On the K-theory of \mathbf{Z}/p^n – announcement

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Quillen introduced higher algebraic K-theory in [27] and computed the K-groups $K_*(\mathbf{F}_q)$ in [26]. Except in low degrees, the computation of the Kgroups of closely related rings, for example $\mathbf{Z}/4$, has remained out of reach. In this paper, we announce new methods for computations of K-groups of such rings and outline new results. A full account will be given in [3].

We are interested in rings of the form \mathcal{O}_K/ϖ^n where K is a finite extension of \mathbf{Q}_p of degree d, \mathcal{O}_K is its ring of integers, and ϖ^n is the *n*th power of a uniformizer ϖ . In particular, $p \in (\varpi^e)$ where e is the degree of ramification of K over \mathbf{Q}_p . When n = 1, \mathcal{O}_K/ϖ^n is the residue field $k = \mathbf{F}_q$ of \mathcal{O}_K , where $q = p^f$ for some f, called the residual degree of the extension.

The problem of computing the K-groups of such rings, and of finite rings in general, was raised by Swan in the Battelle proceedings [13, Prob. 20].

1 History

For any field k, $K_0(k) \cong \mathbb{Z}$ and $K_1(k) \cong k^{\times}$. Quillen showed in [26] that if \mathbf{F}_q is the finite field with $q = p^f$ elements, then for $r \ge 1$,

$$\mathbf{K}_r(\mathbf{F}_q) \cong \begin{cases} 0 & \text{if } r \text{ is even and} \\ \mathbf{Z}/(q^i - 1) & \text{if } r = 2i - 1. \end{cases}$$

Note in particular that there is no *p*-torsion in the K-groups of \mathbf{F}_{q} .

For each prime ℓ and ring R, $K(R, \mathbf{Z}_{\ell})$ denotes the ℓ -completion of the K-theory spectrum of R. In the main case of interest to us, namely when $R = \mathcal{O}_K/\varpi^n$, $K_r(R)$ is finitely generated torsion for r > 0 and $K_r(R, \mathbf{Z}_{\ell})$ is the subgroup of ℓ -primary torsion in $K_r(R)$.

Gabber's rigidity theorem [12] implies that if R is a commutative ring which is henselian with respect to an ideal I and if ℓ is invertible in R, then

$$\operatorname{K}(R; \mathbf{Z}_{\ell}) \simeq \operatorname{K}(R/I; \mathbf{Z}_{\ell}).$$

Quillen introduced higher algebraic K-theory Examples of such henselian pairs are the rings of inte-[27] and computed the K-groups $K_*(\mathbf{F}_q)$ in [26]. gers \mathcal{O}_K as above with the ideal (ϖ) or the quotients accept in low degrees, the computation of the K- \mathcal{O}/ϖ^n , again with the ideal (ϖ) . It follows that for oups of closely related rings, for example $\mathbf{Z}/4$, has $\ell \neq p$ we have

$$\mathrm{K}_*(\mathcal{O}; \mathbf{Z}_\ell) \cong \mathrm{K}_*(\mathcal{O}/\varpi^n; \mathbf{Z}_\ell) \cong \mathrm{K}_*(\mathbf{F}_q; \mathbf{Z}_\ell)$$

so that these ℓ -adic K-groups are all determined by Quillen's computation.

The situation of the *p*-adic K-theory of \mathcal{O}_K or \mathcal{O}_K/ϖ^n is very different. A result of Dundas–Goodwillie–McCarthy [11] implies that $K(\mathcal{O}/\varpi^n; \mathbf{Z}_p) \simeq \tau_{\geq 0} TC(\mathcal{O}/\varpi^n; \mathbf{Z}_p)$, while work of Hesselholt–Madsen [17] and of Panin [25] implies that $K(\mathcal{O}_K; \mathbf{Z}_p) \simeq \tau_{\geq 0} TC(\mathcal{O}_K; \mathbf{Z}_p)$. Here, $TC(\mathcal{O}_K; \mathbf{Z}_p)$ and $TC(\mathcal{O}_K/\varpi^n; \mathbf{Z}_p)$ denote the *p*-adic topological cyclic homology spectra of \mathcal{O}_K and \mathcal{O}_K/ϖ^n , respectively. This theory is built from topological Hochschild homology and is closely connected to *p*adic cohomology theories thanks to the work of [6]. These results make the *p*-adic K-groups amenable to calculation using so-called trace methods.

Hesselholt and Madsen determine the structure of $\mathrm{TC}_*(\mathcal{O}_K; \mathbf{Z}_p) \cong \mathrm{K}_*(\mathcal{O}_K; \mathbf{Z}_p)$ in [18] and thereby verify the Quillen–Lichtenbaum conjecture for \mathcal{O}_K . This conjecture now follows in general from the proof of the Bloch–Kato conjecture due to Rost and Voevodsky; see for example [14], although the *p*-adic ranks of the groups $\mathrm{K}_*(\mathcal{O}_K; \mathbf{Z}_p)$ had previously been computed by Wagoner [31].

The Hesselholt–Madsen approach uses logarithmic de Rham–Witt forms and TR, i.e., the classical approach to trace method computations. These have recently been revisited by Liu–Wang [21] who describe $K_*(\mathcal{O}_K; \mathbf{F}_p)$, the K-groups with mod p coefficients, using new cyclotomic techniques from [6, 24].

The result is that

$$K_r(\mathcal{O}_K; \mathbf{Z}_p) \cong \begin{cases} \mathbf{Z}_p & \text{if } r = 0, \\ H^1_{\text{\acute{e}t}}(\operatorname{Spec} K, \mathbf{Z}_p(i)) & \text{if } r = 2i - 1, \text{ and} \\ H^2_{\text{\acute{e}t}}(\operatorname{Spec} K, \mathbf{Z}_p(i)) & \text{if } r = 2i - 2, \end{cases}$$

where $\mathbf{Z}_{p}(i)$ is the *i*th Tate twist. These cohomology

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groups are determined by Iwasawa theory: for i > 0, 2

$$\begin{aligned} \mathrm{H}^{1}_{\mathrm{\acute{e}t}}(\mathrm{Spec}\,K,\mathbf{Z}_{p}(i)) &\cong \mathbf{Z}_{p}^{d} \oplus \mathbf{Z}/w_{i}, \\ \mathrm{H}^{2}_{\mathrm{\acute{e}t}}(\mathrm{Spec}\,K,\mathbf{Z}_{p}(i)) &\cong \mathbf{Z}/w_{i-1}, \end{aligned}$$

where d is the degree of K over \mathbf{Q}_p and where w_i is the largest pth power p^{ν} such that the exponent of the cyclotomic Galois group $\operatorname{Gal}(K(\mu_{p^{\nu}})/K)$ divides *i*. The number w_i is the *p*-part of a number introduced by Harris–Segal [15], Quillen, and Lichtenbaum in the setting of the Quillen–Lichtenbaum conjecture. See Weibel's book [32, Chap. VI] for more details.

Much less is known about the K-theory of the intermediate rings \mathcal{O}_K/ϖ^n for $1 < n < \infty$. As for fields, $\mathrm{K}_0(\mathcal{O}_K/\varpi^n) \cong \mathbb{Z}$ and $\mathrm{K}_1(\mathcal{O}_K/\varpi^n)$ is isomorphic to the group of units in \mathcal{O}_K/ϖ^n .

In [10], Dennis and Stein determined the structure of $K_2(\mathcal{O}_K/\varpi^n)$. No other work we are aware of has addressed the K-groups of general rings of the form \mathcal{O}_K/ϖ^n .

In special situations, more is known. First, every ring $\mathbf{F}_q[z]/z^n$ is of the form \mathcal{O}_K/ϖ^n for K of ramification degree at least n. The algebraic K-groups of these truncated polynomial rings have been studied by Hesselholt–Madsen in [16] using classical trace method techniques, by Speirs in [28] using the new approach to TC due to Nikolaus–Scholze [24], and by Sulyma in [29] using the approach to TC via syntomic cohomology due to Bhatt–Morrow–Scholze [6] and as outlined by Mathew in [23].

Second, for unramified extension there are some results in low degrees. In the unramified case, where e = 1, \mathcal{O}_K is the ring $W(\mathbf{F}_q)$ of *p*-typical Witt vectors of the residue field. Brun [8] determined the K-groups of \mathbf{Z}/p^n (i.e., when e = 1 and f = 1) up to degree p - 3 and Angeltveit [2] determined the K-groups of $W_n(\mathbf{F}_q) = W(\mathbf{F}_q)/\varpi^n = W(\mathbf{F}_q)/p^n$ up to degree 2p - 2.

Angeltveit also proved an important quantitative result:

$$\frac{\#\mathrm{K}_{2i-1}(W_n(\mathbf{F}_q);\mathbf{Z}_p)}{\#\mathrm{K}_{2i-2}(W_n(\mathbf{F}_q);\mathbf{Z}_p)} = q^{i(n-1)}.$$

Both Brun and Angeltveit use classical trace methods and the *p*-adic filtration on the truncated Witt vectors to translate part of the problem to the cases of truncated polynomial rings where a complete answer is known.

The cases of K₃ of \mathbf{Z}/p^n or $\mathbf{F}_q[z]/z^2$ were also considered earlier in [1] using group homology calculations.

2 New results

As $K(\mathcal{O}/\varpi^k; \mathbf{Z}_p) \simeq \tau_{\geq 0} TC(\mathcal{O}/\varpi^k; \mathbf{Z}_p)$ by [11, 18], it is enough to determine TC of these rings. To do so, we use the filtration on TC constructed by Bhatt–Morrow–Scholze in [6]. If R is a quasisyntomic ring, there is a complete decreasing filtration $F_{syn}^{sxn} TC(R; \mathbf{Z}_p)$ with associated graded pieces

$$\mathbf{F}_{\mathrm{syn}}^{=i} \mathrm{TC}(R; \mathbf{Z}_p) \simeq \mathbf{Z}_p(i)(R)[2i],$$

where $\mathbf{Z}_p(i)(R)$ is the weight *i* syntomic cohomology of *R* introduced in [6]. The syntomic complexes provide a *p*-adic analogue of the motivic filtration on K-theory.

As shown in [4], the weight *i* syntomic cohomology $\mathbf{Z}_p(i)(R)$ is concentrated in [0, i + 1], independent of R; this means that $\mathrm{H}^r(\mathbf{Z}_p(i)(R)) = 0$ for $r \notin [0, i + 1]$. In the special case of \mathcal{O}_K or \mathcal{O}_K/ϖ^n , an argument using the ϖ -adic associated graded implies that in fact the weight *i* syntomic cohomology is in [0, 2]; moreover, for $i \ge 1$, $\mathrm{H}^0(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)) = 0$ so the complex has cohomology concentrated in degrees 1 and 2.

One checks that $\mathrm{H}^2(\mathbf{Z}_p(1)(\mathcal{O}_K/\varpi^n)) = 0$, so the spectral sequence associated to the syntomic filtration on TC collapses at the E₁-page for \mathcal{O}/ϖ^n (or the E₂-page in the reindexing in [6, Thm. 1.12]). Hence,

$$\operatorname{TC}_{2i-1}(\mathcal{O}_K/\varpi^n; \mathbf{Z}_p) \cong \operatorname{H}^1(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n))$$

for $i \ge 1$ and

$$\operatorname{TC}_{2i-2}(\mathcal{O}_K/\varpi^n; \mathbf{Z}_p) \cong \operatorname{H}^2(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n))$$

for $i \ge 2$. Thus, it makes sense to speak of the syntomic weights of the K-groups of \mathcal{O}_K/ϖ^n .

Theorem 2.1. For $i \ge 1$, if the residue field of \mathcal{O}_K has $q = p^f$ elements, then there is an explicit cochain complex

$$\left(\mathbf{Z}_p^{f(in-1)} \xrightarrow{\operatorname{syn}_0} \mathbf{Z}_p^{2f(in-1)} \xrightarrow{\operatorname{syn}_1} \mathbf{Z}_p^{f(in-1)}\right)$$

quasi-isomorphic to $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)$. The terms are free \mathbf{Z}_p -modules of the given ranks in cohomological degrees 0, 1, and 2.

The proof of the existence of this explicit cochain complex model of the syntomic complex will be discussed in Sections 4 and 5.

The groups $K_*(\mathcal{O}_K/\varpi^n)$ are torsion for * > 0. In particular, the complex above is exact rationally. Thus, to find the cohomology of $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)$, and hence the *p*-adic K-groups of \mathcal{O}_K/ϖ^n , it is enough to compute the matrices syn_0 and syn_1 and their elementary divisors. **Theorem 2.2.** The matrices syn_0 and syn_1 are effectively computable. Specifically, they can be determined with enough p-adic precision to guarantee computability of the effective divisors.

We have implemented our algorithm in SAGE [30] in the case where f = 1, i.e., when the residue field is \mathbf{F}_p . Future work will include an implementation for general f.

Corollary 2.3. There is an algorithm to determine the structure of $K_r(\mathcal{O}_K/\varpi^n)$ for any K, n, and r.

Along the way, we extend the result of Angeltveit on the quotients of the orders from the unramified case to any \mathcal{O}_K/ϖ^n .

Corollary 2.4. For any \mathcal{O}_K/ϖ^n ,

$$\frac{\#\mathbf{K}_{2i-1}(\mathbb{O}_K/\varpi^n; \mathbf{Z}_p)}{\#\mathbf{K}_{2i-2}(\mathbb{O}_K/\varpi^n; \mathbf{Z}_p)} = q^{i(n-1)},$$

where $q = p^f$ is the order of the residue field of \mathcal{O}_K .

This corollary is especially powerful thanks to the following theorem.

Theorem 2.5 (Even vanishing theorem). If

$$i \geqslant \frac{p^2}{(p-1)^2} \left(p^{\lceil \frac{n}{e} \rceil} - 1 \right),$$

then $\mathrm{H}^2(\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)) = 0$ and hence

$$\mathbf{K}_{2i-2}(\mathfrak{O}_K/\varpi^n) = 0$$

if additionally $i \ge 2$.

Corollary 2.6. If

$$i \geqslant \frac{p^2}{(p-1)^2} \left(p^{\lceil \frac{n}{e} \rceil} - 1 \right),$$

then $\# \mathbf{K}_{2i-1}(\mathcal{O}_K/\varpi^k) = q^{i(n-1)} \cdot (q^i - 1).$

Corollary 2.7. There is an algorithm to compute the orders of all of the K-groups of $0/\varpi^n$.

Indeed, Theorem 2.5 and Corollary 2.6 reduce the problem to the computation of the cohomology of the syntomic complexes $\mathbf{Z}_p(i)(\mathfrak{O}/\varpi^n)$ for finitely many *i*: those satisfying

$$i < \tfrac{p^2}{(p-1)^2} \left(p^{\lceil \frac{n}{e} \rceil} - 1 \right).$$

This number grows rather quickly, but improvements are possible and will be described in our forthcoming work [3].

3 Computations

We present here four example calculations.

3.1 Z/4

The even vanishing theorem holds in syntomic weights $i \ge 12$. In fact, machine computations show in this case that $K_{2i-2}(\mathbb{Z}/4) = 0$ for all $i \ge 3$, while $K_2(\mathbb{Z}/4) \cong \mathbb{Z}/2$. Corollary 2.4 together with Quillen's calculation implies that

$$\#\mathbf{K}_3(\mathbf{Z}/4) = 8 \cdot (2^2 - 1) \text{ and } \#\mathbf{K}_{2i-1}(\mathbf{Z}/4) = 2^i \cdot (2^i - 1)$$

for $i \ge 3$. This gives the complete calculation of the orders of all K-groups of $\mathbb{Z}/4$.

The precise structure of the decomposition of pprimary part of the K-groups into cyclic groups remains unknown to us. Figure 1 displays a table of the output of our machine computations giving the groups in syntomic weights $i \leq 16$.

3.2 Chain rings of order 8

A chain ring is a commutative ring whose ideals are totally ordered with respect to inclusion. Examples include valuation rings or quotients of valuation rings. Every finite chain ring is of the form \mathcal{O}_K/ϖ^n for some $1 \leq n < \infty$. There are four chain rings of order 8, namely $\mathbf{Z}/8$, $\mathbf{Z}[2^{1/2}]/2^{3/2}$ (so n = 3 in our notation), $\mathbf{F}_2[z]/z^3$, and \mathbf{F}_8 ; see [9]. The 2-adic K-groups $K_n(\mathbf{F}_8; \mathbf{Z}_2)$ vanish for $n \geq 1$. Figure 2 displays the low-degree 2-adic K-groups of the other three chain rings of order 8.

3.3 Quotients of degree 2 totally ramified 2-adic fields

The lmfdb [22] provides tables of p-adic fields based on work of Jones–Roberts [19]. There are 6 totally ramified degree 2 extensions of \mathbf{Q}_2 . In Figure 3, we give low-degree p-adic K-groups of the quotients of these fields.

3.4 Z/9

The even vanishing theorem holds in syntomic weights $i \ge 18$. Figure 4 displays a table of the output of our machine computations in syntomic weights $i \le 18$. In particular, $K_4(\mathbb{Z}/9) \cong \mathbb{Z}/3$ and all other positive even K-groups vanish. In odd degrees,

$$\# K_5(\mathbf{Z}/9) = 81 \cdot (3^3 - 1) \text{ and } \# K_{2i-1}(\mathbf{Z}/9) = 3^i \cdot (3^i - 1)$$

for $i \ge 1$, $i \ne 3$. This gives the complete calculation of the orders of all K-groups of $\mathbb{Z}/9$.

K_1	$\mathbf{Z}/2$	K ₁₇	$(\mathbf{Z}/2)^3 \oplus (\mathbf{Z}/8)^2$
K_2	$\mathbf{Z}/2$	K_{18}	0
K_3	$\mathbf{Z}/8$	K_{19}	$\mathbf{Z}/4 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/32$
K_4	0	K_{20}	0
\mathbf{K}_{5}	$\mathbf{Z}/8$	K_{21}	$(\mathbf{Z}/2)^2 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/32$
K_6	0	K_{22}	0
\mathbf{K}_7	$\mathbf{Z}/2 \oplus \mathbf{Z}/8$	K_{23}	$(\mathbf{Z}/2)^4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/64$
\mathbf{K}_{8}	0	K_{24}	0
\mathbf{K}_{9}	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/8$	K_{25}	$(\mathbf{Z}/2)^4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/16$
K_{10}	0	K_{26}	0
K_{11}	$\mathbf{Z}/2 \oplus \mathbf{Z}/32$	K_{27}	$\mathbf{Z}/2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/16 \oplus \mathbf{Z}/128$
K_{12}	0	K_{28}	0
K_{13}	$\mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/16$	K_{29}	$(\mathbf{Z}/2)^3 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/32$
K_{14}	0	K_{30}	0
K_{15}	$(\mathbf{Z}/2)^3 \oplus \mathbf{Z}/32$	K_{31}	$(\mathbf{Z}/2)^6 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/128$
K_{16}	0	K_{32}	0

Figure 1: The 2-adic K-groups of $\mathbf{Z}/4$ for syntomic weights $1 \leq i \leq 16$; the final zero, $K_{32}(\mathbf{Z}/4;\mathbf{Z}_2) = 0$, is a (null) contribution from syntomic weight 17.

\mathbf{K}_r	$\mathbf{Z}/8$	${f F}_2[z]/z^3$	$\mathbf{Z}_{2}[2^{1/2}]/2^{3/2}$
K_1	$\mathbf{Z}/4$	$\mathbf{Z}/4$	$\mathbf{Z}/4$
K_2	$\mathbf{Z}/2$	0	0
K_3	$\mathbf{Z}/4 \oplus \mathbf{Z}/8$	$\mathbf{Z}/2\oplus\mathbf{Z}/8$	$\mathbf{Z}/2\oplus\mathbf{Z}/8$
K_4	$\mathbf{Z}/2$	0	0
K_5	$\mathbf{Z}/2 \oplus \mathbf{Z}/64$	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/16$	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/16$
K_6	0	0	0
K_7	$(\mathbf{Z}/4)^2$	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/16$	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/16$
K_8	0	0	0
K_9	$\mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/128$	$(\mathbf{Z}/2)^2 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/16$	$(\mathbf{Z}/2)^2 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/16$
K_{10}	0	0	0
K ₁₁	$\mathbf{Z}/8 \oplus \mathbf{Z}/512$	$(\mathbf{Z}/2)^3 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/32$	$(\mathbf{Z}/2)^3 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/32$
K_{12}	0	0	0
K_{13}	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/512$	$(\mathbf{Z}/2)^4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/32$	$(\mathbf{Z}/2)^4 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/32$
K ₁₄	0	0	0
$\overline{\mathrm{K}_{15}}$	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/64 \oplus \mathbf{Z}/256$	$(\mathbf{Z}/2)^4 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/32$	$(\mathbf{Z}/2)^4 \oplus (\mathbf{Z}/4)^2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/32$

Figure 2: The 2-adic K-groups of the displayed chain rings of order 8 for syntomic weights $1 \le i \le 8$. Note that the second and third columns agree. We do not know at present if this continues in all higher weights. The second column agrees with the calculations of [16] (see for example [28, Lem. 2]).

4 rings

Our proofs are motivated by previous work of Krause-Nikolaus [20] and the approach of Liu–Wang [21] There are two main new ideas: the notion of prismatic cohomology relative to a δ -ring and the systematic use of the filtration on the syntomic complexes induced by the ϖ -adic filtration on \mathcal{O}_K/ϖ^n . Similar filtrations have also been used by Angeltveit [2] and Brun [8] in the topological context.

Let $A^0 = W(\mathbf{F}_q)[\![z]\!]$ be the δ -ring with $\delta(z) = 0$

Prismatic cohomology over δ - and hence $\varphi(z) = z^p$. If E(z) is an Eisenstein polynomial for \mathcal{O}_K , then the pair $(A^0, (E(z)))$ is a prism. Bhatt and Scholze show that $\mathbb{A}_{(\mathcal{O}_K/\varpi^n)/A^0}$ is discrete and admits a description as a prismatic envelope $A^0\left\{\frac{\varpi^n}{E(z)}\right\}^{\wedge}$ in the sense of [7, Prop. 3.13]; the prismatic envelope is an explicit pushout in (p, E(z))complete δ -rings over A^0 .

> The main idea is to determine the syntomic complexes $\mathbf{Z}_p(i)(\mathcal{O}/\varpi^n)$ by descent along the map $\mathbb{A}_{\mathcal{O}/\varpi^n} \to \mathbb{A}_{(\mathcal{O}/\varpi^n)/A^0}$ from absolute prismatic cohomology to relative prismatic cohomology. To make sense of this, we introduce prismatic cohomology rel

ative to a δ -ring. Let us outline the definition.

Given an arbitrary derived *p*-complete δ -ring *A* and a derived *p*-complete *A*-algebra *R*, let X = Spf Rand let $(X/A)_{\mathbb{A}}$ be the opposite of the category of commutative diagrams



where (B, J) is a bounded prism and $A \to B$ is a map of δ -rings.

By definition, $\mathbb{A}_{R/A} = \mathrm{R}\Gamma((X/A)_{\mathbb{A}}, \mathbb{O}_{\mathbb{A}})$, where $\mathbb{O}_{\mathbb{A}}$ is the prismatic structure sheaf, which sends a commutative diagram as above to B. Warning: this site-theoretic definition should be derived in general, but gives the correct answer under additional assumptions on R, in particular in the case of $R = \mathcal{O}_K/\varpi^n$ over the multivariable Breuil–Kisin prisms appearing in this paper.

Example 4.1. If $A = \mathbf{Z}_p$ is the initial (derived *p*-complete) δ -ring, then Δ_{R/\mathbf{Z}_p} recovers absolute prismatic cohomology as introduced in [6, 7] and studied further in [5]. More generally, this is true if A is replaced by the ring of *p*-typical Witt vectors of any perfect \mathbf{F}_p -algebra.

Example 4.2. If (A, I) is a prism and R is an A/I-algebra, then $\Delta_{R/A}$ agrees with derived relative prismatic cohomology as studied in [7].

Now, consider the augmented cosimplicial diagram A^{\bullet} where $A^{-1} = W(\mathbf{F}_q), A^0 = W(\mathbf{F}_q)[\![z]\!]$, and $A^s = W(\mathbf{F}_q)[\![z_0, \ldots, z_s]\!]$. This is a completed descent complex for $W(\mathbf{F}_q) \to W(\mathbf{F}_q)[z]$.

In the cosimplicial diagram

$$W(\mathbf{F}_q) \longrightarrow A^0 \rightleftharpoons A^1 \rightleftharpoons A^2 \cdots,$$

the arrows are all δ -ring maps and the entire diagram admits a map to \mathcal{O}_K sending each generator z_j to ϖ . As a result, for any \mathcal{O}_K -algebra R, there is an induced augmented cosimplicial diagram in prismatic cohomology of R relative to the δ -rings A^{\bullet} .

Theorem 4.3. The augmented cosimplicial diagram

$$\mathbb{A}_R \longrightarrow \mathbb{A}_{R/A^0} \longleftrightarrow \mathbb{A}_{R/A^1} \longleftrightarrow \mathbb{A}_{R/A^2} \cdots$$

is a limit diagram for $R = \mathcal{O}_K / \varpi^n$.

Thus, the absolute prismatic cohomology of an \mathcal{O}_K -algebra, such as \mathcal{O}_K/ϖ^n , can be computed by descent using the cosimplicial diagram above.

This does not make sense when speaking of prismatic cohomology as defined in [7] because there is no compatible way to equip the entire cosimplicial diagram with the structure of a cosimplicial prism. For example, if E(z) is an Eisenstein polynomial making $A^0 = W(\mathbf{F}_q)[[z]]$ into a prism, both $E(z_0)$ and $E(z_1)$ are distinguished elements in $A^1 = W(\mathbf{F}_q)[[z_0, z_1]]$ making it into a prism in two different ways.

Proposition 4.4. For any $s \ge 0$, the relative prismatic cohomology $\mathbb{A}_{(\mathcal{O}_K/\varpi^n)/A^s}$ is discrete and is isomorphic to a prismatic envelope

$$A^{s}\left\{\frac{z_{0}^{n}}{E(z_{0})}, \frac{z_{1}-z_{0}}{E(z_{0})}, \dots, \frac{z_{n}-z_{0}}{E(z_{0})}\right\}^{\wedge}$$

The proposition follows immediately from Example 4.2. Note that while prismatic cohomology relative to δ -rings is functorial in arbitrary maps of δ rings, the presentation of a given term $\&_{R/A^s}$ as a prismatic envelope depends on the choice of a prism structure J on A^s making R into an A^s/J -algebra. In the theorem above, we choose to make A^s into a prism with respect to the ideal $(E(z_0))$.

It follows that the cosimplicial diagram appearing in Theorem 4.3 gives a resolution of $\mathbb{A}_{\mathcal{O}_K/\varpi^n}$ as the limit of a cosimplicial diagram of discrete δ -rings.

To give the main idea of the rest of the argument, we illustrate it here for prismatic cohomology instead of the syntomic complexes. The absolute prismatic cohomology of a quasisyntomic ring R admits a Nygaard filtration $\mathbb{N}^{\geq \star} \Delta_R$; Nygaard completion of prismatic cohomology is written $\widehat{\Delta}_R$.

Proposition 4.5. The Nygaard-complete absolute prismatic cohomology groups $H^r(\widehat{\mathbb{A}}_{\mathcal{O}_K/\varpi^n})$ vanish for $r \neq 0, 1$.

The proposition can be proved by computing directly with a Nygaard-complete, Frobenius-twisted variant of the cosimplicial diagram in Theorem 4.3 using the prismatic envelopes of Proposition 4.4. Alternatively, one can argue as follows: the ϖ -adic filtration on \mathcal{O}_K/ϖ^n induces a filtration on $\mathbb{A}_{\mathcal{O}_K/\varpi^n}$ whose completion agrees with $\widehat{\mathbb{A}}_{\mathcal{O}_K/\varpi^n}$, and whose associated graded is the same as that of the corresponding filtration on $\widehat{\mathbb{A}}_{\mathbf{F}_q[z]/z^n}$. This associated graded can be described using crystalline cohomology and vanishes away from cohomological degrees 0, 1. Thus, by dévissage and completeness, the same vanishing holds for $\widehat{\mathbb{A}}_{\mathcal{O}_K/\varpi^n}$.

It follows from the proposition that the cochain complex $A^0 \to A^1 \to A^2 \to \cdots$ associated to the cosimplicial abelian group $\widehat{\mathbb{A}}_{(\mathcal{O}_K/\varpi^n)/A^{\bullet}}$ is exact in degrees ≥ 2 . This reduces the computation of $\widehat{\mathbb{A}}_{\mathcal{O}_K/\varpi^n}$ to a much smaller computation involving prismatic envelopes of \mathcal{O}_K/ϖ^n relative to A^0 , A^1 , and A^2 .

However, we are interested not in the absolute prismatic cohomology of \mathcal{O}_K/ϖ^n but rather in its syntomic cohomology. Relative syntomic cohomology is defined in the setting of prismatic cohomology relative to a δ -ring. We first have to explain the Nygaard filtration and the Breuil–Kisin twist, following [7, 5].

The Frobenius twist $\mathbb{A}_{R/A}^{(1)}$ is defined to be $\mathbb{A}_{R/A} \otimes_A \varphi^A$, the base-change of $\mathbb{A}_{R/A}$ along the Frobenius map on A. The Frobenius twist admits a map $\mathbb{A}_{R/A}^{(1)} \to \mathbb{A}_{R/A}$ and the Nygaard filtration $\mathcal{N}^{\geq \star} \mathbb{A}_{R/A}^{(1)}$ is a filtration which is taken by this map to the Iadic filtration on $\mathbb{A}_{R/A}$. If $\mathbb{A}_{R/A}$ is discrete (as in our examples of interest) then the Nygaard filtration is simply the preimage of the I-adic filtration.

Given a prism (A, I), let I_r be the invertible Amodule $I \cdot \varphi(I) \cdots \varphi^{r-1}(I)$. If (A, I) is transversal, meaning that A/I is *p*-torsion-free, then the canonical map $I_r/I_r^2 \to I_{r-1}/I_{r-1}^2$ is divisible by *p* and the induced map $I_r/I_r^2 \xrightarrow{1/p} I_{r-1}/I_{r-1}^2$ is surjective. The Breuil–Kisin twist is defined to be

$$A\{1\} = \lim \left(\dots \to I_3/I_3^2 \xrightarrow{1/p} I_2/I_2^2 \xrightarrow{1/p} I/I^2 \right).$$

This is an invertible A-module. For a general A-module M, let $M\{1\} = M \otimes_A A\{1\}$.

The relative syntomic cohomology of R over a δ -ring A is

$$\mathbf{Z}_p(i)(R/A) = \operatorname{fib}\left(\mathbb{N}^{\geqslant i}\mathbb{A}_{R/A}^{(1)}\{i\} \xrightarrow{\operatorname{can}-\varphi} \mathbb{A}_{R/A}^{(1)}\{i\}\right),$$

where φ is a Frobenius which exists on $\mathbb{N}^{\geq i} \mathbb{A}_{R/A}^{(1)}\{i\}$. Note that in [6], the syntomic complexes are defined using Nygaard complete prismatic cohomology; however, the two definitions agree by [6, Lem. 7.22] or [4, Cor. 5.31].

It follows along the lines of Theorem 4.3 that, for each $i \ge 0$, the limit of the cosimplicial diagram

$$\mathbf{Z}_p(i)(R/A^0) \Longrightarrow \mathbf{Z}_p(i)(R/A^1) \Longrightarrow \cdots$$

is equivalent to $\mathbf{Z}_p(i)(R)$ when $R = \mathcal{O}_K/\varpi^k$.

The fact that the Nygaard-complete absolute prismatic cohomology $\widehat{\mathbb{A}}_{\mathcal{O}_K/\varpi^n}$ is concentrated in cohomological degrees 0, 1 implies that $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)$ is concentrated in cohomological degrees 0, 1, 2. In fact, it is not hard to show that, for $i \ge 1$, each relative syntomic complex $\mathbf{Z}_p(i)((\mathcal{O}_K/\varpi^n)/A^s)$ is concentrated in cohomological degree 1. Thus, the spectral sequence associated to the limit diagram

$$\mathbf{Z}_p(i)(\mathfrak{O}_K/\varpi^n) \simeq \lim_{\Delta} \mathbf{Z}_p(i)((\mathfrak{O}_K/\varpi^n)/A^{\bullet})$$

implies that $\mathbf{Z}_p(i)(\mathfrak{O}_K/\varpi^n)$ is concentrated in cohomological degrees 1, 2 for $i \ge 1$.

By the same spectral sequence, to determine $\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)$, and hence $K_{2i-2}(\mathcal{O}_K/\varpi^n; \mathbf{Z}_p)$ and $K_{2i-1}(\mathcal{O}_K/\varpi^n; \mathbf{Z}_p)$, it is enough to compute the co-homology of the complex

$$H^{1}(\mathbf{Z}_{p}(i)(R/A^{0}))$$

 $\rightarrow \ker \left(H^{1}(\mathbf{Z}_{p}(i)(R/A^{1})) \rightarrow H^{1}(\mathbf{Z}_{p}(i)(R/A^{2})) \right)$

where $R = \mathcal{O}_K / \varpi^n$. In the next section, we explain how to use the ϖ -adic filtration to reduce this to a finite problem.

5 The syntomic matrices

In the cosimplicial diagram A^{\bullet} , each term is a filtered δ -ring, where in $A^s = W(k)[[z_0, \ldots, z_s]]$ the weight of z_j is 1. A filtered δ -ring is a δ -ring Awith a complete and separated decreasing filtration $\mathcal{F}^{\geq \star}A$ such that $\delta(\mathcal{F}^{\geq i}A) \subseteq \mathcal{F}^{\geq pi}A$. Since each $A^{\bullet} \to \mathcal{O}_K/\varpi^n$ is a filtered map where \mathcal{O}_K/ϖ^n is given the ϖ -adic filtration, all resulting invariants, such as prismatic or syntomic cohomology complexes admit induced filtrations, which we will write for instance as $\mathcal{F}^{\geq \star}\mathbf{Z}_n(i)((\mathcal{O}_K/\varpi^n)/A^{\bullet}).$

Theorem 5.1. For $b \ge in - 1$ and $i \ge 1$, the natural maps

$$\begin{array}{c} \mathcal{F}^{[1,b]}\mathbf{Z}_p(i)(\mathfrak{O}_K/\varpi^n) \\ & \downarrow \\ \mathbf{Z}_p(i)(\mathfrak{O}_K/\varpi^n) \longrightarrow \mathcal{F}^{[0,b]}\mathbf{Z}_p(i)(\mathfrak{O}_K/\varpi^n) \end{array}$$

are equivalences.

The right-hand arrow is easy to handle because $\mathcal{F}^{=0}\mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n) \simeq \mathbf{Z}_p(i)(\mathbf{F}_q) \simeq 0$ for i > 0. For the left-hand arrow, we argue by an explicit study of the interaction between the \mathcal{F} -filtration and the Ny-gaard filtration on each $\mathbb{A}_{(\mathcal{O}_K/\varpi^n)/A^{\bullet}}$.

The entire problem has now been reduced to a finite computation. Set $R = \mathcal{O}_K / \varpi^n$ and consider the commutative diagram

All four terms are finitely generated free \mathbb{Z}_p -modules. The vertical fibers are $\mathbb{Z}_p(i)(R/A^0)$ and $\mathbb{Z}_p(i)(R/A^1)$, respectively. Our approach to the computation avoids the more traditional approach of computing either $\operatorname{TR}(\mathcal{O}_K/\varpi^n)^{F=1}$ or computing $\operatorname{TC}(\mathcal{O}_K/\varpi^n)$ as the fiber of $\operatorname{TC}^-(\mathcal{O}_K/\varpi^n) \xrightarrow{\operatorname{can}-\varpi} \operatorname{TP}(\mathcal{O}_K/\varpi^n)$. It would nevertheless be very interesting to understand $\operatorname{TP}(\mathcal{O}_K/\varpi^n)$.

Since the complexes $\mathcal{F}^{[1,b]} \mathbb{N}^{\geq i} \mathbb{A}_R\{i\}$ and $\mathcal{F}^{[1,b]} \mathbb{A}_R\{i\}$ are torsion for $i \geq 1$ by another use of the ϖ -adic filtration, one can replace

$$\ker \left(\mathcal{F}^{[1,b]} \mathcal{N}^{\geqslant i} \mathbb{A}_{R/A^1}^{(1)} \{i\} \to \mathcal{F}^{[1,b]} \mathcal{N}^{\geqslant i} \mathbb{A}_{R/A^2}^{(1)} \{i\} \right)$$

with the saturation of the image of the top horizontal map, where by saturation we mean the sub- \mathbb{Z}_p -module consisting of elements x such that $p^N x$ is in the image for some N, and similarly for ker $\left(\mathcal{F}^{[1,b]} \mathbb{A}_{R/A^1}^{(1)}\{i\} \to \mathcal{F}^{[1,b]} \mathbb{A}_{R/A^2}^{(1)}\{i\}\right)$. Write S^0 and S^1 for the saturations. The resulting commutative square

consists of free \mathbf{Z}_p -modules of rank bf and the total cohomology computes $\mathcal{F}^{[1,b]}\mathbf{Z}_p(i)(R)$ and hence $\mathbf{Z}_p(i)(R) = \mathbf{Z}_p(i)(\mathcal{O}_K/\varpi^n)$ for $i \ge 1$.

To conclude, we use explicit polynomial presentations of the relevant prismatic envelopes as well as Breuil–Kisin orientations to give explicit bases of all four terms and to compute the maps between them. Taking b = in - 1, the result is the matrices syn₀ and syn₁ and the complex appearing in Theorem 2.1.

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2.2.2.1	$z^2 + 2z -$	+ 2					
$K_r \setminus n$	\mathcal{O}_K/ϖ^2	$0_K/\varpi^3$	\mathcal{O}_K/ϖ^4	\mathcal{O}_K/ϖ^5	\mathcal{O}_K/ϖ^6	\mathcal{O}_K/ϖ^7	\mathcal{O}_K/ϖ^8
K ₁	1	2	1,2	1, 1, 2	1, 2, 2	2, 2, 2	2, 2, 3
K_2			1	1	2	2	2
K ₃	1, 1	1, 3	1, 2, 4	2, 3, 4	1, 3, 3, 5	1, 1, 3, 3, 6	1, 1, 3, 4, 7
K ₄					1	1	2
K ₅	1, 1, 1	1, 1, 4	1, 2, 2, 4	2, 2, 2, 6	1, 2, 2, 4, 7	1, 1, 2, 2, 4, 9	1,2,2,2,6,10
K ₆	1 1 1 1	1194	111994	111445	1119555	1 1 9 9 4 7 7	11122480
Λ7	1, 1, 1, 1, 1	1, 1, 2, 4	1, 1, 1, 2, 3, 4	1, 1, 1, 4, 4, 5	1, 1, 1, 2, 3, 5, 5	1, 1, 2, 3, 4, 7, 7	1, 1, 1, 2, 3, 4, 8, 9
2.2.2.2	$z^2 + 2z -$	- 2					
$K_r \setminus n$	O_K/ϖ^2	\mathcal{O}_K/ϖ^3	\mathcal{O}_K/ϖ^4	\mathcal{O}_K/ϖ^5	$\mathbb{O}_K/arpi^6$	$\mathbb{O}_K/arpi^7$	\mathfrak{O}_K/ϖ^8
K ₁	1	2	1,2	1, 1, 2	1, 2, 2	1,2,3	1,3,3
K ₂			1	1	1	1	1
K ₃	1, 1	1, 3	1, 2, 4	2, 3, 4	1, 3, 3, 4	1, 1, 3, 3, 5	1, 1, 3, 4, 6
K ₄					1	1	2
K ₅	1, 1, 1	1, 1, 4	1, 2, 2, 4	2, 2, 2, 6	1, 2, 2, 4, 7	1, 1, 2, 2, 4, 9	1, 2, 2, 2, 6, 10
K ₆	1 1 1 1	1194	111994	111445	1119555	1 1 2 2 4 7 7	1 1 1 2 2 4 8 0
Λ7	1, 1, 1, 1, 1	1, 1, 2, 4	1, 1, 1, 2, 3, 4	1, 1, 1, 4, 4, 5	1, 1, 1, 2, 3, 5, 5	1, 1, 2, 3, 4, 7, 7	1, 1, 1, 2, 3, 4, 8, 9
2.2.3.1	$z^2 + 14$						
$K_r \setminus n$	\mathcal{O}_K/ϖ^2	\mathcal{O}_K/ϖ^3	\mathcal{O}_K/ϖ^4	\mathbb{O}_K/ϖ^5	$\mathbb{O}_K/arpi^6$	\mathbb{O}_K/ϖ^7	\mathcal{O}_K/ϖ^8
K ₁	1	2	1,2	1, 1, 2	1, 1, 3	1,2,3	1, 2, 4
K ₂			1	1	1	1	1
K ₃	1, 1	1, 3	1, 2, 4	2, 3, 4	1, 3, 3, 4	1, 1, 3, 4, 4	1, 2, 4, 4, 4
K ₄		1 1 4	1.0.0.4		1 0 0 0 7	1	2
K ₅	1, 1, 1	1, 1, 4	1, 2, 2, 4	2, 2, 2, 6	1, 2, 3, 3, 7	1, 1, 2, 3, 3, 9	1, 1, 3, 3, 5, 10
K-	1111	1194	119994	112345	1199455	1 2 2 3 5 5 7	11223569
117	1, 1, 1, 1, 1	1, 1, 2, 4	1, 1, 2, 2, 2, 4	1, 1, 2, 0, 4, 0	1, 1, 2, 2, 4, 0, 0	1, 2, 2, 5, 0, 0, 1	1, 1, 2, 2, 5, 0, 0, 5
2.2.3.2	$z^2 + 6$						
$\begin{array}{c} \textbf{2.2.3.2} \\ \textbf{K}_r \backslash n \end{array}$	$\frac{z^2+6}{\mathfrak{O}_K/\varpi^2}$	\mathcal{O}_K/ϖ^3	\mathcal{O}_K/ϖ^4	\mathcal{O}_K/ϖ^5	$\mathcal{O}_K/arpi^6$	\mathbb{O}_K/ϖ^7	\mathbb{O}_K/ϖ^8
$\begin{array}{c} \textbf{2.2.3.2}\\ \textbf{K}_r \backslash n\\ \hline \textbf{K}_1 \end{array}$	$\frac{z^2 + 6}{\mathcal{O}_K / \varpi^2}$	$0_K/\varpi^3$	\mathbb{O}_K/ϖ^4 1,2	$\mathbb{O}_K/arpi^5$ 1,1,2	$\frac{\mathbb{O}_K/\varpi^6}{1,1,3}$	$\frac{\mathbb{O}_K/\varpi^7}{1,2,3}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \end{array}$
$\begin{array}{c} \textbf{2.2.3.2}\\ \textbf{K}_r \backslash n\\ \hline \textbf{K}_1\\ \hline \textbf{K}_2 \end{array}$	$\begin{array}{c} z^2 + 6 \\ 0_K / \varpi^2 \\ 1 \end{array}$	$0_K/\varpi^3$	$\frac{\mathfrak{O}_K/\varpi^4}{1,2}$	$\begin{array}{c} \mathbb{O}_K/\varpi^5\\ 1,1,2\\ 1\end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ \hline 1,2,3 \\ 1 \end{array}$	$\begin{array}{c c} & \mathcal{O}_K/\varpi^8 \\ \hline & 1,2,4 \\ \hline & 1 \end{array}$
$\begin{array}{c} \textbf{2.2.3.2}\\ \textbf{K}_r \backslash n \\ \hline \textbf{K}_1 \\ \textbf{K}_2 \\ \hline \textbf{K}_3 \\ \textbf{K}_3 \\ \hline K$	$\begin{array}{c} z^2 + 6 \\ \mathbb{O}_K / \overline{\omega}^2 \\ \hline 1 \\ 1, 1 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ \hline 1,2 \\ 1 \\ 1,2,4 \end{array}$	$\begin{array}{c} \mathfrak{O}_K/\varpi^5\\ 1,1,2\\ 1\\ 2,3,4\end{array}$	$rac{\mathbb{O}_K/arpi^6}{1,1,3}$ 1,3,3,4 1,3,3,4	$\frac{\mathbb{O}_{K}/\varpi^{7}}{1,2,3}$ 1,1,3,4,4	$0_K/\varpi^8$ 1,2,4 1,2,3,4,5 1,2,3,4,5
$\begin{array}{c} \textbf{2.2.3.2}\\ \textbf{K}_r \backslash n\\ \hline \textbf{K}_1\\ \textbf{K}_2\\ \hline \textbf{K}_3\\ \hline \textbf{K}_4\\ \hline \textbf{K}_4\\ \hline \textbf{K}_4 \end{array}$	$\begin{array}{c} z^2 + 6\\ \mathbb{O}_K/\varpi^2\\ \hline 1\\ 1,1\\ \hline 1,1\\ \hline \end{array}$	$\begin{array}{c c} \mathbb{O}_K/\varpi^3 \\ \hline \\ 1,3 \\ \hline \\ 1,1.4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \end{array}$	${{{\mathfrak O}_K}/{{\varpi ^6}}} \ {1,1,3} \ {1,3,3,4} \ {1,2,3,3,4} \ {1,2,2,2,7}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,2,2,2 \\ 1 \\ 1,2,2,2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1$
$\begin{array}{c} \textbf{2.2.3.2}\\ \textbf{K}_r \backslash n\\ \hline \textbf{K}_1\\ \textbf{K}_2\\ \textbf{K}_3\\ \hline \textbf{K}_4\\ \hline \textbf{K}_5\\ \textbf{K}_4 \end{array}$	$ \begin{array}{c} z^2 + 6 \\ $	$\begin{array}{c c} \mathbb{O}_K/\varpi^3 \\ \hline \\ 1,3 \\ \hline \\ 1,1,4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ 1,2,2,4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ 2,2,2,6 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1\\ 1,3,3,4 \\ 1\\ 1,2,3,3,7 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ \hline 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ \hline 1 \\ 1,1,2,3,3,9 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ \end{array}$
$\begin{array}{c c} 2.2.3.2 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline \end{array}$	$ \begin{array}{c} z^2 + 6 \\ 0_K / \overline{\omega}^2 \\ \hline \\ 1 \\ 1, 1 \\ 1, 1, 1 \\ \hline \\ 1, 1, 1, 1 \\ \hline \\ 1, 1, 1, 1 \\ \hline \end{array} $	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,3 \\ \hline \\ 1,1,4 \\ \hline \\ 1,1,2,4 \end{array}$	$ \begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ 1,2,2,4 \\ 1,1,2,2,4 \\ 1,1,2,2,2,4 \end{array} $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ 2,2,2,6 \\ \hline 1,1,2,3,4,5 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1\\ 1,3,3,4 \\ 1\\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1\\ 1,2,3,4,5 \\ 2\\ 1,1,3,3,5,10 \\ 1\\ 1,1,2,2,3,5,6,9 \end{array}$
$ \begin{bmatrix} 2.2.3.2 \\ K_r \backslash n \\ \hline \\ K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \\ K_7 \\ \hline \end{bmatrix} $	$\begin{array}{c c} z^2 + 6 \\ & \mathbb{O}_K / \varpi^2 \\ & 1 \\ & 1, 1 \\ & 1, 1, 1 \\ & 1, 1, 1, 1 \end{array}$	$ \begin{array}{c c} $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ 1,2,2,4 \\ 1,1,2,2,2,4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \\ 2,2,2,6 \\ \\ 1,1,2,3,4,5 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1\\ 1,3,3,4 \\ 1\\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1\\ 1,1,3,4,4 \\ 1\\ 1,1,2,3,3,9 \\ 1\\ 1,2,2,3,5,5,7 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \end{array}$
$\begin{tabular}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c c} z^2 + 6 & \\ 0_K / \overline{\varpi}^2 & \\ & 1 & \\ & 1, 1, 1 & \\ & 1, 1, 1, 1 & \\ & 1, 1, 1, 1 & \\ & z^2 + 2 & \end{array}$	$ \begin{array}{c c} $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \hline \\ 2,2,2,6 \\ \hline \\ 1,1,2,3,4,5 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1\\ 1,1,3,4,4 \\ 1\\ 1,1,2,3,3,9 \\ 1\\ 1,2,2,3,5,5,7 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1\\ 1,2,3,4,5 \\ 2\\ 1,1,3,3,5,10 \\ 1\\ 1,1,2,2,3,5,6,9 \end{array}$
$\begin{tabular}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c c} z^2 + 6 & \\ & \bigcirc_K / \varpi^2 & \\ & & 1 & \\ & & \\ & & 1, 1, 1 & \\ & & \\ & & 1, 1, 1, 1 & \\ & & \\ & & 1, 1, 1, 1 & \\ & & z^2 + 2 & \\ & \bigcirc_K / \varpi^2 & \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \\ 1,1,4 \\ 1,1,2,4 \\ \mathbb{O}_{K}/\varpi^{3} \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,4 \\ \\ 0_{K}/\varpi^{4} \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \end{array}$ $\begin{array}{c} 2,2,2,6 \\ 1,1,2,3,4,5 \\ \end{array}$ $\mathbb{O}_{K}/\varpi^{5} \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \mathbb{O}_{K}/\varpi^{6} \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \mathbb{O}_{K}/\varpi^{7} \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline 1,1,2,2,3,5,6,9 \\ \mathbb{O}_{K}/\varpi^{8} \end{array}$
$\begin{tabular}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c c} z^2 + 6 & \\ & \bigcirc_K / \varpi^2 & \\ & & 1 & \\ & & \\ & & 1, 1, 1 & \\ & & \\ & & 1, 1, 1, 1 & \\ & & \\ & & 1, 1, 1, 1 & \\ & & \\ & & z^2 + 2 & \\ & & \bigcirc_K / \varpi^2 & \\ & & 1 & \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \\ 1,1,4 \\ 1,1,2,4 \\ \mathbb{O}_{K}/\varpi^{3} \\ 2 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \\ 2,2,2,6 \\ \\ 1,1,2,3,4,5 \\ \\ \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1, 1, 3 \\ 1 \\ 1, 3, 3, 4 \\ 1 \\ 1, 2, 3, 3, 7 \\ 1, 1, 2, 2, 4, 5, 5 \\ \hline \mathbb{O}_{K}/\varpi^{6} \\ 1, 1, 3 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \mathbb{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \end{array}$
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	$ \begin{array}{c} z^{2} + 6 \\ 0_{K}/\overline{\omega}^{2} \\ 1 \\ 1, 1, 1 \\ 1, 1, 1, 1 \\ z^{2} + 2 \\ 0_{K}/\overline{\omega}^{2} \\ 1 \\ 1, 1, 1, 1, 1 \\ z^{2} + 1 \\ 0_{K}/\overline{\omega}^{2} \\ 1 \\ 1 \\ 1, 1, 1 \\ z^{2} + 1 \\ 1, 1, 1 \\ z^{2} + 1 \\ 1, 1, 1 \\ z^{2} + 1 \\ z^{2} + 1 \\ z^{2} \\ z^$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \\ 1,1,4 \\ 1,1,2,4 \\ \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,2 \\ 1,3 \\ 2 \\ 1,4 \\ 2 \\ 1,4 \\ 2 \\ 1,4 \\ 1,4 \\ 2 \\ 1,4 \\$	$\begin{array}{c} \mathfrak{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,4 \\ \\ \mathfrak{O}_{K}/\varpi^{4} \\ \hline 1,2 \\ 1,2 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \hline \\ 2,2,2,6 \\ \hline \\ 1,1,2,3,4,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1,1,2 \\ 1 \\ 2,2,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \\ 3 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 1$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1 \\ 1,2,4 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,2,3,5,6,1 \\ \hline \end{array}$
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c} z^2 + 6 \\ 0_K / \overline{\varpi}^2 \\ \hline \\ 1 \\ 1, 1 \\ 1, 1, 1 \\ 1, 1, 1, 1 \\ \hline \\ z^2 + 2 \\ 0_K / \overline{\varpi}^2 \\ \hline \\ 1 \\ 1, 1 \\ 1, 1 \\ \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ 1,3 \\ \hline \\ 1,1,4 \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,3 \\ \hline \end{array}$	$\begin{array}{c} \mathfrak{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,4 \\ \\ \mathfrak{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \hline \\ 2,2,2,6 \\ \hline \\ 1,1,2,3,4,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ \hline 1,1,3 \\ 1 \\ 1,3,3,4 \\ \hline 1 \\ 1,2,3,3,7 \\ \hline 1,1,2,2,4,5,5 \\ \hline \mathbb{O}_{K}/\varpi^{6} \\ \hline 1,1,3 \\ 1 \\ 1,3,3,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ \end{array}$	$\begin{array}{c c} & & & & & \\ & & & \\ & & & & \\$
$ \begin{array}{c} 2.2.3.2 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline 2.2.3.3 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_4 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_4 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ $	$ \begin{array}{c} z^2 + 6 \\ \bigcirc_{K} / \overline{\varpi}^2 \\ 1 \\ 1 \\ 1, 1, 1 \\ 1, 1, 1, 1 \\ z^2 + 2 \\ \bigcirc_{K} / \overline{\varpi}^2 \\ 1 \\ 1, $	$ \begin{array}{c c} & \mathbb{O}_{K}/\varpi^{3} \\ & 2 \\ & 1,3 \\ & 1,1,4 \\ & 1,1,2,4 \\ \end{array} $ $ \begin{array}{c} & \mathbb{O}_{K}/\varpi^{3} \\ & 2 \\ & 1,3 \\ & 1,1,4 \\ \end{array} $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ 1,2,2,4 \\ 1,1,2,2,2,4 \\ 1,1,2,2,2,4 \\ 0 \\ K/\varpi^{4} \\ 1,2 \\ 1,2 \\ 1 \\ 1,2,4 \\ 1 \\ 1,2,2 \\ 4 \\ 1 \\ 1,2,2 \\ 4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ 2,2,2,6 \\ 1,1,2,3,4,5 \\ \hline \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ \hline 1, 1, 3 \\ 1 \\ 1, 3, 3, 4 \\ \hline 1 \\ 1, 2, 3, 3, 7 \\ \hline 1, 1, 2, 2, 4, 5, 5 \\ \hline \mathbb{O}_{K}/\varpi^{6} \\ \hline 1, 1, 3 \\ 1 \\ 1, 3, 3, 4 \\ \hline 1 \\ 2, 3, 3, 7 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3 \\ 1 \\ 1,1,2,3,3 \\ \end{array}$	$\begin{array}{c c} & \mathcal{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \mathcal{O}_{K}/\varpi^{8} \\ \hline 1,1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline \end{array}$
$ \begin{array}{c} 2.2.3.2 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline \\ 2.2.3.3 \\ \hline K_r \backslash n \\ \hline \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline \end{array} $	$\begin{array}{c} z^2 + 6 \\ 0_K / \overline{\varpi}^2 \\ 1 \\ 1 \\ 1, 1 \\ 1, 1, 1 \\ 1, 1, 1, 1 \\ z^2 + 2 \\ 0_K / \overline{\varpi}^2 \\ 1 \\ 1, 1 \\ 1, 1, 1 \\ 1, 1, 1 \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ 1,3 \\ \hline \\ 1,1,4 \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ 1,3 \\ 1,1,4 \\ \hline \\ 1,1,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ 1,2,2,4 \\ 1,2,2,4 \\ 1,1,2,2,2,4 \\ \hline \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1,2 \\ 1 \\ 1,2,4 \\ \hline 1,2,2,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ 2, 2, 2, 6 \\ 1, 1, 2, 3, 4, 5 \\ \hline \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline 2, 2, 2, 6 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ \end{array}$	$\begin{array}{c} & \mathbb{O}_{K}/\varpi^{8} \\ & 1,2,4 \\ & 1 \\ & 1,2,3,4,5 \\ & 2 \\ & 1,1,3,3,5,10 \\ & 1 \\ & 1,1,2,2,3,5,6,9 \\ \hline & 0_{K}/\varpi^{8} \\ & 0_{K}/\varpi^{8} \\ & 1,1,2,3,4,5 \\ & 1 \\ & 1,2,3,4,5 \\ & 2 \\ & 1,1,3,3,5,10 \\ \hline & 1 \\ \end{array}$
$ \begin{array}{c} 2.2.3.2 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline 2.2.3.3 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline \end{array} $	$\begin{array}{c c} z^2 + 6 & \\ 0_K / \varpi^2 & \\ \hline 1 & \\ 1, 1 & \\ 1, 1, 1 & \\ 1, 1, 1, 1 & \\ z^2 + 2 & \\ 0_K / \varpi^2 & \\ \hline 1 & \\ 1, 1, 1 & \\ 1, 1, 1 & \\ 1, 1, 1, 1 & \\ \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,3 \\ \hline \\ 1,1,4 \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,3 \\ \hline \\ 1,1,4 \\ \hline \\ 1,1,2,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ 1,2,2,4 \\ 1,1,2,2,4 \\ 0_{K}/\varpi^{4} \\ 1,1,2,2,2,4 \\ 1,2,4 \\ 1,2,2,4 \\ 1,2,2,4 \\ 1,1,2,2,2,4 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ 2, 2, 2, 6 \\ 1, 1, 2, 3, 4, 5 \\ \hline \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ 2, 2, 2, 6 \\ 1, 1, 2, 3, 4, 5 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ \hline 1, 1, 3 \\ 1 \\ 1, 3, 3, 4 \\ \hline 1 \\ 1, 2, 3, 3, 7 \\ \hline 1, 1, 2, 2, 4, 5, 5 \\ \hline \mathbb{O}_{K}/\varpi^{6} \\ \hline 1, 1, 3 \\ 1, 3, 3, 4 \\ \hline 1 \\ 1, 2, 3, 3, 7 \\ \hline 1, 1, 2, 2, 4, 5, 5 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \end{array}$
$ \begin{bmatrix} 2.2.3.2 \\ K_r \backslash n \\ \hline K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \\ K_7 \\ \hline 2.2.3.3 \\ K_r \backslash n \\ \hline K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \\ K_7 \\ \hline \end{bmatrix} $	$\begin{array}{c c} z^2 + 6 & \\ & 0_K / \varpi^2 \\ & 1 \\ & \\ & 1, 1, 1 \\ & \\ & 1, 1, 1, 1 \\ & \\ & 1, 1, 1, 1 \\ \\ z^2 + 2 & \\ & 0_K / \varpi^2 \\ & 1 \\ & \\ & 1, 1, 1 \\ & \\ & 1, 1, 1, 1 \\ \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,3 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,3 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ 1,1,2,2,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ 2, 2, 2, 6 \\ 1, 1, 2, 3, 4, 5 \\ \hline \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline 2, 2, 2, 6 \\ 1, 1, 2, 3, 4, 5 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ \hline 1,1,2,2,4,5,5 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \end{array}$
$\begin{tabular}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c c} z^2+6 & \\ & & \\$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,3 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,3 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ 1,2,2,4 \\ 1,1,2,2,4 \\ 0_{K}/\varpi^{4} \\ 1,2 \\ 1,2,4 \\ 1,2,4 \\ 1,2,2,4 \\ 1,1,2,2,4 \\ 1,1,2,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ 2, 2, 2, 6 \\ 1, 1, 2, 3, 4, 5 \\ \hline \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1, 1, 2 \\ 2, 3, 4 \\ \hline 2, 2, 2, 6 \\ 1, 1, 2, 3, 4, 5 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1\\ 1,2,3,4,5 \\ 2\\ 1,1,3,3,5,10 \\ 1\\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1\\ 1,2,3,4,5 \\ 2\\ 1,1,3,3,5,10 \\ 1\\ 1,1,2,2,3,5,6,9 \\ \hline \end{array}$
$\begin{tabular}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c c} z^2+6 & \\ & \bigcirc_K/\varpi^2 & \\ & & 1 \\ & & \\ & & \\ & & \\ & & \\ 1,1,1 & \\ & \\ & & \\ 1,1,1,1 & \\ & \\ z^2+2 & \bigcirc_K/\varpi^2 & \\ & & 1 \\ & & \\ & & \\ 1,1,1 & \\ & & \\ 1,1,1,1 & \\ & \\ & & \\ z^2+10 & \\ & \bigcirc_K/\varpi^2 & \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,3 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,3 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \end{array} \\ 1,2,2,4 \\ 1,2,2,4 \\ 0_{K}/\varpi^{4} \\ 1,2,2,4 \\ 1,2,2,4 \\ 1,2,2,4 \\ 1,1,2,2,2,4 \\ 0_{K}/\varpi^{4} \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \hline \\ 2,2,2,6 \\ \hline \\ 1,1,2,3,4,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \hline \\ 2,2,2,6 \\ \hline \\ 1,1,2,3,4,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \mathbb{O}_{K}/\varpi^{7} \\ \mathbb{O}_{K}/\varpi^{7} \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ \end{array}$
$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c c} z^2+6 & \\ & 0_K/\varpi^2 \\ & 1 \\ & \\ & 1,1,1 \\ \\ & 1,1,1,1 \\ \\ z^2+2 & \\ & 0_K/\varpi^2 \\ & 1 \\ & 1,1,1,1 \\ \\ & 1,1,1,1 \\ \\ & 1,1,1,1 \\ \\ & z^2+10 \\ & 0_K/\varpi^2 \\ & 1 \\ \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,3 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline 1,1,4 \\ \hline 1,1,2,4 \\ \hline \mathbb{O}_{K}/\varpi^{3} \\ \hline \mathbb{O}_{K}/\varpi^{3} \\ \hline 2 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \end{array} \\ 1,2,2,4 \\ 1,1,2,2,4 \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1,2,4 \\ 1,2,2,4 \\ 1,1,2,2,4 \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,1,2 \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ \hline \\ 1,2,4 \\ 1,2,4 \\ \hline \end{array}$
$ \begin{array}{c} 2.2.3.2 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline 2.2.3.3 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline 2.2.3.4 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_7 \\ \hline \end{pmatrix} $	$\begin{array}{c} z^2 + 6 \\ 0_K / \varpi^2 \\ \hline \\ 1 \\ 1, 1, 1 \\ \hline \\ 1, 1, 1, 1 \\ \hline \\ z^2 + 2 \\ 0_K / \varpi^2 \\ \hline \\ 1 \\ 1, 1, 1, 1 \\ \hline \\ 1, 1, 1, 1 \\ \hline \\ z^2 + 10 \\ 0_K / \varpi^2 \\ \hline \\ 1 \\ \hline \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,3 \\ \hline \\ 1,1,4 \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 1,1,4 \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \end{array}$ $1,2,2,4 \\ 1,1,2,2,2,4 \\ \hline \mathbb{O}_{K}/\varpi^{4} \\ 1,2,2,4 \\ 1,2,2,4 \\ 1,1,2,2,2,4 \\ \hline \mathbb{O}_{K}/\varpi^{4} \\ 1,2,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ \hline \\ 1, 1, 2 \\ 3, 4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ \hline \\ 1, 1, 2 \\ 1 \\ 2, 2 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,2,2,4,5,5 \\ \hline \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1,2,3 \\ 1 \\ 1 \\ 1,2,4 \\ 1 \\ 1 \\ 1,2,3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,4 \\ 1 \\ 1,2,4 \\ 1 \\ 1,2,4 \\ 1 \\ 1,2,4 \\ 1 \\ 1 \\ 1,2,4 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1$
$ \begin{array}{c} 2.2.3.2 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline 2.2.3.3 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline 2.2.3.4 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline K_1 \\ \hline K_1 \\ \hline K_1 \\ \hline K_2 \\ \hline K_1 \\ \hline \hline K_1 \\ \hline K_1 $	$\begin{array}{c} z^2 + 6 \\ 0_K / \varpi^2 \\ \hline \\ 1 \\ 1, 1, 1 \\ \hline \\ 1, 1, 1, 1 \\ \hline \\ 1, 1, 1, 1 \\ \hline \\ z^2 + 2 \\ 0_K / \varpi^2 \\ \hline \\ 1 \\ 1, 1, 1, 1 \\ \hline \\ z^2 + 10 \\ 0_K / \varpi^2 \\ \hline \\ 1 \\ 1, 1 \\ \hline \\ 1, 1 \\ \hline \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,3 \\ \hline \\ 1,1,4 \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 1,1,2,4 \\ \hline \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ \hline \\ 2 \\ \hline \\ 1,3 \\ \hline \\ 1,3 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \end{array} \\ 1,2,2,4 \\ 1,1,2,2,4 \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1,2,2,4 \\ 1,2,2,4 \\ 1,1,2,2,2,4 \\ \end{array} \\ \begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1,2,4 \\ 1,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ \hline \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,3,3,9 \\ 1 \\ 1,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 2 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 2 \\ 2 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3$
$\begin{tabular}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c} z^2 + 6 \\ 0_K / \varpi^2 \\ \hline 1 \\ 1, 1, 1 \\ 1, 1, 1, 1 \\ \hline 1, 1, 1, 1 \\ z^2 + 2 \\ 0_K / \varpi^2 \\ \hline 1 \\ 1, 1, 1, 1 \\ z^2 + 10 \\ 0_K / \varpi^2 \\ \hline 1 \\ 1, 1, 1, 1 \\ z^2 + 10 \\ 0_K / \varpi^2 \\ \hline 1 \\ 1, 1 \\ 1, 1 \\ 1, 1 \\ 1, 1 \\ 1, 1 \\ 1, 1 \\ 1, 1 \\ 1 \\$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \\ 1,1,4 \\ 1,1,2,4 \\ \end{array}$ $\begin{array}{c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ \mathbb{O}_{K}/\varpi^{3} \\ 1,1,4 \\ \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ \mathbb{O}_{K}/\varpi^{3} \\ 1,1,4 \\ \mathbb{O}_{K}/\varpi^{3} \\ 1,3 \\ \mathbb{O}_{K}/\varpi^{3} $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \\ \mathbb{O}_{K}/\varpi^{4} \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \\ 1,2,$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ \hline \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ \hline \\ 1,1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 7 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,9 \\ 1 \\ 1,1,2,3,9 \\ 1 \\ 1,1,2,3,9 \\ 1 \\ 1,1,2,3,9 \\ 1 \\ 1,1,2,3,9 \\ 1 \\ 1 \\ 1,1,2,3,9 \\ 1 \\ 1 \\ 1,1,2,3,9 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 1 \\ 1,2,3,4,5 \\ 1 \\ 1 \\ 1,2,3,4,5 \\ 1 \\ 1 \\ 1,2,3,4,5 \\ 1 \\ 1 \\ 1,2,3,4,5 \\ 1 \\ 1 \\ 1 \\ 1,2,3,4,5 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ $
$ \begin{array}{c} 2.2.3.2 \\ \hline K_r \backslash n \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline 2.2.3.3 \\ \hline K_r \backslash n \\ \hline \\ \hline K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \\ \hline K_5 \\ \hline K_6 \\ \hline K_7 \\ \hline \\ 2.2.3.4 \\ \hline K_r \backslash n \\ \hline \\ \hline \\ \hline K_1 \\ \hline \\ K_2 \\ \hline \\ K_6 \\ \hline \\ K_7 \\ \hline \\ \hline \\ K_1 \\ \hline \\ K_2 \\ \hline \\ K_6 \\ \hline \\ K_7 \\ \hline \\ \hline \\ K_6 \\ \hline \\ K_7 \\ \hline \\ \hline \\ K_6 \\ \hline \\ K_7 \\ \hline \\ \hline \\ K_6 \\ \hline \\ K_7 \\ \hline \\ \hline \\ K_6 \\ \hline \\ K_7 \\ \hline \\ \hline \\ K_7 \\ $	$\begin{array}{c c} z^2+6 & \\ & \bigcirc_K/\varpi^2 & \\ & & 1 \\ & & \\ & & \\ & & \\ 1,1,1 \\ & \\ & & \\ 1,1,1,1 \\ & \\ z^2+2 \\ & \bigcirc_K/\varpi^2 & \\ & & 1 \\ & \\ & & \\ 1,1,1,1 \\ & \\ z^2+10 \\ & \bigcirc_K/\varpi^2 & \\ & & 1 \\ & \\ & & \\ 1,1,1 \\ & \\ & & \\ 1,1,1 \\ & \\ & \\ 1,1,1 \\ & \\ \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \\ 1,1,4 \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \\ 1,1,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,1,2,4 \\ \hline \\ \mathbb{O}_{K}/\varpi^{3} \\ 1,1,4 \\ 1,1,4 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,2,2,4 \\ \\ 0_{K}/\varpi^{4} \\ 1,2,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \\ 0_{K}/\varpi^{4} \\ 1,2,2,4 \\ \\ 1,2,2,4 \\ \\ 1,2,2,4 \\ \\ 1,2,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 1, 1, 2, 3, 4, 5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1, 1, 2 \\ 1 \\ 2, 3, 4 \\ \hline \\ 2, 2, 2, 6 \\ \hline \\ 2, 2, 2, 6 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ \hline \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,2,3,3,7 \\ \hline \\ 1,1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ \hline \\ 1,2,3,3,7 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ $	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \\ \mathbb{O}_{K}/\varpi^{8} \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ 1 \\ \end{array}$
$ \begin{array}{c} 2.2.3.2\\ K_r \backslash n\\ \hline K_1\\ K_2\\ K_3\\ K_4\\ K_5\\ K_6\\ K_7\\ \hline 2.2.3.3\\ K_r \backslash n\\ \hline K_1\\ K_2\\ K_3\\ K_4\\ K_5\\ K_6\\ K_7\\ \hline \hline 2.2.3.4\\ K_r \backslash n\\ \hline K_1\\ K_2\\ K_3\\ K_4\\ K_5\\ K_6\\ K_7\\ \hline \end{array} $	$\begin{array}{c c} z^2+6 & \\ & \bigcirc_K/\varpi^2 & \\ & & 1 & \\ & & \\ & & & \\ 1,1,1 & \\ & & \\ 1,1,1,1 & \\ & & \\ z^2+2 & \\ & \bigcirc_K/\varpi^2 & \\ & & 1 & \\ & & \\ 1,1,1,1 & \\ & & \\ z^2+10 & \\ & \bigcirc_K/\varpi^2 & \\ & & 1 & \\ & & \\ 1,1,1,1 & \\ & & \\ 1,1,1,1 & \\ \end{array}$	$\begin{array}{c c} \mathbb{O}_{K}/\varpi^{3} \\ 2 \\ 1,3 \\ 1,1,4 \\ 0 \\ 1,1,2,4 \\ \end{array}$ $\begin{array}{c} \mathbb{O}_{K}/\varpi^{3} \\ \mathbb{O}_{K}/\varpi^{3} \\ 1,1,4 \\ 0 \\ 1,1,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{4} \\ 1,2 \\ 1 \\ 1,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,4 \\ \\ 0_{K}/\varpi^{4} \\ 1,2 \\ 1,1,2,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \\ 0_{K}/\varpi^{4} \\ 1,2 \\ 1,2,2,4 \\ \\ 1,2,2,4 \\ \\ 1,2,2,4 \\ \\ 1,1,2,2,2,4 \\ \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \end{array} \\ 2,2,2,6 \\ 1,1,2,3,4,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \end{array} \\ \begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1,1,2 \\ 3,4,5 \\ \end{array} \\ \begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \end{array} \\ \begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ \mathbb{O}_{K}/\varpi^{5} \\ 1,1,2 \\ 1 \\ 2,3,4 \\ \end{array} \\ \begin{array}{c} \mathbb{O}_{K}/\varpi^{5} \\ \mathbb{O}_{$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \\ \mathbb{O}_{K}/\varpi^{6} \\ 1,1,3 \\ 1 \\ 1,3,3,4 \\ 1 \\ 1,2,3,3,7 \\ 1,1,2,2,4,5,5 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \\ \mathbb{O}_{K}/\varpi^{7} \\ 1,2,3 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,3,4,4 \\ 1 \\ 1,1,2,3,3,9 \\ 1 \\ 1,2,2,3,5,5,7 \\ \hline \end{array}$	$\begin{array}{c} \mathbb{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline 1,1,2,2,3,5,6,9 \\ \hline \mathbb{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline 1 \\ 1,1,2,2,3,5,6,9 \\ \hline \mathbb{O}_{K}/\varpi^{8} \\ \hline 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline 1 \\ 1,2,3,4,5 \\ 2 \\ 1,2,4 \\ 1 \\ 1,2,3,4,5 \\ 2 \\ 1,1,3,3,5,10 \\ \hline 1 \\ 1,1,2,2,3,5,6,9 \end{array}$

Figure 3: The 2-adic K-groups in syntomic weights i = 1, 2, 3, 4 for the totally ramified degree 2 extensions of \mathbb{Z}_2 . The lmfdb [22] label is given in the top left corner together with an Eisenstein polynomial. The data gives the exponents of the elementary divisors in each degree: for example, the entry 1,3 in the K₃ row of the \mathcal{O}_K/ϖ^3 column means that $K_3(\mathcal{O}_K/\varpi^3; \mathbb{Z}_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$.

K_1	$\mathbf{Z}/3$	K ₁₉	$(\mathbf{Z}/3)^3 \oplus \mathbf{Z}/9 \oplus \mathbf{Z}/243$
K_2	0	K_{20}	0
K_3	$({f Z}/3)^2$	K_{21}	$(\mathbf{Z}/3)^3 \oplus \mathbf{Z}/9 \oplus \mathbf{Z}/729$
K_4	$\mathbf{Z}/3$	K_{22}	0
K_5	$\mathbf{Z}/81$	K_{23}	$\mathbf{Z}/3 \oplus \mathbf{Z}/27 \oplus \mathbf{Z}/6561$
K_6	0	K_{24}	0
\mathbf{K}_7	$\mathbf{Z}/3 \oplus \mathbf{Z}/27$	K_{25}	$(\mathbf{Z}/3)^4 \oplus \mathbf{Z}/9 \oplus \mathbf{Z}/2187$
\mathbf{K}_{8}	0	K_{26}	0
\mathbf{K}_{9}	$\mathbf{Z}/3 \oplus \mathbf{Z}/81$	K_{27}	$(\mathbf{Z}/3)^4 \oplus \mathbf{Z}/9 \oplus \mathbf{Z}/6561$
K_{10}	0	K_{28}	0
K ₁₁	$({f Z}/27)^2$	K_{29}	$\mathbf{Z}/3 \oplus \mathbf{Z}/9 \oplus \mathbf{Z}/27 \oplus \mathbf{Z}/19683$
K_{12}	0	K_{30}	0
K_{13}	$({f Z}/3)^2 \oplus {f Z}/243$	K_{31}	$(\mathbf{Z}/3)^4 \oplus (\mathbf{Z}/9)^2 \oplus \mathbf{Z}/6561$
K_{14}	0	K_{32}	0
K_{15}	$(\mathbf{Z}/3)^2 \oplus \mathbf{Z}/729$	K_{33}	$({f Z}/3)^4 \oplus ({f Z}/9)^2 \oplus {f Z}/19683$
K_{16}	0	K_{34}	0
K_{17}	$\mathbf{Z}/9 \oplus \mathbf{Z}/2187$	K_{35}	$({f Z}/9)^2 \oplus {f Z}/243 \oplus {f Z}/19683$
K_{18}	0	K ₃₆	0

Figure 4: The 3-adic K-groups of $\mathbb{Z}/9$ for syntomic weights $1 \leq i \leq 18$. The contribution of $K_{36}(\mathbb{Z}/9;\mathbb{Z}_3) = 0$ is a (null) group from weight 19.