# On the K-theory of $\mathbf{Z} / p^{n}$ - announcement 

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Quillen introduced higher algebraic K-theory in [27] and computed the K-groups $\mathrm{K}_{*}\left(\mathbf{F}_{q}\right)$ in [26]. Except in low degrees, the computation of the Kgroups of closely related rings, for example $\mathbf{Z} / 4$, has remained out of reach. In this paper, we announce new methods for computations of K-groups of such rings and outline new results. A full account will be given in [3].

We are interested in rings of the form $\mathcal{O}_{K} / \varpi^{n}$ where $K$ is a finite extension of $\mathbf{Q}_{p}$ of degree $d, \mathcal{O}_{K}$ is its ring of integers, and $\varpi^{n}$ is the $n$th power of a uniformizer $\varpi$. In particular, $p \in\left(\varpi^{e}\right)$ where $e$ is the degree of ramification of $K$ over $\mathbf{Q}_{p}$. When $n=1$, $\mathcal{O}_{K} / \varpi^{n}$ is the residue field $k=\mathbf{F}_{q}$ of $\mathcal{O}_{K}$, where $q=p^{f}$ for some $f$, called the residual degree of the extension.

The problem of computing the K-groups of such rings, and of finite rings in general, was raised by Swan in the Battelle proceedings [13, Prob. 20].

## 1 History

For any field $k, \mathrm{~K}_{0}(k) \cong \mathbf{Z}$ and $\mathrm{K}_{1}(k) \cong k^{\times}$. Quillen showed in [26] that if $\mathbf{F}_{q}$ is the finite field with $q=p^{f}$ elements, then for $r \geqslant 1$,

$$
\mathrm{K}_{r}\left(\mathbf{F}_{q}\right) \cong \begin{cases}0 & \text { if } r \text { is even and } \\ \mathbf{Z} /\left(q^{i}-1\right) & \text { if } r=2 i-1\end{cases}
$$

Note in particular that there is no $p$-torsion in the K-groups of $\mathbf{F}_{q}$.

For each prime $\ell$ and ring $R, \mathrm{~K}\left(R, \mathbf{Z}_{\ell}\right)$ denotes the $\ell$-completion of the K-theory spectrum of $R$. In the main case of interest to us, namely when $R=\mathcal{O}_{K} / \varpi^{n}$, $\mathrm{K}_{r}(R)$ is finitely generated torsion for $r>0$ and $\mathrm{K}_{r}\left(R, \mathbf{Z}_{\ell}\right)$ is the subgroup of $\ell$-primary torsion in $\mathrm{K}_{r}(R)$.

Gabber's rigidity theorem [12] implies that if $R$ is a commutative ring which is henselian with respect to an ideal $I$ and if $\ell$ is invertible in $R$, then

$$
\mathrm{K}\left(R ; \mathbf{Z}_{\ell}\right) \simeq \mathrm{K}\left(R / I ; \mathbf{Z}_{\ell}\right)
$$

[^0]Examples of such henselian pairs are the rings of integers $\mathcal{O}_{K}$ as above with the ideal $(\varpi)$ or the quotients $\mathcal{O} / \varpi^{n}$, again with the ideal $(\varpi)$. It follows that for $\ell \neq p$ we have

$$
\mathrm{K}_{*}\left(\mathcal{O} ; \mathbf{Z}_{\ell}\right) \cong \mathrm{K}_{*}\left(\mathcal{O} / \varpi^{n} ; \mathbf{Z}_{\ell}\right) \cong \mathrm{K}_{*}\left(\mathbf{F}_{q} ; \mathbf{Z}_{\ell}\right)
$$

so that these $\ell$-adic K-groups are all determined by Quillen's computation.

The situation of the $p$-adic K-theory of $\mathcal{O}_{K}$ or $\mathcal{O}_{K} / \varpi^{n}$ is very different. A result of Dundas-Goodwillie-McCarthy [11] implies that $\mathrm{K}\left(\mathcal{O} / \varpi^{n} ; \mathbf{Z}_{p}\right) \simeq \tau_{\geqslant 0} \mathrm{TC}\left(\mathcal{O} / \varpi^{n} ; \mathbf{Z}_{p}\right)$, while work of Hesselholt-Madsen [17] and of Panin [25] implies that $\mathrm{K}\left(\mathcal{O}_{K} ; \mathbf{Z}_{p}\right) \simeq \tau_{\geqslant 0} \mathrm{TC}\left(\mathcal{O}_{K} ; \mathbf{Z}_{p}\right)$. Here, $\mathrm{TC}\left(\mathcal{O}_{K} ; \mathbf{Z}_{p}\right)$ and $\mathrm{TC}\left(\mathcal{O}_{K} / \varpi^{n} ; \mathbf{Z}_{p}\right)$ denote the $p$-adic topological cyclic homology spectra of $\mathcal{O}_{K}$ and $\mathcal{O}_{K} / \varpi^{n}$, respectively. This theory is built from topological Hochschild homology and is closely connected to $p$ adic cohomology theories thanks to the work of [6]. These results make the $p$-adic K-groups amenable to calculation using so-called trace methods.

Hesselholt and Madsen determine the structure of $\mathrm{TC}_{*}\left(\mathcal{O}_{K} ; \mathbf{Z}_{p}\right) \cong \mathrm{K}_{*}\left(\mathcal{O}_{K} ; \mathbf{Z}_{p}\right)$ in [18] and thereby verify the Quillen-Lichtenbaum conjecture for $\mathcal{O}_{K}$. This conjecture now follows in general from the proof of the Bloch-Kato conjecture due to Rost and Voevodsky; see for example [14], although the $p$-adic ranks of the groups $\mathrm{K}_{*}\left(\mathcal{O}_{K} ; \mathbf{Z}_{p}\right)$ had previously been computed by Wagoner [31].

The Hesselholt-Madsen approach uses logarithmic de Rham-Witt forms and TR, i.e., the classical approach to trace method computations. These have recently been revisited by Liu-Wang [21] who describe $\mathrm{K}_{*}\left(\mathcal{O}_{\mathrm{K}} ; \mathbf{F}_{p}\right)$, the K-groups with $\bmod p$ coefficients, using new cyclotomic techniques from [6, 24].

The result is that
$\mathrm{K}_{r}\left(\mathcal{O}_{K} ; \mathbf{Z}_{p}\right) \cong \begin{cases}\mathbf{Z}_{p} & \text { if } r=0, \\ \mathrm{H}_{\text {ét }}^{1}\left(\operatorname{Spec} K, \mathbf{Z}_{p}(i)\right) & \text { if } r=2 i-1, \text { and } \\ \mathrm{H}_{\text {ét }}^{2}\left(\operatorname{Spec} K, \mathbf{Z}_{p}(i)\right) & \text { if } r=2 i-2,\end{cases}$
where $\mathbf{Z}_{p}(i)$ is the $i$ th Tate twist. These cohomology
groups are determined by Iwasawa theory: for $i>0$,

$$
\begin{aligned}
& \mathrm{H}_{\text {ét }}^{1}\left(\operatorname{Spec} K, \mathbf{Z}_{p}(i)\right) \cong \mathbf{Z}_{p}^{d} \oplus \mathbf{Z} / w_{i} \\
& \mathrm{H}_{\text {ét }}^{2}\left(\operatorname{Spec} K, \mathbf{Z}_{p}(i)\right) \cong \mathbf{Z} / w_{i-1}
\end{aligned}
$$

where $d$ is the degree of $K$ over $\mathbf{Q}_{p}$ and where $w_{i}$ is the largest $p$ th power $p^{\nu}$ such that the exponent of the cyclotomic Galois group $\operatorname{Gal}\left(K\left(\mu_{p^{\nu}}\right) / K\right)$ divides $i$. The number $w_{i}$ is the $p$-part of a number introduced by Harris-Segal [15], Quillen, and Lichtenbaum in the setting of the Quillen-Lichtenbaum conjecture. See Weibel's book [32, Chap. VI] for more details.

Much less is known about the K-theory of the intermediate rings $\mathcal{O}_{K} / \varpi^{n}$ for $1<n<\infty$. As for fields, $\mathrm{K}_{0}\left(\mathcal{O}_{K} / \varpi^{n}\right) \cong \mathbf{Z}$ and $\mathrm{K}_{1}\left(\mathcal{O}_{K} / \varpi^{n}\right)$ is isomorphic to the group of units in $\mathcal{O}_{K} / \varpi^{n}$.

In [10], Dennis and Stein determined the structure of $K_{2}\left(\mathcal{O}_{K} / \varpi^{n}\right)$. No other work we are aware of has addressed the K-groups of general rings of the form $\mathcal{O}_{K} / \varpi^{n}$.

In special situations, more is known. First, every $\operatorname{ring} \mathbf{F}_{q}[z] / z^{n}$ is of the form $\mathcal{O}_{K} / \varpi^{n}$ for $K$ of ramification degree at least $n$. The algebraic K-groups of these truncated polynomial rings have been studied by Hesselholt-Madsen in [16] using classical trace method techniques, by Speirs in [28] using the new approach to TC due to Nikolaus-Scholze [24], and by Sulyma in [29] using the approach to TC via syntomic cohomology due to Bhatt-Morrow-Scholze [6] and as outlined by Mathew in [23].

Second, for unramified extension there are some results in low degrees. In the unramified case, where $e=1, \mathcal{O}_{K}$ is the ring $W\left(\mathbf{F}_{q}\right)$ of $p$-typical Witt vectors of the residue field. Brun [8] determined the K-groups of $\mathbf{Z} / p^{n}$ (i.e., when $e=1$ and $f=1$ ) up to degree $p-3$ and Angeltveit [2] determined the K-groups of $W_{n}\left(\mathbf{F}_{q}\right)=W\left(\mathbf{F}_{q}\right) / \varpi^{n}=W\left(\mathbf{F}_{q}\right) / p^{n}$ up to degree $2 p-2$.

Angeltveit also proved an important quantitative result:

$$
\frac{\# \mathrm{~K}_{2 i-1}\left(W_{n}\left(\mathbf{F}_{q}\right) ; \mathbf{Z}_{p}\right)}{\# \mathrm{~K}_{2 i-2}\left(W_{n}\left(\mathbf{F}_{q}\right) ; \mathbf{Z}_{p}\right)}=q^{i(n-1)}
$$

Both Brun and Angeltveit use classical trace methods and the $p$-adic filtration on the truncated Witt vectors to translate part of the problem to the cases of truncated polynomial rings where a complete answer is known.

The cases of $\mathrm{K}_{3}$ of $\mathbf{Z} / p^{n}$ or $\mathbf{F}_{q}[z] / z^{2}$ were also considered earlier in [1] using group homology calculations.

## 2 New results

As $\mathrm{K}\left(\mathcal{O} / \varpi^{k} ; \mathbf{Z}_{p}\right) \simeq \tau_{\geqslant 0} \mathrm{TC}\left(\mathcal{O} / \varpi^{k} ; \mathbf{Z}_{p}\right)$ by [11, 18], it is enough to determine TC of these rings. To do so, we use the filtration on TC constructed by Bhatt-Morrow-Scholze in [6]. If $R$ is a quasisyntomic ring, there is a complete decreasing filtration $\mathrm{F} \geqslant \star \mathrm{syn} \mathrm{TC}\left(R ; \mathbf{Z}_{p}\right)$ with associated graded pieces

$$
\mathrm{F}_{\mathrm{syn}}^{=i} \mathrm{TC}\left(R ; \mathbf{Z}_{p}\right) \simeq \mathbf{Z}_{p}(i)(R)[2 i]
$$

where $\mathbf{Z}_{p}(i)(R)$ is the weight $i$ syntomic cohomology of $R$ introduced in [6]. The syntomic complexes provide a $p$-adic analogue of the motivic filtration on K-theory.

As shown in [4], the weight $i$ syntomic cohomology $\mathbf{Z}_{p}(i)(R)$ is concentrated in $[0, i+1]$, independent of $R$; this means that $\mathrm{H}^{r}\left(\mathbf{Z}_{p}(i)(R)\right)=0$ for $r \notin[0, i+1]$. In the special case of $\mathcal{O}_{K}$ or $\mathcal{O}_{K} / \varpi^{n}$, an argument using the $\varpi$-adic associated graded implies that in fact the weight $i$ syntomic cohomology is in $[0,2]$; moreover, for $i \geqslant 1, \mathrm{H}^{0}\left(\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)\right)=0$ so the complex has cohomology concentrated in degrees 1 and 2.

One checks that $H^{2}\left(\mathbf{Z}_{p}(1)\left(\mathcal{O}_{K} / \varpi^{n}\right)\right)=0$, so the spectral sequence associated to the syntomic filtration on TC collapses at the $\mathrm{E}_{1}$-page for $\mathcal{O} / \varpi^{n}$ (or the $\mathrm{E}_{2}$-page in the reindexing in [6, Thm. 1.12]). Hence,

$$
\mathrm{TC}_{2 i-1}\left(\mathcal{O}_{K} / \varpi^{n} ; \mathbf{Z}_{p}\right) \cong \mathrm{H}^{1}\left(\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)\right)
$$

for $i \geqslant 1$ and

$$
\mathrm{TC}_{2 i-2}\left(\mathcal{O}_{K} / \varpi^{n} ; \mathbf{Z}_{p}\right) \cong \mathrm{H}^{2}\left(\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)\right)
$$

for $i \geqslant 2$. Thus, it makes sense to speak of the syntomic weights of the K-groups of $\mathcal{O}_{K} / \varpi^{n}$.

Theorem 2.1. For $i \geqslant 1$, if the residue field of $\mathcal{O}_{K}$ has $q=p^{f}$ elements, then there is an explicit cochain complex

$$
\left(\mathbf{Z}_{p}^{f(i n-1)} \xrightarrow{\operatorname{syn}_{0}} \mathbf{Z}_{p}^{2 f(i n-1)} \xrightarrow{\mathrm{syn}_{1}} \mathbf{Z}_{p}^{f(i n-1)}\right)
$$

quasi-isomorphic to $\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)$. The terms are free $\mathbf{Z}_{p}$-modules of the given ranks in cohomological degrees 0,1 , and 2 .

The proof of the existence of this explicit cochain complex model of the syntomic complex will be discussed in Sections 4 and 5 .

The groups $\mathrm{K}_{*}\left(\mathcal{O}_{K} / \varpi^{n}\right)$ are torsion for $*>0$. In particular, the complex above is exact rationally. Thus, to find the cohomology of $\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)$, and hence the $p$-adic K-groups of $\mathcal{O}_{K} / \varpi^{n}$, it is enough to compute the matrices $\operatorname{syn}_{0}$ and $\operatorname{syn}_{1}$ and their elementary divisors.

Theorem 2.2. The matrices $\operatorname{syn}_{0}$ and $\operatorname{syn}_{1}$ are effectively computable. Specifically, they can be determined with enough p-adic precision to guarantee computability of the effective divisors.

We have implemented our algorithm in SAGE [30] in the case where $f=1$, i.e., when the residue field is $\mathbf{F}_{p}$. Future work will include an implementation for general $f$.

Corollary 2.3. There is an algorithm to determine the structure of $\mathrm{K}_{r}\left(\mathcal{O}_{K} / \varpi^{n}\right)$ for any $K$, $n$, and $r$.

Along the way, we extend the result of Angeltveit on the quotients of the orders from the unramified case to any $\mathcal{O}_{K} / \varpi^{n}$.

Corollary 2.4. For any $\mathcal{O}_{K} / \varpi^{n}$,

$$
\frac{\# \mathrm{~K}_{2 i-1}\left(\mathcal{O}_{K} / \varpi^{n} ; \mathbf{Z}_{p}\right)}{\# \mathrm{~K}_{2 i-2}\left(\mathcal{O}_{K} / \varpi^{n} ; \mathbf{Z}_{p}\right)}=q^{i(n-1)}
$$

where $q=p^{f}$ is the order of the residue field of $\mathcal{O}_{K}$.
This corollary is especially powerful thanks to the following theorem.

Theorem 2.5 (Even vanishing theorem). If

$$
i \geqslant \frac{p^{2}}{(p-1)^{2}}\left(p^{\left\lceil\frac{n}{e}\right\rceil}-1\right)
$$

then $\mathrm{H}^{2}\left(\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)\right)=0$ and hence

$$
\mathrm{K}_{2 i-2}\left(\mathcal{O}_{K} / \varpi^{n}\right)=0
$$

if additionally $i \geqslant 2$.
Corollary 2.6. If

$$
i \geqslant \frac{p^{2}}{(p-1)^{2}}\left(p^{\left\lceil\frac{n}{e}\right\rceil}-1\right)
$$

then $\# \mathrm{~K}_{2 i-1}\left(\mathcal{O}_{K} / \varpi^{k}\right)=q^{i(n-1)} \cdot\left(q^{i}-1\right)$.
Corollary 2.7. There is an algorithm to compute the orders of all of the K-groups of $\mathcal{O} / \varpi^{n}$.

Indeed, Theorem 2.5 and Corollary 2.6 reduce the problem to the computation of the cohomology of the syntomic complexes $\mathbf{Z}_{p}(i)\left(\mathcal{O} / \varpi^{n}\right)$ for finitely many $i$ : those satisfying

$$
i<\frac{p^{2}}{(p-1)^{2}}\left(p^{\left\lceil\frac{n}{e}\right\rceil}-1\right) .
$$

This number grows rather quickly, but improvements are possible and will be described in our forthcoming work [3].

## 3 Computations

We present here four example calculations.

## $3.1 \mathrm{Z} / 4$

The even vanishing theorem holds in syntomic weights $i \geqslant 12$. In fact, machine computations show in this case that $\mathrm{K}_{2 i-2}(\mathbf{Z} / 4)=0$ for all $i \geqslant 3$, while $\mathrm{K}_{2}(\mathbf{Z} / 4) \cong \mathbf{Z} / 2$. Corollary 2.4 together with Quillen's calculation implies that
$\# \mathrm{~K}_{3}(\mathbf{Z} / 4)=8 \cdot\left(2^{2}-1\right)$ and $\# \mathrm{~K}_{2 i-1}(\mathbf{Z} / 4)=2^{i} \cdot\left(2^{i}-1\right)$
for $i \geqslant 3$. This gives the complete calculation of the orders of all K-groups of $\mathbf{Z} / 4$.

The precise structure of the decomposition of $p$ primary part of the K-groups into cyclic groups remains unknown to us. Figure 1 displays a table of the output of our machine computations giving the groups in syntomic weights $i \leqslant 16$.

### 3.2 Chain rings of order 8

A chain ring is a commutative ring whose ideals are totally ordered with respect to inclusion. Examples include valuation rings or quotients of valuation rings. Every finite chain ring is of the form $\mathcal{O}_{K} / \varpi^{n}$ for some $1 \leqslant n<\infty$. There are four chain rings of order 8 , namely $\mathbf{Z} / 8, \mathbf{Z}\left[2^{1 / 2}\right] / 2^{3 / 2}$ (so $n=3$ in our notation), $\mathbf{F}_{2}[z] / z^{3}$, and $\mathbf{F}_{8}$; see [9]. The 2-adic K-groups $\mathrm{K}_{n}\left(\mathbf{F}_{8} ; \mathbf{Z}_{2}\right)$ vanish for $n \geqslant 1$. Figure 2 displays the low-degree 2 -adic K-groups of the other three chain rings of order 8 .

### 3.3 Quotients of degree 2 totally ramified 2-adic fields

The 1 mfdb [22] provides tables of $p$-adic fields based on work of Jones-Roberts [19]. There are 6 totally ramified degree 2 extensions of $\mathbf{Q}_{2}$. In Figure 3, we give low-degree $p$-adic K-groups of the quotients of these fields.

## $3.4 \mathrm{Z} / 9$

The even vanishing theorem holds in syntomic weights $i \geqslant 18$. Figure 4 displays a table of the output of our machine computations in syntomic weights $i \leqslant 18$. In particular, $\mathrm{K}_{4}(\mathbf{Z} / 9) \cong \mathbf{Z} / 3$ and all other positive even K-groups vanish. In odd degrees,
$\# \mathrm{~K}_{5}(\mathbf{Z} / 9)=81 \cdot\left(3^{3}-1\right)$ and $\# \mathrm{~K}_{2 i-1}(\mathbf{Z} / 9)=3^{i} \cdot\left(3^{i}-1\right)$
for $i \geqslant 1, i \neq 3$. This gives the complete calculation of the orders of all K-groups of $\mathbf{Z} / 9$.

| $\mathrm{K}_{1}$ | Z/2 | $\mathrm{K}_{17}$ | $(\mathbf{Z} / 2)^{3} \oplus(\mathbf{Z} / 8)^{2}$ |
| :---: | :---: | :---: | :---: |
| $\mathrm{K}_{2}$ | Z/2 | $\mathrm{K}_{18}$ | 0 |
| $\mathrm{K}_{3}$ | Z/8 | $\mathrm{K}_{19}$ | $\mathbf{Z} / 4 \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 32$ |
| $\mathrm{K}_{4}$ | 0 | $\mathrm{K}_{20}$ | (Z/2) 0 |
| $\mathrm{K}_{5}$ | Z/8 | $\mathrm{K}_{21}$ | $(\mathbf{Z} / 2)^{2} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 32$ |
| $\mathrm{K}_{6}$ | 0 | $\mathrm{K}_{22}$ | 0 |
| $\mathrm{K}_{7}$ | $\mathbf{Z} / 2 \oplus \mathbf{Z} / 8$ | $\mathrm{K}_{23}$ | $(\mathbf{Z} / 2)^{4} \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 64$ |
| $\mathrm{K}_{8}$ | 0 | $\mathrm{K}_{24}$ | , |
| $\mathrm{K}_{9}$ | $(\mathbf{Z} / 2)^{2} \oplus \mathbf{Z} / 8$ | $\mathrm{K}_{25}$ | $(\mathbf{Z} / 2)^{4} \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 16$ |
| $\mathrm{K}_{10}$ | 0 | $\mathrm{K}_{26}$ | (Z) 0 |
| $\mathrm{K}_{11}$ | $\mathbf{Z} / 2 \oplus \mathbf{Z} / 32$ | $\mathrm{K}_{27}$ | $\mathbf{Z} / 2 \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 16 \oplus \mathbf{Z} / 128$ |
| $\mathrm{K}_{12}$ | 0 | $\mathrm{K}_{28}$ | (Z/2) 0 |
| $\mathrm{K}_{13}$ | $\mathbf{Z} / 2 \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 16$ | $\mathrm{K}_{29}$ | $(\mathbf{Z} / 2)^{3} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 32$ |
| $\mathrm{K}_{14}$ | 0 | $\mathrm{K}_{30}$ | (Z/2) ${ }^{6}$ Z/8 ${ }^{\text {Z/128 }}$ |
| $\mathrm{K}_{15}$ | $(\mathbf{Z} / 2)^{3} \oplus \mathbf{Z} / 32$ | $\mathrm{K}_{31}$ | $(\mathbf{Z} / 2)^{6} \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 128$ |
| $\mathrm{K}_{16}$ | 0 | $\mathrm{K}_{32}$ | 0 |

Figure 1: The 2-adic K-groups of $\mathbf{Z} / 4$ for syntomic weights $1 \leqslant i \leqslant 16$; the final zero, $\mathrm{K}_{32}\left(\mathbf{Z} / 4 ; \mathbf{Z}_{2}\right)=0$, is a (null) contribution from syntomic weight 17.

| $\mathrm{K}_{r}$ | Z/8 | $\mathbf{F}_{2}[z] / z^{3}$ | $\mathbf{Z}_{2}\left[2^{1 / 2}\right] / 2^{3 / 2}$ |
| :---: | :---: | :---: | :---: |
| $\mathrm{K}_{1}$ | Z/4 | Z/4 | Z/4 |
| $\mathrm{K}_{2}$ | Z/2 | 0 | 0 |
| $\mathrm{K}_{3}$ | $\mathbf{Z / 4 \oplus} \mathbf{Z} / 8$ | $\mathbf{Z / 2 \oplus} \mathbf{Z} / 8$ | $\mathbf{Z} / 2 \oplus \mathbf{Z} / 8$ |
| $\mathrm{K}_{4}$ | Z/2 | 0 | 0 |
| $\mathrm{K}_{5}$ | $\mathbf{Z / 2 \oplus \mathbf { Z } / 6 4}$ | $(\mathbf{Z} / 2)^{2} \oplus \mathbf{Z} / 16$ | $(\mathbf{Z} / 2)^{2} \oplus \mathbf{Z} / 16$ |
| $\mathrm{K}_{6}$ | 0 | 0 |  |
| $\mathrm{K}_{7}$ | $(\mathbf{Z} / 4)^{2}$ | $(\mathbf{Z} / 2)^{2} \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 16$ | $(\mathbf{Z} / 2)^{2} \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 16$ |
| $\mathrm{K}_{8}$ | 0 | 0 | 0 |
| $\mathrm{K}_{9}$ | $\mathbf{Z} / 2 \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 128$ | $(\mathbf{Z} / 2)^{2} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 16$ | $(\mathbf{Z} / 2)^{2} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 16$ |
| $\mathrm{K}_{10}$ | 0 | 0 | 0 |
| $\mathrm{K}_{11}$ | $\mathbf{Z / 8 \oplus \mathbf { Z } / 5 1 2}$ | $(\mathbf{Z} / 2)^{3} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 32$ | $(\mathbf{Z} / 2)^{3} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 32$ |
| $\mathrm{K}_{12}$ | 0 | 0 | 0 |
| $\mathrm{K}_{13}$ | $(\mathbf{Z} / 2)^{2} \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 512$ | $(\mathbf{Z} / 2)^{4} \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 32$ | $(\mathbf{Z} / 2)^{4} \oplus \mathbf{Z} / 4 \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 32$ |
| $\mathrm{K}_{14}$ | 0 | 0 | 0 |
| $\mathrm{K}_{15}$ | $(\mathbf{Z} / 2)^{2} \oplus \mathbf{Z} / 64 \oplus \mathbf{Z} / 256$ | $(\mathbf{Z} / 2)^{4} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 32$ | $(\mathbf{Z} / 2)^{4} \oplus(\mathbf{Z} / 4)^{2} \oplus \mathbf{Z} / 8 \oplus \mathbf{Z} / 32$ |

Figure 2: The 2 -adic K-groups of the displayed chain rings of order 8 for syntomic weights $1 \leqslant i \leqslant 8$. Note that the second and third columns agree. We do not know at present if this continues in all higher weights. The second column agrees with the calculations of [16] (see for example [28, Lem. 2]).

## 4 Prismatic cohomology over $\delta$ rings

Our proofs are motivated by previous work of KrauseNikolaus [20] and the approach of Liu-Wang [21]. There are two main new ideas: the notion of prismatic cohomology relative to a $\delta$-ring and the systematic use of the filtration on the syntomic complexes induced by the $\varpi$-adic filtration on $\mathcal{O}_{K} / \varpi^{n}$. Similar filtrations have also been used by Angeltveit [2] and Brun [8] in the topological context.
Let $A^{0}=W\left(\mathbf{F}_{q}\right) \llbracket z \rrbracket$ be the $\delta$-ring with $\delta(z)=0$
and hence $\varphi(z)=z^{p}$. If $E(z)$ is an Eisenstein polynomial for $\mathcal{O}_{K}$, then the pair $\left(A^{0},(E(z))\right)$ is a prism. Bhatt and Scholze show that $\triangle_{\left(\mathcal{O}_{K} / \varpi^{n}\right) / A^{0}}$ is discrete and admits a description as a prismatic envelope $A^{0}\left\{\frac{\sigma^{n}}{E(z)}\right\}^{\wedge}$ in the sense of [7, Prop. 3.13]; the prismatic envelope is an explicit pushout in $(p, E(z))$ complete $\delta$-rings over $A^{0}$.

The main idea is to determine the syntomic complexes $\mathbf{Z}_{p}(i)\left(\mathcal{O} / \varpi^{n}\right)$ by descent along the map $\Delta_{\mathcal{O} / \varpi^{n}} \rightarrow \Delta_{\left(\mathcal{O} / \varpi^{n}\right) / A^{0}}$ from absolute prismatic cohomology to relative prismatic cohomology. To make sense of this, we introduce prismatic cohomology rel-
ative to a $\delta$-ring. Let us outline the definition.
Given an arbitrary derived $p$-complete $\delta$-ring $A$ and a derived $p$-complete $A$-algebra $R$, let $X=\operatorname{Spf} R$ and let $(X / A)_{\triangle}$ be the opposite of the category of commutative diagrams

where $(B, J)$ is a bounded prism and $A \rightarrow B$ is a map of $\delta$-rings.

By definition, $\Delta_{R / A}=\mathrm{R} \Gamma\left((X / A)_{\Delta}, \mathcal{O}_{\triangle}\right)$, where $\mathcal{O}_{\Delta}$ is the prismatic structure sheaf, which sends a commutative diagram as above to $B$. Warning: this site-theoretic definition should be derived in general, but gives the correct answer under additional assumptions on $R$, in particular in the case of $R=\mathcal{O}_{K} / \varpi^{n}$ over the multivariable Breuil-Kisin prisms appearing in this paper.

Example 4.1. If $A=\mathbf{Z}_{p}$ is the initial (derived $p$ complete) $\delta$-ring, then $\mathbb{}_{R / \mathbf{Z}_{p}}$ recovers absolute prismatic cohomology as introduced in $[6,7]$ and studied further in [5]. More generally, this is true if $A$ is replaced by the ring of $p$-typical Witt vectors of any perfect $\mathbf{F}_{p}$-algebra.

Example 4.2. If $(A, I)$ is a prism and $R$ is an $A / I$ algebra, then $\triangle_{R / A}$ agrees with derived relative prismatic cohomology as studied in [7].

Now, consider the augmented cosimplicial diagram $A^{\bullet}$ where $A^{-1}=W\left(\mathbf{F}_{q}\right), A^{0}=W\left(\mathbf{F}_{q}\right) \llbracket z \rrbracket$, and $A^{s}=W\left(\mathbf{F}_{q}\right) \llbracket z_{0}, \ldots, z_{s} \rrbracket$. This is a completed descent complex for $W\left(\mathbf{F}_{q}\right) \rightarrow W\left(\mathbf{F}_{q}\right)[z]$.

In the cosimplicial diagram

$$
W\left(\mathbf{F}_{q}\right) \longrightarrow A^{0} \rightleftarrows A^{1} \rightleftarrows A^{2} \cdots
$$

the arrows are all $\delta$-ring maps and the entire diagram admits a map to $\mathcal{O}_{K}$ sending each generator $z_{j}$ to $\varpi$. As a result, for any $\mathcal{O}_{K}$-algebra $R$, there is an induced augmented cosimplicial diagram in prismatic cohomology of $R$ relative to the $\delta$-rings $A^{\bullet}$.

Theorem 4.3. The augmented cosimplicial diagram

$$
\Delta_{R} \longrightarrow \Delta_{R / A^{0}} \rightleftarrows \Delta_{R / A^{1}} \rightleftarrows \Delta_{R / A^{2}} \cdots
$$

is a limit diagram for $R=\mathcal{O}_{K} / \varpi^{n}$.
Thus, the absolute prismatic cohomology of an $\mathcal{O}_{K^{-}}$ algebra, such as $\mathcal{O}_{K} / \varpi^{n}$, can be computed by descent using the cosimplicial diagram above.

This does not make sense when speaking of prismatic cohomology as defined in [7] because there is no compatible way to equip the entire cosimplicial diagram with the structure of a cosimplicial prism. For example, if $E(z)$ is an Eisenstein polynomial making $A^{0}=W\left(\mathbf{F}_{q}\right) \llbracket z \rrbracket$ into a prism, both $E\left(z_{0}\right)$ and $E\left(z_{1}\right)$ are distinguished elements in $A^{1}=W\left(\mathbf{F}_{q}\right) \llbracket z_{0}, z_{1} \rrbracket$ making it into a prism in two different ways.

Proposition 4.4. For any $s \geqslant 0$, the relative prismatic cohomology $\triangle_{\left(\mathcal{O}_{K} / \varpi^{n}\right) / A^{s}}$ is discrete and is isomorphic to a prismatic envelope

$$
A^{s}\left\{\frac{z_{0}^{n}}{E\left(z_{0}\right)}, \frac{z_{1}-z_{0}}{E\left(z_{0}\right)}, \ldots, \frac{z_{n}-z_{0}}{E\left(z_{0}\right)}\right\}^{\wedge}
$$

The proposition follows immediately from Example 4.2. Note that while prismatic cohomology relative to $\delta$-rings is functorial in arbitrary maps of $\delta$ rings, the presentation of a given term $\Delta_{R / A^{s}}$ as a prismatic envelope depends on the choice of a prism structure $J$ on $A^{s}$ making $R$ into an $A^{s} / J$-algebra. In the theorem above, we choose to make $A^{s}$ into a prism with respect to the ideal $\left(E\left(z_{0}\right)\right)$.

It follows that the cosimplicial diagram appearing in Theorem 4.3 gives a resolution of $\Delta_{\mathcal{O}_{K} / \varpi^{n}}$ as the limit of a cosimplicial diagram of discrete $\delta$-rings.

To give the main idea of the rest of the argument, we illustrate it here for prismatic cohomology instead of the syntomic complexes. The absolute prismatic cohomology of a quasisyntomic ring $R$ admits a Ny gaard filtration $\mathcal{N} \geqslant \star \Delta_{R}$; Nygaard completion of prismatic cohomology is written $\widehat{\triangle}_{R}$.

Proposition 4.5. The Nygaard-complete absolute prismatic cohomology groups $\mathrm{H}^{r}\left(\widehat{\mathbb{}}_{\mathcal{O}_{K} / \varpi^{n}}\right)$ vanish for $r \neq 0,1$.

The proposition can be proved by computing directly with a Nygaard-complete, Frobenius-twisted variant of the cosimplicial diagram in Theorem 4.3 using the prismatic envelopes of Proposition 4.4. Alternatively, one can argue as follows: the $\varpi$-adic filtration on $\mathcal{O}_{K} / \varpi^{n}$ induces a filtration on $\triangle_{\mathcal{O}_{K} / \varpi^{n}}$ whose completion agrees with $\widehat{\mathbb{}}_{\mathcal{O}_{K} / \varpi^{n}}$, and whose associated graded is the same as that of the corresponding filtration on $\widehat{\triangle}_{\mathbf{F}_{q}[z] / z^{n}}$. This associated graded can be described using crystalline cohomology and vanishes away from cohomological degrees 0,1 . Thus, by dévissage and completeness, the same vanishing holds for $\widehat{\triangle}_{\mathcal{O}_{K} / \varpi^{n}}$.

It follows from the proposition that the cochain complex $A^{0} \rightarrow A^{1} \rightarrow A^{2} \rightarrow \cdots$ associated to the cosimplicial abelian group $\widehat{\Delta}_{\left(\mathcal{O}_{K} / \varpi^{n}\right) / A} \bullet$ is exact in degrees $\geqslant 2$. This reduces the computation of $\widehat{\Delta}_{\mathcal{O}_{K} / \varpi^{n}}$
to a much smaller computation involving prismatic envelopes of $\mathcal{O}_{K} / \varpi^{n}$ relative to $A^{0}, A^{1}$, and $A^{2}$.

However, we are interested not in the absolute prismatic cohomology of $\mathcal{O}_{K} / \varpi^{n}$ but rather in its syntomic cohomology. Relative syntomic cohomology is defined in the setting of prismatic cohomology relative to a $\delta$-ring. We first have to explain the Nygaard filtration and the Breuil-Kisin twist, following [7, 5].

The Frobenius twist $\Delta_{R / A}^{(1)}$ is defined to be $\triangle_{R / A} \otimes_{A}$ ${ }_{\varphi} A$, the base-change of $\triangle_{R / A}$ along the Frobenius map on $A$. The Frobenius twist admits a map $\Delta_{R / A}^{(1)} \rightarrow \triangle_{R / A}$ and the Nygaard filtration $\mathcal{N} \geqslant \star \Delta_{R / A}^{(1)}$ is a filtration which is taken by this map to the $I$ adic filtration on $\triangle_{R / A}$. If $\triangle_{R / A}$ is discrete (as in our examples of interest) then the Nygaard filtration is simply the preimage of the $I$-adic filtration.

Given a prism $(A, I)$, let $I_{r}$ be the invertible $A$ module $I \cdot \varphi(I) \cdots \varphi^{r-1}(I)$. If $(A, I)$ is transversal, meaning that $A / I$ is $p$-torsion-free, then the canonical map $I_{r} / I_{r}^{2} \rightarrow I_{r-1} / I_{r-1}^{2}$ is divisible by $p$ and the induced map $I_{r} / I_{r}^{2} \xrightarrow{1 / p} I_{r-1} / I_{r-1}^{2}$ is surjective. The Breuil-Kisin twist is defined to be

$$
A\{1\}=\lim \left(\cdots \rightarrow I_{3} / I_{3}^{2} \xrightarrow{1 / p} I_{2} / I_{2}^{2} \xrightarrow{1 / p} I / I^{2}\right) .
$$

This is an invertible $A$-module. For a general $A$ module $M$, let $M\{1\}=M \otimes_{A} A\{1\}$.

The relative syntomic cohomology of $R$ over a $\delta$ $\operatorname{ring} A$ is

$$
\mathbf{Z}_{p}(i)(R / A)=\operatorname{fib}\left(\mathcal{N} \geqslant i \triangle_{R / A}^{(1)}\{i\} \xrightarrow{\text { can }-\varphi} \triangle_{R / A}^{(1)}\{i\}\right)
$$

where $\varphi$ is a Frobenius which exists on $\mathcal{N} \geqslant i \triangle_{R / A}^{(1)}\{i\}$. Note that in [6], the syntomic complexes are defined using Nygaard complete prismatic cohomology; however, the two definitions agree by [6, Lem. 7.22] or [4, Cor. 5.31].

It follows along the lines of Theorem 4.3 that, for each $i \geqslant 0$, the limit of the cosimplicial diagram

$$
\mathbf{Z}_{p}(i)\left(R / A^{0}\right) \rightleftarrows \mathbf{Z}_{p}(i)\left(R / A^{1}\right) \rightleftarrows \cdots
$$

is equivalent to $\mathbf{Z}_{p}(i)(R)$ when $R=\mathcal{O}_{K} / \varpi^{k}$.
The fact that the Nygaard-complete absolute prismatic cohomology $\widehat{\mathbb{}}_{\mathcal{O}_{K} / \varpi^{n}}$ is concentrated in cohomological degrees 0,1 implies that $\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)$ is concentrated in cohomological degrees $0,1,2$. In fact, it is not hard to show that, for $i \geqslant 1$, each relative syntomic complex $\mathbf{Z}_{p}(i)\left(\left(\mathcal{O}_{K} / \varpi^{n}\right) / A^{s}\right)$ is concentrated in cohomological degree 1. Thus, the spectral sequence associated to the limit diagram

$$
\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right) \simeq \lim _{\Delta} \mathbf{Z}_{p}(i)\left(\left(\mathcal{O}_{K} / \varpi^{n}\right) / A^{\bullet}\right)
$$

implies that $\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)$ is concentrated in cohomological degrees 1,2 for $i \geqslant 1$.

By the same spectral sequence, to determine $\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)$, and hence $\mathrm{K}_{2 i-2}\left(\mathcal{O}_{K} / \varpi^{n} ; \mathbf{Z}_{p}\right)$ and $\mathrm{K}_{2 i-1}\left(\mathcal{O}_{K} / \varpi^{n} ; \mathbf{Z}_{p}\right)$, it is enough to compute the cohomology of the complex

$$
\begin{gathered}
\mathrm{H}^{1}\left(\mathbf{Z}_{p}(i)\left(R / A^{0}\right)\right. \\
\rightarrow \operatorname{ker}\left(\mathrm{H}^{1}\left(\mathbf{Z}_{p}(i)\left(R / A^{1}\right)\right) \rightarrow \mathrm{H}^{1}\left(\mathbf{Z}_{p}(i)\left(R / A^{2}\right)\right)\right)
\end{gathered}
$$

where $R=\mathcal{O}_{K} / \varpi^{n}$. In the next section, we explain how to use the $\varpi$-adic filtration to reduce this to a finite problem.

## 5 The syntomic matrices

In the cosimplicial diagram $A^{\bullet}$, each term is a filtered $\delta$-ring, where in $A^{s}=W(k) \llbracket z_{0}, \ldots, z_{s} \rrbracket$ the weight of $z_{j}$ is 1 . A filtered $\delta$-ring is a $\delta$-ring $A$ with a complete and separated decreasing filtration $\mathcal{F} \geqslant \star A$ such that $\delta\left(\mathcal{F}^{\geqslant i} A\right) \subseteq \mathcal{F} \geqslant p^{i} A$. Since each $A^{\bullet} \rightarrow \mathcal{O}_{K} / \varpi^{n}$ is a filtered map where $\mathcal{O}_{K} / \varpi^{n}$ is given the $\varpi$-adic filtration, all resulting invariants, such as prismatic or syntomic cohomology complexes admit induced filtrations, which we will write for instance as $\mathcal{F} \geqslant \star \mathbf{Z}_{p}(i)\left(\left(\mathcal{O}_{K} / \varpi^{n}\right) / A^{\bullet}\right)$.
Theorem 5.1. For $b \geqslant i n-1$ and $i \geqslant 1$, the natural maps

$$
\begin{gathered}
\mathcal{F}^{[1, b]} \mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right) \\
\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right) \longrightarrow \mathcal{F}^{[0, b]} \mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)
\end{gathered}
$$

are equivalences.
The right-hand arrow is easy to handle because $\mathcal{F}^{=0} \mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right) \simeq \mathbf{Z}_{p}(i)\left(\mathbf{F}_{q}\right) \simeq 0$ for $i>0$. For the left-hand arrow, we argue by an explicit study of the interaction between the $\mathcal{F}$-filtration and the Ny gaard filtration on each $\triangle_{\left(\mathcal{O}_{K} / \varpi^{n}\right) / A \bullet}$.

The entire problem has now been reduced to a finite computation. Set $R=\mathcal{O}_{K} / \varpi^{n}$ and consider the commutative diagram


All four terms are finitely generated free $\mathbf{Z}_{p}$-modules. The vertical fibers are $\mathbf{Z}_{p}(i)\left(R / A^{0}\right)$ and $\mathbf{Z}_{p}(i)\left(R / A^{1}\right)$, respectively.

Our approach to the computation avoids the more traditional approach of computing either $\operatorname{TR}\left(\mathcal{O}_{K} / \varpi^{n}\right)^{F=1}$ or computing $\operatorname{TC}\left(\mathcal{O}_{K} / \varpi^{n}\right)$ as the fiber of $\mathrm{TC}^{-}\left(\mathcal{O}_{K} / \varpi^{n}\right) \xrightarrow{\text { can }-\varpi} \mathrm{TP}\left(\mathcal{O}_{K} / \varpi^{n}\right)$. It would nevertheless be very interesting to understand $\operatorname{TP}\left(\mathcal{O}_{K} / \varpi^{n}\right)$.

Since the complexes $\mathcal{F}^{[1, b]} \mathcal{N} \geqslant i \triangle_{R}\{i\}$ and $\mathcal{F}^{[1, b]} \triangle_{R}\{i\}$ are torsion for $i \geqslant 1$ by another use of the $\varpi$-adic filtration, one can replace

$$
\operatorname{ker}\left(\mathcal{F}^{[1, b]} \mathcal{N}^{\geqslant i} \Delta_{R / A^{1}}^{(1)}\{i\} \rightarrow \mathcal{F}^{[1, b]} \mathcal{N} \geqslant i \Delta_{R / A^{2}}^{(1)}\{i\}\right)
$$

with the saturation of the image of the top horizontal map, where by saturation we mean the sub- $\mathbf{Z}_{p}$-module consisting of elements $x$ such that $p^{N} x$ is in the image for some $N$, and similarly for $\operatorname{ker}\left(\mathcal{F}^{[1, b]} \triangle_{R / A^{1}}^{(1)}\{i\} \rightarrow \mathcal{F}^{[1, b]} \triangle_{R / A^{2}}^{(1)}\{i\}\right)$. Write $S^{0}$ and $S^{1}$ for the saturations. The resulting commutative square

consists of free $\mathbf{Z}_{p}$-modules of rank $b f$ and the total cohomology computes $\mathcal{F}^{[1, b]} \mathbf{Z}_{p}(i)(R)$ and hence $\mathbf{Z}_{p}(i)(R)=\mathbf{Z}_{p}(i)\left(\mathcal{O}_{K} / \varpi^{n}\right)$ for $i \geqslant 1$.

To conclude, we use explicit polynomial presentations of the relevant prismatic envelopes as well as Breuil-Kisin orientations to give explicit bases of all four terms and to compute the maps between them. Taking $b=i n-1$, the result is the matrices $\operatorname{syn}_{0}$ and $\operatorname{syn}_{1}$ and the complex appearing in Theorem 2.1.

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## References

[1] Janet E. Aisbett, Emilio Lluis-Puebla, and Victor Snaith, On $K_{*}(\mathbf{Z} / n)$ and $K_{*}\left(\mathbf{F}_{q}[t] /\left(t^{2}\right)\right)$, Mem. Amer. Math. Soc. 57 (1985), no. 329, vi +200 , With an appendix by Christophe Soulé. MR 803974
[2] Vigleik Angeltveit, On the algebraic K-theory of Witt vectors of finite length, arXiv preprint arXiv:1101.1866 (2011).
[3] Benjamin Antieau, Achim Krause, and Thomas Nikolaus, On the $K$-theory of $\mathbf{Z} / p^{n}$, forthcoming.
[4] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus, On the Beilinson fiber square, arXiv preprint arXiv:2003.12541 (2020).
[5] Bhargav Bhatt and Jacob Lurie, Absolute prismatic cohomology, arXiv preprint arXiv:2201.06120 (2022).
[6] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, Topological Hochschild homology and integral p-adic Hodge theory, Publ. Math. Inst. Hautes Études Sci. 129 (2019), 199-310. MR 3949030
[7] Bhargav Bhatt and Peter Scholze, Prisms and prismatic cohomology, arXiv preprint arXiv:1905.08229 (2019).
[8] Morten Brun, Filtered topological cyclic homology and relative $K$-theory of nilpotent ideals, Algebr. Geom. Topol. 1 (2001), 201-230. MR 1823499
[9] W. Edwin Clark and Joseph J. Liang, Enumeration of finite commutative chain rings, J. Algebra 27 (1973), 445-453. MR 337910
[10] R. Keith Dennis and Michael R. Stein, $K_{2}$ of discrete valuation rings, Advances in Math. 18 (1975), no. 2, 182-238. MR 437620
[11] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy, The local structure of algebraic Ktheory, Algebra and Applications, vol. 18, SpringerVerlag London, Ltd., London, 2013. MR 3013261
[12] Ofer Gabber, K-theory of Henselian local rings and Henselian pairs, Algebraic $K$-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 59-70. MR 1156502
[13] S. M. Gersten, Problems about higher K-functors, Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), 1973, pp. 43-56. Lecture Notes in Math., Vol. 341. MR 0338125
[14] Christian Haesemeyer and Charles A. Weibel, The norm residue theorem in motivic cohomology, Annals of Mathematics Studies, vol. 200, Princeton University Press, Princeton, NJ, 2019. MR 3931681
[15] B. Harris and G. Segal, $K_{i}$ groups of rings of algebraic integers, Ann. of Math. (2) 101 (1975), 20-33. MR 387379
[16] Lars Hesselholt and Ib Madsen, Cyclic polytopes and the K-theory of truncated polynomial algebras, Invent. Math. 130 (1997), no. 1, 73-97. MR 1471886
[17] _ On the K-theory of finite algebras over Witt vectors of perfect fields, Topology 36 (1997), no. 1, 29-101. MR 1410465
[18] _, On the K-theory of local fields, Ann. of Math. (2) 158 (2003), no. 1, 1-113. MR 1998478
[19] John W. Jones and David P. Roberts, A database of local fields, J. Symbolic Comput. 41 (2006), no. 1, 80-97. MR 2194887
[20] Achim Krause and Thomas Nikolaus, Bökstedt periodicity and quotients of $D V R s$, arXiv preprint arXiv:1907.03477 (2019).
[21] Ruochuan Liu and Guozhen Wang, Topological cyclic homology of local fields, arXiv preprint arXiv:2012.15014 (2020).
[22] The LMFDB Collaboration, The L-functions and modular forms database, http://www.lmfdb.org, 2022, [Online; accessed 22 March 2022].
[23] Akhil Mathew, Some recent advances in topological Hochschild homology, arXiv preprint arXiv:2101.00668 (2021).
[24] Thomas Nikolaus and Peter Scholze, On topological cyclic homology, Acta Math. 221 (2018), no. 2, 203409. MR 3904731
[25] I.A. Panin, On a theorem of Hurewicz and K-theory of complete discrete valuation rings, Mathematics of the USSR-Izvestiya 29 (1987), no. 1, 119-131.
[26] Daniel Quillen, On the cohomology and $K$-theory of the general linear groups over a finite field, Ann. of Math. (2) 96 (1972), 552-586. MR 0315016
[27] _, Higher algebraic K-theory. I, Algebraic $K$ theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), 1973, pp. 85147. Lecture Notes in Math., Vol. 341. MR 0338129
[28] Martin Speirs, On the K-theory of truncated polynomial algebras, revisited, Adv. Math. 366 (2020), 107083, 18. MR 4070307
[29] Yuri JF Sulyma, Floor, ceiling, slopes, and K-theory, arXiv preprint arXiv:2110.04978 (2021).
[30] The Sage Developers, SageMath, the Sage Mathematics Software System (Version 9.4), 2021, https://www.sagemath.org.
[31] J. B. Wagoner, Continuous cohomology and p-adic K-theory, (1976), 241-248. Lecture Notes in Math., Vol. 551. MR 0498502
[32] Charles A. Weibel, The K-book, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013. MR 3076731

| 2.2 .2 .1 | $z^{2}+2 z+2$ |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\mathrm{~K}_{r} \backslash n$ | $\mathcal{O}_{K} / \varpi^{2}$ | $\mathcal{O}_{K} / \varpi^{3}$ | $\mathcal{O}_{K} / \varpi^{4}$ | $\mathcal{O}_{K} / \varpi^{5}$ | $\mathcal{O}_{K} / \varpi^{6}$ | $\mathcal{O}_{K} / \varpi^{7}$ | $\mathcal{O}_{K} / \varpi^{8}$ |
| $\mathrm{~K}_{1}$ | 1 | 2 | 1,2 | $1,1,2$ | $1,2,2$ | $2,2,2$ | $2,2,3$ |
| $\mathrm{~K}_{2}$ |  |  | 1 | 1 | 2 | 2 |  |
| $\mathrm{~K}_{3}$ | 1,1 | 1,3 | $1,2,4$ | $2,3,4$ | $1,3,3,5$ | $1,1,3,3,6$ | $1,1,3,4,7$ |
| $\mathrm{~K}_{4}$ |  |  |  |  | 1 | 1 | 2 |
| $\mathrm{~K}_{5}$ | $1,1,1$ | $1,1,4$ | $1,2,2,4$ | $2,2,2,6$ | $1,2,2,4,7$ | $1,1,2,2,4,9$ | $1,2,2,2,6,10$ |
| $\mathrm{~K}_{6}$ |  |  |  |  |  | 1 | 1 |
| $\mathrm{~K}_{7}$ | $1,1,1,1$ | $1,1,2,4$ | $1,1,1,2,3,4$ | $1,1,1,4,4,5$ | $1,1,1,2,5,5,5$ | $1,1,2,3,4,7,7$ | $1,1,1,2,3,4,8,9$ |


| 2.2 .2 .2 | $z^{2}+2 z-2$ | $\mathcal{O}_{K} / \varpi^{4}$ | $\mathcal{O}_{K} / \varpi^{5}$ | $\mathcal{O}_{K} / \varpi^{6}$ | $\mathcal{O}_{K} / \varpi^{7}$ | $\mathcal{O}_{K} / \varpi^{8}$ |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\mathrm{~K}_{r} \backslash n$ | $\mathcal{O}_{K} / \varpi^{2}$ | $\mathcal{O}_{K} / \varpi^{3}$ | 2 | 1,2 | $1,1,2$ | $1,2,2$ | $1,2,3$ |
| $\mathrm{~K}_{1}$ | 1 | 2 | 1 | 1 | 1 | $1,3,3$ |  |
| $\mathrm{~K}_{2}$ |  |  | $1,2,4$ | $2,3,4$ | $1,3,3,4$ | $1,1,3,3,5$ | 1 |
| $\mathrm{~K}_{3}$ | 1,1 | 1,3 |  |  | 1 | 1 | $1,1,3,4,6$ |
| $\mathrm{~K}_{4}$ |  |  |  |  | 2, |  |  |
| $\mathrm{~K}_{5}$ | $1,1,1$ | $1,1,4$ | $1,2,2,4$ | $2,2,2,6$ | $1,2,2,4,7$ | $1,1,2,2,4,9$ | $1,2,2,2,6,10$ |
| $\mathrm{~K}_{6}$ |  |  |  |  |  | 1 | 1 |
| $\mathrm{~K}_{7}$ | $1,1,1,1$ | $1,1,2,4$ | $1,1,1,2,3,4$ | $1,1,1,4,4,5$ | $1,1,1,2,5,5,5$ | $1,1,2,3,4,7,7$ | $1,1,1,2,3,4,8,9$ |


| 2.2 .3 .1 |  | $z^{2}+14$ |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\mathrm{~K}_{r} \backslash n$ | $\mathcal{O}_{K} / \varpi^{2}$ | $\mathcal{O}_{K} / \varpi^{3}$ | $\mathcal{O}_{K} / \varpi^{4}$ | $\mathcal{O}_{K} / \varpi^{5}$ | $\mathcal{O}_{K} / \varpi^{6}$ | $\mathcal{O}_{K} / \varpi^{7}$ | $\mathcal{O}_{K} / \varpi^{8}$ |
| $\mathrm{~K}_{1}$ | 1 | 2 | 1,2 | $1,1,2$ | $1,1,3$ | $1,2,3$ | $1,2,4$ |
| $\mathrm{~K}_{2}$ |  |  | 1 | 1 | 1 | 1 | 1 |
| $\mathrm{~K}_{3}$ | 1,1 | 1,3 | $1,2,4$ | $2,3,4$ | $1,3,3,4$ | $1,1,3,4,4$ | $1,2,4,4,4$ |
| $\mathrm{~K}_{4}$ |  |  |  |  | 1 | 1 | 2 |
| $\mathrm{~K}_{5}$ | $1,1,1$ | $1,1,4$ | $1,2,2,4$ | $2,2,2,6$ | $1,2,3,3,7$ | $1,1,2,3,3,9$ | $1,1,3,3,5,10$ |
| $\mathrm{~K}_{6}$ |  |  |  |  |  | 1 | 1 |
| $\mathrm{~K}_{7}$ | $1,1,1,1$ | $1,1,2,4$ | $1,1,2,2,2,4$ | $1,1,2,3,4,5$ | $1,1,2,2,4,5,5$ | $1,2,2,3,5,5,7$ | $1,1,2,2,3,5,6,9$ |


| 2.2.3.2 | $z^{2}+6$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{K}_{r} \backslash n$ | $\mathcal{O}_{K} / \varpi^{2}$ | $\mathcal{O}_{K} / \varpi^{3}$ | $\mathcal{O}_{K} / \varpi^{4}$ | $\mathcal{O}_{K} / \varpi^{5}$ | $\mathcal{O}_{K} / \varpi^{6}$ | $\mathcal{O}_{K} / \varpi^{7}$ | $\mathcal{O}_{K} / \varpi^{8}$ |
| $\mathrm{K}_{1}$ | 1 | 2 | 1,2 | 1,1,2 | 1, 1, 3 | 1,2,3 | 1,2,4 |
| $\mathrm{K}_{2}$ |  |  | 1 | 1 | 1 | 1 | 1 |
| $\mathrm{K}_{3}$ | 1,1 | 1,3 | 1,2,4 | 2, 3, 4 | 1,3,3,4 | 1, 1, 3, 4, 4 | 1,2,3,4,5 |
| $\mathrm{K}_{4}$ |  |  |  |  | 1 | 1 | 2 |
| $\mathrm{K}_{5}$ | 1, 1, 1 | 1, 1, 4 | 1,2,2,4 | 2, 2, 2, 6 | 1,2,3,3,7 | 1, 1, 2, 3, 3, 9 | 1, 1, 3, 3, 5, 10 |
| $\mathrm{K}_{6}$ |  |  |  |  |  | 1 | 1 |
| $\mathrm{K}_{7}$ | 1, 1, 1, 1 | 1, 1, 2, 4 | $1,1,2,2,2,4$ | $1,1,2,3,4,5$ | $1,1,2,2,4,5,5$ | $1,2,2,3,5,5,7$ | $1,1,2,2,3,5,6,9$ |


| 2.2 .3 .3 |  | $z^{2}+2$ |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\mathrm{~K}_{r} \backslash n$ | $\mathcal{O}_{K} / \varpi^{2}$ | $\mathcal{O}_{K} / \varpi^{3}$ | $\mathcal{O}_{K} / \varpi^{4}$ | $\mathcal{O}_{K} / \varpi^{5}$ | $\mathcal{O}_{K} / \varpi^{6}$ | $\mathcal{O}_{K} / \varpi^{7}$ | $\mathcal{O}_{K} / \varpi^{8}$ |
| $\mathrm{~K}_{1}$ | 1 | 2 | 1,2 | $1,1,2$ | $1,1,3$ | $1,2,3$ | $1,2,4$ |
| $\mathrm{~K}_{2}$ |  |  | 1 | 1 | 1 | 1 | 1 |
| $\mathrm{~K}_{3}$ | 1,1 | 1,3 | $1,2,4$ | $2,3,4$ | $1,3,3,4$ | $1,1,3,4,4$ | $1,2,3,4,5$ |
| $\mathrm{~K}_{4}$ |  |  |  |  | 1 | 1 | 2 |
| $\mathrm{~K}_{5}$ | $1,1,1$ | $1,1,4$ | $1,2,2,4$ | $2,2,2,6$ | $1,2,3,3,7$ | $1,1,2,3,3,9$ | $1,1,3,3,5,10$ |
| $\mathrm{~K}_{6}$ |  |  |  |  |  | 1 | 1 |
| $\mathrm{~K}_{7}$ | $1,1,1,1$ | $1,1,2,4$ | $1,1,2,2,2,4$ | $1,1,2,3,4,5$ | $1,1,2,2,4,5,5$ | $1,2,2,3,5,5,7$ | $1,1,2,2,3,5,6,9$ |


|  |  |  |  |  |  |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 2.2 .3 .4 | $z^{2}+10$ |  |  |  |  |  |  |
| $\mathrm{~K}_{r} \backslash n$ | $\mathcal{O}_{K} / \varpi^{2}$ | $\mathcal{O}_{K} / \varpi^{3}$ | $\mathcal{O}_{K} / \varpi^{4}$ | $\mathcal{O}_{K} / \varpi^{5}$ | $\mathcal{O}_{K} / \varpi^{6}$ | $\mathcal{O}_{K} / \varpi^{7}$ | $\mathcal{O}_{K} / \varpi^{8}$ |
| $\mathrm{~K}_{1}$ | 1 | 2 | 1,2 | $1,1,2$ | $1,1,3$ | $1,2,3$ | $1,2,4$ |
| $\mathrm{~K}_{2}$ |  |  | 1 | 1 | 1 | 1 | 1 |
| $\mathrm{~K}_{3}$ | 1,1 | 1,3 | $1,2,4$ | $2,3,4$ | $1,3,3,4$ | $1,1,3,4,4$ | $1,2,3,4,5$ |
| $\mathrm{~K}_{4}$ |  |  |  |  | 1 | 1 | 2 |
| $\mathrm{~K}_{5}$ | $1,1,1$ | $1,1,4$ | $1,2,2,4$ | $2,2,2,6$ | $1,2,3,3,7$ | $1,1,2,3,3,9$ | $1,1,3,3,5,10$ |
| $\mathrm{~K}_{6}$ |  |  |  |  |  | 1 | 1 |
| $\mathrm{~K}_{7}$ | $1,1,1,1$ | $1,1,2,4$ | $1,1,2,2,2,4$ | $1,1,2,3,4,5$ | $1,1,2,2,4,5,5$ | $1,2,2,3,5,5,7$ | $1,1,2,2,3,5,6,9$ |

Figure 3: The 2 -adic K-groups in syntomic weights $i=1,2,3,4$ for the totally ramified degree 2 extensions of $\mathbf{Z}_{2}$. The lmfdb [22] label is given in the top left corner together with an Eisenstein polynomial. The data gives the exponents of the elementary divisors in each degree: for example, the entry 1,3 in the $\mathrm{K}_{3}$ row of the $\mathcal{O}_{K} / \varpi^{3}$ column means that $\mathrm{K}_{3}\left(\mathcal{O}_{K} / \varpi^{3} ; \mathbf{Z}_{2}\right) \cong \mathbf{Z} / 2 \oplus \mathbf{Z} / 8$.

| $\mathrm{K}_{1}$ | $\mathbf{Z} / 3$ | $\mathrm{~K}_{19}$ | $(\mathbf{Z} / 3)^{3} \oplus \mathbf{Z} / 9 \oplus \mathbf{Z} / 243$ |
| ---: | ---: | :--- | ---: |
| $\mathrm{~K}_{2}$ | 0 | $\mathrm{~K}_{20}$ | $(\mathbf{Z} / 3)^{3} \oplus \mathbf{Z} / 9 \oplus \mathbf{Z} / 729$ |
| $\mathrm{~K}_{3}$ | $(\mathbf{Z} / 3)^{2}$ | $\mathrm{~K}_{21}$ | $(\mathbf{Z}$ |
| $\mathrm{K}_{4}$ | $\mathbf{Z} / 3$ | $\mathrm{~K}_{22}$ | 0 |
| $\mathrm{~K}_{5}$ | $\mathbf{Z} / 81$ | $\mathrm{~K}_{23}$ | $\mathbf{Z} / 3 \oplus \mathbf{Z} / 27 \oplus \mathbf{Z} / 6561$ |
| $\mathrm{~K}_{6}$ | 0 | $\mathrm{~K}_{24}$ | 0 |
| $\mathrm{~K}_{7}$ | $\mathbf{Z} / 3 \oplus \mathbf{Z} / 27$ | $\mathrm{~K}_{25}$ | $(\mathbf{Z} / 3)^{4} \oplus \mathbf{Z} / 9 \oplus \mathbf{Z} / 2187$ |
| $\mathrm{~K}_{8}$ | 0 | $\mathrm{~K}_{26}$ | 0 |
| $\mathrm{~K}_{9}$ | $\mathbf{Z} / 3 \oplus \mathbf{Z} / 81$ | $\mathrm{~K}_{27}$ | $(\mathbf{Z} / 3)^{4} \oplus \mathbf{Z} / 9 \oplus \mathbf{Z} / 6561$ |
| $\mathrm{~K}_{10}$ | 0 | $\mathrm{~K}_{28}$ | 0 |
| $\mathrm{~K}_{11}$ | $(\mathbf{Z} / 27)^{2}$ | $\mathrm{~K}_{29}$ | $\mathbf{Z} / 3 \oplus \mathbf{Z} / 9 \oplus \mathbf{Z} / 27 \oplus \mathbf{Z} / 19683$ |
| $\mathrm{~K}_{12}$ | 0 | $\mathrm{~K}_{30}$ | 0 |
| $\mathrm{~K}_{13}$ | $(\mathbf{Z} / 3)^{2} \oplus \mathbf{Z} / 243$ | $\mathrm{~K}_{31}$ | $(\mathbf{Z} / 3)^{4} \oplus(\mathbf{Z} / 9)^{2} \oplus \mathbf{Z} / 6561$ |
| $\mathrm{~K}_{14}$ | 0 | $\mathrm{~K}_{32}$ | 0 |
| $\mathrm{~K}_{15}$ | $(\mathbf{Z} / 3)^{2} \oplus \mathbf{Z} / 729$ | $\mathrm{~K}_{33}$ | $(\mathbf{Z} / 3)^{4} \oplus(\mathbf{Z} / 9)^{2} \oplus \mathbf{Z} / 19683$ |
| $\mathrm{~K}_{16}$ | 0 | $\mathrm{~K}_{34}$ | 0 |
| $\mathrm{~K}_{17}$ | $\mathbf{Z} / 9 \oplus \mathbf{Z} / 2187$ | $\mathrm{~K}_{35}$ | $(\mathbf{Z} / 9)^{2} \oplus \mathbf{Z} / 243 \oplus \mathbf{Z} / 19683$ |
| $\mathrm{~K}_{18}$ | 0 | $\mathrm{~K}_{36}$ | 0 |

Figure 4: The 3 -adic K-groups of $\mathbf{Z} / 9$ for syntomic weights $1 \leqslant i \leqslant 18$. The contribution of $K_{36}\left(\mathbf{Z} / 9 ; \mathbf{Z}_{3}\right)=0$ is a (null) group from weight 19.


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