, F\right)(X) & \simeq \operatorname{Hom}_{\operatorname{Exc}_{d}\left(s p^{c}, s p\right)}\left(D_{d} h_{\mathbb{S}}, F(X \otimes-)\right) \\
& \simeq \operatorname{Hom}_{\operatorname{Homog}_{d}\left(s p^{c}, s p\right)}\left(D_{d} h_{\mathbb{S}}, D_{d} F(X \otimes-)\right)
\end{aligned}
$$

where we have used that $D_{d}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is right adjoint to the inclusion.

By Proposition 6.23, there is an equivalence

$$
\operatorname{Homog}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \leftrightarrows \operatorname{Fun}\left(B \Sigma_{d}, \mathcal{S} p\right)
$$

given by assigning to a $d$-homogeneous functor $F$ the Goodwillie derivative $\partial_{d} F$ with $\Sigma_{d}$-action. Now we observe that $\partial_{d}\left(D_{d} h_{\mathbb{S}}\right) \simeq \mathbb{S}$ with trivial $\Sigma_{d^{\prime}}$-action, and $\partial_{d}\left(D_{d} F(X \otimes-)\right) \simeq\left(\partial_{d}(F) \otimes X^{\otimes d}\right)$ by (6.29).

We deduce from the above discussion that

$$
\begin{aligned}
\operatorname{hom}_{\operatorname{Exc}_{d}\left(S_{\left.p^{c}, S p\right)}\right.}\left(D_{d} h_{\mathfrak{S}}, F\right)(X) & \simeq \operatorname{Hom}_{S_{p}}\left(\partial_{d} D_{d} h_{\mathbb{S}}, \partial_{d} D_{d} F(X \otimes-)\right)^{h \Sigma_{d}} \\
& \simeq \operatorname{Hom}_{\mathcal{S} p}\left(\mathbb{S}, \partial_{d} F \otimes X^{\otimes d}\right)^{h \Sigma_{d}} \\
& \simeq\left(\partial_{d} F \otimes X^{\otimes d}\right)^{h \Sigma_{d}}
\end{aligned}
$$

as required.
Remark 6.39. We can also identify the Tate square of (6.13). The existence of such a pullback square was originally shown by McCarthy [McC01, Proposition 4]; see also [Kuh04, Proposition 1.9].

Proposition 6.40 (Kuhn-McCarthy). For any reduced functor $F: \mathcal{S} p^{c} \rightarrow \mathcal{S} p$ there is a pullback square of the form


The fiber of the vertical maps is $D_{d} F(X) \simeq\left(\partial_{d} F \otimes X^{\otimes d}\right)_{h \Sigma_{d}}$. In particular, the Tate construction associated to the Thomason closed subset $Y=\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right) \subseteq$ $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ is given by

$$
t_{Y}(F)(X) \simeq\left(\partial_{d} F \otimes X^{\otimes d}\right)^{t \Sigma_{d}}
$$

Proof. By Proposition 6.23 and Lemma 6.26 we have $D_{d} F(X) \simeq\left(\partial_{d} F \otimes X^{\otimes d}\right)_{h \Sigma_{d}}$. Using Remark 6.12, we can identify the bottom right-hand corner of the Tate square as the $(d-1)$-excisive approximation to the functor $X \mapsto\left(\partial_{d} F \otimes X^{\otimes d}\right)^{h \Sigma_{d}}$. A direct computation shows that this functor is $X \mapsto\left(\partial_{d} F \otimes X^{\otimes d}\right)^{t \Sigma_{d}}$ (see, for example, [Kuh07, Lemma 5.2] or [McC01, Proof of Proposition 4]) and the result follows.

Remark 6.41. We have already introduced the notation

$$
t_{d}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)
$$

for the Tate construction associated to $\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right) \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$; see Notation 6.18. By Proposition 6.40, the $d$-excisive functor $t_{d}(F)$ has the following explicit description:

$$
t_{d}(F)(X) \simeq\left(\partial_{d} F \otimes X^{\otimes d}\right)^{t \Sigma_{d}}
$$

Remark 6.42. The proof of Proposition 6.40 shows that $t_{d}(F)$ is actually $(d-1)$ excisive, but we are considering it as a $d$-excisive functor via the natural inclusion.
Remark 6.43. Combining the Tate construction $t_{d}$ with the inflation functor $i_{d}$ (Definition 2.44) and the Goodwillie derivatives $\partial_{k}$ (Definition 6.31) yields:

Definition 6.44. The Tate-derivatives on the category of $d$-excisive functors are the functors $\mathcal{S} p \rightarrow \mathcal{S} p$ defined for each $1 \leq k \leq d$ as the composite

$$
\mathcal{S} p \xrightarrow{i_{d}} \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{t_{d}} \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{\partial_{k}} \mathcal{S} p .
$$

Explicitly, a spectrum $A$ is sent to the $k$-th derivative of the functor

$$
X \mapsto\left(A \otimes X^{\otimes d}\right)^{t \Sigma_{d}}
$$

Remark 6.45. Understanding the chromatic shifting behaviour of the Tate-derivatives will play a crucial role in Part III.

## Part II. The primes of $d$-excisive functors

7. The Balmer spectrum of $d$-excisive functors as a set

We begin by computing the underlying set of the Balmer spectrum of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Remark 7.1. For each $1 \leq k \leq d$, the $k$-th Goodwillie derivative

$$
\partial_{k}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p
$$

is a colimit-preserving symmetric monoidal functor (a.k.a. geometric functor) by Lemma 6.32. Since $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is rigidly-compactly generated (Corollary 5.34), it restricts to a functor

$$
\partial_{k}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c} \rightarrow \mathcal{S} p^{c}
$$

between compact objects. By pulling back prime ideals of $\mathcal{S} p^{c}$ (whose spectrum was described in Example 6.5), we produce a family of prime ideals of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ :
Definition 7.2. Let $\mathcal{P}_{d}([k], p, h):=\left\{x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c} \mid \partial_{k}(x) \in \mathcal{C}_{p, h}\right.$ in $\left.\mathcal{S} p^{c}\right\}$.
The following lemma shows that these are all the prime ideals in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Lemma 7.3. Let $\partial:=\prod_{1 \leq k \leq d} \partial_{k}$. The induced map

$$
\begin{equation*}
\operatorname{Spc}(\partial): \coprod_{1 \leq k \leq d} \operatorname{Spc}\left(\mathcal{S} p^{c}\right) \rightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \tag{7.4}
\end{equation*}
$$

is surjective.
Proof. This follows from Lemmas 6.32 and 6.34 and [BCHS23a, Theorem 1.3].
Lemma 7.5. For each $1 \leq k \leq d$, the composite

$$
\mathcal{S} p \xrightarrow{i_{d}} \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{\partial_{k}} \mathcal{S} p
$$

is equivalent to the identity functor.
Proof. It suffices to prove the claim when $k=d$, since the composite

$$
\mathcal{S} p \xrightarrow{i_{d}} \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{P_{k}} \operatorname{Exc}_{k}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)
$$

is equivalent to $i_{k}$. In other words, we must show that $\partial_{d}\left(i_{d}(A)\right) \simeq A$. Recall that $i_{d}(A) \simeq P_{d}\left(F_{A}\right)$, where $F_{A}(X)=A \otimes h_{\mathbb{S}}(X)=A \otimes \Sigma^{\infty} \Omega^{\infty} X$ (Definition 2.44). When $A=\mathbb{S}$ the claim is clear using, for example, that both functors are symmetric monoidal. One then computes $\partial_{d}\left(P_{d} F_{A}\right) \simeq A \otimes \partial_{d}\left(P_{d} F_{\Im}\right) \simeq A$ and the result follows.

Lemma 7.6. The functor $i_{d}: \mathcal{S} p \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ induces a map

$$
\operatorname{Spc}\left(i_{d}\right): \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \rightarrow \operatorname{Spc}\left(\mathcal{S} p^{c}\right)
$$

which sends $\mathcal{P}_{d}([k], p, h)$ to $\mathcal{C}_{p, h}$ for all $1 \leq k \leq d$.
Proof. This follows from Lemma 7.5 and the definitions.
Lemma 7.7. For each $1 \leq k \leq d$, the functor $\partial_{k}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ induces an embedding

$$
\operatorname{Spc}\left(\partial_{k}\right): \operatorname{Spc}\left(\mathcal{S} p^{c}\right) \hookrightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)
$$

which sends $\mathcal{C}_{p, h}$ to $\mathcal{P}_{d}([k], p, h)$.

Proof. The splitting $\partial_{k} \circ i_{d} \simeq \operatorname{Id}_{s_{p}}$ established by Lemma 7.5 implies by Remark 6.2 that $\mathcal{C}_{p, n} \subseteq \mathcal{C}_{q, m}$ if and only if $\mathcal{P}_{d}([k], p, n) \subseteq \mathcal{P}_{d}([k], q, m)$. It follows that the map $\operatorname{Spc}\left(\partial_{k}\right)$ is injective and, in fact, an embedding.

Proposition 7.8. Let $1 \leq m \leq d$. The functor $P_{m}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{m}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ induces an open embedding

$$
\operatorname{Spc}\left(P_{m}\right): \operatorname{Spc}\left(\operatorname{Exc}_{m}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \hookrightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)
$$

which sends $\mathcal{P}_{m}([k], p, h)$ to $\mathcal{P}_{d}([k], p, h)$ for each $1 \leq k \leq m$.
Proof. By Corollary 6.21 the functor $P_{m}$ is the finite localization on $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ whose kernel is the ideal generated by $\left\{P_{d} h_{\mathscr{S}}(i) \mid m+1 \leq i \leq d\right\}$. Any finite localization induces an embedding on spectra by [Bal05, Proposition 3.11]. Moreover, the complement of its image is the closed set

$$
\bigcup_{m+1 \leq i \leq d} \operatorname{supp}\left(P_{d} h_{\Im}(i)\right) \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)
$$

This establishes that $\operatorname{Spc}\left(P_{m}\right)$ is an open embedding. The fact that $\mathcal{P}_{m}([k], p, h)$ maps to $\mathcal{P}_{d}([k], p, h)$ follows directly from the definitions.

The following is our key computational result for determining the underlying set of the spectrum of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.
Proposition 7.9. For $1 \leq i \leq d$ and $j \geq 1$ we have

$$
\partial_{i}\left(P_{d}\left(h_{\mathbb{S}}(j)\right)\right) \cong \mathbb{S}^{\oplus|\operatorname{surj}(i, j)|}
$$

where the right-hand side is to be interpreted as 0 if $j>i$.
Proof. The map $h_{\mathbb{S}}(j) \rightarrow P_{d}\left(h_{\mathbb{S}}(j)\right)$ induces an equivalence of $\partial_{i}$ for $i \leq d$. So it is enough to prove that

$$
\partial_{i}\left(h_{\mathbb{S}}(j)\right) \cong \mathbb{S}^{\oplus|\operatorname{surj}(i, j)|}
$$

Recall that $h_{\mathbb{S}}(j)$ is the $j$-th cross-effect of the functor $x \mapsto h_{x}$, evaluated at $\mathbb{S}, \mathbb{S}, \ldots, \mathbb{S}$. It is well-known that for $x \in \mathcal{S} p^{c}$ there is an equivalence $\partial_{i}\left(h_{x}\right) \simeq \mathbb{D}\left(x^{\otimes i}\right)$, with $\Sigma_{i}$ action given by permuting the factors of $x$; see, for example, [AC15, Lemma 5.10]. In particular, we deduce that

$$
\begin{equation*}
\partial_{i}\left(h_{\mathscr{S} \oplus j}\right) \simeq \mathbb{S}^{\oplus j^{i}} \tag{7.10}
\end{equation*}
$$

The case of $h_{\mathbb{S}}(j)$ follows by writing $h_{\mathbb{S}}(j)$ as the total fiber of a $j$-cube. Let us demonstrate this in the case $i=j=3$, showing that $\partial_{3}\left(h_{\mathbb{S}}(3)\right) \simeq \mathbb{S}^{\oplus 6}$. We can write $h_{\mathbb{S}}(3)$ as the total fiber of the 3-cube


Applying $\partial_{3}$ and using (7.10) gives the 3-cube


The total fiber of this diagram, which is $\partial_{3}\left(h_{\mathbb{S}}(3)\right)$, is $\mathbb{S}^{\oplus(27-3 \cdot 8+3 \cdot 1)}=\mathbb{S}^{\oplus 6}$. In the general case, the same type of argument gives that $\partial_{i}\left(h_{\mathbb{S}}(j)\right)$ is a sum of
$j^{i}-\binom{j}{j-1}(j-1)^{i}+\binom{j}{j-2}(j-2)^{i}+\cdots+(-1)^{j-1} j=\sum_{k=0}^{j-1}(-1)^{k}\binom{j}{j-k}(j-k)^{i}$ copies of the sphere spectrum. But this is precisely the number of surjections from $i$ to $j$; see (5.21). The result then easily follows.

Remark 7.11. Armed with Proposition 7.9, we have analogs of [BS17, Propositions 4.11 and 4.12]:

Proposition 7.12. Let $1 \leq i, j \leq d$. Then $P_{d}\left(h_{\mathbb{S}}(j)\right) \in \mathcal{P}_{d}([i], p, h)$ if and only if $i<j$.
Proof. By definition, $\mathcal{P}_{d}([i], p, h)=\left(\partial_{i}\right)^{-1}\left(\mathcal{C}_{p, h}\right)$. Now recall that $\mathbb{S} \notin \mathcal{P}$ and $0 \in \mathcal{P}$ for every prime $\mathcal{P}=\mathcal{C}_{p, h}$ in $\operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ and invoke Proposition 7.9.

Corollary 7.13. If $\mathcal{P}_{d}([i], p, h) \subseteq \mathcal{P}_{d}\left([j], p^{\prime}, h^{\prime}\right)$ then $i \geq j$.
Proof. Invoking Proposition 7.12 twice, $P_{d}\left(h_{\mathbb{S}}(j)\right) \notin \mathcal{P}_{d}\left([j], p^{\prime}, h^{\prime}\right)$ implies that $P_{d}\left(h_{\mathbb{S}}(j)\right) \notin \mathcal{P}_{d}([i], p, h)$ by the hypothesis, so that $i \geq j$.

Theorem 7.14. The map $\operatorname{Spc}(\partial)$ in (7.4) is a bijection: Every prime ideal in $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ is of the form $\mathcal{P}_{d}([i], p, h)$ for some triple $(i, p, h)$ consisting of an integer $1 \leq i \leq d$, a prime number $p$ or $p=0$, and a chromatic height $1 \leq h \leq \infty$. Moreover, we have $\mathcal{P}_{d}([i], p, h)=\mathcal{P}_{d}\left([j], p^{\prime}, h^{\prime}\right)$ if and only if $i=j$ and $\mathcal{C}_{p, h}=\mathcal{C}_{p^{\prime}, h^{\prime}}$ in $\mathcal{S} p^{c}$ (i.e., $h=h^{\prime}$ and if $h=h^{\prime}>1$, then also $p=p^{\prime}$ ).

Proof. We established that $\operatorname{Spc}(\partial)$ is surjective in Lemma 7.3, so it remains to show that it is injective, i.e., if $\mathcal{P}_{d}([i], p, h)=\mathcal{P}_{d}\left([j], p^{\prime}, h^{\prime}\right)$ then $i=j$ and $\mathcal{C}_{p, h}=\mathcal{C}_{p^{\prime}, h^{\prime}}$. Indeed, Corollary 7.13 implies $i=j$ while Lemma 7.6 implies $\mathcal{C}_{p, h}=\mathcal{C}_{p^{\prime}, h^{\prime}}$.

Corollary 7.15. Let $1 \leq k \leq d$. Then

$$
\operatorname{supp}\left(P_{d}\left(h_{\mathbb{S}}(k)\right)\right)=\left\{\mathcal{P}_{d}([i], p, h) \mid i \geq k\right\}
$$

Proof. By definition,

$$
\operatorname{supp}\left(P_{d}\left(h_{\mathbb{S}}(k)\right)\right)=\left\{\mathcal{P} \in \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \mid P_{d}\left(h_{\mathbb{S}}(k)\right) \notin \mathcal{P}\right\}
$$

By Theorem 7.14 we know every prime is of the form $\mathcal{P}([i], p, h)$ and so the result follows from Proposition 7.12.

Remark 7.16. It follows from Theorem 7.14 and Lemma 7.7 that $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ consists of $d$ disjoint embedded copies of $\operatorname{Spc}\left(S p^{c}\right)$. Understanding the topological interactions between these $d$ pieces will be the topic of Part III.
Remark 7.17. The $p$-localization $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}$ is the finite localization associated to $i_{d}\left(f_{p, \infty}\right)$ in the notation of Example 6.15. It induces an identification

$$
\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}^{c}\right) \xrightarrow{\sim}\left\{\mathcal{P}_{d}([k], p, h) \mid \text { all } k, h\right\} \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)
$$

on Balmer spectra. This readily follows from Lemma 7.6 and the definitions. Moreover, it follows from Lemma 7.5 that if $F$ is a $p$-local $d$-excisive functor then its derivative $\partial_{k}(F) \in \mathcal{S} p$ is also $p$-local for each $1 \leq k \leq d$. This is almost tautological if one makes the identification $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)} \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p_{(p)}\right)$ provided by Remark 2.46. From this perspective, $p$-localization is just changing coefficients along $\mathcal{S} p \rightarrow \mathcal{S} p_{(p)}$.
Remark 7.18. Via the abstract nilpotence theorem of $\left[\mathrm{BCH}^{+} 23\right.$, Theorem 2.25], the joint conservativity of the derivatives on $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ established in Lemma 6.34 implies the following nilpotence theorem:

Theorem 7.19. The collection of functors $\left\{\partial_{k}\right\}_{1 \leq k \leq d}$ detects tensor-nilpotence of morphisms with compact source in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right): A$ map $\alpha: F \rightarrow G$ in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ with $F$ compact satisfies $\alpha^{\circledast m}=0$ for some $m \geq 0$ if and only if for all $1 \leq k \leq d$ there exists $m_{k} \geq 0$ such that $\partial_{k}(\alpha)^{\otimes m_{k}}=0$.

Remark 7.20. The computation of Theorem 7.14 along with Theorem 7.19 show that Balmer's homological spectrum $\operatorname{Spc}^{h}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ (see [Bal20]) agrees with the tensor triangular spectrum. (Balmer has conjectured that this holds for any rigidly-compactly generated category; see [Bal20, Remark 5.15].) More specifically, we have the following:

Corollary 7.21. The natural comparison map

$$
\phi: \operatorname{Spc}^{h}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \rightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)
$$

of [Bal20, Remark 3.4] is a homeomorphism.
Proof. We will verify the conditions of [Bal20, Theorem 5.6]. This will establish that $\phi$ is a bijection and hence a homeomorphism by [BHS23a, Theorem A]. The argument is analogous to the corresponding argument for the equivariant stable homotopy category given in [Bal20, Corollary 5.10]. We must show that for every $\mathcal{P}=$ $\mathcal{P}_{d}([k], p, h) \in \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ there exists a coproduct-preserving homological symmetric monoidal functor to a tensor abelian category $\mathcal{A}_{\mathcal{P}}$ satisfying the three conditions of [Bal20, Theorem 5.6]. From the definition of the prime ideal $\mathcal{P}_{d}([k], p, h)$ we take this to be the composite

$$
\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \xrightarrow{\partial_{k}} \mathcal{S} p \xrightarrow{K(p, h)} \mathcal{A}_{p, h}:=\mathbb{F}_{p}\left[v_{h}^{ \pm 1}\right]-\operatorname{grMod}
$$

where $K(p, h)$ is Morava $K$-theory at the prime $p$ and height $h$ and the graded field $K(p, h)_{*}(\mathbb{S})=\mathbb{F}_{p}\left[v_{h}^{ \pm 1}\right]$ has $v_{h}$ in degree $2\left(p^{h}-1\right)$. When $h=0$ we take $K(p, h)$ to be rational homology (for all primes $p$ ) and the target to be graded rational vector spaces, while for $h=\infty$ we take $K(p, h)$ to be $\bmod p$ homology and the target to be graded $\mathbb{F}_{p}$-vector spaces. Conditions (1) and (2) of [Bal20, Theorem 5.6] are then straightforward to verify, while (3) follows from the nilpotence theorem of

Theorem 7.19 together with the classical nilpotence theorem [DHS88, HS98] for the stable homotopy category.

## 8. The Goodwillie-Burnside Ring

In this section we introduce, for each integer $d \geq 1$, a commutative ring $A(d)$ which we call the Goodwillie-Burnside ring. The name is justified by Theorem 9.23 in the next section which identifies $A(d)$ with the endomorphism ring of the unit in the category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. It is the analog in Goodwillie calculus of the Burnside ring in equivariant homotopy theory and, following Dress [Dre69], we are able to completely describe its Zariski spectrum.

Remark 8.1. Yoshida [Yos87] introduces a Burnside ring for any finite category $\mathcal{C}$ satisfying certain assumptions. For example, if $\mathcal{C}$ is the orbit category of a finite group, then Yoshida's abstract Burnside ring is the usual Burnside ring of a finite group. Our construction of the Goodwillie-Burnside ring can be regarded as a special case of Yoshida's construction, namely as the abstract Burnside ring associated to the following category.

Definition 8.2. Let $\mathrm{Epi}_{\leq d}$ denote the category of non-empty sets of cardinality at most $d$ and surjective maps.

Remark 8.3. Glasman [Gla18] has shown that the category of $d$-excisive functors from finite spectra to spectra is equivalent to the category of spectral Mackey functors on a category built from $\mathrm{Epi}_{\leq d}$. This provides a conceptual reason for the appearance of $\mathrm{Epi}_{\leq d}$ in the study of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$.

Remark 8.4. Note that when $d=2$, there is an equivalence of categories

$$
\mathrm{Epi}_{\leq 2} \simeq \mathcal{O}_{C_{2}}
$$

where the latter denotes the orbit category of the cyclic group $C_{2}$. This is reflected in the fact that the ring $A(2)$ we construct in this section is isomorphic to $A\left(C_{2}\right)$, the Burnside ring of the group $C_{2}$.

Construction 8.5. Let $d \geq 1$ be an integer. Let $A(d):=\left\langle x_{1}, \ldots, x_{d}\right\rangle$ denote the free abelian group on generators $x_{1}, \ldots, x_{d}$ and let $\mathbb{Z}^{d}:=\left\langle y_{1}, \ldots, y_{d}\right\rangle$ denote the free abelian group on generators $y_{1}, \ldots, y_{d}$. Thus $A(d)$ and $\mathbb{Z}^{d}$ are isomorphic as abelian groups, but they will be endowed with different ring structures. The ring $\mathbb{Z}^{d}=\prod_{k=1}^{d} \mathbb{Z}$ is the product of $d$ copies of $\mathbb{Z}$. Thus the product structure on $\mathbb{Z}^{d}$ is determined by the relations

$$
y_{i} y_{j}=\left\{\begin{array}{cc}
y_{i} & i=j  \tag{8.6}\\
0 & i \neq j
\end{array}\right.
$$

To describe the ring structure on $A(d)$ we first define a group homomorphism $\phi: A(d) \rightarrow \mathbb{Z}^{d}$ by the following formula:

$$
\begin{equation*}
\phi\left(x_{i}\right):=\sum_{k=1}^{d}|\operatorname{surj}(k, i)| y_{k}=\sum_{k=i}^{d}|\operatorname{surj}(k, i)| y_{k} . \tag{8.7}
\end{equation*}
$$

We call $\phi$ the Burnside homomorphism.
Remark 8.8. Recall from Definition 4.9 that a subset of $[i] \times[j]$ is called good if it projects surjectively onto $[i]$ and $[j]$.

Definition 8.9. Let $\mu(i, j, l)$ be the number of good subsets of $[i] \times[j]$ of cardinality $l$.
Theorem 8.10. The homomorphism $\phi$ of (8.7) is injective. Moreover, there is an exact sequence

$$
0 \rightarrow A(d) \xrightarrow{\phi} \prod_{1 \leq i \leq d} \mathbb{Z} \xrightarrow{\psi} \prod_{1 \leq i \leq d} \mathbb{Z} / i!\mathbb{Z} \rightarrow 0
$$

The abelian group $A(d)$ possesses a unique ring structure which makes $\phi$ a ring homomorphism. The multiplication in $A(d)$ is determined by the formula

$$
\begin{equation*}
x_{i} x_{j}=\sum_{1 \leq l \leq d} \mu(i, j, l) x_{l} . \tag{8.11}
\end{equation*}
$$

Proof. This is a consequence of Yoshida's work on abstract Burnside rings [Yos87], but we will give a direct proof.

The formula (8.7) means that as a homomorphism from $\mathbb{Z}^{d}$ to itself, with our chosen bases, $\phi$ is represented by a lower triangular matrix, whose diagonal entries are $\operatorname{surj}(i, i)=i$ ! for $i=1, \ldots, d$. It follows that $\phi$ is injective, and the cokernel of $\phi$ is isomorphic to $\prod_{1 \leq i \leq d} \mathbb{Z} / i!\mathbb{Z}$.

From the injectivity of $\phi$ it follows that there exists at most one product structure on $A(d)$ for which $\phi$ is a ring homomorphism. It remains to find a formula for multiplication in $A(d)$ that is respected by $\phi$. Since $x_{1}, \ldots, x_{d}$ form an abelian group basis of $A(d)$, for any ring structure on $A(d)$ there exist unique integers $\nu(i, j, l)$ such that

$$
\begin{equation*}
x_{i} x_{j}=\sum_{l=1}^{d} \nu(i, j, l) x_{l} \tag{8.12}
\end{equation*}
$$

The map $\phi$ is a ring homomorphism if and only if

$$
\phi\left(x_{i}\right) \phi\left(x_{j}\right)=\phi\left(x_{i} x_{j}\right)
$$

for all $1 \leq i, j \leq n$. Substituting the formula for $\phi$, and using the formulas for the product in $\mathbb{Z}^{d}$ (8.6) and the product in $A(d)$ (8.12), we obtain the relations

$$
\sum_{k=1}^{d}|\operatorname{surj}(k, i)||\operatorname{surj}(k, j)| y_{k}=\sum_{k=1}^{d}\left(\sum_{l=1}^{d} \nu(i, j, l)|\operatorname{surj}(k, l)|\right) y_{k}
$$

It follows that $\phi$ is a ring homomorphism if and only if the numbers $\nu(i, j, l)$ satisfy the following relations, for all $1 \leq i, j, k \leq d$ :

$$
|\operatorname{surj}(k, i)||\operatorname{surj}(k, j)|=\sum_{l=1}^{d} \nu(i, j, l)|\operatorname{surj}(k, l)| .
$$

Since the multiplication on $A(d)$ is unique (if it exists), there is at most one set of numbers $\nu(i, j, k)$ that satisfies these relations. So if we show that the numbers $\mu(i, j, k)$ satisfy these relations, then $\mu(i, j, l)=\nu(i, j, l)$ for all $i, j, l$, and we are done.

We have to prove that for all $i, j, k$ the following equality holds

$$
|\operatorname{surj}(k, i)||\operatorname{surj}(k, j)|=\sum_{l=1}^{d} \mu(i, j, l)|\operatorname{surj}(k, l)| .
$$

The number $|\operatorname{surj}(k, i) \| \operatorname{surj}(k, j)|$ is the same as $|\operatorname{surj}([k],[i]) \times \operatorname{surj}([k],[j])|$. Let $[i]^{[k]}$ denote the set of all functions from $[k]$ to $[i]$. Then $\operatorname{surj}([k],[i]) \subseteq[i]^{[k]}$ and

$$
\operatorname{surj}([k],[i]) \times \operatorname{surj}([k],[j]) \subseteq[i]^{[k]} \times[j]^{[k]} \cong([i] \times[j])^{[k]}
$$

A function $\alpha:[k] \rightarrow[i] \times[j]$, considered as an element of $([i] \times[j])^{[k]}$, is in the image of $\operatorname{surj}([k],[i]) \times \operatorname{surj}([k],[j])$ if and only if the image of $\alpha$ is a good subset of $[i] \times[j]$. It follows that the number of elements of $\operatorname{surj}([k],[i]) \times \operatorname{surj}([k],[j])$ is the sum, indexed by good subsets of $[i] \times[j]$, of the numbers of surjections from $[k]$ onto each good subset. Grouping the good subsets by cardinality, and recalling that $\mu(i, j, l)$ is the number of good subsets with $l$ elements, we obtain the desired formula.

Corollary 8.13. The ring $A(d)$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}\left[x_{1}, \ldots, x_{d}\right]$ by the ideal I generated by the relations (8.11). Furthermore, $x_{1}=1$, so $A(d)$ is also the quotient of $\mathbb{Z}\left[x_{2}, \ldots, x_{d}\right]$ by the relations (8.11).

Proof. It follows from Theorem 8.10 that there is a surjective ring homomorphism $\mathbb{Z}\left[x_{1}, \ldots, x_{d}\right] / I \rightarrow A(d)$. It is easy to see from the definition of $I$ that $x_{1}, \ldots, x_{d}$ generate $\mathbb{Z}\left[x_{1}, \ldots, x_{d}\right] / I$ as an abelian group. Since $A(d)$ is a free abelian group of rank $d$, it follows that the surjective homomorphism is in fact an isomorphism.

Remark 8.14. There does not seem to be a closed formula for $\mu(i, j, k)$. Nevertheless, we can make a few observations about these numbers, and also write some summation formulas.

To begin with, clearly $\mu(i, j, k) \neq 0$ if and only if $\max (i, j) \leq k \leq i j$. Moreover, if $i \leq j$ then $\mu(i, j, j)=|\operatorname{surj}(j, i)|$. At the other end of the range, $\mu(i, j, i j)=1$.

Note that $\mu(i, j, k)$ can be characterized as the number of labelled bipartite graphs with $k$ edges and no isolated vertices, and with parts of size $i$ and $j$. In this guise, there are some formulas for this number, or closely related ones, scattered in the literature [Har58, Lee61, AO18]. For example, one can use the inclusionexclusion principle to write a summation formula for $\mu(i, j, k)$. We learned the following formula from Achim Krause. It can be found in [Lee61] as the Corollary to Lemma 3.

Lemma 8.15. There is an equality

$$
\mu(i, j, k)=\sum_{s \geq 0, t \geq 0}(-1)^{s+t}\binom{(i-s)(j-t)}{k}\binom{i}{s}\binom{j}{t} .
$$

Proof. There are altogether $\binom{i j}{k}$ subsets of $[i] \times[j]$ of cardinality $k$. Let us say that a subset of $[i] \times[j]$ is $b a d$ if it is not good. We are going to apply the inclusion-exclusion principle to analyze the number of bad subsets, and ultimately arrive at the number of good ones.

A subset $U \subseteq[i] \times[j]$ is bad if there exists a row or a column of the array $[i] \times[j]$ that is disjoint from $U$. Suppose we have subsets $S \subseteq[i]$ and $T \subseteq[j]$. Let $B_{S, T}$ be the number of bad subsets of $[i] \times[j]$ with $k$ elements with the property that their projections onto $[i]$ and $[j]$ are disjoint from $S$ and $T$ respectively. In other words, $B_{S, T}$ is the number of subsets of $([i] \backslash S) \times([j] \backslash T)$ of cardinality $k$. It follows that $B_{S, T}$ has $\binom{(i-|S|)(j-|T|)}{k}$ elements.

It is clear that the union of $B_{S, T}$ where $S, T$ range over all subsets of $[i]$ and $[j]$ with $S \cup T \neq \varnothing$ is the set of bad subsets of $[i] \times[j]$ of cardinality $k$. It is also
clear that $B_{S, T} \cap B_{S^{\prime}, T^{\prime}}=B_{S \cup S^{\prime}, T \cup T^{\prime}}$. The maximal elements of the family of subsets $B_{S, T}$ are those where $S \cup T$ consists of a single element. The set $B_{S, T}$ is the intersection of $|S|+|T|$ maximal elements.

Applying the inclusion-exclusion principle to the family of subsets $B_{S, T}$ we obtain that the number of elements in the complement of the union of all the sets $B_{S, T}$, which is $\mu(i, j, k)$, is given by the following formula:

$$
\begin{aligned}
\mu(i, j, k) & =\sum_{S \subseteq[i], T \subseteq[j]}(-1)^{|S|+|T|}\left|B_{S, T}\right| \\
& =\sum_{S \subseteq[i], T \subseteq[j]}(-1)^{|S|+|T|}\binom{(i-|S|)(j-|T|)}{k} \\
& =\sum_{s \geq 0, t \geq 0}(-1)^{s+t}\binom{(i-s)(j-t)}{k}\binom{i}{s}\binom{j}{t} .
\end{aligned}
$$

Remark 8.16. For an explicit way to compute the ring structure, we define a $d \times d$ matrix $M$ by

$$
M_{i j}:=|\operatorname{surj}(i, j)| .
$$

Then one can show that

$$
\begin{equation*}
\mu(i, j, k)=\sum_{m=1}^{d} M_{m i} \cdot M_{m j} \cdot M_{k m}^{-1} \tag{8.17}
\end{equation*}
$$

see for example [YOT18, Equation (1.6)]. Moreover, we can give an explicit description of the inverse matrix $M^{-1}$. Indeed, recall that the number of surjections from $[i]$ to $[j]$ can be expressed in terms of Stirling numbers of the second kind:

$$
M_{i j}=|\operatorname{surj}(i, j)|=j!\left\{\begin{array}{l}
i \\
j
\end{array}\right\}
$$

see for example [Big89, Theorem 5.3.1]. We can therefore write $M=L U$ where $L_{i j}=\left\{\begin{array}{l}i \\ j\end{array}\right\}$ and $U$ is the diagonal matrix with $U_{i i}=i$ !. The inverse of $L$ is the matrix $L_{i j}^{-1}=s(i, j)$ where $s(-,-)$ denotes a Stirling number of the first kind (this follows from [Sta12, Proposition 1.4.1]). Therefore,

$$
M_{i j}^{-1}=\frac{1}{i!} s(i, j)
$$

This gives the formula

$$
\mu(i, j, k)=\sum_{m=1}^{d} i!\left\{\begin{array}{c}
m  \tag{8.18}\\
i
\end{array}\right\} j!\left\{\begin{array}{c}
m \\
j
\end{array}\right\} \frac{1}{k!} s(k, m) .
$$

Note that the terms in the sum can only be non-zero for $\max (i, j) \leq m \leq k$.
Example 8.19. We now give an explicit example of how to compute the ring structure in the $d=3$ case.

We can write this ring as a quotient of $\mathbb{Z}\left[x_{2}, x_{3}\right]$ by Corollary 8.13. The matrix of Remark 8.16 for $d=3$ is:

| 1 | 0 | 0 |
| :--- | :--- | :--- |
| 1 | 2 | 0 |
| 1 | 6 | 6 |

The generators $x_{2}, x_{3}$ correspond to the second and third columns of the matrix, and the product of two generators is given by component-wise multiplication of column vectors, which are decomposed as a linear combination of the columns. In particular, we see that $A(3)$ is the quotient of $\mathbb{Z}\left[x_{2}, x_{3}\right]$ by the following relations:

$$
\begin{aligned}
(0,0,6) \cdot(0,0,6)=(0,0,36)=6 \cdot(0,0,6) & \Longrightarrow x_{3}^{2}=6 x_{3} \\
(0,2,6) \cdot(0,0,6)=(0,0,36)=6 \cdot(0,0,6) & \Longrightarrow x_{2} x_{3}=6 x_{3} \\
(0,2,6) \cdot(0,2,6)=(0,4,12)=2 \cdot(0,2,6)+4 \cdot(0,0,6) & \Longrightarrow x_{2}^{2}=2 x_{2}+4 x_{3}
\end{aligned}
$$

Remark 8.20. We will now determine all the primes ideals in $A(d)$ and describe the Zariski spectrum $\operatorname{Spec}(A(d))$. These ideas closely follow those for the classical Burnside ring [Dre69, Dre73].

Definition 8.21. Let $\phi_{i}: A(d) \rightarrow \mathbb{Z}$ denote the homomorphism defined by $\phi_{i}\left(x_{j}\right):=$ $|\operatorname{surj}(i, j)|$.

Remark 8.22. In other words, the map $\phi: A(d) \rightarrow \prod_{1 \leq i \leq d} \mathbb{Z}$ from Construction 8.5 is the product of the maps $\phi_{i}$ for $1 \leq i \leq d$. In particular, each $\phi_{i}$ is a ring homomorphism by Theorem 8.10.

Definition 8.23. Let $\mathfrak{p}([i], p)$ (for $p$ a prime or 0 and $1 \leq i \leq d$ ) be the preimage of the prime ideal $(p)$ under the map $\phi_{i}: A(d) \rightarrow \mathbb{Z}$.

Proposition 8.24. Every prime ideal of $A(d)$ is of the form $\mathfrak{p}([i], p)$ for some $p$ and $1 \leq i \leq d$.

Proof. We observe that the injection $\phi: A(d) \hookrightarrow \prod_{1 \leq k \leq d} \mathbb{Z}$ of Theorem 8.10 is an integral extension since $\prod_{1 \leq k \leq d} \mathbb{Z}$ is additively generated by idempotent elements. It then follows from the going-up theorem that $\operatorname{Spec}\left(\prod_{1 \leq k \leq d} \mathbb{Z}\right) \rightarrow \operatorname{Spec}(A(d))$ is surjective; see e.g. [Kap74, Section 1.6].

Lemma 8.25. The maximal ideals of $A(d)$ are $\mathfrak{p}([i], p)$ for $p>0$, while the minimal prime ideals are $\mathfrak{p}([i], 0)$. In particular, $A(d)$ has Krull dimension 1. Moreover, we have $\mathfrak{p}([i], 0) \subseteq \mathfrak{p}([j], p)$ if and only if $\mathfrak{p}([i], p)=\mathfrak{p}([j], p)$.
Proof. This is the same as the classical proof for the Burnside ring: If $p>0$, then the quotient of $A(d)$ by $\mathfrak{p}([i], p)$ is $\mathbb{Z} / p$, so that $\mathfrak{p}([i], p)$ is maximal. If $p=0$, then the quotient is $\mathbb{Z}$, and there cannot be any containment among the ideals $\mathfrak{p}([i], 0)$ for varying $i$, as such an inclusion would correspond to a surjective ring homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$. We deduce that $\mathfrak{p}([i], 0)$ is minimal for each $i$. This then shows that $A(d)$ has Krull dimension 1.

Suppose then that $\mathfrak{p}([i], p)=\mathfrak{p}([j], p)$. Then clearly $\mathfrak{p}([i], 0) \subseteq \mathfrak{p}([i], p)=\mathfrak{p}([j], p)$. Conversely, we will show that if $\mathfrak{p}([i], p) \neq \mathfrak{p}([j], p)$, then $\mathfrak{p}([i], 0) \nsubseteq \mathfrak{p}([j], p)$. From the previous paragraph, both $\mathfrak{p}([j], p)$ and $\mathfrak{p}([i], p)$ are maximal ideals; in particular, there exists $a \in A(d)$ with $a \in \mathfrak{p}([i], p)$ and $a \notin \mathfrak{p}([j], p)$. Now consider the element $b=a-\phi_{i}(a) \cdot 1$. We see that $b \in \mathfrak{p}([i], 0)$ but $b \notin \mathfrak{p}([j], p)$, and so $\mathfrak{p}([i], 0) \nsubseteq \mathfrak{p}([j], p)$, as claimed.

Proposition 8.26. Suppose $p, q$ are primes and $1 \leq i, j \leq n$. We have $\mathfrak{p}([i], p)=$ $\mathfrak{p}([j], q)$ in $A(d)$ if and only if $p=q$ and $p-1 \mid j-i$.
Proof. $(\Leftarrow)$ Suppose first that $p=q$ and $p-1$ divides $j-i$. We claim that in this case $\phi_{i} \equiv \phi_{j} \bmod (p)$, so that $\mathfrak{p}([i], p)=\mathfrak{p}([j], p)$. To prove that $\phi_{i} \equiv \phi_{j} \bmod (p)$,
we need to prove that $\phi_{i}\left(x_{\ell}\right) \equiv \phi_{j}\left(x_{\ell}\right) \bmod (p)$, for all $\ell$. This in turn means that we need to prove that $|\operatorname{surj}(i, \ell)| \equiv|\operatorname{surj}(j, \ell)| \bmod (p)$ for all $\ell$. Note that the symmetric group $\Sigma_{\ell}$ acts freely on the set of surjections $\operatorname{surj}(-, \ell)$. It follows that both numbers $|\operatorname{surj}(i, \ell)|$ and $|\operatorname{surj}(j, l)|$ are divisible by $\ell$ !. So if $\ell$ is at least $p$ then both numbers are divisible by $p$ and we are done. It remains to prove the case when $1 \leq \ell \leq p$. In this case we use (5.21) to see that:

$$
\begin{align*}
& |\operatorname{surj}(i, \ell)|=\ell^{i}-\binom{\ell}{\ell-1}(\ell-1)^{i}+\cdots \pm\binom{\ell}{k} k^{i} \cdots \\
& |\operatorname{surj}(j, \ell)|=\ell^{j}-\binom{\ell}{\ell-1}(\ell-1)^{j}+\cdots \pm\binom{\ell}{k} k^{j} \cdots \tag{8.27}
\end{align*}
$$

But if $p-1$ divides $j-i$ then $k^{i} \equiv k^{j} \bmod (p)$ for all $k$ and we are done.
$(\Rightarrow)$ For the converse, suppose first that $i$ and $j$ are arbitrary, and $p \neq q$ are distinct primes. Then $\phi_{i}(p)=\phi_{j}(p)=p \not \equiv 0 \bmod (q)$. It follows that $p \in \mathfrak{p}([i], p)$ but $p \notin \mathfrak{p}([j], q)$, so $\mathfrak{p}([i], p) \neq \mathfrak{p}([j], q)$.

Now suppose that $p=q$ and $p-1 \nmid j-i$. Then there exists a positive integer $a$ such that $a^{i} \not \equiv a^{j} \bmod (p)$. For example, a primitive root modulo $p$ satisfies this. Let $\ell$ be the smallest positive integer for which $\ell^{i} \not \equiv \ell^{j} \bmod (p)$.

We claim that $\ell \leq \max (i, j)$ and therefore $\ell \leq d$. Indeed, suppose first that $p \leq \max (i, j)$. Then $\ell$ is bounded above by the primitive roots modulo $p$, which are all smaller than $\max (i, j)$. Now suppose that $p>\max (i, j)$. Without loss of generality, suppose $i<j$. For integers $0<x<p$, the condition $x^{i} \equiv x^{j} \bmod (p)$ is equivalent to $x^{j-i} \equiv 1 \bmod (p)$. This equation has $\operatorname{gcd}(j-i, p-1)$ solutions between 1 and $p$. It follows that among the numbers $1, \ldots, j-i+1$ there exists at least one $x$ for which $x^{j-i} \not \equiv 1 \bmod (p)$ and thus $x^{i} \not \equiv x^{j} \bmod (p)$. It follows that $l \leq j-i+1 \leq j$. This proves the claim. It follows that $x_{\ell} \in A(d)$.

Since $\ell$ is the smallest positive integer for which $\ell^{i} \not \equiv \ell^{j} \bmod (p)$, it follows, using the formulas in (8.27), that

$$
|\operatorname{surj}(i, l)| \not \equiv|\operatorname{surj}(j, l)| \bmod (p)
$$

Consider the expression $x_{\ell}-|\operatorname{surj}(i, l)| \in A(n)$. Then

$$
\phi_{i}\left(x_{\ell}-|\operatorname{surj}(i, l)|\right)=|\operatorname{surj}(i, l)|-|\operatorname{surj}(i, l)| \equiv 0 \bmod (p)
$$

while

$$
\phi_{j}\left(x_{\ell}-|\operatorname{surj}(i, l)|\right)=|\operatorname{surj}(j, l)|-|\operatorname{surj}(i, l)| \not \equiv 0 \bmod (p)
$$

It follows that $\mathfrak{p}([i], p) \neq \mathfrak{p}([j], p)$.
In summary:
Theorem 8.28. Let $1 \leq i, j \leq d$. Then:
(a) The prime ideals of $A(d)$ are precisely the $\mathfrak{p}([i], p)$ defined in Definition 8.23. The prime ideals $\mathfrak{p}([i], p)$ for $p>0$ are maximal prime ideals, and the prime ideals $\mathfrak{p}([i], 0)$ are minimal prime ideals.
(b) For $p, q>0$ and $i \neq j$ we have $\mathfrak{p}([i], p)=\mathfrak{p}([j], q)$ if and only if $p=q$ and $p-1 \mid j-i$.
(c) We have an inclusion $\mathfrak{p}([i], p) \subseteq \mathfrak{p}([j], q)$ if and only if any of the following are satisfied:
(i) $p=q>0$ and $\mathfrak{p}([i], p)=\mathfrak{p}([j], p)$
(ii) $p=q=0$ and $i=j$
(iii) $p=0, q>0$ and $\mathfrak{p}([i], q)=\mathfrak{p}([j], q)$.

Remark 8.29. Since the ring $A(d)$ is noetherian (Corollary 8.13), the Zariski topology is completely determined by the inclusions among prime ideals. Thus, Theorem 8.28 provides a complete description of the Zariski spectrum of $A(d)$. It consists of $d$ copies of $\operatorname{Spec}(\mathbb{Z})$ with certain closed points glued together, namely, the closed point $p$ in the $i$-th copy is glued together with the closed point $p$ in the $j$-th copy precisely when $p-1 \mid j-i$.

Example 8.30. The Zariski spectrum of $A(3)$ is shown in Figure 3.


Figure 3. The Zariski spectrum of the ring $A(3)$.

## 9. The comparison map to the Goodwillie-Burnside Ring

We now investigate the endomorphism ring of the unit object in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ and show that it is isomorphic to the Goodwillie-Burnside ring defined in Section 8. This identification is significant due to the following comparison map constructed by Balmer [Bal10b]:

Definition 9.1. For an essentially small tensor triangulated category $\mathcal{K}$, the comparison map

$$
\rho_{\mathcal{K}}: \operatorname{Spc}(\mathcal{K}) \rightarrow \operatorname{Spec}\left(\operatorname{End}_{\mathcal{K}}(\mathbb{1})\right)
$$

is the continuous map defined by $\mathcal{P} \mapsto\{f: \mathbb{1} \rightarrow \mathbb{1} \mid \operatorname{cofib}(f) \notin \mathcal{P}\}$.
Remark 9.2. This map is always inclusion-reversing: If $\mathcal{P} \subseteq \mathcal{Q}$ are two primes of $\mathcal{K}$, then $\rho_{\mathcal{K}}(\mathcal{Q}) \subseteq \rho_{\mathcal{K}}(\mathcal{P})$.

Remark 9.3. Applied to (the homotopy category of) a symmetric monoidal stable $\infty$-category $\mathcal{C}$, note that $\operatorname{End}_{h \mathcal{C}}(\mathbb{1})=\pi_{0}\left(\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})\right)$.

Example 9.4. The comparison map

$$
\operatorname{Spc}\left(S p^{c}\right) \rightarrow \operatorname{Spec}\left(\pi_{0} \operatorname{Hom}_{\mathcal{S} p}(\mathbb{S}, \mathbb{S})\right) \cong \operatorname{Spec}(\mathbb{Z})
$$

sends $\mathcal{C}_{p, h}$ to $(p)$ and $\mathcal{C}_{0,1}$ to (0); see Example 6.5.
Remark 9.5. Recall from Theorem 2.38 that the monoidal unit of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ is the $d$-excisive functor $P_{d} h_{\mathcal{S}}$. We are therefore interested in the following ring:

Definition 9.6. We let

$$
R(d):=\pi_{0} \operatorname{Hom}_{\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, s p\right)}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right)
$$

denote the endomorphism ring of the unit object in the category of $d$-excisive functors from finite spectra to spectra.

Remark 9.7. For typesetting reasons, we'll sometimes drop or simplify the subscript indicating the ambient category and just write, for example, $\operatorname{Hom}\left(P_{d} h_{\varsigma}, P_{d} h_{\S}\right)$ or $\operatorname{Hom}_{\operatorname{Exc}_{d}}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right)$ instead of $\operatorname{Hom}_{\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \delta p\right)}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right)$.

Remark 9.8. By Lemma 5.4 and Example 5.6 we have

$$
\operatorname{Hom}_{\operatorname{Exc}_{d}\left(S_{p^{c}}, \mathcal{S p}\right)}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right) \simeq P_{d} \Sigma^{\infty} \Omega^{\infty}(\mathbb{S})
$$

A well-known theorem of Goodwillie shows that the Goodwillie tower of the functor $X \mapsto \Sigma^{\infty} \Omega^{\infty}(X)$ splits when evaluated on connected suspension spectra [Kuh07, Example 6.1], with $\left(D_{d} \Sigma^{\infty} \Omega^{\infty}\right)(X)=X_{h \Sigma_{d}}^{\otimes d}$. In particular, we have the following analog of the tom Dieck splitting in the functor calculus world:

$$
P_{d} \Sigma^{\infty} \Omega^{\infty}(\mathbb{S}) \cong \bigoplus_{1 \leq i \leq d}\left(\Sigma^{\infty} S^{0}\right)_{h \Sigma_{i}}
$$

This implies that $R(d) \cong \prod_{1 \leq i \leq d} \mathbb{Z}$ as an abelian group. Our next goal is to analyze how this interacts with the derivatives, and then use this to determine $R(d)$ as a commutative ring.

Remark 9.9. Recall that $\partial_{i}\left(P_{d} h_{\mathbb{S}}\right) \simeq \mathbb{S}$ for $1 \leq i \leq d$ by Proposition 7.9. Hence, the $i$-th derivative functor induces a map, which we also denote by $\partial_{i}$ :

$$
\partial_{i}: \operatorname{Hom}_{\operatorname{Exc}_{d}\left(S p^{c}, S p\right)}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right) \rightarrow \operatorname{Hom}_{\mathcal{S}_{p}}\left(\partial_{i} h_{\mathbb{S}}, \partial_{i} h_{\mathbb{S}}\right) \simeq \mathbb{S}
$$

Applying $\pi_{0}$ we obtain a ring homomorphism $\theta_{d}^{i}: R(d) \rightarrow \mathbb{Z}$. Taken together, we obtain a ring homomorphism $\theta_{d}:=\prod_{i=1}^{d} \theta_{d}^{i}$ :

$$
\theta_{d}: R(d) \rightarrow \prod_{i=1}^{d} \mathbb{Z}
$$

Similarly, the functor $P_{d-1}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ induces a map, which we also denote simply by $P_{d-1}$ :

$$
P_{d-1}: \operatorname{Hom}_{\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right) \rightarrow \operatorname{Hom}_{\operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \delta p\right)}\left(P_{d-1} h_{\mathbb{S}}, P_{d-1} h_{\mathbb{S}}\right)
$$

Applying $\pi_{0}$ we obtain a ring homomorphism $\pi_{0}\left(P_{d-1}\right): R(d) \rightarrow R(d-1)$.
Proposition 9.10. There is an isomorphism of abelian groups $R(d) \cong \mathbb{Z}^{d}$. The ring homomorphism $\theta_{d}$ is injective and the ring homomorphism $\pi_{0}\left(P_{d-1}\right)$ is surjective for all $d$. Furthermore, there is a commutative diagram, where the columns are split short exact sequences of abelian groups:


Here the vertical maps on the right are the inclusion of the last factor and projection onto the first $d-1$ factors respectively.
Proof. The homomorphism $\theta_{d}$ is defined by taking the first $d$ derivatives of a natural transformation $P_{d} h_{\mathbb{S}} \rightarrow P_{d} h_{\mathbb{S}}$. Since $\partial_{i} P_{d-1} \simeq \partial_{i}$ for $i \leq d-1$, and $\partial_{d} P_{d-1} \simeq 0$, it follows that the bottom square of (9.11) commutes. We will see that the top row of
the diagram is the kernel of the vertical homomorphisms in the bottom square, so the entire diagram commutes.

For each $d \geq 1$, consider the following commutative diagram, where the vertical maps are induced by the natural transformation $p_{d-1}: P_{d} \rightarrow P_{d-1}$ :


The horizontal maps are equivalences by the Yoneda lemma and the universal property of $P_{d}$. In particular, $R(d) \cong \pi_{0}\left(P_{d} h_{\mathbb{S}}(\mathbb{S})\right)$, and the homomorphism $\pi_{0}\left(P_{d-1}\right): R(d) \rightarrow R(d-1)$ can be identified with the natural homomorphism $\pi_{0}\left(p_{d-1}\right): \pi_{0}\left(P_{d} h_{\mathbb{S}}(\mathbb{S})\right) \rightarrow \pi_{0}\left(P_{d-1} h_{\mathbb{S}}(\mathbb{S})\right)$.

Now consider the fiber sequence

$$
\begin{equation*}
D_{d} h_{\mathbb{S}}(\mathbb{S}) \rightarrow P_{d} h_{\mathbb{S}}(\mathbb{S}) \rightarrow P_{d-1} h_{\mathbb{S}}(\mathbb{S}) \tag{9.12}
\end{equation*}
$$

Recall (for example via Proposition 6.23) that $D_{d} h_{\mathbb{S}}(X) \simeq X_{h \Sigma_{d}}^{\otimes d}$. Taking $X=\mathbb{S}$, we have $D_{d} h_{\mathbb{S}}(\mathbb{S}) \simeq \Sigma^{\infty} B \Sigma_{d+}$ and $\pi_{0}\left(D_{d} h_{\mathbb{S}}(\mathbb{S})\right) \cong \mathbb{Z}$. Therefore, applying $\pi_{0}$ to the fibration sequence (9.12) yields an exact sequence

$$
\begin{equation*}
\mathbb{Z} \rightarrow R(d) \xrightarrow{\pi_{0}\left(p_{d-1}\right)} R(d-1) \tag{9.13}
\end{equation*}
$$

We claim that this sequence is in fact short exact. It will be the left column of (9.11).
We will provide two proofs that the sequence above is short exact. Recall that $h_{\mathbb{S}} \simeq \Sigma^{\infty} \Omega^{\infty}$, so $h_{\mathbb{S}}(\mathbb{S}) \simeq \Sigma^{\infty} \Omega^{\infty} \Sigma^{\infty} S^{0}$. Since $\Sigma^{\infty}$ is a left adjoint, composing with $\Sigma^{\infty}$ commutes with $P_{d}$. It follows that the Taylor tower of the functor $h_{\mathcal{S}}$ evaluated at $\mathbb{S}$ is the same as the Taylor tower of the functor $\Sigma^{\infty} \Omega^{\infty} \Sigma^{\infty}$ evaluated at the space $S^{0}$.

It is well-known that the Taylor tower of the functor $\Sigma^{\infty} \Omega^{\infty} \Sigma^{\infty}$ splits, because of the Snaith splitting. It follows that the fiber sequence (9.12) splits, and therefore it induces a split short exact sequence on $\pi_{0}$.

We now give another proof that (9.13) is short exact which will, moreover, establish that the homomorphism at the top of (9.11) is multiplication by $d$ !.

The functor $\partial_{d}: \operatorname{Fun}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ factors through the category of spectra with an action of $\Sigma_{d}$. It follows that the map

$$
\partial_{d}: \operatorname{Hom}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right) \rightarrow \operatorname{Hom}_{\mathcal{S} p}\left(\partial_{d} h_{\mathbb{S}}, \partial_{d} h_{\mathbb{S}}\right) \simeq \operatorname{Hom}_{\mathcal{S} p}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{S}
$$

factors through $\operatorname{Hom}_{\mathcal{S} p}(\mathbb{S}, \mathbb{S})^{h \Sigma_{d}}$. The map fits into the following diagram


Here the square at the bottom is a special case of the Kuhn-McCarthy pullback square (Proposition 6.40) applied to the functor $P_{d} h_{\mathbb{S}}$ and evaluated at $\mathbb{S}$. To see this, use the identification $\operatorname{Hom}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right) \simeq P_{d} h_{\mathbb{S}}(\mathbb{S})$, and see [AC15, Proposition 4.14]
which, together with its proof, shows that the top map in the Kuhn-McCarthy square is indeed induced by our map $\partial_{d}$.

It follows that the map $\mathbb{S}_{h \Sigma_{d}} \rightarrow \mathbb{S}$ from the top left corner to the far right spot in the diagram is the transfer map, whose effect on $\pi_{0}$ is the homomorphism $\mathbb{Z} \xrightarrow{d!} \mathbb{Z}$. In particular, this homomorphism is injective. It follows that the map $S_{h \Sigma_{d}} \rightarrow \operatorname{Hom}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right)$ induces a monomorphism on $\pi_{0}$. But this is the same as the map $D_{d} h_{\mathbb{S}}(\mathbb{S}) \rightarrow P_{d} h_{\mathbb{S}}(\mathbb{S})$ in (9.12). Note also that $\pi_{-1}\left(\mathbb{S}_{h \Sigma_{d}}\right)=0$, and it follows that the map $p_{d-1}: P_{d} h_{\mathbb{S}}(\mathbb{S}) \rightarrow P_{d-1} h_{\mathbb{S}}(\mathbb{S})$ induces a surjective homomorphism on $\pi_{0}$. We have proved (for the second time) that the fibration sequence (9.12) induces a short exact sequence on $\pi_{0}$. Furthermore, we have proved that the map $\partial_{d}: \operatorname{Hom}\left(P_{d} h_{\mathbb{S}}, P_{d} h_{\mathbb{S}}\right) \rightarrow \mathbb{S}$ induces multiplication by $d!$ from the kernel of $\pi_{0}\left(p_{d-1}\right)$ (as defined in $(9.11))$ to $\pi_{0}\left(\operatorname{Hom}_{\mathcal{S} p}\left(\partial_{d} h_{\mathbb{S}}, \partial_{d} h_{\mathbb{S}}\right)\right) \cong \mathbb{Z}$.

We have shown that the left column of (9.11) is a short exact sequence and that the upper horizontal homomorphism is multiplication by $d$ !. By induction on $d$ it follows that $\theta_{d}$ is a monomorphism for all $d$, and $R(d) \cong \mathbb{Z}^{d}$. For the basis of the induction one can take the case $d=0$, in which case $\theta_{0}$ is the homomorphism from the trivial group to itself. Or one can begin with $d=1$, in which case $\theta_{1}$ is the isomorphism $\pi_{0}(\mathbb{S}) \xrightarrow{\cong} \mathbb{Z}$ given by degree.

Remark 9.14. Now that we have shown that $R(d) \cong \mathbb{Z}^{d}$, our next task is to find an explicit basis for $R(d)$. We have just defined an injective homomorphism $\prod_{i=1}^{d} \theta_{d}^{i}: R(d) \rightarrow \mathbb{Z}^{d}$. The following lemma will help us to identify a basis of $R(d)$
Lemma 9.15. Suppose that $b_{1}, \ldots, b_{d}$ are elements of $R(d)$ that satisfy the following for all $i, j \leq d$ :

$$
\theta_{d}^{i}\left(b_{j}\right)=\left\{\begin{array}{ll}
i! & i=j \\
0 & i<j
\end{array} .\right.
$$

Then $b_{1}, \ldots, b_{d}$ is a basis of $R(d)$.
Proof. By Proposition 9.10 we have a split short exact sequence

$$
0 \rightarrow \mathbb{Z} \rightarrow R(d) \xrightarrow{\pi_{0}\left(P_{d-1}\right)} R(d-1) \rightarrow 0
$$

It is enough to show that $b_{d}$ is in the image of a generator of $\mathbb{Z}$ and the images of $b_{1}, \ldots, b_{d-1}$ form a basis of $R(d-1)$. Let $\bar{b}_{i}$ be the image of $b_{i}$ in $R(d-1)$. By the commutativity of the bottom half of (9.11), $\theta_{d-1}^{i}\left(\bar{b}_{j}\right)=\theta_{d}^{i}\left(b_{j}\right)$ for $i, j \leq d-1$. Arguing by induction on $d$, we can conclude that $\bar{b}_{1}, \ldots, \bar{b}_{d-1}$ form a basis for $R(d-1)$. Once again, for a basis of the induction one can take the case $d=0$ or $d=1$, which are both immediate.

Now consider $b_{d}$. By assumption, $\theta_{d}^{i}\left(b_{d}\right)=0$ for $i<d$. It follows that $b_{d} \in$ $\operatorname{ker}\left(\pi_{0}\left(p_{d-1}\right)\right)$ and therefore $b_{d}$ is in the image of the homomorphism $\mathbb{Z} \rightarrow R(d)$, that is, in the upper left corner of (9.11). Furthermore, the assumption $\theta^{d}\left(b_{d}\right)=d$ ! means that $b_{d}$ is the image of a generator of $\mathbb{Z}$.

Remark 9.16. Now we are ready to construct an explicit basis of $R(d)$. Recall that $h_{\mathbb{S}}(m)$ is the value of the $m$-th cross-effect (or equivalently the $m$ th co-cross-effect) of the contravariant functor $x \mapsto h_{x}$ at ( $\mathbb{S}, \ldots, \mathbb{S}$ ); see Section 4.

Definition 9.17. Define $\lambda_{m}: h_{\mathbb{S}} \rightarrow h_{\mathbb{S}}$ to be the following composition

$$
h_{\mathbb{S}} \rightarrow h_{\mathbb{S} \oplus m} \rightarrow h_{\mathbb{S}}(m) \rightarrow h_{\mathbb{S} \oplus m} \rightarrow h_{\mathbb{S}} .
$$

Here the first and the last map are induced by the fold and the diagonal maps between $\mathbb{S}$ and $\bigoplus_{m} \mathbb{S}$. The second and the third map are a special case of the natural map from $F\left(x_{1} \oplus \cdots \oplus x_{m}\right)$ to $\mathrm{cr}_{m} F\left(x_{1}, \ldots, x_{m}\right)$ and back, with $F(x)=h_{x}$; see (3.7).

Remark 9.18. By Lemma 3.29, $\lambda_{m}$ is equivalent to the composition

$$
h_{\mathbb{S}} \rightarrow h_{\mathbb{S} \oplus m} \xrightarrow{\bigcirc_{t \in[m]}^{Q_{1}}\left(1-h_{\psi([m] \backslash\{t\})}\right)} h_{\mathbb{S} \oplus m} \rightarrow h_{\mathbb{S}} .
$$

Let us remind the reader that $h_{\psi([m] \backslash\{t\})}$ is the idempotent map $h_{\mathbb{S} \oplus m} \rightarrow h_{\mathbb{S} \oplus m}$ induced by collapsing to a point the $t$-th summand of $\underbrace{\mathbb{S} \cdots \oplus \mathbb{S}}_{m}$.

Remark 9.19. The maps $\lambda_{m}$ are natural transformations from $h_{\mathbb{S}}$ to itself. To shorten notation, let us also denote by $\lambda_{m}$ the induced map $P_{d}\left(\lambda_{m}\right): P_{d} h_{\mathbb{S}} \rightarrow P_{d} h_{\mathbb{S}}$ for all $d \geq m$. Let us denote by $\left[\lambda_{m}\right]$ the homotopy class of $\lambda_{m}$, considered as an element of $R(d)$.

We want to show that $\left[\lambda_{1}\right], \ldots,\left[\lambda_{d}\right]$ form a basis of $R(d)$. It follows from Lemma 9.15 that to do this we need to analyze the induced maps $\partial_{i} \lambda_{m}: \partial_{i} h_{\mathbb{S}} \rightarrow \partial_{i} h_{\mathbb{S}}$. We already observed that $\partial_{i} h_{\mathbb{S}} \simeq \mathbb{D}\left(\mathbb{S}^{\otimes i}\right) \simeq \mathbb{S}$ and therefore the homotopy type of $\partial_{i} \lambda_{m}$ is determined by a single integer - the degree.

Lemma 9.20. The degree of $\partial_{i} \lambda_{m}$ is $|\operatorname{surj}(i, m)|$. In particular, $\partial_{i} \lambda_{m}$ is null for $i<m$ and $\partial_{i} \lambda_{i}$ has degree $i$ !.

Proof. Using Remark 9.18 and the identification $\partial_{i} h_{x} \simeq \mathbb{D}\left(x^{\otimes i}\right)$, we see that the map $\partial_{i} \lambda_{m}: h_{\mathbb{S}} \rightarrow h_{\mathbb{S}}$ is the Spanier-Whitehead dual of the following composition of maps

$$
\begin{equation*}
\mathbb{S}^{\otimes i} \xrightarrow[m]{\Delta_{m}^{\otimes i}}\left(\mathbb{S}^{\oplus m}\right)^{\otimes i} \xrightarrow{\substack{\bigcirc \\ t \in[m]}}\left(1-\psi([m] \backslash\{t\})^{\otimes i}\right)\left(\mathbb{S}^{\oplus m}\right)^{\otimes i} \xrightarrow[m]{\nabla_{m}^{\otimes i}} \mathbb{S}^{\otimes i} \tag{9.21}
\end{equation*}
$$

where $\Delta_{m}^{\otimes i}$ is the $i$-th smash power of the diagonal map $\mathbb{S} \rightarrow \bigoplus_{m} S, \nabla_{m}^{\otimes i}$ is the $i$-th smash power of the fold map $\bigoplus_{m} \mathbb{S} \rightarrow \mathbb{S}$, and the map in the middle is the composition of maps of the form $1-\psi([m] \backslash\{t\})^{\otimes i}$, where 1 is the identity on $\left(\mathbb{S}^{\oplus m}\right)^{\otimes i}$ and $\psi([m] \backslash\{t\})^{\otimes i}$ is the $i$-th smash power of the self map of $\mathbb{S}^{\oplus m}$ that collapses the $t$-th copy of $\mathbb{S}$ to a point.

There is an obvious identification $\left(\mathbb{S}^{\oplus m}\right)^{\otimes i} \cong \mathbb{S}^{\oplus m^{i}}$, where we identify $m^{i}$ with the set of functions $[i] \rightarrow[m]$. Under this identification, the map $\Delta_{m}^{\otimes i}$ in (9.21) becomes the diagonal map $\mathbb{S} \rightarrow \mathbb{S}^{\oplus m^{i}}$ and the map $\nabla_{i}^{\otimes m}$ in the same line becomes the folding map $\mathbb{S}^{\oplus m^{i}} \rightarrow \mathbb{S}$. Given a $t \in[m]$, the map $\psi([m] \backslash\{t\})^{\otimes i}$ becomes the map $\mathbb{S}^{\oplus m^{i}} \rightarrow \mathbb{S}^{\oplus m^{i}}$ that collapses to a point all the copies of $\mathbb{S}$ that are labeled by maps $[i] \rightarrow[m]$ whose image contains $t$. It follows that $1-\psi([m] \backslash\{t\})^{\otimes i}$ is equivalent to the map $\mathbb{S}^{\oplus m^{i}} \rightarrow \mathbb{S}^{\oplus} m^{i}$ that collapses to a point all the copies of $\mathbb{S}$ that are labeled by maps $[i] \rightarrow[m]$ whose image does not contain $t$. It follows that $\bigcirc_{t \in[m]}\left(1-\psi([m] \backslash\{t\})^{\otimes i}\right)$ is equivalent to the map $\mathbb{S}^{\oplus m^{i}} \rightarrow \mathbb{S}^{\oplus m^{i}}$ that collapses to a point all the copies of $\mathbb{S}$ that are labeled by non-surjective maps $[i] \rightarrow[m]$.

We conclude that the composition of maps on line (9.21) is equivalent to the composition

$$
\mathbb{S} \rightarrow \bigoplus_{m^{i}} \mathbb{S} \rightarrow \bigoplus_{m^{i}} \mathbb{S} \rightarrow \mathbb{S}
$$

where the first map is the diagonal, the middle map collapses to a point all copies of $\mathbb{S}$ labeled by non-surjective maps from $[i]$ to $[m]$, and the third map is the fold map. It is clear that this composition has degree $|\operatorname{surj}(i, m)|$ and therefore $\partial_{i} \lambda_{m}$, which is the Spanier-Whitehead dual of this composition, also has degree $|\operatorname{surj}(i, m)|$.

The motivation for defining the elements $\left[\lambda_{i}\right]$ stems from the following lemma.
Lemma 9.22. The elements $\left[\lambda_{1}\right] \ldots,\left[\lambda_{d}\right]$ form an additive basis of $R(d)$.
Proof. By Lemma 9.15 it is enough to prove that $\lambda_{j}$ induces multiplication by $j$ ! on $\partial_{j}$ and induces the zero map on $\partial_{i}$ for $i<j$. This is a special case of Lemma 9.20.

Now we are ready to prove the main result of this section. Recall that $\mu(i, j, l)$ is the number of good subsets of $[i] \times[j]$ of cardinality $l$.

Theorem 9.23. The ring $R(d)$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}\left[\left[\lambda_{2}\right], \ldots,\left[\lambda_{d}\right]\right]$ by the relations

$$
\left[\lambda_{i}\right]\left[\lambda_{j}\right]=\sum_{l=1}^{d} \mu(i, j, l)\left[\lambda_{i}\right]\left[\lambda_{j}\right]
$$

In particular, there is an isomorphism $R(d) \cong A(d)$ between $R(d)$ and the GoodwillieBurnside ring given by sending $\left[\lambda_{i}\right]$ to $x_{i}$.

Proof. We will show that there is an isomorphism $R(d) \xrightarrow{\cong} A(d)$ that takes $\left[\lambda_{i}\right]$ to $x_{i}$ for $i=1, \ldots, d$. The first part of the theorem then follows from Corollary 8.13.

By Lemma $9.22, R(d)$ is the free abelian group on the set $\left[\lambda_{1}\right], \ldots,\left[\lambda_{d}\right]$. Let $\mathbb{Z}^{d}$ denote the product ring, with generators $y_{1}, \ldots, y_{d}$. Goodwillie derivatives induce a ring homomorphism $R(d) \rightarrow \mathbb{Z}^{d}$. By Lemma 9.20, this homomorphism sends $\left[\lambda_{j}\right]$ to $\sum_{i=1}^{d}|\operatorname{surj}(i, j)| y_{j}$. By Theorem 8.10, there is a unique ring structure on $R(d)$ that makes this map into a ring homomorphism, and this ring structure is isomorphic to $A(d)$.

Remark 9.24. Because of this theorem, we will now cease to use $R(d)$ and instead only use the notation $A(d)$.
Proposition 9.25. For each $1 \leq k \leq d$, the comparison map of Definition 9.1

$$
\rho: \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) \rightarrow \operatorname{Spec}(A(d))
$$

sends $\mathcal{P}([k], 0,1)$ to $\mathfrak{p}([k], 0)$ and $\mathcal{P}([k], p, h)$ to $\mathfrak{p}([k], p)$ for all $h>1$.
Proof. It follows from Lemma 9.20 that the functor $\partial_{k}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ induces the homomorphism $\phi_{k}: A(d) \rightarrow \mathbb{Z}$ that satisfies $\phi_{k}\left(\left[\lambda_{m}\right]\right)=|\operatorname{surj}(k, m)|$, which was used to define the prime ideal $\mathfrak{p}([k],(p))$; see Definition 8.23.

By naturality of the comparison map ([Bal10b, Theorem 5.3(c)]) we deduce that the diagram

commutes. The result then follows from Example 9.4 and the definitions of the various prime ideals.

## Part III. The spectrum of $d$-excisive functors

We determined the underlying set of the Balmer spectrum $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ for any $d \geq 1$ in Theorem 7.14. Our next goal is to completely compute its topology. Recall that each prime tt-ideal $\mathcal{P} \in \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ is of the form $\mathcal{P}=\mathcal{P}_{d}([k], p, h)$ for some triple $(k, p, h)$ consisting of an integer $1 \leq k \leq d$, a prime number $p$ or $p=0$, and a chromatic height $1 \leq h \leq \infty$. As we will see in Proposition 11.1, the topology of the Balmer spectrum is completely determined by the inclusions among prime tt-ideals

$$
\mathcal{P}_{d}([k], p, h) \stackrel{?}{\subseteq} \mathcal{P}_{d}\left([l], q, h^{\prime}\right)
$$

Consequently, our primary goal is to describe precisely when such an inclusion occurs in terms of a numerical formula involving the above variables.

We achieve this in three main steps, each of which requires a different set of techniques:
(a) First, we study the chromatic blueshift behaviour of the Tate-derivatives $\partial_{k} t_{d} i_{d}: \mathcal{S} p \rightarrow \mathcal{S} p$ introduced in Definition 6.44. Using results from [AC15], this problem can be translated into an analogous blueshift question in stable equivariant homotopy theory for the family $\mathcal{F}_{\text {nt }}$ of non-transitive subgroups of products of symmetric groups $\Sigma_{n}$. While the spectrum $\operatorname{Spc}\left(\mathcal{S} p_{\Sigma_{n}}^{c}\right)$ is not known for any $n \geq 4$, we are able to give a complete answer (Theorem 10.33) by reducing to the main result of $\left[\mathrm{BHN}^{+} 19\right]$.
(b) We then explain how the blueshift behaviour of the Tate-derivatives provides "elementary" inclusions among the prime tt-ideals of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$; see Lemma 11.15. Combining general tt-geometric techniques developed in [BS17] together with our computation of the spectrum of the GoodwillieBurnside ring (Theorem 8.28), we then show in Theorem 11.22 that all inclusions among prime tt-ideals are given by certain minimal chains of elementary inclusions.
(c) It then remains to understand the nature of these minimal chains, a combinatorial problem that is-in light of Remark 8.3-ultimately controlled by the structure of the poset $\mathrm{Epi}_{\leq d}$. This in turn translates into an elementary number-theoretic problem about the existence of $p$-power partitions, whose solution is given in Proposition 11.38. The final answer which precisely describes the inclusions among the prime tt-ideals of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S p}\right)^{c}$ is then assembled in Theorem 11.39.
In Section 12, we express the topology of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ more explicitly in terms of type functions and state the resulting classification of tt-ideals (Theorem 12.5). We then conclude with two further results: a calculus analogue (Corollary 12.19) of transchromatic Smith-Floyd theory inspired by [KL20] and a computation (Theorem 12.27) of the Balmer spectrum of the category $\operatorname{Exc}_{d}\left(\mathcal{S}^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)$ obtained by changing coefficients along $\mathcal{S} p \rightarrow \operatorname{Mod}_{\mathrm{HZ}}$ which is inspired by [PSW22].

## 10. Blueshift of Tate-Derivatives

In this section we study the chromatic behaviour of the Tate-derivatives on the category of $d$-excisive functors, which were introduced in Definition 6.44:

$$
\begin{equation*}
\partial_{l} t_{d} i_{d}: \mathcal{S} p \rightarrow \mathcal{S} p, \quad A \mapsto \partial_{l}\left(X \mapsto\left(A \otimes X^{\otimes d}\right)^{t \Sigma_{d}}\right) \tag{10.1}
\end{equation*}
$$

Our goal is to determine the precise blueshift at each prime $p$, i.e., the extent to which the functor (10.1) shifts chromatic height. We can summarize our approach in two steps:
(a) We use results in [AC15] to express the Tate-derivatives in terms of geometric fixed point functors for families of non-transitive subgroups of products of symmetric groups. This translates our question to a problem in stable equivariant homotopy theory.
(b) We reduce the problem further to the case of cyclic groups, for which we can apply the main theorems of $[\mathrm{Kuh} 04]$ and $\left[\mathrm{BHN}^{+} 19\right]$ to resolve the analogous blueshift question.
The answer is given in Theorem 10.33 below.

## Preliminaries on stable equivariant homotopy theory.

Notation 10.2. For a finite group $G$, we denote the symmetric monoidal stable $\infty$-category of genuine $G$-spectra by $\mathcal{S} p_{G}$. It is rigidly-compactly generated by the orbits $\Sigma^{\infty} G / H_{+}$associated to the (conjugacy classes of) subgroups $H \leq G$. For a construction of $\mathcal{S} p_{G}$ and further discussion of its basic properties, see [MNN17] and [BS17]. Each group homomorphism $f: H \rightarrow G$ induces a geometric functor $f^{*}: \mathcal{S} p_{G} \rightarrow \mathcal{S} p_{H}$ which has a lax symmetric monoidal right adjoint $f_{*}$. In particular, for the unique homomorphism $p: G \rightarrow e$ to the trivial group, we have the inflation functor $\operatorname{infl}_{e}^{G}:=p^{*}: \mathcal{S} p \rightarrow \mathcal{S} p_{G}$ whose right adjoint $(-)^{G}:=p_{*}$ is the (categorical) fixed points functor.

Definition 10.3. Let $G$ be a finite group. A family of subgroups $\mathcal{F}$ is a collection of subgroups of $G$ which is closed under conjugation and passage to subgroups.

Example 10.4. Let $G=\Sigma_{d}$ be the symmetric group on $d$ letters. A subgroup $H \subseteq \Sigma_{d}$ is called non-transitive if the induced permutation $H$-action on a set of $d$-elements is non-transitive. The collection of non-transitive subgroups of $\Sigma_{d}$ forms a family $\mathcal{F}_{\mathrm{nt}}(d)$ which will play a distinguished role in what follows.

Remark 10.5. Let $d=\sum_{i=0}^{m} a_{i} p^{i}$ be the expansion of $d$ in base $p$, so that $0 \leq a_{i}<p$ for all $i$. Let $C_{p}^{2 i}:=C_{p} \imath \ldots \prec C_{p}$ denote the $i$-fold wreath product of cyclic groups of order $p$. The $p$-Sylow subgroups of $\Sigma_{d}$ are of the form

$$
S_{d, p} \cong \prod_{i=0}^{m}\left(C_{p}^{i i}\right)^{\times a_{i}}
$$

(See [Rot95, p. 176] or [Kal48].) It follows that $\Sigma_{d}$ has a transitive $p$-Sylow subgroup if and only if $d$ is a power of $p$. This is the underlying reason for the sparsity of blueshift numbers we will encounter in the description of the topology later on.

Remark 10.6. We can pull back a family $\mathcal{F}$ of subgroups of $G$ along any homomorphism $f: H \rightarrow G$ to obtain a family

$$
f^{-1} \mathcal{F}:=\{K \subseteq H \mid f(K) \in \mathcal{F}\}
$$

of subgroups of $H$. One also readily observes that a union or intersection of a collection of families of subgroups of $G$ is again a family of subgroups of $G$.

Definition 10.7. We let $E \mathcal{F}$ denote the universal $G$-space for the family $\mathcal{F}$, which is characterized (up to homotopy equivalence) by

$$
E \mathcal{F}^{H} \simeq \begin{cases}* & H \in \mathcal{F} \\ \varnothing & H \notin \mathcal{F}\end{cases}
$$

We define the pointed $G$-space $\tilde{E} \mathcal{F}$ via the cofiber sequence of pointed spaces

$$
\begin{equation*}
E \mathcal{F}_{+} \rightarrow S_{G}^{0} \rightarrow \tilde{E} \mathcal{F} \tag{10.8}
\end{equation*}
$$

It is characterized by

$$
\tilde{E} \mathcal{F}^{H} \simeq \begin{cases}S^{0} & H \notin \mathcal{F} \\ * & H \in \mathcal{F}\end{cases}
$$

Example 10.9. If $\mathcal{F}_{\text {triv }}$ is the family consisting of only the trivial subgroup of $G$, then we obtain the familiar universal $G$-space $E \mathcal{F}_{\text {triv }} \simeq E G$. If $\mathcal{F}_{\text {all }}$ is the family of all subgroups of $G$, then $E \mathcal{F}_{\text {all }} \simeq S_{G}^{0}$ and $\tilde{E} \mathcal{F}_{\text {all }} \simeq *$.

Example 10.10. Let $\bar{\rho}_{d}$ denote the reduced standard $(d-1)$-dimensional real representation of $\Sigma_{d}$ and write $S^{\bar{\rho}_{d}}$ for the corresponding $\Sigma_{d}$-representation sphere, i.e., the one-point compactification of $\bar{\rho}_{d}$ with induced $\Sigma_{d^{-}}$-action. Let $S^{n \bar{\rho}_{d}}$ be its $n$-fold smash power. The spaces $S^{n \bar{\rho}_{d}}$ form a directed system as $n$ varies, and

$$
S^{\infty \bar{\rho}_{d}}:=\operatorname{colim}_{n} S^{n \bar{\rho}_{d}}
$$

is a model for the $\Sigma_{d}$-space $\tilde{E} \mathcal{F}_{\mathrm{nt}}(d)$ for the family $\mathcal{F}_{\mathrm{nt}}(d)$ of Example 10.4.
Lemma 10.11. Let $G$ and $H$ be finite groups.
(a) Let $f: H \rightarrow G$ be a group homomorphism and $\mathcal{F}$ a family of subgroups of $G$. Then there are equivalences of $H$-spaces

$$
f^{*}\left(E \mathcal{F}_{+}\right) \simeq\left(E f^{-1} \mathcal{F}\right)_{+} \quad \text { and } \quad f^{*}(\tilde{E} \mathcal{F}) \simeq\left(\tilde{E} f^{-1} \mathcal{F}\right)
$$

(b) Consider two families $\mathcal{F}_{1}, \mathcal{F}_{2}$ of subgroups of $G$. Then there are equivalences of $G$-spaces

$$
\left(E \mathcal{F}_{1}\right)_{+} \wedge\left(E \mathcal{F}_{2}\right)_{+} \simeq E\left(\mathcal{F}_{1} \cap \mathcal{F}_{2}\right)_{+} \quad \text { and } \quad\left(\tilde{E} \mathcal{F}_{1}\right) \wedge\left(\tilde{E} \mathcal{F}_{2}\right) \simeq \tilde{E}\left(\mathcal{F}_{1} \cup \mathcal{F}_{2}\right)
$$

Proof. Both pairs of identities can be readily checked by testing on fixed points.
Remark 10.12. Applying $\Sigma^{\infty}$ to the cofiber sequence (10.8) we obtain an exact triangle

$$
\Sigma^{\infty} E \mathcal{F}_{+} \rightarrow \Sigma^{\infty} S_{G}^{0} \rightarrow \Sigma^{\infty} \tilde{E} \mathcal{F}
$$

in $\mathcal{S} p_{G}$ which can be identified with the idempotent triangle (Remark 6.6)

$$
e_{Y} \rightarrow \mathbb{1} \rightarrow f_{Y}
$$

associated to the Thomason subset $Y:=\bigcup_{H \in \mathcal{F}} \operatorname{supp}\left(\Sigma^{\infty} G / H_{+}\right) \subseteq \operatorname{Spc}\left(\mathcal{S} p_{G}^{c}\right)$; cf. [BS17, Example 5.14]. By slight abuse of notation, we will drop the $\Sigma^{\infty}$ and write $E \mathcal{F}+$ and $\tilde{E} \mathcal{F}$ for the corresponding suspension spectra in $\mathcal{S} p_{G}$. Note that the stable versions of the identities of Lemma 10.11 hold as well, since $\Sigma^{\infty}$ commutes with the smash product $\wedge$ and change of groups $f^{*}$. Alternatively, they follow from the general behaviour of Balmer-Favi idempotents using the above identification; cf. Proposition 6.14 and [BHS23b, Lemma 1.27].

Remark 10.13. We now define geometric fixed points with respect to a family of subgroups. When applied to the family $\mathcal{F}=\mathcal{P}(G)$ of all proper subgroups of $G$, the definition recovers the usual geometric fixed points for $G$.
Definition 10.14. The geometric fixed points of a $G$-spectrum $X$ with respect to a family $\mathcal{F}$ of subgroups of $G$ is the non-equivariant spectrum

$$
\Phi^{\mathcal{F}}(X):=(X \otimes \tilde{E} \mathcal{F})^{G}
$$

Lemma 10.15. Suppose $\mathcal{F}_{1} \subseteq \mathcal{F}_{2}$ are families of subgroups of $G$. There is a monoidal natural transformation $\Phi^{\mathcal{F}_{1}} \rightarrow \Phi^{\mathcal{F}_{2}}$ of lax symmetric monoidal functors.
Proof. The desired natural transformation is obtained from the map $\tilde{E} \mathcal{F}_{1} \rightarrow \tilde{E} \mathcal{F}_{2}$ of equivariant commutative ring spectra by applying the lax symmetric monoidal functor of fixed points.

Definition 10.16. Let $\mathcal{F}$ be a family of subgroups of $G$. The $\mathcal{F}$-Tate construction on a $G$-spectrum $X$ is defined as the $\mathcal{F}$-geometric fixed points

$$
X^{t \mathcal{F}}:=\Phi^{\mathcal{F}}(\underline{X})
$$

of the Borel-completion $\underline{X}:=F\left(E G_{+}, X\right)$ of $X$. For a non-equivariant spectrum $Y$, we write

$$
Y^{t \mathcal{F}}:=\left(\operatorname{infl}_{e}^{G} Y\right)^{t \mathcal{F}}
$$

for the $\mathcal{F}$-Tate construction on $Y$ equipped with a trivial $G$-action via inflation.
Example 10.17. Taking the trivial family $\mathcal{F}=\mathcal{F}_{\text {triv }}$, we obtain the usual Tate construction $X^{t G}:=(\underline{X} \otimes \tilde{E} G)^{G}$.

Lemma 10.18. Let $f: H \rightarrow G$ be a group homomorphism and $\mathcal{F}$ a family of subgroups of $G$. Then:
(a) There is a monoidal natural transformation

$$
\Phi^{\mathcal{F}} \rightarrow \Phi^{f^{-1} \mathcal{F}} \circ f^{*}
$$

of symmetric monoidal functors $\mathcal{S} p_{G} \rightarrow \mathcal{S} p$.
(b) There is a monoidal natural transformation

$$
(-)^{t \mathcal{F}} \rightarrow f^{*}(-)^{t\left(f^{-1} \mathcal{F}\right)}
$$

of lax symmetric monoidal functors $\mathcal{S} p_{G} \rightarrow \mathcal{S} p$.
(c) In particular, precomposing with $\operatorname{infl}_{e}^{G}$, there is a monoidal natural transformation

$$
(-)^{t \mathcal{F}} \rightarrow(-)^{t\left(f^{-1} \mathcal{F}\right)}
$$

of lax symmetric monoidal functors $\mathcal{S} p \rightarrow \mathcal{S} p$.
Proof. The natural transformation in part (a) is given by the composite
$\Phi^{\mathcal{F}}(X)=(\tilde{E} \mathcal{F} \otimes X)^{G} \rightarrow\left(f_{*} f^{*}(\tilde{E} \mathcal{F} \otimes X)\right)^{G} \simeq\left(f^{*}(\tilde{E} \mathcal{F}) \otimes f^{*}(X)\right)^{H} \simeq \Phi^{f^{-1} \mathcal{F}}\left(f^{*}(X)\right)$
where the middle map is the unit of the $\left(f^{*}, f_{*}\right)$-adjunction, the first equivalence follows from the relation $\left(f_{*} X\right)^{G} \simeq X^{H}$ for any $X \in \mathcal{S} p_{H}$, and the last from Lemma 10.11(a). For part (b), let $N:=\operatorname{ker} f$ denote the kernel of $f$ and observe that $f^{*}\left(E G_{+}\right) \simeq E \mathcal{F}[\leq N]_{+}$is the universal $H$-space for the family of all subgroups contained in $N$, again by Lemma 10.11(a). Since $\mathcal{F}_{\text {triv }} \subseteq \mathcal{F}[\leq N]$, Lemma 10.11(b) implies there is a canonical morphism $\theta: E H_{+} \rightarrow E \mathcal{F}[\leq N]_{+}$given by

$$
E H_{+} \simeq E H_{+} \otimes E \mathcal{F}[\leq N]_{+} \rightarrow \mathbb{1} \otimes E \mathcal{F}[\leq N]_{+} \simeq E \mathcal{F}[\leq N]_{+}
$$

which induces a map

$$
\operatorname{hom}\left(E \mathcal{F}[\leq N]_{+}, Y\right) \xrightarrow{\operatorname{hom}(\theta, 1)} \operatorname{hom}\left(E H_{+}, Y\right)
$$

for any $H$-spectrum $Y$. The natural transformation in (b) is then induced from the one in part (a) using the natural transformation

$$
\begin{equation*}
f^{*} \operatorname{hom}\left(E G_{+}, X\right) \rightarrow \operatorname{hom}\left(f^{*}\left(E G_{+}\right), f^{*}(X)\right) \xrightarrow{\operatorname{hom}(\theta, 1)} \operatorname{hom}\left(E H_{+}, f^{*}(X)\right) . \tag{10.19}
\end{equation*}
$$

We note in passing that when $f: H \hookrightarrow G$ is the inclusion of a subgroup, the first map in (10.19) is an equivalence by $[\operatorname{BDS} 16,(3.12)]$ and the second map is also an equivalence since in this case $f^{*}\left(E G_{+}\right) \simeq E H_{+}$(i.e., $\mathcal{F}[\leq N]=\mathcal{F}_{\text {triv }}$ ). Part (c) follows from part (b) since $f^{*} \circ \operatorname{infl}_{e}^{G} \simeq \operatorname{infl}_{e}^{H}$.

Remark 10.20. Let $\mathcal{F}$ be a family of subgroups of $G$ and consider some $H \in \mathcal{F}$ of index $[G: H]$ in $G$. If $X$ is a $G$-spectrum on which $[G: H]$ is invertible, then a standard transfer argument shows that $X^{t \mathcal{F}}=0$. Indeed, we may apply [GL20, Corollary A.9] with $\mathcal{F}^{\prime}=\mathcal{F}_{\text {all }}$ to see that $X^{t \mathcal{F}} \simeq X^{t \mathcal{F}_{\text {all }}}=0$. For example, if $X$ is a $p$-local $G$-spectrum (such as $X=\operatorname{infl}_{e}^{G} Y$ for a $p$-local spectrum $Y$ ) and $p$ does not divide $[G: H]$ then $X^{t \mathcal{F}}=0$.

Derivatives of the Tate construction. Let $d>l>0$ be positive integers. Our next goal is to express the Tate-derivative $\partial_{l} t_{d} i_{d}: \mathcal{S} p \rightarrow \mathcal{S} p$ of Definition 6.44 in terms of the equivariant Tate constructions of Definition 10.16. This will be based on a description of the Tate-derivatives due to [AC15]. We first introduce some notation.

Definition 10.21. For a partition $\lambda=\left(d_{1}, \ldots, d_{l}\right) \vdash d$ of $d$ of length $l$, we let

$$
\Sigma_{\lambda}:=\Sigma_{d_{1}} \times \ldots \times \Sigma_{d_{l}}
$$

denote the corresponding product of symmetric groups. The projection onto the $i$-th factor will be denoted by $\pi_{i}: \Sigma_{\lambda} \rightarrow \Sigma_{d_{i}}$. Furthermore, we write

$$
\begin{equation*}
\mathcal{F}_{\mathrm{nt}}(\lambda):=\pi_{1}^{-1} \mathcal{F}_{\mathrm{nt}}\left(d_{1}\right) \cup \ldots \cup \pi_{l}^{-1} \mathcal{F}_{\mathrm{nt}}\left(d_{l}\right) \tag{10.22}
\end{equation*}
$$

for the family of subgroups $K$ of $\Sigma_{\lambda}$ with the property that at least one of the projections $\pi_{i} K \subseteq \Sigma_{d_{i}}$ is non-transitive. Note that $\mathcal{F}_{\mathrm{nt}}(\lambda)$ does indeed form a family by Remark 10.6.

Example 10.23. Consider $\lambda=(2,2) \vdash 4$. Then $\mathcal{F}_{\text {nt }}(\lambda)$ consists of the three subgroups $e, \Sigma_{2} \times e, e \times \Sigma_{2}$ in $\Sigma_{2} \times \Sigma_{2}$. In particular, the diagonal $\Sigma_{2}$ is not part of the family.

We now recall a construction from [AC15] that appears in their description of the Tate-derivatives.

Definition 10.24. Let $\lambda=\left(d_{1}, \ldots, d_{l}\right) \vdash d$ be a partition of $d$ of length $l$. Continuing Example 10.10, we define the $\Sigma_{\lambda}$-space

$$
S_{(\lambda)}^{\infty(d-l)}:=\operatorname{colim}_{L \rightarrow \infty} S^{L \bar{\rho}_{d_{1}}} \wedge \ldots \wedge S^{L \bar{\rho}_{d_{l}}}
$$

where $\Sigma_{\lambda}$ acts on each smash factor $S^{L \bar{\rho}_{d_{i}}}$ through its projection to $\Sigma_{d_{i}}$. We also write $S_{(\lambda)}^{\infty(d-l)}$ for the associated $\Sigma_{\lambda}$-suspension spectrum.

Lemma 10.25. Let $\lambda=\left(d_{1}, \ldots, d_{l}\right) \vdash d$ be a partition of $d$ of length $l$. Then we have an equivalence of $\Sigma_{\lambda}$-spectra

$$
S_{(\lambda)}^{\infty(d-l)} \simeq \bigotimes_{i=1}^{l} S^{\infty \bar{\rho}_{d_{i}}} \simeq \tilde{E} \mathcal{F}_{\mathrm{nt}}(\lambda)
$$

where $\mathcal{F}_{\mathrm{nt}}(\lambda)$ is the family (10.22) of $\Sigma_{\lambda}$.
Proof. The first equivalence holds because the equivariant suspension spectrum functor is symmetric monoidal. The second equivalence then follows from Example 10.10 and Lemma 10.11.

Proposition 10.26. Let $A$ be a spectrum. Then there is an equivalence

$$
\partial_{l}\left(X \mapsto\left(A \otimes X^{\otimes d}\right)^{h \Sigma_{d}}\right) \simeq \prod_{\lambda \vdash d:|\lambda|=l} A^{t \mathcal{F}_{\mathrm{nt}}(\lambda)}
$$

where the product ranges over partitions of $d$ of length $l$ and $\mathcal{F}_{\mathrm{nt}}(\lambda)$ is the family defined in (10.22).

Proof. This result is essentially a reformulation of [AC15, Proposition 5.2 and Remark 5.3]. There the authors write $K_{l} A$ for the $l$-th derivative of the functor $X \mapsto\left(A \otimes X^{\otimes d}\right)^{h \Sigma_{d}}$ and describe it by the formula

$$
\partial_{l}\left(X \mapsto\left(A \otimes X^{\otimes d}\right)^{h \Sigma_{d}}\right) \simeq \prod_{\lambda \vdash d:|\lambda|=l} \operatorname{colim}_{L \rightarrow \infty}\left(A \otimes S^{L(d-l)}\right)^{h \Sigma_{\lambda}}
$$

Here, $S^{L(d-l)}$ is the spectrum with $\Sigma_{d}$-action constructed in [AC15, Definition 5.1]. Restricting the action along $\Sigma_{\lambda} \subseteq \Sigma_{d}$ for some partition $\lambda=\left(d_{1}, \ldots, d_{l}\right) \vdash d$, we may identify $S^{L(d-l)}$ with $S^{L \bar{\rho}_{d_{1}}} \otimes \ldots \otimes S^{L \bar{\rho}_{d_{l}}}$ as spectra with $\Sigma_{\lambda}$-action. To obtain the proposition, we can thus rewrite the factors in the product as follows:

$$
\begin{aligned}
\operatorname{colim}_{L \rightarrow \infty}\left(A \otimes S^{L(d-l)}\right)^{h \Sigma_{\lambda}} & \simeq \operatorname{colim}_{L \rightarrow \infty} \operatorname{hom}\left(E \Sigma_{\lambda+},\left(\operatorname{infl}_{e}^{\Sigma_{\lambda}} A\right) \otimes \bigotimes_{i=1}^{l} S^{L \bar{\rho}_{d_{i}}}\right)^{\Sigma_{\lambda}} \\
& \simeq \operatorname{colim}_{L \rightarrow \infty}\left(\operatorname{hom}\left(E \Sigma_{d+}, \inf _{e}^{\Sigma_{\lambda}} A\right) \otimes \bigotimes_{i=1}^{l} S^{L \bar{\rho}_{d_{i}}}\right)^{\Sigma_{\lambda}} \\
& \simeq\left(\operatorname{hom}\left(E \Sigma_{d+}, \inf _{e}^{\Sigma_{\lambda}} A\right) \otimes S_{(\lambda)}^{\infty(d-l)}\right)^{\Sigma_{\lambda}} \\
& \simeq\left(\underline{A} \otimes \tilde{E} \mathcal{F}_{\mathrm{nt}}(\lambda)\right)^{\Sigma_{\lambda}} \\
& \simeq \Phi^{\mathcal{F}_{\mathrm{nt}}(\lambda)}(\underline{A})=A^{t \mathcal{F}_{\mathrm{nt}}(\lambda)}
\end{aligned}
$$

where the penultimate equivalence substitutes the identification of Lemma 10.25.
Blueshift. If $M$ is a spectrum, we can restrict the homology theory represented by $M$ to $p$-local finite spectra. Its kernel will then be a thick subcategory of $\mathcal{S} p_{(p)}^{c}$, so we can use the thick subcategory theorem to define a notion of height for $M$ :

Definition 10.27. If $M$ is a spectrum, then its height at the prime $p$ is defined as

$$
\operatorname{height}_{p}(M):=\inf \left\{h \in \mathbb{N} \cup\{-1, \infty\} \mid M \otimes \mathcal{C}_{p, h+1}=0\right\}
$$

where $\mathcal{C}_{p, h+1}$ is the prime given by the vanishing of $K(p, h)$, as in Example 6.5.
Example 10.28. By definition, we have height ${ }_{p}(0)=-1$ and $\operatorname{height}_{p}(\mathbb{S})=\infty$ for all primes $p$. In fact, any spectrum that is not $p$-local has infinite $p$-height using the convention that $\inf (\varnothing)=\infty$.

Remark 10.29. Let $L_{p, h}^{f}$ be the finite localization on $\mathcal{S} p$ which localizes away from $\mathcal{C}_{p, h+1}$ (Example 6.15). If we further set $L_{p,-1}^{f} X:=0$ and $L_{p, \infty}^{f} X:=X_{(p)}$ for all $X \in \mathcal{S} p$, we can rewrite Definition 10.27 as

$$
\operatorname{height}_{p}(M):=\inf \left\{h \in \mathbb{N} \cup\{-1, \infty\} \mid M \simeq L_{p, h}^{f}(M)\right\} .
$$

This implies that the notion of height defined here is compatible with, and in fact extends, the one appearing in $\left[\mathrm{BHN}^{+} 19\right]$ using the fact that the telescope conjecture holds for ring spectra ([LMMT22, Lemma 2.3]). In particular, we see that $\operatorname{height}_{p}\left(L_{p, h}^{f} \mathbb{S}\right)=h$ and $\operatorname{height}_{p}(K(p, h))=h$ for all $h$.
Lemma 10.30. Let $p$ be a prime. Height has the following properties:
(a) If $R \rightarrow S$ is a map of ring spectra, then $\operatorname{height}_{p}(R) \geq \operatorname{height}_{p}(S)$.
(b) If $M, N$ are spectra, then $\operatorname{height}_{p}(M \oplus N)=\sup \left(\operatorname{height}_{p}(M)\right.$, $\left.\operatorname{height}_{p}(N)\right)$.

Proof. Both statements follow from unwinding the definitions. For instance, for the first one, observe that $S$ is an $R$-module, so $R \otimes X=0$ implies $S \otimes X=0$ for any spectrum $X$.

Definition 10.31. Let $d \geq 1$ be an integer and $p$ a prime. For any $h \geq 0$, we define a $d$-excisive functor via inflation (Definition 2.44) from the finite local sphere of height $h$ :

$$
\mathcal{L}_{p, h}^{f}:=i_{d} L_{p, h}^{f} \subseteq
$$

Definition 10.32. A $p$-power partition of $d$ of length $l$ is a partition

$$
\lambda=\left(d_{1}, \ldots, d_{l}\right) \vdash d
$$

of $d$ of length $l$ such that each $d_{i}$ is a power of $p$. In this definition, we include $p^{0}=1$ as a power of $p$. We will write $\mathcal{P}_{p}(d ; l)$ for the set of such partitions.

We are ready to state and prove the main theorem of this section, which captures the chromatic behaviour of the Tate-derivatives.

Theorem 10.33. Let $p$ be a prime and fix integers $d \geq l>0$ and $h \geq 0$. Then $\partial_{l} t_{d} \mathcal{L}_{p, h}^{f}$ is contractible unless $d>l$ and there exists a $p$-power partition of $d$ of length $l$. If such a partition exists, then

$$
\operatorname{height}_{p}\left(\partial_{l} t_{d} \mathcal{L}_{p, h}^{f}\right)=h-1
$$

Proof. Let $\lambda=\left(d_{1}, \ldots, d_{l}\right)$ be a partition of $d$ of length $l$. The Tate construction $t_{d}(F)$ of any $d$-excisive functor $F$ is $(d-1)$-excisive, so the theorem holds for $d=l$. Therefore, we may assume $l<d$ for the remainder of the proof. The vertical maps in the Tate square of Proposition 6.40 are $\partial_{i}$-equivalences for all $i<d$, so we may replace the Tate fixed points by homotopy fixed points in our formula. In light of Proposition 10.26 and Lemma 10.30 (b), it then suffices to verify the claim for

$$
\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\mathrm{nt}}(\lambda)}:=\Phi^{\mathcal{F}_{\mathrm{nt}}(\lambda)}\left(\operatorname{infl}_{e}^{\Sigma_{\lambda}} L_{p, h}^{f} \mathbb{S}\right)
$$

in place of the Tate-derivative $\partial_{l} t_{d} \mathcal{L}_{p, h}^{f}$.
We begin with the vanishing claim. To this end, suppose $\lambda$ is not a $p$-power partition, so that there exists some $d_{i}$ which is not a power of $p$, say $d_{1}$. Let $S$ be a $p$-Sylow subgroup of $\Sigma_{d_{1}}$, which is then a non-transitive subgroup of $\Sigma_{d_{1}}$ by Example 10.4. It follows that $S^{\prime}:=S \times \prod_{i=2}^{l} \Sigma_{d_{i}} \subseteq \Sigma_{\lambda}$ is contained in $\mathcal{F}_{\mathrm{nt}}(\lambda)$.

Since $L_{p, h}^{f} S$ is $p$-local and [ $\left.\Sigma_{\lambda}: S^{\prime}\right]$ is prime to $p$, it follows from Remark 10.20 that $\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\text {nt }}(\lambda)}=0$. This proves the vanishing part of the theorem.

For the rest of this proof, we will assume that $\lambda$ is a $p$-power partition of $d$ of length $l$, say $d_{i}=p^{e_{i}}$ for some $e_{i} \geq 0$. Note that, since $l<d$ by assumption, at least one of the $e_{i}$ must be positive. Write $\mathcal{F}_{\text {triv }}$ for the family on $\Sigma_{\lambda}$ consisting only of the trivial subgroup and recall that $X^{t \mathcal{F}_{\text {triv }}}=X^{t \Sigma_{\lambda}}$ is the usual Tate construction (Example 10.17). Lemma 10.15 therefore supplies a map of commutative ring spectra

$$
\left(L_{p, h}^{f} \mathbb{S}\right)^{t \Sigma_{\lambda}}=\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\text {triv }}} \rightarrow\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\text {nt }}(\lambda)}
$$

Kuhn's blueshift theorem [Kuh04, Theorem 1.5] shows that the height of the domain is at most $h-1$, so via Lemma 10.30(a) we get the upper bound

$$
\begin{equation*}
\operatorname{height}_{p}\left(\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\mathrm{nt}}(\lambda)}\right) \leq h-1 \tag{10.34}
\end{equation*}
$$

As for the lower bound, let $\sigma_{i}$ be the long cycle in $\Sigma_{p^{e_{i}}}$ and consider the cyclic $p$-subgroup $\langle\sigma\rangle$ of $\Sigma_{\lambda}$ generated by $\sigma=\left(\sigma_{1}, \ldots, \sigma_{l}\right) \in \Sigma_{\lambda}$. In particular, the order of $\langle\sigma\rangle$ is the maximum of the $p^{e_{i}}$. By construction, $\pi_{i}\langle\sigma\rangle=\left\langle\sigma_{i}\right\rangle$ is a transitive subgroup in $\Sigma_{p^{e_{i}}}$ and, as at least one of the $e_{i}$ 's is positive, $\langle\sigma\rangle$ is not the trivial group. Therefore, $\langle\sigma\rangle \notin \mathcal{F}_{\mathrm{nt}}(\lambda)$.

With this preparation in hand, we can apply Lemma 10.18 once more to obtain a map of commutative ring spectra

$$
\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\mathrm{nt}}(\lambda)} \rightarrow\left(L_{p, h}^{f} \mathbb{S}\right)^{t\left(\mathcal{F}_{\mathrm{nt}}(\lambda) \cap\langle\sigma\rangle\right)}
$$

The projection of every proper subgroup of $\langle\sigma\rangle$ has order less than $p^{e_{i}}$ and thus cannot act transitively on $\Sigma_{p^{e_{i}}}$, so it is not a member of $\mathcal{F}_{\mathrm{nt}}(\lambda)$. We conclude that $\mathcal{F}_{\mathrm{nt}}(\lambda) \cap\langle\sigma\rangle$ is precisely the family of proper subgroups of $\langle\sigma\rangle$. Hence

$$
\left(L_{p, h}^{f} \mathbb{S}\right)^{t\left(\mathcal{F}_{\mathrm{nt}}(\lambda) \cap\langle\sigma\rangle\right)}=\Phi^{\langle\sigma\rangle}\left(\underline{\inf _{e}^{\langle\sigma\rangle} L_{p, h}^{f} \mathbb{S}}\right)
$$

By $\left[\mathrm{BHN}^{+} 19\right.$, Theorem 1.5], the height of the latter ring spectrum is $h-1$, so

$$
\begin{equation*}
\operatorname{height}_{p}\left(\left(L_{p, h}^{f} \mathbb{S}\right)^{t \mathcal{F}_{\mathrm{nt}}(\lambda)}\right) \geq h-1 \tag{10.35}
\end{equation*}
$$

using Lemma 10.30(a) again. Combining (10.34) and (10.35), we obtain the desired formula for the height of $\left(L_{p, h}^{f} S\right)^{t \mathcal{F}_{\text {nt }}(\lambda)}$ and hence for the height of $\partial_{l} t_{d} \mathcal{L}_{p, h}^{f}$.
Remark 10.36. The theorem shows that the height-shifting behaviour of $\partial_{l} t_{d} i_{d}$ is independent of the input height $h$.

## 11. The topology of the Balmer spectrum

We now determine the topology of the spectrum of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$. The architecture of our proof is modelled on the strategy given in the context of equivariant stable homotopy theory in [BS17]. First we establish that the topology is entirely determined by the inclusions among prime tt-ideals and then proceed to understanding this poset structure. The key idea is that these inclusions are controlled by the blueshift behaviour of the Tate-derivatives computed in the previous section, together with the combinatorics of chains of $p$-power partitions.
Proposition 11.1. Every closed subset of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ is a finite union of irreducible closed subsets. In particular, the topology of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ is determined by the inclusions among prime tt-ideals.

Proof. The proof that every closed subset is a finite union of irreducible closed subsets is similar to the proof of [BS17, Proposition 6.1]. The key ingredient (Lemma 7.3) is that $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ is covered by finitely many spectra for which the statement is correct; see [Bal10a, Corollary 9.5(e)]. The second statement follows from the first because the irreducible closed subsets correspond to the tt-primes and are given by $\overline{\{\mathcal{P}\}}=\{Q \mid Q \subseteq \mathcal{P}\}$.
Remark 11.2. In the proofs of the following results, we will repeatedly use the fact that the map $\operatorname{Spc}(F)$ on Balmer spectra induced by a tt-functor $F$ preserves inclusions among prime tt-ideals; see Remark 6.2.

Proposition 11.3. For every pair of integers $1 \leq l \leq d$, primes $p, q$, and chromatic integers $1 \leq h, h^{\prime} \leq \infty$, the following are equivalent:
(a) $\mathcal{P}_{d}\left([l], q, h^{\prime}\right) \subseteq \mathcal{P}_{d}([l], p, h)$;
(b) $\mathcal{C}_{q, h^{\prime}} \subseteq \mathcal{C}_{p, h}$;
(c) $h^{\prime} \geq h$ and, either $h=1$ or $q=p$.

Proof. $(a) \Longleftrightarrow(b)$ follows from Lemma 7.7 and Lemma 7.6 while $(b) \Longleftrightarrow(c)$ follows from the computation of the spectrum of $\mathcal{S} p^{c}$.

Corollary 11.4. Suppose $\mathcal{P}_{d}\left([k], q, h^{\prime}\right) \subseteq \mathcal{P}_{d}([l], p, h)$ for integers $1 \leq l, k \leq d$, primes $p, q$ and chromatic integers $1 \leq h, h^{\prime} \leq \infty$. Then $\mathcal{C}_{q, h^{\prime}} \subseteq \mathcal{C}_{p, h}$ in $\mathcal{S} p^{c}$ and so $h^{\prime} \geq h$. If $h>1$, then $p=q$.

Proof. This is an immediate consequence of Lemma 7.6 and Proposition 11.3.
Remark 11.5. For any pair of primes $p, q$, we have $\mathcal{P}_{d}([l], p, 1)=\mathcal{P}_{d}([l], q, 1)$ in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ since $\mathcal{C}_{p, 1}=\mathcal{C}_{q, 1}$ in $\mathcal{S} p^{c}$. Hence, to determine the inclusions

$$
\mathcal{P}_{d}\left([k], q, h^{\prime}\right) \subseteq \mathcal{P}_{d}([l], p, h)
$$

among tt-primes, Corollary 11.4 allows us to restrict attention to the case $p=q$.
Remark 11.6. We now bring to bear the constraints on such inclusions supplied by the comparison map to the spectrum of the Goodwillie-Burnside ring (studied in Sections 8 and 9):

Proposition 11.7. Fix a prime number $p$. Suppose that $\mathcal{P}_{d}\left([k], p, h^{\prime}\right) \subseteq \mathcal{P}_{d}([l], p, h)$ for integers $1 \leq k, l \leq d$ and chromatic integers $1 \leq h, h^{\prime} \leq \infty$. Then:
(a) $p-1 \mid k-l \geq 0$; and
(b) if $h^{\prime}=1$ then $h=1$ and $k=l$, so that the two tt-primes are equal.

Proof. Corollary 7.13 implies $k \geq l$ while Proposition 9.25 implies that we have an inclusion $\mathfrak{p}([k], p) \supseteq \mathfrak{p}([l], q)$ in the Goodwillie-Burnside ring and hence $p-1 \mid k-l$ by Theorem 8.28. This verifies (a). For (b), note that $\mathcal{P}_{d}([k], p, 1) \subseteq \mathcal{P}_{d}([l], p, h)$ implies $h=1$ by Corollary 11.4. It follows from Proposition 9.25 that $\mathfrak{p}([k], 0) \supseteq \mathfrak{p}([l], 0)$ and hence $k=l$ by Theorem 8.28.

Remark 11.8. In summary, to determine the topology of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$, it is enough to determine the minimal number $\beth \geq 0$ such that

$$
\begin{equation*}
\mathcal{P}_{d}([k], p, h+\beth) \subseteq \mathcal{P}_{d}([l], p, h) \tag{11.9}
\end{equation*}
$$

whenever $p-1 \mid k-l \geq 0$ and for any $1 \leq h \leq \infty$. Moreover, since the map

$$
\operatorname{Spc}\left(P_{k}\right): \operatorname{Spc}\left(\operatorname{Exc}_{k}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)\right) \hookrightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)\right)
$$

is an open embedding (Proposition 7.8), the inclusion (11.9) is equivalent to the inclusion

$$
\mathcal{P}_{k}([k], p, h+\beth) \subseteq \mathcal{P}_{k}([l], p, h)
$$

This leads to:
Definition 11.10. For each $p-1 \mid k-l \geq 0$ and $h \geq 1$, define

$$
\beth_{p}^{\operatorname{geom}}(k, l ; h):=\min \left\{\beth \mid \mathcal{P}_{d}([k], p, h+\beth) \subseteq \mathcal{P}_{d}([l], p, h)\right\} .
$$

It will follow from Proposition 11.19 below that the collection of such inclusions is nonempty, so that it is a well-defined natural number. Moreover, as noted above, it doesn't depend on the ambient $d \geq k$. These geometric blueshift numbers determine the topology of the spectrum.

Example 11.11. It follows from Proposition 11.3 that $\beth_{p}^{\text {geom }}(l, l ; h)=0$ for all $l$.
Remark 11.12. Our next goal is to relate the geometric blueshift to the blueshift of the Tate-derivatives.

Definition 11.13. For each prime number $p$, integers $k \geq l \geq 1$ and $h \geq 1$, the Tate blueshift number is defined as

$$
\beth_{p}^{\mathrm{Tate}}(k, l ; h):=\operatorname{height}_{p}\left(\mathcal{L}_{p, h-1}^{f}\right)-\operatorname{height}_{p}\left(\partial_{l} t_{k} \mathcal{L}_{p, h-1}^{f}\right)
$$

with height ${ }_{p}(0)=-1$. (Recall Definition 10.27.) Thus:

$$
\beth_{p}^{\text {Tate }}(k, l ; h)= \begin{cases}(h-1)-\operatorname{height}_{p}\left(\partial_{l} t_{k} \mathcal{L}_{p, h-1}^{f}\right) & \text { if } \partial_{l} t_{k} \mathcal{L}_{p, h-1}^{f} \neq 0 \\ h & \text { if } \partial_{l} t_{k} \mathcal{L}_{p, h-1}^{f}=0\end{cases}
$$

By definition, we have

$$
\begin{equation*}
\mathcal{C}_{p, h-\beth_{p}^{\text {Tate }}(k, l ; h)}=\operatorname{ker}\left(-\otimes \partial_{l} t_{k} \mathcal{L}_{p, h-1}^{f}: \mathcal{S}_{(p)}^{c} \rightarrow \mathcal{S} p\right) \tag{11.14}
\end{equation*}
$$

with the convention that $\mathcal{C}_{p, 0}$ is the whole category of finite $p$-local spectra.
The next lemma establishes the first relation between the two types of blueshift:
Lemma 11.15. If $h>\beth_{p}^{\text {Tate }}(k, l ; h)$, then

$$
\mathcal{P}_{d}([k], p, h) \subseteq \mathcal{P}_{d}\left([l], p, h-\beth_{p}^{\text {Tate }}(k, l ; h)\right)
$$

Proof. Let $x \in \mathcal{P}_{d}([k], p, h)$, i.e., $\partial_{k}(x) \in \mathcal{C}_{p, h}$, meaning $K(p, h-1)_{*} \partial_{k}(x)=0$. Equivalently, we have

$$
0=L_{p, h-1}^{f} \mathbb{S} \otimes \partial_{k}(x) \simeq \partial_{k}\left(i_{d}\left(L_{p, h-1}^{f} \mathbb{S}\right) \circledast x\right)
$$

hence $D_{k} \mathbb{1} \circledast i_{d}\left(L_{p, h-1}^{f} \mathbb{S}\right) \circledast x=0$. Since $t_{k}(-) \simeq \operatorname{hom}\left(P_{k-1} \mathbb{1}, \Sigma D_{k} \mathbb{1} \circledast-\right)$, we deduce that $t_{k}\left(i_{d}\left(L_{p, h-1}^{f} \mathbb{S}\right) \circledast x\right)=0$. Consequently,

$$
0=\partial_{l} t_{k}\left(i_{d} L_{p, h-1}^{f} \mathbb{S} \circledast x\right) \simeq \partial_{l}\left(t_{k}\left(i_{d} L_{p, h-1}^{f} \mathbb{S}\right) \circledast x\right) \simeq \partial_{l}\left(t_{k}\left(i_{d} L_{p, h-1}^{f} \mathbb{S}\right)\right) \otimes \partial_{l}(x)
$$

It follows that $\partial_{l}(x) \in \mathcal{C}_{h-\beth_{p}^{\text {Tate }}(k, l ; h)}$ by the definition of Tate blueshift numbers (11.14). In other words, $x \in \mathcal{P}_{d}\left([l], h-\beth_{p}^{\text {Tate }}(k, l ; h)\right)$.

Remark 11.16. We computed the Tate blueshifts in Theorem 10.33; namely,

$$
\beth_{p}^{\text {Tate }}(k, l ; h)= \begin{cases}1 & \text { if there exists a } p \text {-power partition of } k \text { of length } l \\ h & \text { otherwise }\end{cases}
$$

In particular, it does not depend on $h$ when there exists a $p$-power partition of $k$ of length $l$ and, in such a situation, we will drop the $h$ from the notation. Although the reader may wish to substitute the value 1 in what follows, we have opted to keep the discussion in terms of $\beth_{p}^{\text {Tate }}(k, l)$ for conceptual clarity. In any case, in order to amplify the connection between the geometric blueshift and the Tate blueshift we need to introduce the following auxiliary notion:

Definition 11.17. Let $p$ be a prime number and pick integers $k>l \geq 1$. A chain of $p$-power partitions between $k$ and $l$ is a sequence of integers

$$
\left(k=l_{s}>l_{s-1}>\cdots>l_{1}>l_{0}=l\right)
$$

with $s>0$ such that for each consecutive pair $\left(l_{\alpha}, l_{\alpha-1}\right)$ there exists a $p$-power partition of $l_{\alpha}$ of length $l_{\alpha-1}$ in the sense of Definition 10.32. We call $s$ the length of the chain. We write $\mathrm{Ch}_{p}(k, l)$ for the set of chains of $p$-power partitions between $k$ and $l$; in symbols:

$$
\mathrm{Ch}_{p}(k, l)=\left\{k=l_{s}>l_{s-1}>\ldots>l_{1}>l_{0}=l \mid s>0 \text { and } \forall \alpha: \mathcal{P}_{p}\left(l_{\alpha} ; l_{\alpha-1}\right) \neq \varnothing\right\} .
$$

Remark 11.18. If there exists a $p$-power partition of $k$ of length $l$ then $p-1 \mid k-l>0$. The converse is not true, in general. (Consider, for example, $p=2, k=3, l=1$.) However, if $p-1 \mid k-l>0$ then there exists a chain of $p$-power partitions from $k$ to $l$, that is, $\mathrm{Ch}_{p}(k, l) \neq \varnothing$. Indeed, the statement is true when $k-l=p-1$, since we can write $k=p+\sum_{\alpha=1}^{l-1} p^{0}$. The general case follows from this, by using the chain

$$
(k>k-(p-1)>\ldots>k-a(p-1)=l) \in \mathrm{Ch}_{p}(k, l)
$$

where $a$ the natural number such that $k-l=a(p-1)$.
Proposition 11.19. If $\left(k=l_{s}>\cdots>l_{0}=l\right) \in \mathrm{Ch}_{p}(k, l)$ is a chain of p-power partitions, then

$$
\mathcal{P}_{d}\left(k, p, h+\sum_{\alpha=1}^{s} \beth_{p}^{\mathrm{Tate}}\left(l_{\alpha}, l_{\alpha-1}\right)\right) \subseteq \mathcal{P}_{d}(l, p, h)
$$

for each $h \geq 1$.
Proof. By definition, for each $\alpha=1, \ldots, s$, there exists a $p$-power partition of $l_{\alpha}$ of length $l_{\alpha-1}$, hence by Theorem 10.33, $\beth_{p}^{\text {Tate }}\left(l_{\alpha}, l_{\alpha-1} ; h\right)$ is independent of $h$; see Remark 11.16. It then follows from Lemma 11.15 that we have an inclusion of prime ideals

$$
\mathcal{P}_{d}\left(l_{\alpha}, p, h^{\prime}+\beth_{p}^{\text {Tate }}\left(l_{\alpha}, l_{\alpha-1}\right)\right) \subseteq \mathcal{P}_{d}\left(l_{\alpha-1}, p, h^{\prime}\right)
$$

for all $1 \leq \alpha \leq s$ and all heights $h^{\prime} \geq 1$. Iteratively combining these inclusions, we deduce that

$$
\mathcal{P}_{d}\left(k, p, h+\sum_{\alpha=1}^{s} \beth_{p}^{\text {Tate }}\left(l_{\alpha}, l_{\alpha-1}\right)\right) \subseteq \mathcal{P}_{d}(l, p, h)
$$

as desired.
Remark 11.20. It follows from Remark 11.18 and Proposition 11.19 that the set of inclusions in Definition 11.10 is nonempty, hence the geometric blueshift is well-defined for all $h \geq 1$. Although we have not defined the geometric blueshift for $h=\infty$, it follows that $\mathcal{P}_{d}([k], p, n) \subseteq \mathcal{P}_{d}([l], p, \infty)$ if and only if $n=\infty$ and $p-1 \mid k-l \geq 0$.

Definition 11.21. Define the $p$-distance $\delta_{p}(k, l)$ between $k>l$ as the smallest length of a chain of $p$-power partitions between $k$ and $l$, or $\infty$ if no such chain exists; in symbols:

$$
\delta_{p}(k, l):=\inf \left(\left\{s \mid\left(l_{s}>\ldots>l_{0}\right) \in \mathrm{Ch}_{p}(k, l)\right\}\right)
$$

We also set $\delta_{p}(l, l):=0$.
Theorem 11.22. For each $p-1 \mid k-l \geq 0$ and $h \geq 1$, we have

$$
\begin{equation*}
\beth_{p}^{\text {geom }}(k, l ; h)=\delta_{p}(k, l) \tag{11.23}
\end{equation*}
$$

In particular, the geometric blueshift $\beth_{p}^{\text {geom }}(k, l ; h)$ does not depend on $h$.
Proof. The $k=l$ case is immediate from Example 11.11, so we may assume $k>l$. Recall from Remark 11.18 that $p-1 \mid k-l>0 \operatorname{implies} \mathrm{Ch}_{p}(k, l) \neq \varnothing$, so $\delta_{p}(k, l)$ is a well-defined natural number. By Proposition 11.19 and Theorem 10.33, we have

$$
\beth_{p}^{\text {geom }}(k, l ; h) \leq \sum_{\alpha=1}^{s} \beth_{p}^{\text {Tate }}\left(l_{\alpha}, l_{\alpha-1}\right)=s
$$

for any $\left(l_{s}>\cdots>l_{0}\right) \in \mathrm{Ch}_{p}(k, l)$. Hence

$$
\beth_{p}^{\text {geom }}(k, l ; h) \leq \delta_{p}(k, l)
$$

It remains to prove that $\beth_{p}^{\text {geom }}(k, l ; h) \geq \delta_{p}(k, l)$ or, in other words, to prove that if $n<\delta_{p}(k, l)$ then $\mathcal{P}_{d}([l], p, h+n) \nsubseteq \mathcal{P}_{d}([k], p, h)$. Due to the vertical inclusions among tt-primes (Proposition 11.3), it suffices to consider the extremal case $n=\delta_{p}(k, l)-1$. Moreover, recall that we can assume $d=k$ (Remark 11.8). In other words, it suffices to prove the following claim:
$(\mathrm{P}(d))$ For all $p-1 \mid d-l \geq 0$ and $1 \leq h<\infty$,

$$
\mathcal{P}_{d}\left([d], p, h+\delta_{p}(d, l)-1\right) \nsubseteq \mathcal{P}_{d}([l], p, h)
$$

We prove this by strong induction on $d$. The base case $d=1$ holds trivially, so let $d>1$ and suppose $(\mathrm{P}(i))$ has been verified for all $1 \leq i<d$. Set $n:=\delta_{p}(d, l)-1$ for notational convenience and write $\operatorname{Exc}_{d}=\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$. Recall that $\mathcal{L}_{h+n-1}^{f}:=$ $i_{d} L_{p, h+n-1}^{f} \mathbb{S} \in \operatorname{Exc}_{d}$ is the right idempotent for the finite localization

$$
\begin{equation*}
(-)_{\leq h+n}: \operatorname{Exc}_{d} \rightarrow \mathcal{L}_{h+n-1}^{f} \circledast \operatorname{Exc}_{d}=:\left(\operatorname{Exc}_{d}\right)_{\leq h+n} \tag{11.24}
\end{equation*}
$$

which truncates below height $h+n$; see Example 6.15 and Definition 2.44. That is,

$$
\operatorname{Spc}\left(\left(\operatorname{Exc}_{d}\right)_{\leq h+n}^{c}\right) \hookrightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)
$$

is a homeomorphism onto $V_{\leq h+n}:=\left\{\mathcal{P}_{d}([k], p, m) \mid 1 \leq m \leq h+n\right\} \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$. Since the space $V_{\leq h+n}$ is finite, it follows that there exists $z \in \operatorname{Exc}_{d}^{c}$ such that

$$
\begin{equation*}
\operatorname{supp}(z) \cap V_{\leq h+n}=\overline{\left\{\mathcal{P}_{d}([l], p, h)\right\}} \cap V_{\leq h+n} \tag{11.25}
\end{equation*}
$$

Indeed, the complement $W$ of $\overline{\left\{\mathcal{P}_{d}([l], p, h)\right\}} \cap V_{\leq h+n}$ in $V_{\leq h+n}$ is finite and open, hence also finite and open when viewed as a subset of $\operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$. It follows from [Bal05, Proposition 2.14] that there exists $z \in \operatorname{Exc}_{d}^{c}$ with $\operatorname{supp}(z)=W^{c}$ in $\operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$. Intersecting with $V_{\leq h+n}$ then gives Equation (11.25).

In order to prove $(\mathrm{P}(d))$ we will first prove the following auxiliary claim:

$$
\begin{equation*}
t_{d}\left(\mathcal{L}_{h+n-1}^{f} \circledast z\right)=0 \tag{11.26}
\end{equation*}
$$

We can test this on derivatives (Lemma 6.34). In fact, since the Tate construction $t_{d}$ of a $d$-excisive functor is $(d-1)$-excisive, it is enough to establish

$$
\partial_{i}\left(t_{d}\left(\mathcal{L}_{h+n-1}^{f} \circledast z\right)\right)=0
$$

for $1 \leq i \leq d-1$. Since $\partial_{i}$ is symmetric monoidal and $z$ is dualizable, we need to show

$$
\begin{equation*}
\partial_{i}\left(t_{d}\left(\mathcal{L}_{h+n-1}^{f}\right)\right) \otimes \partial_{i} z=0 \tag{11.27}
\end{equation*}
$$

in $\mathcal{S} p$ for each $1 \leq i \leq d-1$. This is trivially true if $d$ does not have a $p$-power partition of length $i$ by Theorem 10.33. Thus, assume $d$ has a $p$-power partition of length $i$. Then, also invoking Theorem 10.33, we have

$$
\operatorname{ker}\left(-\otimes \partial_{i} t_{d} \mathcal{L}_{n+h-1}^{f}\right) \cap \operatorname{Spc}\left(\mathcal{S} p_{(p)}^{c}\right)=\mathcal{C}_{p, n+h-\beth^{\mathrm{Tate}}(d, i)}=\mathcal{C}_{p, n+h-1}
$$

and (11.27) would follow from the claim that

$$
\begin{equation*}
\partial_{i} z \in \mathcal{C}_{p, n+h-1} \tag{11.28}
\end{equation*}
$$

In order to verify (11.28) and hence (11.26), we distinguish two cases. Suppose first that $\mathcal{P}_{d}([i], p, n+h) \nsubseteq \mathcal{P}_{d}([l], p, h)$. Then

$$
\mathcal{P}_{d}([i], p, n+h) \notin \overline{\left\{\mathcal{P}_{d}([l], p, h)\right\}} \cap V_{\leq h+n}=\operatorname{supp}(z) \cap V_{\leq h+n}
$$

so in particular $\mathcal{P}_{d}([i], p, n+h) \notin \operatorname{supp}(z)$. This is equivalent to the statement that $z \in \mathcal{P}_{d}([i], p, n+h)$, hence $\partial_{i} z \in \mathcal{C}_{p, h+n} \subseteq \mathcal{C}_{p, h+n-1}$, as desired.

On the other hand, if $\mathcal{P}_{d}([i], p, n+h) \subseteq \mathcal{P}_{d}([l], p, h)$ then Proposition 11.7 implies $p-1 \mid i-l \geq 0$. Then, invoking the inductive hypothesis $(\mathrm{P}(i))$, we have

$$
\begin{equation*}
\mathcal{P}_{i}\left([i], p, h+\partial_{p}(i, l)-1\right) \nsubseteq \mathcal{P}_{i}([l], p, h) \tag{11.29}
\end{equation*}
$$

The desired claim (11.28) is equivalent to $z \in \mathcal{P}_{d}([i], p, n+h-1)$ or, in other words, to $\mathcal{P}_{d}([i], p, n+h-1) \notin \operatorname{supp}(z)$. If this were not the case then we would have

$$
\mathcal{P}_{d}([i], p, n+h-1) \in \operatorname{supp}(z) \cap V_{\leq n+h}
$$

which would imply

$$
\mathcal{P}_{i}([i], p, n+h-1) \subseteq \mathcal{P}_{i}([l], p, h)
$$

This would then imply that

$$
\mathcal{P}_{i}\left([i], p, h+\delta_{p}(i, l)-1\right) \nsubseteq \mathcal{P}_{i}([i], p, n+h-1)
$$

because of (11.29), so that $h+\delta_{p}(i, l)-1<n+h-1$ by Proposition 11.3. Substituting the definition $n=\delta_{p}(d, l)-1$ this would mean that

$$
\delta_{p}(i, l)+1<\delta_{p}(d, l) \leq \delta_{p}(d, i)+\delta_{p}(i, l)
$$

Hence $1<\delta_{p}(d, i)=1$ which is a contradiction. This completes the verification of (11.26).

Recall that Tate construction $t_{d}: \operatorname{Exc}_{d} \rightarrow \mathrm{Exc}_{d}$ is the Tate construction associated to the Thomason closed subset $\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right)$ whose open complement is $X_{d-1}:=$ $\left\{\mathcal{P}_{d}([k], p, m) \mid 1 \leq k \leq d-1\right\} \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$. We can also consider the Tate construction $t_{Y}:\left(\operatorname{Exc}_{d}\right)_{\leq h+n} \rightarrow\left(\operatorname{Exc}_{d}\right)_{\leq h+n}$ on the truncated category associated to the preimage $Y:=\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)\right) \cap V_{\leq h+n}=\operatorname{supp}\left(P_{d} h_{\mathbb{S}}(d)_{\leq h+n}\right)$. Since the right adjoint of the localization (11.24) is conservative, the Tate vanishing (11.26) implies the vanishing

$$
\begin{equation*}
t_{Y}\left(z_{\leq h+n}\right)=0 \tag{11.30}
\end{equation*}
$$

by Remark 6.17. By [PSW22, Proposition 2.29], the Tate vanishing (11.30) translates into a disconnectedness statement for $\operatorname{supp}\left(z_{\leq h+n}\right)$ : namely, we have a decomposition

$$
\operatorname{supp}\left(z_{\leq h+n}\right)=Z_{1} \sqcup Z_{2}
$$

in $\operatorname{Spc}\left(\left(\operatorname{Exc}_{d}\right)_{\leq h+n}^{c}\right) \cong V_{\leq h+n}$ with $Z_{1} \cap X_{d-1}=\varnothing$ and $Z_{2} \subseteq X_{d-1}$. Therefore, we have $\mathcal{P}_{d}([l], p, h) \in Z_{2}$. If $\mathcal{P}_{d}([d], p, n+h) \subseteq \mathcal{P}_{d}([l], p, h)$ then we would have $\mathcal{P}_{d}([d], p, n+h) \in Z_{2}$ which contradicts $Z_{2} \subseteq X_{d-1}$. This establishes $(\mathrm{P}(d))$ for each $l<d$, and it is trivially true for $l=d$.

## The combinatorics of $p$-power partition chains.

Remark 11.31. Armed with Theorem 11.22, we see that the task of giving an explicit description of the topology of the Balmer spectrum reduces to understanding the combinatorics of chains of $p$-power partitions. We have already seen (Remark 11.18) that such a chain between $k>l$ exists (that is, $\mathrm{Ch}_{p}(k, l) \neq \varnothing$ ) if and only if $k-l$ is divisible by $p-1$. We will discover the surprising fact that in this case there always exists a chain of length at most 2 (see Proposition 11.38 below).

Definition 11.32. Let $p$ be a prime and consider integers $k \geq l \geq 1$. We define the weight of a $p$-power partition $k=\sum_{i=0}^{r} a_{i} p^{i}$ of $k$ (with $a_{i} \geq 0$ for all $i$ ) as the sum of the coefficients: $w(a):=\sum_{i=0}^{r} a_{i}$ for $a=\left(a_{0}, \ldots, a_{r}\right)$.
Remark 11.33. The weight has the following two properties:
(a) Since $p^{i} \equiv 1 \bmod (p-1)$, it follows that $\sum_{i=0}^{r} a_{i} p^{i} \equiv w(a) \bmod (p-1)$.
(b) Among all $p$-power partitions of $k$, the expansion in base $p$ is the unique one that minimizes its weight. Indeed, suppose $k=\sum_{i=0}^{r} a_{i} p^{i}$ is some $p$-power partition of $k$. If one of the coefficients $a_{j} \geq p$, then the $p$-power partition $k=\sum_{i=0}^{r} a_{i}^{\prime} p^{i}$ with

$$
a_{i}^{\prime}:= \begin{cases}a_{j}-p & \text { if } i=j \\ a_{j}+1 & \text { if } i=j+1 \\ a_{i} & \text { otherwise }\end{cases}
$$

has smaller weight. Because the base $p$ expansion of any integer is unique, the claim follows.
Motivated by the second property, we define:
Definition 11.34. For any prime $p$ and integer $k \geq 1$, let $s_{p}(k)$ be the weight of the base $p$ expansion of $k$, i.e., the sum of the coefficients of the base $p$ expansion of $k$.

Remark 11.35. The following two lemmas provide the number-theoretic input to our formula for the geometric blueshift numbers $\beth_{p}^{\text {geom }}(k, l)$, treating separately the cases when $l \geq s_{p}(k)$ and $l<s_{p}(k)$. Moreover, the first lemma completes our observations in Remark 11.18.

Lemma 11.36. There exists a p-power partition of $k$ of length $l$ (that is, one can write $k$ as a sum of $l$ powers of $p$ ) if and only if $p-1 \mid k-l \geq 0$ and $l \geq s_{p}(k)$.
Proof. $(\Rightarrow)$ : Suppose $k$ can be written as a sum of $l$ powers of $p$, say $k=\sum_{j=1}^{l} p^{e(j)}$ for some $e(j) \geq 0$. We have $k \geq l$ and $k-l=\sum_{j=1}^{l}\left(p^{e(j)}-1\right)$ is divisible by $p-1$. Property (b) of Remark 11.33 implies that $l \geq s_{p}(k)$.
$(\Leftarrow)$ : For the converse, let $k=\sum_{i=0}^{r} a_{i} p^{i}$ be the expansion of $k$ in base $p$. Property (a) of Remark 11.33 shows that $0 \equiv k-l \equiv s_{p}(k)-l \bmod (p-1)$. If
$l=s_{p}(k)$, then the expansion in base $p$ gives a $p$-power expansion of length $l$. Consider then the case $l>s_{p}(k)$. Starting from the expansion of $k$ in base $p$, we have to argue that we can increase the length of the corresponding $p$-power partition

$$
(\underbrace{p^{r}, \ldots, p^{r}}_{a_{r}}, \underbrace{p^{r-1}, \ldots, p^{r-1}}_{a_{r-1}}, \ldots, \underbrace{p^{0}, \ldots, p^{0}}_{a_{0}}) \vdash k
$$

by multiples of $(p-1)$ until we reach $l$. The case $k=l$ is clear, so assume $k>l$ and pick some $j>0$ with $a_{j}>0$, which exists because $k>p-1$ by our divisibility assumption on $k-l$. Setting

$$
a_{i}^{\prime}:= \begin{cases}a_{j}-1 & \text { if } i=j \\ a_{j-1}+p & \text { if } i=j-1 \\ a_{i} & \text { otherwise }\end{cases}
$$

gives a presentation $k=\sum_{i=0}^{r} a_{i}^{\prime} p^{i}$ of $k$ whose corresponding $p$-power partition is of length $s_{p}(k)+(p-1)$. Continuing this process if necessary until we reach weight $l$, we obtain the claim.

Lemma 11.37. If $0<l<s_{p}(k)$ and $p-1$ divides $k-l$, then there exists an integer $i<k$ that is congruent to $k$ and $l$ modulo $p-1$ and which satisfies $i \geq s_{p}(k)$ and $l \geq s_{p}(i)$.

Proof. Consider $k=\sum_{i=0}^{r} a_{i} p^{i}$, the expansion of $k$ in base $p$ with $a_{r} \neq 0$. If $k<p$, then $s_{p}(k)=k$ and $p-1$ cannot divide $k-l$, so we may assume $k \geq p$, i.e., $r>0$. Let $1 \leq x \leq p-1$ be the integer that is congruent to $k$ modulo $p-1$ and set $i:=p^{r}+(x-1)$. We will verify that this choice of $i$ satisfies the conditions given in the statement. Since $r>0$, we have

$$
i \equiv x \equiv k \equiv l \quad \bmod p-1
$$

Next, $s_{p}(i)=x$ is the smallest positive integer congruent to $l$ modulo $p-1$, hence $l \geq s_{p}(i)$. By construction, $k \geq i$; in fact, since $k \neq i$ as $s_{p}(k)>l \geq s_{p}(i)$, we get $k>i$. It remains to see that $i \geq s_{p}(k)$, for which we introduce a case distinction. Suppose first that $r \geq 2$. Then $i \geq p^{r} \geq(r+1)(p-1) \geq s_{p}(k)$. If $r=1$ instead, then $s_{p}(k) \leq 2(p-1)$. If $s_{p}(k)<p$, then $x=s_{p}(k)$ and thus $i=p+x>s_{p}(k)$. Finally, if $p \leq s_{p}(k) \leq 2(p-1)$, then $x=s_{p}(k)-(p-1)$, so $i=p+(x-1)=s_{p}(k)$. We have shown that $i \geq s_{p}(k)$ in all cases.

Proposition 11.38. Let $p$ be a prime and consider integers $k \geq l \geq 1$. Then:

$$
\delta_{p}(k, l)= \begin{cases}0 & \text { if } k=l \\ 1 & \text { if } p-1 \mid k-l>0 \text { and } l \geq s_{p}(k) \\ 2 & \text { if } p-1 \mid k-l>0 \text { and } l<s_{p}(k) \\ \infty & \text { otherwise }\end{cases}
$$

Proof. This follows from Lemma 11.36 and Lemma 11.37.
Combining everything together we have:
Theorem 11.39. Let $p, q$ be prime numbers, $1 \leq k, l \leq d$ integers, and suppose $1 \leq h, h^{\prime} \leq \infty$. There is an inclusion $\mathcal{P}_{d}\left([k], p, h^{\prime}\right) \subseteq \mathcal{P}_{d}([l], q, h)$ if and only if the following conditions hold:
(a) $p-1 \mid k-l \geq 0$;
(b) $h^{\prime} \geq h+\delta_{p}(k, l) ;$ and
(c) if $h>1$, then $p=q$.

Proof. This is the culmination of Theorem 11.22, Proposition 11.38, Corollary 11.4, and Proposition 11.7 bearing in mind Remark 11.20 for the $h=\infty$ case.
Remark 11.40. A complete depiction of the spectrum for $d=3$ is displayed in Figure 1 on page 5 . For larger $d$ such pictures start to become unwieldy and are more easily drawn one prime at a time. For example, the $d=4$ case is displayed in Figure 4 below. The mathematical justification for working $p$-locally is as follows: Recall from Remark 7.17 that the $p$-localization $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}$ is a finite localization which induces an identification

$$
\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}^{c}\right) \xrightarrow{\sim}\left\{\mathcal{P}_{d}([k], p, n) \mid \text { all } k, n\right\} \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right) .
$$

Moreover, since all nontrivial inclusions occur $p$-locally (Remark 11.5), the spectra of the $p$-localizations completely describe the spectrum of the unlocalized category.


Figure 4. The $p$-local part of the Balmer spectrum of 4-excisive functors from finite spectra to spectra.

## 12. Applications

The classification of tt-ideals. We can now give the classification of thick ideals in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$. Throughout this section we fix $d \geq 1$ and abbreviate $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ by $\operatorname{Exc}_{d}^{c}$ when convenient.

Proposition 12.1. Let $Z=\overline{\left\{\mathcal{P}_{1}\right\}} \cup \cdots \cup \overline{\left\{\mathcal{P}_{N}\right\}}$ be a closed subset of $\operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$ (see Proposition 11.1). Assume that this is irredundant, that is, $\mathcal{P}_{i} \nsubseteq \mathcal{P}_{j}$ for all $1 \leq i \neq j \leq N$. Let $\mathcal{P}_{i}=\mathcal{P}_{d}\left(\left[k_{i}\right], p_{i}, m_{i}\right)$ for all $i=1, \ldots, N$. Then the complement of $Z$ is quasi-compact if and only if all the chromatic integers $m_{1}, \ldots, m_{N}$ are finite.
Proof. We follow the approach of [BS17, Proposition 10.1].
$(\Leftarrow)$ Suppose the $m_{1}, \ldots, m_{N}$ are finite. We claim that $Z$ has a quasi-compact complement. Since the supports of compact objects form a basis of closed sets, it suffices to prove that whenever $Z=\cap_{i \in I} \operatorname{supp}\left(x_{i}\right)$ for a collection $\left\{x_{i}\right\}_{i \in I}$ of objects in $\operatorname{Exc}_{d}^{c}$, there exists a finite subset $J \subseteq I$ such that $Z=\cap_{i \in J} \operatorname{supp}\left(x_{i}\right)$. For each $l \in[d]=\{1, \ldots, d\}$, let $\varphi_{l}: \operatorname{Spc}\left(\operatorname{Sp}^{c}\right) \rightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$ be the map on spectra
induced by the $l$ th derivative $\partial_{l}: \operatorname{Exc}_{d} \rightarrow \mathcal{S} p$. Since $\operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)=\cup_{1 \leq l \leq d} \operatorname{im} \varphi_{l}$ by Lemma 7.3, it suffices to prove that for each $l$ there exists a finite subset $J_{l} \subseteq I$ such that

$$
\cap_{j \in J_{l}} \operatorname{supp}\left(x_{j}\right) \cap \operatorname{im} \varphi_{l} \subseteq Z \cap \operatorname{im} \varphi_{l}
$$

(since we can then take $J:=\cup_{1 \leq l \leq d} J_{l}$ ). Moreover, since $\varphi_{l}$ is an embedding of $\operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ onto $\operatorname{im} \varphi_{l}$ (Lemma 7.7), this reduces to the claim that

$$
\varphi_{l}^{-1}\left(\cap_{j \in J_{l}} \operatorname{supp}\left(x_{j}\right)\right) \subseteq \varphi_{l}^{-1}(Z)
$$

Now it follows from Theorem 11.39 that $\varphi_{l}^{-1}\left(\overline{\left\{\mathcal{P}_{d}\left(\left[k_{i}\right], p_{i}, m_{i}\right)\right\}}\right)$ is either empty or of the form $\overline{\left\{\mathcal{C}_{p_{i}, n_{i}}\right\}}$ for some finite $n_{i} \geq m_{i}$. This closed subset of $\operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ has quasi-compact complement by [Bal10b, Corollary 9.5(d)]. We conclude that indeed there is a finite $J_{l} \subseteq I$ such that

$$
\cap_{j \in J_{l}} \operatorname{supp}\left(\partial_{l} x_{j}\right) \subseteq \varphi_{l}^{-1}(Z)
$$

as desired.
$(\Rightarrow)$ Conversely, suppose that one of the $m_{i}=\infty$. Write $Z=Z^{\prime} \cup \overline{\left\{\mathcal{P}_{d}([k], p, \infty)\right\}}$, regrouping the other irreducibles under $Z^{\prime}$. Since $\mathcal{C}_{p, \infty}=\cap_{1 \leq n<\infty} \mathcal{C}_{p, n}$, it follows that $\mathcal{P}_{d}([k], p, \infty)=\cap_{1 \leq n<\infty} \mathcal{P}_{d}([k], p, n)$. Hence $Z=\cap_{1 \leq n<\infty}\left(Z^{\prime} \cup \overline{\left\{\mathcal{P}_{d}([k], p, n)\right\}}\right)$. If the complement of $Z$ is quasi-compact then there is a finite $n$ such that $Z=Z^{\prime} \cup$ $\overline{\left\{\mathcal{P}_{d}([k], p, n)\right\}}$. But since $\mathcal{P}_{d}([k], p, n) \nsubseteq \mathcal{P}_{d}([k], p, \infty)$, it follows that $\mathcal{P}_{d}([k], p, n) \in Z^{\prime}$ and hence that $\mathcal{P}_{d}([k], p, \infty) \in Z^{\prime}$. But this implies that $\mathcal{P}_{d}([k], p, \infty)$ is contained in one of the other irreducibles of $Z$, contradicting the assumption that the $\mathcal{P}_{1}, \ldots, \mathcal{P}_{N}$ were irredundant.

Corollary 12.2. The Thomason subsets of $\operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$ are the arbitrary unions of $\overline{\left\{\mathcal{P}_{d}([k], p, m)\right\}}=\left\{Q \mid \mathcal{Q} \subseteq \mathcal{P}_{d}([k], p, m)\right\}$ for arbitrary $1 \leq k \leq d$, $p$ prime and finite chromatic integer $1 \leq m<\infty$.

Proof. This is an immediate consequence of Proposition 12.1, Proposition 11.1 and Theorem 7.14.

Notation 12.3. Let $\mathbb{P}$ denote the set of prime numbers and let

$$
\mathbb{N}_{\infty}:=\{0,1,2, \ldots\} \cup\{\infty\}
$$

with the natural strict ordering $<$ with the understanding that $\infty \nless \infty$ and $\infty+n=\infty$ for any finite $n$.
Definition 12.4. Let $d \geq 1$ be an integer. Say that a function $f:[d] \times \mathbb{P} \rightarrow \mathbb{N}_{\infty}$ is admissible if it satisfies the following two conditions:
(a) $f(k, p) \leq \delta_{p}(k, l)+f(l, p)$ for all $p-1 \mid k-l \geq 0$; and
(b) if $f(k, p)=0$ for some prime $p$ then $f(k, q)=0$ for every prime $q$.

Recall that the function $\delta_{p}(k, l)$ was computed in Proposition 11.38.
Theorem 12.5 (Classification of tt-ideals). Let $d \geq 1$ be an integer. There is a one-to-one correspondence between the set of admissible functions $[d] \times \mathbb{P} \rightarrow \mathbb{N}_{\infty}$ and the Thomason subsets of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)$ which maps $f$ to

$$
Y_{f}:=\left\{\mathcal{P}_{d}([k], p, m) \mid m>f(k, p)\right\} .
$$

Consequently, there is a one-to-one correspondence between admissible functions and tt-ideals in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ which maps $f$ to
$\mathcal{J}_{f}:=\left\{x \in \operatorname{Exc}_{d}^{c} \mid \partial_{k}(x) \in \mathcal{C}_{p, f(k, p)}\right.$ for each $(k, p) \in[d] \times \mathbb{P}$ such that $\left.f(k, p)>0\right\}$.

Proof. We will use repeatedly that Thomason subsets are closed under specialization. With this in mind, let $Y \subseteq \operatorname{Spc}\left(\operatorname{Exc}_{d}^{c}\right)$ be a Thomason subset. Then

$$
\begin{equation*}
Y=\bigcup_{(k, p) \in[d] \times \mathbb{P}} X(k, p) \tag{12.6}
\end{equation*}
$$

where $X(k, p):=\left\{\mathcal{P}_{d}([k], p, n) \mid \mathcal{P}_{d}([k], p, n) \in Y\right\}$. The nontrivial inclusion $\subseteq$ in (12.6) follows from Theorem 7.14. Then consider a pair $(k, p) \in[d] \times \mathbb{P}$. If $X(k, p)$ is nonempty then $\mathcal{P}_{d}([k], p, \infty) \in X(k, p)$. Since $Y$ is Thomason, Corollary 12.2 implies that $\mathcal{P}_{d}([k], p, \infty) \subseteq \mathcal{P}_{d}([l], q, h) \in Y$ for some $(l, q) \in[l] \times \mathbb{P}$ and finite $h$. It then follows from Theorem 11.39 that $\mathcal{P}_{d}\left([k], p, h+\delta_{p}(k, l)\right) \subseteq \mathcal{P}_{d}([l], q, h)$ and, moreover, that $h+\delta_{p}(k, l)$ is finite. This establishes that if $X(k, p)$ is nonempty then there is a finite $n \geq 1$ such that $\mathcal{P}_{d}([k], p, n) \in X(k, p)$. It follows that every nonempty $X(k, p)$ is of the form $\left\{\mathcal{P}_{d}([k], p, n) \mid n_{k, p} \leq n \leq \infty\right\}$ for some (uniquely determined) finite $n_{k, p} \geq 1$.

We then define a function $f:[d] \times \mathbb{P} \rightarrow \mathbb{N}_{\infty}$ by setting $f(k, p)=\infty$ when $X(k, p)=\varnothing$ and $f(k, p)=n_{k, p}-1$ when $X(k, p)$ is nonempty. With this definition, $X(k, p)=\left\{\mathcal{P}_{d}([k], p, n) \mid n>f(k, p)\right\}$. We claim that the function $f$ is admissible. Indeed, condition (b) follows simply from the fact that $\mathcal{P}_{d}([k], p, 1)=\mathcal{P}_{d}([k], q, 1)$ for any two primes $p$ and $q$. On the other hand, condition (a) follows from Theorem 11.39, including the cases where one side of the inequality is equal to infinity.

In this way we obtain an injective map from the set of Thomason subsets to the set of admissible functions. We claim that every admissible function arises in this way. Indeed, given an admissible function $f$, consider the subset

$$
Y:=\bigcup_{\substack{(l, q) \in[d] \times \mathbb{P}: \\ f(l, q)<\infty}} \overline{\left\{\mathcal{P}_{d}([l], q, f(l, q)+1)\right\}} .
$$

It is Thomason by Corollary 12.2. We claim that $\mathcal{P}_{d}([k], p, n) \in Y$ if and only if $n>f(k, p)$. The $(\Leftarrow)$ direction is immediate from the definition of $Y$. On the other hand, if $\mathcal{P}_{d}([k], p, n) \in Y$ then $\mathcal{P}_{d}([k], p, n) \subseteq \mathcal{P}_{d}([l], q, f(l, q)+1)$ for some $(l, q)$ such that $f(l, q)<\infty$. Theorem 11.39 then implies that $p-1 \mid k-l \geq 0$ and that $n>f(l, q)+\delta_{p}(k, l)$ and, moreover, that $p=q$ if $f(l, q) \neq 0$. Since $f$ is admissible, it follows that $n>f(k, p)$ as desired.

This establishes the claimed bijection between admissible functions and Thomason subsets. The translation from Thomason subsets to tt-ideals is [Bal05, Theorem 4.10] just unravelling the definitions.

Remark 12.7. The above bijective correspondence is order-preserving when the set of admissible functions $[d] \times \mathbb{P} \rightarrow \mathbb{N}_{\infty}$ is endowed with the pointwise order induced from the order on $\mathbb{N}_{\infty}$.

Remark 12.8. Following the perspective of [BGH20], we can define the type function of a compact $d$-excisive functor $x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ as the function which collects the types of the derivatives $\partial_{k} x \in \mathcal{S} p^{c}$ of $x$ at each prime $p$, i.e., the function type $_{x}:[d] \times \mathbb{P} \rightarrow \mathbb{N}_{\infty}$ defined by

$$
\operatorname{type}_{x}(k, p):=\inf \left\{h \in \mathbb{N}_{\infty} \mid K(p, h)_{*} \partial_{k} x \neq 0\right\}
$$

with the convention that $\inf \varnothing=\infty$. We can extend the notion from objects $x$ to tt-ideals $\mathcal{J} \subseteq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ as follows:

$$
\text { type }_{\mathcal{J}}:[d] \times \mathbb{P} \rightarrow \mathbb{N}_{\infty}, \quad(l, p) \mapsto \inf \left\{\operatorname{type}_{x}(l, p) \mid x \in \mathcal{J}\right\}
$$

One readily checks from the proof of Theorem 12.5 that the function $\mathcal{J} \mapsto$ type $_{\mathcal{J}}$ which sends a tt-ideal to its type function is the inverse of the bijection $f \mapsto \mathcal{J}_{f}$.

Definition 12.9. The classification of tt-ideals simplifies $p$-locally. Say that a function $f:[d] \rightarrow \mathbb{N}_{\infty}$ is $p$-admissible if

$$
f(k) \leq \delta_{p}(k, l)+f(l)
$$

for all $p-1 \mid k-l \geq 0$.
Corollary 12.10. Let $d \geq 1$ be an integer and $p$ a prime. There is a one-to-one correspondence between the set of p-admissible functions $[d] \rightarrow \mathbb{N}_{\infty}$ and the Thomason subsets of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}^{c}\right)$ which maps $f$ to $Y_{f}:=\left\{\mathcal{P}_{d}([k], p, m) \mid m>f(k)\right\}$.

Example 12.11. Note that the empty Thomason subset corresponds to the constant $p$-admissible function $f \equiv \infty$ while the whole space corresponds to the constant function $f \equiv 0$.

Remark 12.12. In the situation of Corollary 12.10 , we observe that Thomason subsets are automatically closed. Hence every every tt-ideal of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}^{c}$ is generated by a single compact object; see [San13, Lemma 2.3]. Since Remark 2.46 supplies a geometric equivalence

$$
\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)} \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p_{(p)}\right)
$$

we thus obtain the statement of our main theorem on page 2 .
Remark 12.13. The behaviour of the function $\delta_{p}(k, l)$ depends on the behaviour of the function $s_{p}(k)$ which can be quite erratic. Further information about the latter function can be found in [AS03, Section 3.2]. Its generating function has a remarkable closed form formula; see [AWR09].

Transchromatic Smith and Floyd theory in functor calculus. A famous theorem of P.A. Smith [Smi41] states that if $H$ is a subgroup of a $p$-group $G$ and $X$ is a finite-dimensional based $G$-CW-complex, then

$$
\tilde{H}_{*}\left(X^{H} ; \mathbb{F}_{p}\right)=0 \Longrightarrow \tilde{H}_{*}\left(X^{G} ; \mathbb{F}_{p}\right)=0
$$

This was later extended by Floyd [Flo52], who showed that if $H$ is a subgroup of a $p$-group $G$ and $X$ is a finite-dimensional $G$-CW complex with $\operatorname{dim}_{\mathbb{F}_{p}} H_{*}\left(X^{H} ; \mathbb{F}_{p}\right)$ finite, then

$$
\operatorname{dim}_{\mathbb{F}_{p}} H_{*}\left(X^{H} ; \mathbb{F}_{p}\right) \geq \operatorname{dim}_{\mathbb{F}_{p}} H_{*}\left(X^{G} ; \mathbb{F}_{p}\right)
$$

Noting that $\bmod p$ homology corresponds to $K(p, \infty)$, i.e., Morava $K$-theory at height $\infty$, Hausmann and Kuhn-Lloyd [KL20] considered versions of the Smith and Floyd theorems for Morava $K$-theory $K(p, n)$. They showed that understanding such "transchromatic" Smith and Floyd theorems is equivalent to understanding the topology of the Balmer spectrum of the category of compact $G$-spectra. Our aim in this section is to relate the topology of the Balmer spectrum of compact $d$-excisive functors to analogs of the Smith and Floyd inequalities in functor calculus.

Definition 12.14. Let $d \geq 1$ be an integer and $p$ a prime. For each $1 \leq k, l \leq d$ and $0 \leq n, h \leq \infty$, we say that

- Smith ${ }_{d, p}(k, l ; n, h)$ holds if: for all $x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$, we have

$$
K(p, n)_{*}\left(\partial_{k} x\right)=0 \Longrightarrow K(p, h)_{*}\left(\partial_{l} x\right)=0
$$

- $\operatorname{Floyd}_{d, p}(k, l ; n, h)$ holds if: for all $x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$, we have

$$
\operatorname{dim}_{K(p, n)_{*}} K(p, n)_{*}\left(\partial_{k} x\right) \geq \operatorname{dim}_{K(p, h)_{*}} K(p, h)_{*}\left(\partial_{l} x\right)
$$

Remark 12.15. Recall from Remark 7.17 that $p$-localization

$$
\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}
$$

is the finite localization associated to $i_{d}\left(f_{p, \infty}\right)$. Hence Lemma 7.5 implies that the Goodwillie derivative $\partial_{k}$ commutes with $p$-localization. Since $K(p, n)$ is itself $p$-local, it follows that $K(p, n)_{*}\left(\partial_{k} x\right) \simeq K(p, n)_{*}\left(\partial_{k}\left(x_{(p)}\right)\right)$ where $x_{(p)}$ denotes the $p$-localization of $x$. Thus, the statements in Definition 12.14 could be equivalently formulated by instead letting $x$ range over all compact $p$-local functors $x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}^{c}$. To see this, just recall that if $x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)_{(p)}^{c}$ then $x \oplus \Sigma x$ is the $p$-localization of a compact $x^{\prime} \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$.

Proposition 12.16. For all $1 \leq k, l \leq d$ and $0 \leq n, h \leq \infty$, we have

$$
\operatorname{Smith}_{d, p}(k, l ; n, h) \Longleftrightarrow \operatorname{Floyd}_{d, p}(k, l ; n, h)
$$

Proof. The $(\Longleftarrow)$ implication is clear. In order to establish the converse, we will prove the contrapositive: Suppose there exists some $x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ with

$$
\operatorname{dim}_{K(p, n)_{*}} K(p, n)_{*}\left(\partial_{k} x\right)<\operatorname{dim}_{K(p, h)_{*}} K(p, h)_{*}\left(\partial_{l} x\right)
$$

We may replace $x$ with its $p$-localization (Remark 12.15). We will employ the methods of J. Smith [Rav92, Appendix C], as expanded upon in the work of KuhnLloyd [KL20], to construct a compact $p$-local $d$-excisive functor $y$ from $x$ which falsifies the corresponding statement $\operatorname{Smith}_{d}(k, l ; n, h)$. Note that, for any $m \geq 1$, the symmetric group acts on $x^{\circledast m}$ by permuting the factors, giving rise to a map of rings

$$
\mathbb{Z}_{(p)}\left[\Sigma_{m}\right] \rightarrow \pi_{0} \operatorname{End}\left(x^{\circledast m}\right)
$$

In particular, any idempotent $e \in \mathbb{Z}_{(p)}\left[\Sigma_{m}\right]$ induces a self-map $e: x^{\circledast m} \rightarrow x^{\circledast m}$, and we write $e x^{\circledast m}$ for the corresponding retract of $x^{\circledast m}$. The monoidality of $\partial_{l}(-)$ and $K(p, h)_{*}(-)$ imply that this construction is compatible with its algebraic counterpart:

$$
K(p, h)_{*}\left(\partial_{l}\left(e x^{\circledast m}\right)\right) \cong e\left(K(p, h)_{*}\left(\partial_{l} x\right)\right)^{\otimes m}
$$

Set $V_{*}:=K(p, h)_{*}\left(\partial_{l} x\right)$ as a graded module over $K(p, h)_{*}$ and similarly $W_{*}:=$ $K(p, n)_{*}\left(\partial_{k} x\right)$ as a graded $K(p, n)_{*}$-module. The results of [KL20, Section 6.4] provide a choice of integer $m$ and idempotent $e_{m} \in \mathbb{Z}_{(p)}\left[\Sigma_{m}\right]$ such that ${ }^{1}$

$$
e_{m} V_{*}^{\otimes m} \neq 0 \quad \text { and } \quad e_{m} W_{*}^{\otimes m}=0
$$

It follows that $y:=e_{m} x^{\circledast m}$ witnesses the failure of $\operatorname{Smith}_{d}(k, l ; n, h)$, as desired.
Remark 12.17. The relation with the topology of the Balmer spectrum is the following:

Proposition 12.18. Let $1 \leq k, l \leq d$ and $0 \leq n, h \leq \infty$. Then $\operatorname{Smith}_{d, p}(k, l ; n, h)$ holds if and only if $\mathcal{P}_{d}([k], p, n+1) \subseteq \mathcal{P}_{d}([l], p, h+1)$ in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$.

[^1]Proof. Unravelling the definitions one sees that $\operatorname{Smith}_{d, p}(k, l ; n, h)$ is equivalent to the statement that for every compact $x, \mathcal{P}_{d}([l], p, h+1) \in \operatorname{supp}(x)$ implies $\mathcal{P}_{d}([k], p, n+1) \in \operatorname{supp}(x)$. This is equivalent to $\mathcal{P}_{d}([k], p, n+1) \subseteq \mathcal{P}_{d}([l], p, h+1)$ since the supports of compact objects form a basis of closed sets for the Balmer topology.

Corollary 12.19. Consider integers $1 \leq l \leq k \leq d$ satisfying $p-1 \mid k-l$, a height $0 \leq h \leq \infty$, and let $n \geq h+\delta_{p}(k, l)$. For any compact $x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$, the following inequality holds:

$$
\operatorname{dim}_{K(p, n)_{*}} K(p, n)_{*}\left(\partial_{k} x\right) \geq \operatorname{dim}_{K(p, h)_{*}} K(p, h)_{*}\left(\partial_{l} x\right)
$$

Proof. This is a direct consequence of Proposition 12.16, Proposition 12.18 and Theorem 11.39.

Calculus with coefficients. The aim of this subsection is to explain how our computation of the spectrum of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$ can be extended to other coefficient categories $\mathcal{D}$ with a particular focus on the case $\mathcal{D}=\operatorname{Mod}_{\mathrm{HZ}}$. In the context of stable equivariant homotopy theory, this is the subject of [PSW22, $\mathrm{BHS} 23 \mathrm{~b}, \mathrm{BCH}^{+} 23$ ].

Remark 12.20 . Let $\mathcal{D}$ be a rigidly-compactly generated $\mathrm{tt}-\infty$-category and fix some integer $d \geq 1$. The category $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right)$ of reduced $d$-excisive functors from finite spectra to $\mathcal{D}$ may then again be equipped with the symmetric monoidal structure afforded by localized Day convolution, and as such forms a rigidly-compactly generated tt - $\infty$-category. As explained in Remark 2.46, there is a canonical symmetric monoidal equivalence $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right) \simeq \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \otimes \mathcal{D}$ and we obtain $\mathcal{D}$-linear derivatives $\partial_{k}^{\mathcal{D}}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right) \rightarrow \mathcal{D}$ for all $k \in[d]$.

Let $i_{d}: \mathcal{S} p \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ be the functor defined in Definition 2.44 and write $i_{d}^{\mathcal{D}}$ for its $\mathcal{D}$-linear analogue. For any $1 \leq k \leq d$, we thus obtain a commutative diagram

in which the vertical functors are induced by base-change along the canonical geometric functor $F: \mathcal{S} p \rightarrow \mathcal{D}$ (Remark 2.43). Since the top composite is an equivalence (Lemma 7.5), by base-change the bottom horizontal composite is one as well. This setup allows us to generalize Theorem 7.14:

Proposition 12.21. Suppose $\mathcal{D}$ is a rigidly-compactly generated tt- $\infty$-category. The (D-linear) derivatives induce a bijection

$$
\varphi^{\mathcal{D}}:=\operatorname{Spc}\left(\left(\partial_{k}^{\mathcal{D}}\right)_{1 \leq k \leq d}\right): \coprod_{1 \leq k \leq d} \operatorname{Spc}\left(\mathcal{D}^{c}\right) \xrightarrow{\sim} \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right)^{c}\right)
$$

whose restriction to each component is an embedding.
Proof. By induction on the $\mathcal{D}$-linear Taylor tower, we see that the functors $\partial_{k}^{\mathcal{D}}$ are jointly conservative, so $\varphi^{\mathcal{D}}$ is surjective by [BCHS23a, Theorem 1.3]. Consider the
following commutative diagram


Here $j_{k}$ denotes the inclusion of the $k$-th component and the bottom map is the identity because $\partial_{k}^{\mathcal{D}} \circ i_{d}^{\mathcal{D}} \simeq \mathrm{id}$ by Remark 12.20. This implies that $\operatorname{Spc}\left(\partial_{k}^{\mathcal{D}}\right)=\varphi^{\mathcal{D}} \circ j_{k}$ is an embedding (as in the proof of Lemma 7.7) and it remains to show that the images of the different components are disjoint.

To this end, consider the following commutative diagram

$$
\begin{align*}
& \amalg_{1 \leq k \leq d} \operatorname{Spc}\left(\mathcal{D}^{c}\right) \xrightarrow{\varphi^{\mathcal{D}}} \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(S p^{c}, \mathcal{D}\right)^{c}\right) \\
& \amalg^{\operatorname{Spc}(F)} \downarrow  \tag{12.22}\\
& \amalg_{1 \leq k \leq d} \operatorname{Spc}\left(S p^{c}\right) \xrightarrow{\varphi} \operatorname{Spc}\left(F_{d}\right) \\
& \left.\operatorname{Sxc}\left(S p^{c}, S p\right)^{c}\right) .
\end{align*}
$$

Since the composite through the lower left corner separates components, so does $\varphi^{\mathcal{D}}$ as desired.

Remark 12.23. Since a finite union of noetherian subspaces is a noetherian space, Proposition 12.21 implies that $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right)^{c}\right)$ is noetherian whenever $\operatorname{Spc}\left(\mathcal{D}^{c}\right)$ is.
Remark 12.24. We now completely determine the topology of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{D}\right)^{c}\right)$ in the special case that $\mathcal{D}=\operatorname{Mod}_{\mathrm{HZ}}$. This argument is meant as a proof of concept; we will return to a more systematic discussion elsewhere.
Remark 12.25. Repeating the arguments given in, and leading up to, [PSW22, Corollary 4.11], one can show that there is a geometric equivalence

$$
\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right) \simeq \operatorname{Mod}_{\operatorname{Exc}_{d}\left(s p^{c}, s p\right)}\left(i_{d} \mathrm{HZ}\right)
$$

induced by base-change along $\mathbb{S} \rightarrow \mathrm{H} \mathbb{Z}$.
Notation 12.26. Henceforth, in functors such as $\partial$ and $i_{d}$, we will abbreviate superscripts ' $\operatorname{Mod}_{H \mathbb{Z}}$ ' by ' $\mathbb{Z}$ '. Let $1 \leq k \leq d$ and write $\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p})$ for the prime tt-ideal in $\operatorname{Exc}_{d}\left(\mathcal{S}^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right)^{c}$ that corresponds to $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z}) \cong \operatorname{Spc}\left(\operatorname{Mod}_{\mathrm{H} \mathbb{Z}}^{c}\right)$ under the bijection of Proposition 12.21. Explicitly, this means that

$$
\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p})=\left\{x \in \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right)^{c} \mid \kappa(\mathfrak{p}) \otimes \partial_{k}^{\mathbb{Z}}(x)=0\right\}
$$

where $\kappa(\mathfrak{p})$ denotes the residue field of $\mathbb{Z}$ at $\mathfrak{p}$.
Theorem 12.27. Let $1 \leq k, l \leq d$ be integers and consider two prime ideals $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(\mathbb{Z})$. Then there is an inclusion $\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p}) \subseteq \mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{q})$ if and only if one of the following two conditions is satisfied:
(a) $\mathfrak{p}=(p)$ for some prime $p, \mathfrak{q}=(p)$ or $\mathfrak{q}=(0)$, and $p-1 \mid k-l \geq 0$;
(b) $\mathfrak{p}=(0)=\mathfrak{q}$ and $k=l$.

Moreover, $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S p}^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right)^{c}\right)$ is noetherian, so the topology is determined by these inclusions. Finally, base-change $\mathcal{S} p \rightarrow \operatorname{Mod}_{\mathrm{HZ}}$ induces a map

$$
\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{H} Z}\right)^{c}\right) \rightarrow \operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}\right)
$$

which is a homeomorphism onto its image. It maps $\mathcal{P}_{d}^{\mathbb{Z}}([k],(p))$ to $\mathcal{P}_{d}([k], p, \infty)$ and maps $\mathcal{P}_{d}^{\mathbb{Z}}([k],(0))$ to $\mathcal{P}_{d}([k], 0,1)$.

Proof. The spectrum of $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)^{c}$ is noetherian by Remark 12.23, so in order to understand its topology, it is enough to determine all inclusions among prime tt-ideals. Suppose first that we are given an inclusion $\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p}) \subseteq \mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{q})$. The commutative square (12.22) shows that $\operatorname{Spc}(F): \operatorname{Spc}\left(\operatorname{Mod}_{\mathrm{HZ}}^{c}\right) \rightarrow \operatorname{Spc}\left(\mathcal{S} p^{c}\right)$ satisfies

$$
\operatorname{Spc}(F)\left(\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p})\right)=\mathcal{P}_{d}([k], \operatorname{Spc}(F)(\mathfrak{p})),
$$

where $\operatorname{Spc}(F)(\mathfrak{p})$ can be identified with a pair $(p, h) \in \mathbb{P} \times\{1, \infty\}$. Since $\operatorname{Spc}\left(F_{d}\right)$ preserves inclusions (Remark 6.2), we thus obtain an inclusion

$$
\mathcal{P}_{d}([k], \operatorname{Spc}(F)(\mathfrak{p})) \subseteq \mathcal{P}_{d}([l], \operatorname{Spc}(F)(\mathfrak{q}))
$$

of prime tt-ideals in $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)^{c}$. Our classification theorem ${ }^{1}$ Theorem 11.39 then implies that either condition $(a)$ or (b) must hold.

For the converse, assume first that $(b)$ is satisfied. In this case, we have an equality $\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p})=\mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{q})$ as desired. Similarly, if $(a)$ holds with $\mathfrak{q}=(0)$, then $\mathcal{P}_{d}^{\mathbb{Z}}\left([l], \mathfrak{q}^{\prime}\right) \subseteq \mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{q})$ for any $\mathfrak{q}^{\prime} \in \operatorname{Spec}(\mathbb{Z})$, in particular for $\mathfrak{p}=(p)$. In order to show $\mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p}) \subseteq \mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{q})$, we are therefore reduced to verifying the following claim:

$$
p-1 \mid k-l \geq 0 \Longrightarrow \mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p}) \subseteq \mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{p})
$$

To this end, suppose first that there exists a $p$-power partition $\lambda \vdash k$ of length $l$. Consider some $x \in \mathcal{P}_{d}^{\mathbb{Z}}([k], \mathfrak{p})$, i.e., a compact functor $x \in \operatorname{Exc}_{d}\left(\mathcal{S p}^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right)$ with

$$
\begin{equation*}
0=\mathbb{F}_{p} \otimes \partial_{k}^{\mathbb{Z}}(x) \simeq \partial_{k}^{\mathbb{Z}}\left(i_{d}^{\mathbb{Z}}\left(\mathbb{F}_{p}\right) \circledast x\right) \tag{12.28}
\end{equation*}
$$

Let $t_{k}^{\mathbb{Z}}$ be the Tate construction (Definition 6.11) internal to $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right)$ with respect to the subset

$$
Y_{k}:=\left\{\mathcal{P}_{d}^{\mathbb{Z}}([i], \mathfrak{p}) \mid i \geq k\right\}=\operatorname{Spc}\left(F_{d}\right)^{-1}\left(\left\{\mathcal{P}_{d}([i], p, h) \mid i \geq k\right\}\right)
$$

Note that $Y_{k}$ is a Thomason subset since it is the pullback of a Thomason subset along a spectral map. As in the proof of Lemma 11.15, we then deduce formally from (12.28) that $t_{k}^{\mathbb{Z}}\left(i_{d}^{\mathbb{Z}}\left(\mathbb{F}_{p}\right) \circledast x\right)=0$. Using the compatibility of the Tate construction with base-change (Proposition 6.14) as well as Remark 12.20 and the dualizability of $x$, we compute

$$
\begin{aligned}
0=\partial_{l}^{\mathbb{Z}} t_{k}^{\mathbb{Z}}\left(i_{d}^{\mathbb{Z}}\left(\mathbb{F}_{p}\right) \circledast x\right) & \simeq\left(\partial_{l}^{\mathbb{Z}} t_{k}^{\mathbb{Z}} i_{d}^{\mathbb{Z}}\left(\mathbb{F}_{p}\right)\right) \otimes \partial_{l}^{\mathbb{Z}}(x) \\
& \left.\simeq\left(\partial_{l}^{\mathbb{Z}} t_{k}^{\mathbb{Z}} i_{d}^{\mathbb{Z}}(\mathbb{Z} \otimes \mathbb{S} / p)\right)\right) \otimes \partial_{l}^{\mathbb{Z}}(x) \\
& \simeq\left(\partial_{l}^{\mathbb{Z}} t_{k}^{\mathbb{Z}} F_{d} i_{d}(\mathbb{S} / p)\right) \otimes \partial_{l}^{\mathbb{Z}}(x) \\
& \simeq F\left(\partial_{l} t_{k}\left(i_{d}(\mathbb{S} / p)\right)\right) \otimes \partial_{l}^{\mathbb{Z}}(x) \\
& \simeq\left(\mathbb{F}_{p} \otimes \partial_{l} t_{k} i_{d}(\mathbb{S})\right) \otimes \partial_{l}^{\mathbb{Z}}(x) .
\end{aligned}
$$

In order to show that $x \in \mathcal{P}_{d}^{\mathbb{Z}}([l], \mathfrak{p})$, it thus suffices to prove that $\mathbb{F}_{p} \otimes \partial_{l} t_{k} i_{d}(\mathbb{S})$ is non-trivial, so that it contains $\mathbb{F}_{p}$ as a retract. To this end, observe that we have ring maps

$$
\partial_{l} t_{k} i_{d}(\mathbb{S}) \rightarrow \partial_{l} t_{k} i_{d}\left(\mathbb{F}_{p}\right) \quad \text { and } \quad \mathbb{F}_{p}^{t \mathcal{F}_{\mathrm{nt}}(\lambda)} \rightarrow \mathbb{F}_{p}^{t C}
$$

where $C$ is a non-trivial cyclic $p$-group and the second map is the one we constructed in the proof of Theorem 10.33. On the one hand, the target of the second map is non-zero, hence $\mathbb{F}_{p}{ }^{t \mathcal{F}_{\text {nt }}(\lambda)}$ is also non-zero. On the other hand, Proposition 10.26

[^2]implies that $\mathbb{F}_{p}{ }^{t \mathcal{F}_{\text {nt }}(\lambda)}$ is a retract of $\partial_{l} t_{k} i_{d}\left(\mathbb{F}_{p}\right)$, so the target of the first map is non-zero and $\mathbb{F}_{p}$-linear, and we conclude that $\mathbb{F}_{p} \otimes \partial_{l} t_{k} i_{d}(\mathbb{S}) \neq 0$, as claimed.

This finishes the proof under the assumption that there is a $p$-power partition of $k$ of length $l$. In general, our assumption that $p-1 \mid k-l \geq 0$ guarantees the existence a chain of $p$-power partitions (see Remark 11.18) and we conclude by concatenating the inclusions just established.

Remark 12.29. The $d=3$ case is shown in Figure 5. Observe that it is homeomorphic to the top and bottom chromatic layers of Figure 1. The relations between these spaces are completely analogous to the situation in equivariant stable homotopy theory established in [PSW22].


Figure 5. The Balmer spectrum $\operatorname{Spc}\left(\operatorname{Exc}_{3}\left(\mathcal{S}^{c}, \operatorname{Mod}_{H Z}\right)^{c}\right)$.
Corollary 12.30. Let $d \geq 1$. Say that a subset $Y \subseteq[d] \times \operatorname{Spec}(\mathbb{Z})$ is admissible if it satisfies the following closure properties:
(a) If $(l, 0) \in Y$ then $(k, p) \in Y$ for all $p-1 \mid k-l \geq 0$.
(b) If $(l, p) \in Y$ then $(k, p) \in Y$ for all $p-1 \mid k-l \geq 0$.

There is an inclusion-preserving bijection between the set of admissible subsets of $[d] \times \operatorname{Spec}(\mathbb{Z})$ and the collection of tt-ideals of $\operatorname{Exc}_{d}\left(\operatorname{Sp}^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right)^{c}$ given by

$$
Y \mapsto\left\{x \in \operatorname{Exc}_{d}\left(\mathcal{S}^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)^{c} \mid \partial_{k}^{\mathbb{Z}}(x) \in \mathfrak{p} \text { if }(k, \mathfrak{p}) \notin Y\right\}
$$

with inverse

$$
\mathcal{C} \mapsto\left\{(k, \mathfrak{p}) \in[d] \times \operatorname{Spec}(\mathbb{Z}) \mid \partial_{k}^{\mathbb{Z}}(x) \notin \mathfrak{p} \text { for some } x \in \mathcal{C}\right\}
$$

Proof. As a set, we can identify $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S p}^{c}, \operatorname{Mod}_{H \mathbb{Z}}\right)^{c}\right)$ with $[d] \times \operatorname{Spec}(\mathbb{Z})$ by Proposition 12.21. Since the space is noetherian, a subset is Thomason if and only if it is specialization closed. According to Theorem 12.27, a subset is specialization closed precisely when it has the closure properties $(a)$ and $(b)$. In other words, the admissible subsets are precisely the Thomason subsets. With this in hand, the rest is just a formulation of the correspondence between Thomason subsets and tt-ideals [Bal05, Theorem 4.10].

Remark 12.31. By base-change along the initial geometric functor $F: \mathcal{S} p \rightarrow \operatorname{Mod}_{\mathrm{HZ}}$, we have that $P_{d-1}^{\mathbb{Z}}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{H Z}\right) \rightarrow \operatorname{Exc}_{d-1}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)$ is a finite localization, while $\partial_{d}^{\mathbb{Z}}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right) \rightarrow \operatorname{Mod}_{\mathrm{HZ}}$ is finite étale (this follows either by direct verification, or from Remark 6.33 and [San22, Example 5.7]). We are thus in a situation where we can apply the techniques of [BHS23b] and [BCHS23b] to show that $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S p}^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)^{c}\right)$ is stratified and costratified over its spectrum and that the telescope conjecture holds for $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{HZ}}\right)$.
Example 12.32. We can also consider $\operatorname{Exc}_{d}\left(\mathcal{S}^{c}{ }^{c}, \operatorname{Mod}_{\mathrm{HF}}\right)$ for a field $\mathbb{F}$. It follows from the above techniques that $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S p}^{c}, \mathrm{HQ}\right)^{c}\right)$ is the discrete space with $d$ points, and can be identified via $\mathbb{Z} \rightarrow \mathbb{Q}$ with the subspace of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \operatorname{Mod}_{\mathrm{H} \mathbb{Z}}\right)^{c}\right)$
consisting of $\left\{\mathcal{P}_{d}^{\mathbb{Z}}([k], 0) \mid 1 \leq k \leq d\right\}$. On the other hand, $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S p}^{c}, \operatorname{Mod}_{\mathrm{HF}_{p}}\right)^{c}\right)$ identifies with the subspace $\left\{\mathcal{P}_{d}^{\mathbb{Z}}([k], p) \mid 1 \leq k \leq d\right\}$ of $\operatorname{Spc}\left(\operatorname{Exc}_{d}\left(\mathcal{S}^{c}, \operatorname{Mod}_{H \mathbb{Z}}\right)^{c}\right)$. In other words, it is the set $\{1,2, \ldots, d\}$ equipped with the topology given by $k \in \overline{\{l\}}$ if and only if $p-1 \mid k-l \geq 0$. Finite spectral spaces correspond to finite posets (via the map which sends a space to its specialization poset, see [DST19, 1.1.16]). From this perspective, it is the poset $\left([d], \leq_{p}\right)$ where $k \leq_{p} l$ means that $p-1 \mid k-l \geq 0$.

## Appendix A. Equivariant homotopy and Goodwillie calculus dictionary

For readers familiar with equivariant homotopy theory, we include here a 'dictionary' for translating between equivariant homotopy theory and Goodwillie calculus. When $d=2$ there is an equivalence of categories $\mathcal{S} p_{C_{2}} \simeq \operatorname{Exc}_{2}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ [Gla18] and in this case the analogies described by the table are genuine correspondences.

| Equivariant homotopy | Goodwillie calculus |
| :---: | :---: |
| $\delta p_{G}$ | $\operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ |
| Burnside ring $A(G)$ | Goodwillie-Burnside ring $A(d)$ |
| Conjugacy class of subgroups $H \leq G$ | Integer $1 \leq i \leq d$ |
| Compact generator $\Sigma^{\infty} G / H_{+}$ | Compact generator $P_{d} h_{s}(i)$ |
| Fixed points functor $(-)^{H}: S p_{G} \rightarrow$ Sp | Cross-effect $\operatorname{cr}_{i}(-)(\mathcal{S}, \ldots, \mathcal{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ |
| Restriction to the trivial group res ${ }_{e}^{G}: \mathcal{S} p_{G} \rightarrow \boldsymbol{S} p$ | $d$-th cross-effect $\operatorname{cr}_{d}(-)(S, \ldots, S): \operatorname{Exc}_{d}\left(S p^{c}, \delta p\right) \rightarrow \delta p$ |
| 'Partial' geometric fixed points $\tilde{\Phi}^{H}: S p_{G} \rightarrow$ Sp $p_{W_{G} H}$ | $P_{i}: \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \operatorname{Exc}_{i}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ |
| Geometric fixed points $\Phi^{H}=\operatorname{res}_{e}^{W_{G} H} \circ \Phi^{H}: \mathcal{S} p_{G} \rightarrow \mathcal{S} p$ | $i$-th derivative $\partial_{i}=\operatorname{cr}_{i} P_{i}(-)(\mathbb{S}, \ldots, \mathbb{S}): \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right) \rightarrow \mathcal{S} p$ |
| Inflation $\mathrm{inf}_{1}^{G}: \mathcal{S} p \rightarrow \mathcal{S} p_{G}$ | $i_{d}: \mathcal{S} p \rightarrow \operatorname{Exc}_{d}\left(\mathcal{S} p^{c}, \mathcal{S} p\right)$ |
| Free $G$-spectra | $d$-homogeneous functors |
| Tom Dieck splitting: $\left(\mathbb{S}_{G}^{0}\right)^{G} \simeq \bigoplus_{(H)} \Sigma^{\infty}\left(S^{0}\right)_{h W_{G} H}$ | Snaith splitting: $\mathrm{cr}_{1} P_{d} h_{\mathbb{S}} \simeq \bigoplus_{1 \leq i \leq d} \Sigma^{\infty}\left(S^{0}\right)_{h \Sigma_{i}}$ |
| Double coset formula: $\Sigma^{\infty} G / H_{+} \otimes \Sigma^{\infty} G / K_{+} \simeq \bigoplus_{g \in H \backslash G / K} \Sigma^{\infty} G /\left(H^{g} \cap K\right)_{+}$ | Good subset decomposition: $P_{d} h_{\Im}(i) \circledast P_{d} h_{\Im}(j) \simeq \bigoplus_{\substack{\mathcal{U} \subseteq[i] \times[j] \\ \mathcal{U} \text { good } \\\|\mathcal{U}\| \leq d}} P_{d} h_{s}(\|U\|)$ |
| Tate square: | Kuhn-McCarthy square: |

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[^0]:    ${ }^{1}$ Note that the definition of localization given in [HPS97] automatically assumes compatibility with the symmetric monoidal structure.

[^1]:    ${ }^{1}$ There is a subtlety at the prime $p=2$ related to the exotic multiplicative structure of $K(n)$, which however can be 'filtered away' as in [KL20, Section 6.3].

[^2]:    ${ }^{1}$ In fact, we do not require the full strength of this theorem; it is enough to combine Corollary 11.4, Remark 11.5, and Proposition 11.7.

