Picard sheaves, local Brauer groups, and topological modular forms

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Abstract

We prove that the Brauer group of TMF is isomorphic to the Brauer group of the derived moduli stack of elliptic curves. Then, we compute the local Brauer group, i.e., the subgroup of the Brauer group of elements trivialized by some étale cover of the moduli stack, up to a finite 2-torsion group.

Key Words. Brauer groups, topological modular forms, moduli of elliptic curves, elliptic cohomology.

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Contents

1 Introduction

The Brauer group $\mathrm{Br}(R)$ of an $\mathbb{E}_\infty$-ring spectrum R was introduced by Baker–Richter–Szymik [8] following previous work of Baker–Lazarev [6] and Toën [67]. The group classifies Azumaya algebras over R up to Morita equivalence; equivalently it classifies invertible R-linear stable ∞-categories. These can be seen as twisted versions of R-modules and thus $\mathrm{Br}(R)$ classifies all possible twists of Mod_R. One can actually replace Mod_R here by any symmetric monoidal ∞-category, like quasi-coherent sheaves on a scheme or stack. In the most classical case of vector spaces over a field

k, Azumaya algebras are just central simple algebras (i.e. matrix algebras over a central division algebra) and the corresponding Brauer group was introduced by Brauer around 1930.

Classically, Brauer groups can often be computed as étale cohomology groups. They thus allow cohomological control of natural occurrences of Azumaya algebras (e.g. as endomorphism algebras of representations [60, Section 12.2]) or twisted sheaves (like in the theory of moduli of stable sheaves [16]). Another ∞-categorical example is given by the relevance of twists of parametrized spectra in Seiberg–Witten Floer homotopy theory [26]. On the other hand, Brauer groups also allow algebraic or geometric interpretations of cohomology classes, as utilized e.g. in the classic Artin–Mumford example of a non-rational unirational variety [5] or the Merkurjev–Suslin theorem [31]. Brauer groups give also one of the approaches to class field theory [58, 69, 54] and form the basis of the Brauer–Manin obstruction for rational points [21]. Thus, the study of Brauer groups of ring spectra might be interesting for possible theories of étale cohomology on $\mathbb{E}_\infty$-ring spectra, and can be seen as a contribution to the nascent subject of arithmetic of $\mathbb{E}_\infty$-ring spectra. Moreover, the Brauer space provides a natural delooping of the Picard space, like the Picard space is a natural delooping of the space of units of an $\mathbb{E}_\infty$-ring spectrum.

When R is a connective $\mathbb{E}_\infty$-ring spectrum, $\mathrm{Br}(R)$ depends only on $\pi_0 R$ and

$$\mathrm{Br}(R) \cong \mathrm{H}^1(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times \mathrm{H}^2(\operatorname{Spec} \pi_0 R, \mathbb{G}_m), \tag{1.1}$$

where all cohomology is étale unless otherwise specified; see [3, 67]. For example, for a prime p, we have $\mathrm{Br}(\mathbb{S}[1/p]) \cong \mathbb{Z}/2$, so there is a "twisted form" of finite spectra after inverting p. These twisted forms are ∞-categories of modules for spherical quaternion algebras. In case that R is a classical ring, $\mathrm{Br}(R)$ might actually be larger than the classical Brauer group of R because of the presence of derived, non-classical Azumaya algebras.

The role of connectivity is to ensure (by an argument of Toën) that Brauer classes on connective $\mathbb{E}_\infty$-rings are étale-locally trivial. This fact enables the cohomological calculation of the Brauer group as in (1.1). We will show in Example 5.7 that this *fails* in general for nonconnective ring spectra. Thus we will differentiate between $\mathrm{Br}(R)$ and its subgroup $\mathrm{LBr}(R)$ of Brauer classes that are *étale-locally trivial*, i.e. become trivial after some faithful étale extension in the sense of [46, Definition 7.5.0.4]. Two of our main themes are that $\mathrm{LBr}(R)$ is quite computable (up to the general difficulty of computing differentials), and that sometimes we may enlarge $\mathrm{LBr}(R)$ by allowing more general extensions to kill Brauer classes. We can say something about the resulting subgroups of $\mathrm{Br}(R)$, which may or may not coincide with $\mathrm{LBr}(R)$.

Our main examples are real K-theory and topological modular forms. Let us begin with the former.

Theorem 1.2 (Theorem 3.13)**.** *There is an isomorphism* $\mathrm{LBr}(\mathrm{KO}) \cong \mathbb{Z}/2$.

The nontriviality of $\mathrm{Br}(\mathrm{KO})$ goes back to [30], where Gepner and Lawson compute the subgroup $\mathrm{Br}(\mathrm{KU}|\mathrm{KO}) \subseteq \mathrm{Br}(\mathrm{KO})$ of classes split by the faithful $\mathbb{Z}/2$-Galois extension $\mathrm{KO} \to \mathrm{KU}$ to be $\mathbb{Z}/2$. It is not hard to check that $\mathrm{LBr}(\mathrm{KU}) = 0$ and thus we find in fact that $\mathrm{LBr}(\mathrm{KO}) = \mathrm{Br}(\mathrm{KU}|\mathrm{KO})$ as subgroups of $\mathrm{Br}(\mathrm{KO})$ although one a priori might expect $\mathrm{Br}(\mathrm{KU}|\mathrm{KO})$ to be bigger. In particular, we show that the non-trivial class $\alpha \in \mathrm{Br}(\mathrm{KU}|\mathrm{KO})$ is split by the faithful étale extension $\mathrm{KO} \to \mathrm{KO}[\frac{1}{2}, \zeta_4] \times \mathrm{KO}[\frac{1}{3}, \zeta_3]$.

Regarding the spectrum TMF of topological modular forms, we recall that Goerss, Hopkins and Miller have defined a sheaf of $\mathbb{E}_\infty$-ring spectra $\mathcal{O}$ on the moduli stack $\mathcal{M}$ of elliptic curves [25]. The pair $(\mathcal{M}, \mathcal{O})$ defines a nonconnective spectral Deligne–Mumford stack in the sense of [47] and TMF is the spectrum of global sections of $\mathcal{O}$. We may define $\mathrm{Br}(\mathcal{M}, \mathcal{O})$ as the Brauer group of $\mathrm{QCoh}(\mathcal{M}, \mathcal{O})$,[1] which coincides with $\mathrm{Mod}_{\mathrm{TMF}}$ as $(\mathcal{M}, \mathcal{O})$ is 0-affine by [49]. On the other hand, we may define $\mathrm{LBr}(\mathcal{M}, \mathcal{O})$ as the subgroup of Brauer classes that become trivial after pulling back to an étale cover of $\mathcal{M}$, and this group is potentially bigger than $\mathrm{LBr}(\mathrm{TMF})$. As by [48, Theorem 10.4], all faithful Galois extensions of localizations of TMF arise from étale covers of $\mathcal{M}$, this local Brauer group $\mathrm{LBr}(\mathcal{M}, \mathcal{O})$ is a natural analogue of $\mathrm{Br}(\mathrm{KU}|\mathrm{KO})$ above.

Theorem 1.3 (Theorem 8.2, Theorem 8.3,Theorem 8.7)**.** *After inverting* 2*, the inclusion* $\mathrm{LBr}(\mathrm{TMF}) \subset \mathrm{LBr}(\mathcal{M}, \mathcal{O})$ *becomes an equality and both groups are isomorphic to* $\mathbb{Z}/3$.

After localizing at 2*, the inclusion* $\mathrm{LBr}(\mathrm{TMF}) \subset \mathrm{LBr}(\mathcal{M}, \mathcal{O})$ *has finite cokernel and both groups admit surjections to* $(\mathbb{Z}/2)^\infty$ *with kernel of order at most* 8*. In particular,* $\mathrm{Br}(\mathrm{TMF})$ *is an infinitely generated torsion abelian group.*

For the (partial) determination of $\mathrm{LBr}(\mathrm{KO})$ and $\mathrm{LBr}(\mathrm{TMF})$ our most important tool is an exact sequence for $\mathrm{LBr}(R)$ (with mild assumptions on $\pi_0 R$) of the form

$$\mathrm{Br}(\pi_0 R) \to \mathrm{LBr}(R) \to \mathrm{H}^1(\operatorname{Spec} \pi_0 R, \pi_0 \mathbf{Pic}_R),$$

[1] Our actual definition of the Brauer group of a nonconnective spectral DM stack in Definition 4.10 is slightly different, but coincides in this case.

which will be proven in a more precise form in Proposition 2.25. Here, $\pi_0\mathbf{Pic}_R$ is the *Picard sheaf* of R. It arises as the étale sheafification of the presheaf sending each étale extension A of $\pi_0 R$ to $\mathrm{Pic}(R_A)$, where $R \to R_A$ is the unique étale extension realizing $\pi_0 R \to A$. We determine the Picard sheaf of KO in Proposition 3.8 and give a partial determination of the Picard sheaf of TMF in Theorem 6.5. The main method is a sheafy version of the Picard spectral sequence of [50]. The remaining uncertainties lie in our inability to compute long differentials in the sheafy Picard spectral sequence or, essentially equivalently, in our inability to compute $\mathrm{Pic}(\mathrm{TMF}_{(2)}[\zeta_{2^n-1}])$ for $n \geq 2$. For possible subleties arising in such computations, we refer to Remark 3.12.

For the (partial) determination of $\mathrm{LBr}(\mathscr{M}, \mathscr{O})$ a crucial point is to compare the (local) Brauer groups of a nonconnective spectral Deligne–Mumford stack $(\mathscr{X}, \mathscr{O})$ with the following variant: The *cohomological* Brauer group $\mathrm{Br}'(\mathscr{X}, \mathscr{O})$ is defined using descent from the affine case and we also obtain a subgroup $\mathrm{LBr}'(\mathscr{X}, \mathscr{O})$. The Brauer group of $(\mathscr{X}, \mathscr{O})$ is the subgroup $\mathrm{Br}(\mathscr{X}, \mathscr{O}) \subseteq \mathrm{Br}'(\mathscr{X}, \mathscr{O})$ of Brauer classes representable by Azumaya algebras. If $(\mathscr{X}, \mathscr{O}) \simeq \mathrm{Spec}\, R$ is affine, then $\mathrm{Br}(R) \cong \mathrm{Br}(\mathrm{Spec}\, R) \cong \mathrm{Br}'(\mathrm{Spec}\, R)$ and likewise for LBr, but Br and Br′ might be different in general. In many cases of interest we show however (extending work of Toën [67] and Hall–Rydh [35]) that the cohomological Brauer agrees with the usual Brauer group. This applies in particular to $(\mathscr{M}, \mathscr{O})$.

Theorem 1.4 (Br $=$ Br′, Theorem 4.17)**.** *If $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ is a nonconnective spectral DM stack satisfying some mild conditions stated in the body of the paper, then* $\mathrm{Br}(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) \simeq \mathrm{Br}'(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ *and* $\mathrm{LBr}(\mathscr{X}, \mathscr{O}_{\mathscr{X}}) \simeq \mathrm{LBr}'(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$.

Since by definition, Br′ and LBr′ are approachable via descent, this result allows us to calculate $\mathrm{LBr}(\mathscr{M}, \mathscr{O})$ via the Picard spectral sequence of [50], where it will be visible in the (-1)-column. Up to two differentials, we can analyze this column using [50] and the vanishing of the classical Brauer group of $\mathscr{M}$ from [4].

Question 1.5. Is the inclusion $\mathrm{LBr}(\mathscr{M}, \mathscr{O}) \subset \mathrm{Br}(\mathrm{TMF})$ an equality?

A similar question can be asked for every 0-affine nonconnective spectral DM stack $(\mathscr{X}, \mathscr{O})$ where $\mathscr{O}$ is even-periodic and the underlying stack of $\mathscr{X}$ is regular noetherian. Here we would replace LBr by a Brauer–Wall type extension as in Remark 2.30 (which makes no difference in the case of $(\mathscr{M}, \mathscr{O})$). Inspired by the case of KO, we also want to pose the question:

Question 1.6. For $(\mathscr{X}, \mathscr{O})$ as above, is $\mathrm{LBr}(\mathscr{O}(\mathscr{X})) \subset \mathrm{LBr}(\mathscr{X}, \mathscr{O})$ always an equality?

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Conventions. We will always use the étale topology. Thus, H^s means *étale* cohomology if applied to a scheme or Deligne–Mumford stack, and $\mathrm{R}^s f_*$ will like-wise refer to the s-th higher direct image with respect to the étale topology. The notation Γ will always refer to the global sections of some étale sheaf with values in an appropriate ∞-category; in particular, when applied to a sheaf of abelian groups $\mathscr{F}$ on a site, we will view $\mathscr{F}$ as a sheaf of spaces so that $\pi_{-i}\Gamma(\mathscr{F})$ is the i-th cohomology group of $\mathscr{F}$. Moreover, if $\mathscr{F}$ is a sheaf of spaces or spectra, $\pi_i\mathscr{F}$ will always refer to the étale sheafification of the presheaf of homotopy groups.

Generally, we will work in an ∞-categorical context. In particular, a *commutative ring spectrum* will mean for us a commutative algebra in the ∞-category of spectra, i.e. what is also called an $\mathbb{E}_\infty$-ring (spectrum). If R is a commutative ring spectrum, we will always equip $\mathrm{Spec}\, R$ with the étale topology. In an ∞-category $\mathscr{C}$, we will write $\mathrm{Map}_{\mathscr{C}}(x, y)$ for the mapping space between $x, y \in \mathscr{C}$; if $\mathscr{C}$ is a stable ∞-category or an ∞-category of quasi-coherent sheaves, then we will write $\mathbf{Map}_{\mathscr{C}}(x, y)$ or simply $\mathbf{Map}(x, y)$ for the mapping spectrum or the internal mapping spectrum.

The following infinity categories will be used in some of the theoretical results:

- Pr^{L}, the infinity category of presentable ∞-categories and left adjoint morphisms;
- $\widehat{\mathrm{Cat}}_{\infty}$, the infinity category of possibly large ∞-categories.

See [44] for details.

2 The local Brauer group in the affine case

After reminding the reader about the classical Brauer group of a commutative ring, we recall in this section the definition of the Brauer group and Brauer space of a commutative ring spectrum and introduce the notion of the local Brauer group. We will prove several basic properties (in particular that Brauer spaces define an étale hypersheaf) and provide basic tools for the computation of local Brauer groups.

2.1 The classical Brauer group

In this subsection, we will give a short introduction to the classical Brauer group. For more background we refer for example to [31], [21] and the series of articles starting with [32].

Let R be a commutative ring. An R-algebra A is called **Azumaya** if one of the following equivalent conditions holds:

1. A is finitely generated, faithful, and projective as an R-module and the map
$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A), \qquad a \otimes b \mapsto (x \mapsto axb)$$
is an isomorphism.
2. étale locally, A is isomorphic to the matrix algebra $\mathrm{Mat}_n(R)$.

Two Azumaya algebras A and B are called **Morita equivalent** if their module categories are equivalent.

Definition 2.1. The **classical Brauer group** $\mathrm{Br}^{\mathrm{cl}}(R)$ of R is the set of Azumaya algebras over R up to Morita equivalence.

Remark 2.2. Instead of working with Morita equivalence classes of Azumaya algebras, one can also directly define the Brauer groups via the module categories. This is the approach we will take in Definition 2.11.

In the case that R is regular noetherian, $\mathrm{Br}^{\mathrm{cl}}(R)$ coincides with what we later introduce as $\mathrm{Br}(R)$; thus we will drop the superscript in this case. Moreover, a result of Gabber identifies $\mathrm{Br}(R)$ in the regular noetherian case with $\mathrm{H}^2(\mathrm{Spec}\, R; \mathbb{G}_m)$ [42, Corollary 3.1.4.2]. As $\mathrm{Pic}(R) \cong \mathrm{H}^1(\mathrm{Spec}\, R; \mathbb{G}_m)$, this gives one perspective on why Brauer groups are a higher variant of Picard groups.

If $R = k$ is a field, every finite-dimensional division k-algebra with center k is Azumaya. Conversely, every Azumaya k-algebra is Morita equivalent to a unique such. Thus, $\mathrm{Br}(k)$ is in bijection with isomorphism classes of finite-dimensional division k-algebras with center k. For example, $\mathrm{Br}(\mathbb{R}) \cong \{[\mathbb{R}], [\mathbb{H}]\} \cong \mathbb{Z}/2$ and $\mathrm{Br}(\mathbb{C}) = 0$. In contrast, the Brauer group of a non-archimedean local field K (like $\mathbb{Q}_p$) is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

It will be important for our later calculations to understand the Brauer groups of rings like $\mathbb{Z}$ or $\mathbb{Z}[\frac{1}{2}, \zeta_4]$. More generally, we consider a number field K and let R be a localization of the ring of integers of K. In this case, by [34, Proposition 2.1], there is an exact sequence
$$0 \to \mathrm{Br}(R) \to \mathrm{Br}(K) \to \bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, R^{(1)}} \mathrm{Br}(\mathrm{Spec}\, K_{\mathfrak{p}}),$$
where $\mathrm{Spec}\, R^{(1)}$ denotes the set of closed points of $\mathrm{Spec}\, R$ and $K_{\mathfrak{p}}$ denotes the completion. This exact sequence is compatible with the exact sequence
$$0 \to \mathrm{Br}(K) \to \bigoplus_{\mathfrak{p}} \mathrm{Br}(\mathrm{Spec}\, K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{2.3}$$
of class field theory (see [56, Theorem 8.1.17]). The sum ranges over the finite and the infinite places of K, and the map $\mathrm{Br}(\mathrm{Spec}\, K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is the isomorphism described above when $\mathfrak{p}$ is a finite place, the natural inclusion $\mathbb{Z}/2 \to \mathbb{Q}/\mathbb{Z}$ when $K_{\mathfrak{p}} \cong \mathbb{R}$, and the natural map $0 \to \mathbb{Q}/\mathbb{Z}$ when $K_{\mathfrak{p}} \cong \mathbb{C}$.

Example 2.4. One can deduce the following vanishing results from the exact sequences above:

(1) $\mathrm{Br}(\mathbb{Z}) = 0$;

(2) $\mathrm{Br}(\mathbb{Z}[\frac{1}{6}]) \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/2$;

(3) $\mathrm{Br}(\mathbb{Z}[\frac{1}{p}, \zeta_{p^n}]) = 0$ for a prime p and a natural number n

We will only give an argument for the last vanishing and only if $p^n \geq 3$. Let $R = \mathbb{Z}[\frac{1}{p}, \zeta_{p^n}]$. The field $\mathbb{Q}(\zeta_{p^n})$ is totally imaginary and there is a unique prime ideal $\mathfrak{p} \subset \mathbb{Z}[\zeta_{p^n}]$ lying over $(p) \subset \mathbb{Z}$ (since p is totally ramified). Thus for every place $\mathfrak{q}$ of $\mathbb{Q}(\zeta_{p^n})$ we have either $\mathrm{Br}(K_{\mathfrak{q}}) = 0$, $\mathfrak{q} \in \operatorname{Spec} R^{(1)}$, or $\mathfrak{q} = \mathfrak{p}$. Thus, $\mathrm{Br}(\mathbb{Z}[\frac{1}{p}, \zeta_{p^n}])$ can be identified with the kernel of the map $\mathrm{Br}(\mathbb{Q}(\zeta_{p^n})_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$, which is zero.

Brauer groups have several nice properties, three of which we will summarize in the next theorem.

Theorem 2.5. *Let R be a regular noetherian ring.*

(1) *If* $\operatorname{Spec} R[\frac{1}{f}] \subset \operatorname{Spec} R$ *is dense, then* $\mathrm{Br}(R) \to \mathrm{Br}(R[\frac{1}{f}])$ *is injective. If* $\frac{1}{p} \in R$ *and f is a non-zero divisor, we have more precisely a short exact sequence*

$$0 \to \mathrm{Br}(R)_{(p)} \to \mathrm{Br}(R[\frac{1}{f}])_{(p)} \to \mathrm{H}^1(R/f; \mathbb{Q}/\mathbb{Z})_{(p)} \to 0.$$

If there is a ring homomorphism right inverse of $R \to R/f$, the sequence is split, sending $[\chi]$ to the Brauer class of the cyclic algebra (χ, f).

(2) *If* $\operatorname{Spec} R[\frac{1}{p}] \subset \operatorname{Spec} R$ *is dense, then* $\mathrm{Br}(R)_{(p)} \to \mathrm{Br}(R[x])_{(p)}$ *is an isomorphism.*

Proof. The first point of the first point follows from [42, Proposition 3.1.3.3]. The rest is contained in [4, Propositions 2.14 and 2.16].

For the second point, a proof in the case $\frac{1}{p} \in R$ can be found e.g. in [4, Proposition 2.5]. To show that $\mathrm{Br}(R)_{(p)} \cong \mathrm{Br}(R[x])_{(p)}$ in general, consider the diagram

$$\begin{tikzcd}
\mathrm{Br}(R[x])_{(p)} \arrow[r] \arrow[d] & \mathrm{Br}(R[\frac{1}{p}][x])_{(p)} \arrow[d, "\cong"] \\
\mathrm{Br}(R)_{(p)} \arrow[r] & \mathrm{Br}(R[\frac{1}{p}])_{(p)}
\end{tikzcd}$$

induced by the morphism $R[x] \to X$, sending x to 0. The right vertical morphism is an isomorphism since $\frac{1}{p} \in R[\frac{1}{p}]$. The horizontal arrows are injections by the first point. Thus, $\mathrm{Br}(R[x])_{(p)} \to \mathrm{Br}(R)_{(p)}$ must be an injection as well. On the other hand, it is a split surjection, using the map $R \to R[x]$. This implies that it is an isomorphism. □

Corollary 2.6. *Let $S \subset \mathbb{Q}$ be a subring. Then the morphism*

$$\Phi : \mathrm{Br}(S) \oplus \mathrm{H}^1(S; \mathbb{Q}/\mathbb{Z}) \to \mathrm{Br}(S[j^{\pm 1}])$$

sending $[\chi] \in \mathrm{H}^1(S; \mathbb{Q}/\mathbb{Z})$ *to the cyclic algebra* $[(\chi, j)]$ *is an isomorphism. In particular,* $\mathrm{Br}(\mathbb{Z}[j^{\pm 1}]) = 0$.

Proof. Let p be a prime and assume first that $S \subset \mathbb{Q}$ with $\frac{1}{p} \in S$. By Theorem 2.5, we obtain a split short exact sequence

$$0 \to \mathrm{Br}(S[j])_{(p)} \to \mathrm{Br}(S[j^{\pm 1}])_{(p)} \to \mathrm{H}^1(S; \mathbb{Q}/\mathbb{Z})_{(p)} \to 0.$$

By $\mathbb{A}^1$-invariance, $\mathrm{Br}(S[j]) \cong \mathrm{Br}(S)$. This proves the claim p-locally if $\frac{1}{p} \in S$. Thus we obtain it for $S = \mathbb{Q}$ without localization. As for a general $S \subset \mathbb{Q}$ the maps $\mathrm{Br}(S[j^{\pm 1}]) \to \mathrm{Br}(\mathbb{Q}[j^{\pm 1}])$ and $\mathrm{H}^1(S; \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ are injections, Φ is an injection in general.

Next we will show the statement in the case $S = \mathbb{Z}_{(p)}$. Let $(a,\chi) \in \mathrm{Br}(\mathbb{Q}) \oplus \mathrm{H}^1(\mathbb{Q};\mathbb{Q}/\mathbb{Z})$. If $\Phi(a,\chi)$ lies in $\mathrm{Br}(\mathbb{Z}_{(p)}[j^{\pm 1}])$, the image $\Phi_u(a,\chi)$ of the class $j_u^*\Phi(a,\chi) \in \mathrm{Br}(\mathbb{Z}_{(p)})$ (for an arbitrary $u \in \mathbb{Z}_{(p)}^\times \subset \mathbb{Z}_p^\times$ inducing $j_u\colon \operatorname{Spec}\mathbb{Z}_{(p)} \to \operatorname{Spec}\mathbb{Z}_{(p)}[j^{\pm 1}]$) must have image zero in $\mathrm{Br}(\mathbb{Q}_p)$ since $\mathrm{Br}(\mathbb{Z}_p) = 0$.

$$\begin{tikzcd}
\mathrm{Br}(\mathbb{Z}_{(p)}[j^{\pm 1}]) \arrow[r, "j_u^*"] \arrow[d] & \mathrm{Br}(\mathbb{Z}_{(p)}) \arrow[r] \arrow[d] & \mathrm{Br}(\mathbb{Z}_p) = 0 \arrow[d] \\
\mathrm{Br}(\mathbb{Q}[j^{\pm 1}]) \arrow[r] & \mathrm{Br}(\mathbb{Q}) \arrow[r] & \mathrm{Br}(\mathbb{Q}_p)
\end{tikzcd}$$

Assume now that $(a,\chi) \notin \mathrm{Br}(\mathbb{Z}_{(p)}) \oplus \mathrm{H}^1(\mathbb{Z}_{(p)};\mathbb{Q}/\mathbb{Z})$. If $a \notin \mathrm{Br}(\mathbb{Z}_{(p)})$, then its image $b \in \mathrm{Br}(\mathbb{Q}_p)$ is non-trivial and thus $\Phi_1(a,\chi) = b \neq 0$. Now suppose $a \in \mathrm{Br}(\mathbb{Z}_{(p)})$, but $\chi \notin \mathrm{H}^1(\mathbb{Z}_{(p)};\mathbb{Q}/\mathbb{Z})$. Then the corresponding extension K of $\mathbb{Q}_p$ must be ramified. Hence the image $N(K) = N_{K|\mathbb{Q}_p}(K^\times) \subset \mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z}$ cannot contain $\mathbb{Z}_p^\times$. As $N(K)$ is of finite index, the surjections $\mathbb{Z}_{(p)}^\times \to (\mathbb{Z}/p^n)^\times$ imply that there exists a $u \in \mathbb{Z}_{(p)}^\times$ such that $u \notin N(K)$. By [31, Corollary 4.7.4], the class $\Phi_u(a,\chi) = [(\chi,u)]$ is thus nonzero in $\mathrm{Br}(\mathbb{Q}_p)$. This shows that Φ is indeed surjective if $S = \mathbb{Z}_{(p)}$.

For $S \subsetneq \mathbb{Q}$ general, we have

$$\mathrm{Br}(S) \oplus \mathrm{H}^1(S;\mathbb{Q}/\mathbb{Z}) = \left(\mathrm{Br}(S[\tfrac{1}{p}]) \oplus \mathrm{H}^1(S[\tfrac{1}{p}];\mathbb{Q}/\mathbb{Z})\right) \cap \left(\mathrm{Br}(\mathbb{Z}_{(p)}) \oplus \mathrm{H}^1(\mathbb{Z}_{(p)};\mathbb{Q}/\mathbb{Z})\right)$$

as subgroups of $\mathrm{Br}(\mathbb{Q}) \oplus \mathrm{H}^1(\mathbb{Q};\mathbb{Q}/\mathbb{Z})$. As we have for every p with $\frac{1}{p} \neq S$ a containment

$$\mathrm{Br}(S[j^{\pm 1}])_{(p)} \subset \mathrm{Br}(S[\tfrac{1}{p}, j^{\pm 1}])_{(p)} \cap \mathrm{Br}(\mathbb{Z}_{(p)}[j^{\pm 1}])_{(p)}$$

we see that $\Phi_{(p)}$ is actually surjective for every $S \subset \mathbb{Q}$ and every prime p, which proves the claim. □

In some of our examples below, we will also use the following classical results, which will help us compute Brauer groups of various ring spectra. The first is Grothendieck's rigidity result for the Brauer group [32, Corollaire 6.2]

Theorem 2.7. *Suppose R is Hensel local with residue field k; then* $\mathrm{Br}(R) \cong \mathrm{Br}(k)$. *If R is also regular, then* $\mathrm{Br}(R) \cong \mathrm{H}^2(\operatorname{Spec} R, \mathbb{G}_m)$ *so that* $\mathrm{H}^2(\operatorname{Spec} R, \mathbb{G}_m) \cong \mathrm{H}^2(\operatorname{Spec} k, \mathbb{G}_m)$.

The next is a corollary of the affine analogue of proper base change as proved in Gabber–Huber [29, 39], see also [11, Corollary 1.18(a)].

Theorem 2.8. *If R is a Hensel local ring with residue field k, then* $\mathrm{H}^*(\operatorname{Spec} R, \mathscr{A}) \cong \mathrm{H}^*(\operatorname{Spec} k, i^*\mathscr{A})$ *for all torsion étale sheaves $\mathscr{A}$ on* $\operatorname{Spec} R$, *where* $i\colon \operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$.

The next result can be found in [23, Corollaire II.3.5, Proposition II.3.6] or [53, Corollary II.3.6] and we will use it several times in the setting of closed immersions.

Theorem 2.9. *Let $f\colon X \to Y$ be a finite morphism of schemes. If $\mathscr{F}$ is an étale sheaf on X, then $\mathrm{R}^q f_* = 0$ for $q > 0$ and hence $\mathrm{H}^i(X;\mathscr{F}) \cong \mathrm{H}^i(Y; f_*\mathscr{F})$ for all $i \geq 0$.*

2.2 Brauer groups of ring spectra

In this subsection, we will recall the Brauer group and Brauer space of a commutative ring spectrum, which were first introduced by [8] and [65]. For our purposes, an approach will be convenient that sees Brauer groups as categorified Picard groups. Let us thus first recall the definition of the Picard group and Picard space.

If $\mathscr{C}$ is a symmetric monoidal ∞-category, its underlying ∞-groupoid $\iota\mathscr{C}$ naturally admits the structure of an $\mathbb{E}_\infty$-space and the counit map $\iota\mathscr{C} \to \mathscr{C}$ is symmetric monoidal. We define the **Picard space** $\mathbf{Pic}(\mathscr{C})$ to be the maximal grouplike $\mathbb{E}_\infty$-groupoid in $\iota\mathscr{C}$. In other words, $\mathbf{Pic}(\mathscr{C})$ is the space of $\otimes$-invertible objects of $\mathscr{C}$ and equivalences. The **Picard group** of $\mathscr{C}$ is $\mathrm{Pic}(\mathscr{C}) = \pi_0\mathbf{Pic}(\mathscr{C})$. We refer to [50] for more background on Picard groups and spaces.

Example 2.10. If R is a commutative ring spectrum, its **Picard space** $\mathbf{Pic}(R)$ is $\mathbf{Pic}(\mathrm{Mod}_R)$ and its **Picard group** is $\mathrm{Pic}(R) = \pi_0\mathbf{Pic}(R)$.

Next we introduce the Brauer group $\mathrm{Br}(R)$ of a commutative ring spectrum R as a categorification of the Picard group. In the case that R is a regular noetherian ring, this will agree with the classical Brauer group (see Remark 2.26).

Definition 2.11. Let R be a commutative ring spectrum and let Cat_R denote the presentably symmetric monoidal ∞-category of compactly generated R-linear stable ∞-categories and compact object-preserving left adjoint functors.[2]

(a) We let $\mathbf{Br}(R) = \mathbf{Pic}(\mathrm{Cat}_R)$ denote the **Brauer space of** R. The **Brauer group** of a commutative ring spectrum R is $\mathrm{Br}(R) = \pi_0\mathbf{Br}(R)$.

(b) If A is an R-algebra, we say that A is an **Azumaya algebra over** R if Mod_A defines a point of $\mathbf{Br}(R)$.

(c) An Azumaya R-algebra A is **trivial** if $\mathrm{Mod}_A \simeq \mathrm{Mod}_R$, i.e., if A is R-linearly (derived) Morita equivalent to R.

(d) An Azumaya R algebra A is **étale-locally trivial** if there is an étale cover $R \to S$ such that $S \otimes_R A$ is trivial.

This definition of an Azumaya algebra is due to Toën [67]. It agrees with the original definition of an Azumaya algebra in this setting due to [8]; see [3] for more details.

Lemma 2.12. *If R is a commutative ring spectrum, then there is a natural equivalence* $\mathbf{Pic}(R) \simeq \Omega\mathbf{Br}(R)$*, where* $\Omega\mathbf{Br}(R)$ *is computed via loops based at the trivial Brauer class.*

Proof. By construction, $\Omega\mathbf{Br}(R)$ is the space of autoequivalences of the unit object of Cat_R. The unit object is Mod_R and the autoequivalences must be R-linear, so they correspond to tensoring with invertible R-modules. □

We will prove in the next section that $R \mapsto \mathbf{Br}(R)$ is a Postnikov complete étale sheaf. To do so, we first establish that $R \mapsto \mathrm{Cat}_R$ is an étale sheaf (with values in Pr^{L}). The result was discovered in the context of the present project, but appeared first in [2, Thm. 2.16].

Proposition 2.13. *The presheaf $R \mapsto \mathrm{Cat}_R$ is an étale sheaf with values in* Pr^{L}.

Proof. The fact that each Cat_R is presentable follows from [12, Corollary 4.25]. That for any map $R \to S$ of commutative ring spectra, the induced functor $\mathrm{Cat}_R \to \mathrm{Cat}_S$ is a left adjoint follows because $\mathrm{Cat}_S \simeq \mathrm{Mod}_{\mathrm{Mod}_S}(\mathrm{Cat}_R)$. Thus, since the forgetful functor $\mathrm{Pr}^{\mathrm{L}} \to \widehat{\mathrm{Cat}}_\infty$ preserves limits by [44, Prop. 5.5.3.13], it suffices to see that $R \mapsto \mathrm{Cat}_R$ is an étale sheaf with values in $\widehat{\mathrm{Cat}}_\infty$. This is part of [2, Thm. 2.16]. □

Corollary 2.14. *The construction $R \mapsto \mathbf{Br}(R)$ is an étale sheaf on* $\mathrm{CAlg}^{\mathrm{op}}$.

Proof. The construction $\mathscr{C} \mapsto \mathbf{Pic}(\mathscr{C})$ preserves limits as a functor from symmetric monoidal ∞-categories to spaces [50, Proposition 2.2.3]. As limits of symmetric monoidal ∞-categories are computed on the level of underlying ∞-categories, the result follows from Proposition 2.13. □

2.3 The local Brauer group

While in classical algebras, Azumaya algebras are always étale-locally Morita equivalent to the ground ring, this is no longer true in the spectral setting. In this subsection (and actually the whole article), we will concentrate on those which *are* étale-locally trivial.

Definition 2.15. Let $\pi_0\mathbf{Br}$ denote the étale sheaf of connected components of $\mathbf{Br}$. We let $\mathbf{LBr}$ be the fiber of the natural map $\mathbf{Br} \to \pi_0\mathbf{Br}$ in étale sheaves. The space $\mathbf{LBr}(R)$ is the **local Brauer space** of R and $\mathrm{LBr}(R) = \pi_0(\mathbf{LBr}(R))$ is the **local Brauer group** of R.

Remark 2.16. Thanks to Lemma 2.12, we could equivalently have defined $\mathbf{LBr}$ as $\mathbf{BPic}$, the étale classifying space of $\mathbf{Pic}$, computed in étale sheaves. However, note that the functor $R \mapsto \mathrm{B}\mathbf{Pic}(R)$, sending R to the classifying space of its Picard space, is not a sheaf, and $\mathbf{BPic}$ is its sheafification.

The name 'local Brauer group' is short-hand for 'locally-trivial Brauer group', which is justified by the following lemma.

[2] Background on this can for example be found in [3, Section 3.1], where the notation $\mathrm{Cat}_{R,\omega}$ is used.

Lemma 2.17. *Let R be a commutative ring spectrum.*

(a) *The natural map* $\mathrm{LBr}(R) \to \mathrm{Br}(R)$ *is an injection and hence* $\mathbf{LBr}(R) \to \mathbf{Br}(R)$ *is the inclusion of a subspace of connected components.*

(b) *An element* $\alpha \in \mathrm{Br}(R)$ *is contained in* $\mathrm{LBr}(R)$ *if and only if there is a faithful étale map* $R \to S$ *such that* α *maps to zero in* $\mathrm{Br}(S)$.

Proof. For (a), use the fiber sequence

$$\mathbf{LBr}(R) \to \mathbf{Br}(R) \to \Gamma(\operatorname{Spec} R, \pi_0\mathbf{Br})$$

of spaces. Recall here that Γ denotes the *space* (as opposed to the set) of global sections of an étale sheaf and $\pi_0\mathbf{Br}$ denotes the étale-sheafified homotopy group. Since $\pi_i\Gamma(\operatorname{Spec} R, \pi_0\mathbf{Br}) = 0$ for $i > 0$, the first claim follows from the long exact sequence in homotopy. Thus we can identify $\mathrm{LBr}(R)$ as a subgroup of $\mathrm{Br}(R)$.

If $\alpha \in \mathrm{LBr}(R)$, then α is étale-locally trivial since $\pi_0\mathbf{LBr} = 0$. Conversely, if $\alpha \in \mathrm{Br}(R)$ is such that there exists a faithful étale map $R \to S$ such that $\alpha_S = 0 \in \mathrm{Br}(S)$, then the image of α in $\pi_0\mathrm{R}\Gamma(\operatorname{Spec} R, \pi_0\mathbf{Br})$ is zero. Thus, $\alpha \in \mathrm{LBr}(R)$. This proves (b). □

If R is a commutative ring spectrum, we will always equip $\operatorname{Spec} R$ with the étale topology. The small étale sites of $\operatorname{Spec} R$ and $\operatorname{Spec} \pi_0 R$ agree, so we can compute cohomology of sheaves of abelian groups on $\operatorname{Spec} R$ via étale cohomology on $\operatorname{Spec} \pi_0 R$: given a sheaf of abelian groups $\mathscr{A} \in \mathrm{Shv}_{\mathrm{Sp}}(\operatorname{Spec} R)^{\heartsuit}$, we have

$$\Gamma(\operatorname{Spec} R, \mathscr{A}) \simeq \Gamma(\operatorname{Spec} \pi_0 R, \mathscr{A})$$

and thus

$$\pi_{-i}\Gamma(\operatorname{Spec} R, \mathscr{A}) \cong \pi_{-i}\Gamma(\operatorname{Spec} \pi_0 R, \mathscr{A}) \cong \mathrm{H}^i(\operatorname{Spec} \pi_0 R; \mathscr{A})$$

This will be used constantly below.

Specifically, to compute $\mathbf{LBr}(R)$, we can restrict $\mathbf{LBr}$ and $\mathbf{Br}$ to étale sheaves $\mathbf{LBr}_{\mathscr{O}}$ and $\mathbf{Br}_{\mathscr{O}}$ on the small étale site of R. Note that these are already sheaves and no additional sheafification is necessary. While we may and will view the $\mathbf{LBr}_{\mathscr{O}}$ and $\mathbf{Br}_{\mathscr{O}}$ as sheaves on $\operatorname{Spec} \pi_0 R$, they certainly depend crucially on R, not only on $\pi_0 R$. We use the notation $\mathbf{Pic}_{\mathscr{O}}$ also for the restriction of $\mathbf{Pic}$ to $\operatorname{Spec} R$.

Lemma 2.18. *Let* $\mathbf{LBr}_{\mathscr{O}}$ *be the local Brauer space sheaf constructed above on* $\operatorname{Spec} R$ *for a commutative ring spectrum* R. *The homotopy sheaves of* $\mathbf{LBr}_{\mathscr{O}}$ *are given by*

$$\pi_t\mathbf{LBr}_{\mathscr{O}} \cong \begin{cases} 0 & \text{if } t = 0, \\ \pi_0\mathbf{Pic}_{\mathscr{O}} & \text{if } t = 1, \\ \mathbb{G}_m & \text{if } t = 2, \text{ and} \\ \pi_{t-2}\mathscr{O} & \text{if } t \geq 3, \end{cases}$$

where $\mathbb{G}_m$ *is the étale sheaf* $\mathbb{G}_m(S) \cong (\pi_0 S)^{\times}$ *for an étale commutative* R*-algebra* S. *In particular,* $\pi_t\mathbf{LBr}_{\mathscr{O}}$ *is quasi-coherent for* $t \geq 3$.

Proof. This follows from Lemma 2.12, using that étale sheafification commutes with restriction along the morphism $\mathrm{CAlg}_R^{\text{ét}} \to \mathrm{CAlg}$ from commutative étale R-algebras to all commutative ring spectra. □

Remark 2.19. Analysis of $\pi_1\mathbf{LBr}_{\mathscr{O}} \cong \pi_0\mathbf{Pic}_{\mathscr{O}}$ is often the most difficult part of local Brauer group computations.

Definition 2.20. Let $\mathscr{C}$ be a prestable ∞-category in the sense of [47, Appendix C] having all limits, which is automatically the nonnegative part of a t-structure on a stable ∞-category. We say that $X \in \mathscr{C}$ is **∞-connective** if $\mathrm{Map}(X, Y) \simeq 0$ for every truncated object Y. An object Y of $\mathscr{C}$ is **hypercomplete** if $\mathrm{Map}(X, Y) \simeq 0$ for every ∞-connective object X. Finally, Y is **Postnikov complete** if the natural map $Y \to \lim_n \tau_{\leq n} Y$ is an equivalence; this occurs if and only if $\lim_n \tau_{\geq n+1} Y \simeq 0$ as fiber sequences are closed under limits.

Postnikov complete objects are hypercomplete, but the converse is not always true. The significance of Postnikov completeness is that it allows us to compute global sections by using descent spectral sequences. As our prestable $\mathscr{C}$ we will use sheaves with values in grouplike $\mathbb{E}_\infty$-spaces (i.e., connective spectra). Note that the forgetful functor from grouplike $\mathbb{E}_\infty$-spaces to spaces preserves and detects limits.

Proposition 2.21. *The assignments $R \mapsto \mathbf{LBr}(R)$ and $R \mapsto \mathbf{Br}(R)$ define Postnikov complete étale sheaves of grouplike $\mathbb{E}_\infty$-spaces on $\mathrm{CAlg}^{\mathrm{op}}$. Similarly, $\mathbf{LBr}_{\mathscr{O}}$ and $\mathbf{Br}_{\mathscr{O}}$ are Postnikov complete étale sheaves on $\operatorname{Spec} R$ for any commutative ring spectrum R.*

Proof. The proofs of all four cases are the same, so we give it only for $\mathbf{Br}$ on $\mathrm{CAlg}^{\mathrm{op}}$. As $\mathbf{Br}$ is Postnikov complete if and only if $\lim_n \tau_{\geq n+1}\mathbf{Br} \simeq *$, it is enough to prove Postnikov completeness for any sufficiently connective cover of $\mathbf{Br}$. We prove that $\tau_{\geq 3}\mathbf{Br}$ is Postnikov complete in two steps. First, $R \mapsto \mathrm{Mod}_R$ satisfies hyperdescent (see [47, Corollary D.6.3.3]), so Pic preserving limits as a functor from symmetric monoidal ∞-categories implies $\mathbf{Pic}$ being hypercomplete.

This implies that $\mathbf{LBr} \simeq \mathbf{BPic}$ is hypercomplete: On 1-connective étale sheaves, Ω is fully faithful. If X is an ∞-connective étale sheaf, then

$$\mathrm{Map}(X, \mathbf{LBr}) \simeq \mathrm{Map}(\Omega X, \mathbf{Pic}) \simeq 0,$$

since ΩX is still ∞-connective.

Second, the fact that $\mathbf{LBr}$ is hypercomplete implies that its 3-connective cover $\tau_{\geq 3}\mathbf{LBr}$ is hypercomplete. However, the homotopy sheaves $\pi_i\tau_{\geq 3}\mathbf{LBr}$ are all quasi-coherent by Lemma 2.18. Therefore, there are enough objects (affines for example) of cohomological dimension ≤ 0 with $\{\pi_*\tau_{\geq 3}\mathbf{LBr}\}$-coefficients in the sense of [19, Def. 2.8]. By [19, Prop. 2.10], it follows that $\tau_{\geq 3}\mathbf{LBr}$ is in fact Postnikov complete, which is what we wanted to show. □

Remark 2.22. A form of Proposition 2.21 was proved in [3, Section 7] in the special case of connective commutative ring spectra using a different argument, although the proof in the connective case and of [45, Proposition 6.5], which is used in the proof of Proposition 2.13, are closely related under the hood. The main point of [3] is that when R is connective, every Azumaya R-algebra is étale locally trivial. This is not true in general, as Example 5.7 below shows.

As the sheaves $\mathbf{Pic}$, $\mathbf{LBr}$ and $\mathbf{Br}$ take values in grouplike $\mathbb{E}_\infty$-spaces, we can deloop them to presheaves of spectra. Sheafifying these results in sheaves $\mathbf{pic}$, $\mathbf{lbr}$ and $\mathbf{br}$ and the restrictions $\mathbf{pic}_{\mathscr{O}}$, $\mathbf{lbr}_{\mathscr{O}}$ and $\mathbf{br}_{\mathscr{O}}$ when working on the étale site of $\operatorname{Spec} R$. Note that by construction, $\mathbf{lbr}_{\mathscr{O}} \simeq \mathbf{pic}[1]$. Note moreover that $\pi_n\mathbf{lbr} \cong \pi_n\mathbf{LBr}$, but the global sections can acquire additional negative homotopy groups.

Corollary 2.23. *There is a convergent spectral sequence*

$$\mathrm{E}_2^{s,t} \cong \mathrm{H}^s(\operatorname{Spec} \pi_0 R, \pi_t\mathbf{LBr}_{\mathscr{O}}) \quad\Longrightarrow\quad \pi_{t-s}\mathbf{lbr}_{\mathscr{O}}(\operatorname{Spec} R) \underset{t-s\geq 0}{\cong} \pi_{t-s}\mathbf{LBr}(R) \tag{2.24}$$

with differentials $d_r : \mathrm{E}_r^{s,t} \to E_r^{s+r,t+r-1}$, which degenerates at the E_3-page.

Proof. This spectral sequence is the descent spectral sequence for the sheaf $\mathbf{lbr}_{\mathscr{O}}$ of spectra, associated to the tower of global sections of the truncations of $\mathbf{lbr}_{\mathscr{O}}$. Convergence follows from the fact that $\pi_t\mathbf{LBr}_{\mathscr{O}}$ is quasi-coherent for $t \geq 3$ by Lemma 2.18 so that $\mathrm{E}_2^{s,t} = 0$ for $t \geq 3$ and $s \geq 1$ and hence the spectral sequence degenerates at the E_3-page. □

The next proposition is our main tool to attack the local Brauer group of a commutative ring spectrum. Recall to that purpose that a commutative ring spectrum R is **weakly $2k$-periodic** if $\pi_{2k}R$ is an invertible $\pi_0 R$-module and $\pi_{2k}R \otimes_{\pi_0 R} \pi_n R \to \pi_{2k+n}R$ is an isomorphism for all $n \in \mathbb{Z}$.

Proposition 2.25. *Let R be a commutative ring spectrum.*

(i) *There is a natural exact sequence*

$$0 \to \mathrm{H}^1(\operatorname{Spec} \pi_0 R, \mathbb{G}_m) \to \mathrm{Pic}(R) \to \mathrm{H}^0(\operatorname{Spec} \pi_0 R, \pi_1\mathbf{LBr}_{\mathscr{O}}) \to$$
$$\mathrm{H}^2(\operatorname{Spec} \pi_0 R, \mathbb{G}_m) \to \mathrm{LBr}(R) \to \mathrm{H}^1(\operatorname{Spec} \pi_0 R, \pi_1\mathbf{LBr}_{\mathscr{O}}) \to \mathrm{H}^3(\operatorname{Spec} \pi_0 R, \mathbb{G}_m).$$

(ii) *If R is connective, then there are natural identifications* $\mathrm{LBr}(R) = \mathrm{Br}(R) \cong \mathrm{H}^1(\operatorname{Spec} \pi_0 R, \mathbb{Z}) \times \mathrm{H}^2(\operatorname{Spec} \pi_0 R, \mathbb{G}_m)$.

(iii) *Fix $k \geq 1$. If R is weakly $2k$-periodic, with $\pi_i R = 0$ for i not divisible by $2k$, and such that $\pi_0 R$ is regular noetherian, then there is a natural exact sequence*

$$0 \to \mathrm{H}^2(\operatorname{Spec} \pi_0 R, \mathbb{G}_m) \to \mathrm{LBr}(R) \to \mathrm{H}^1(\operatorname{Spec} \pi_0 R, \mathbb{Z}/2k) \to \mathrm{H}^3(\operatorname{Spec} \pi_0 R, \mathbb{G}_m).$$

Proof. Part (i) is the exact sequence of low-degree terms of the spectral sequence (2.24) using that $\pi_0 \mathbf{LBr}_{\mathcal{O}} = 0$ and that the spectral sequence degenerates at the E_3-page.

Part (ii) is the content of [3, Theorem 5.11 and Corollary 7.13]. Note that the exact sequence from part (i) splits in the case as $\mathbb{Z}$ is the free grouplike $\mathbb{E}_1$-space, which lets us split the map $\mathbf{LBr}_{\mathcal{O}} \to \mathbf{B}\pi_1\mathbf{LBr}_{\mathcal{O}} \simeq \mathbf{B}\mathbb{Z}$ in the connective case. Here we use that $\mathrm{Pic}(R) \cong \mathrm{Pic}(\pi_* R) \cong \mathrm{Pic}(\pi_0 R) \times \mathbb{Z}$ by a result of [7, Theorem 21] and [50, Theorem 2.4.4] and that the functor $R \mapsto \mathrm{Pic}(\pi_0 R) \times \mathbb{Z}$ sheafifies in the étale topology to the constant sheaf $\mathbb{Z}$ since every Picard group element is étale locally trivial.

For Part (iii), we first claim that $\pi_1 \mathbf{LBr}_{\mathcal{O}} \cong \pi_0 \mathbf{Pic}_{\mathcal{O}} \cong \mathbb{Z}/2k$ when R satisfies the conditions of (iii). Indeed, étale-locally we can assume that R is $2k$-periodic and thus we obtain $\mathrm{Pic}(R) \cong \mathrm{Pic}(\pi_0 R) \times \mathbb{Z}/2k$ by [7, Theorem 37] when $k = 1$ and [50, Corollary 2.4.7] in general. As above, this sheafifies to $\mathbb{Z}/2k$. Hence $\mathrm{Pic}(R) \to \mathrm{H}^0(\operatorname{Spec} \pi_0 R, \mathbb{Z}/2k)$ is surjective. □

Remark 2.26. If R is a classical commutative ring, then $\mathrm{LBr}(R) = \mathrm{Br}(R)$ differs from the classical notion of the Brauer group, because we allow derived Azumaya algebras. In this case,

$$\mathrm{Br}(R) \cong \mathrm{H}^1(\operatorname{Spec} R, \mathbb{Z}) \times \mathrm{H}^2(\operatorname{Spec} R, \mathbb{G}_m)$$

by part (ii) of Proposition 2.25 or by [67, Theorem 1.1]. The Brauer group of ordinary Azumaya algebras is given by $\mathrm{H}^2(\operatorname{Spec} R, \mathbb{G}_m)_{\mathrm{tors}}$, by Gabber [28]. As before, we write $\mathrm{Br}^{\mathrm{cl}}(R)$ for the **classical Brauer group** of ordinary Azumaya algebras when R is a commutative ring. If R is regular noetherian, then $\mathrm{Br}^{\mathrm{cl}}(R) = \mathrm{Br}(R)$ since in this case the $\mathrm{H}^1(\operatorname{Spec} R, \mathbb{Z})$ term vanishes because R is normal (see [24, 2.1]) and since $\mathrm{H}^2(\operatorname{Spec} R, \mathbb{G}_m)$ is all torsion (see [33, Cor. 1.8]).

Example 2.27. When $\pi_0 R = \mathbb{Z}$ and R is either connective or satisfies the conditions of (iii) in Proposition 2.25, then the lemma implies that $\mathrm{LBr}(R) = 0$. Indeed, $\mathrm{H}^1(\operatorname{Spec} \mathbb{Z}, \mathbb{Z}) = 0 = \mathrm{H}^1(\operatorname{Spec} \mathbb{Z}, \mathbb{Z}/2k)$ and Grothendieck proved that $\mathrm{H}^2(\operatorname{Spec} \mathbb{Z}, \mathbb{G}_m) = 0$ in [34]. This covers the sphere spectrum $\mathbb{S}$, the complex cobordism ring MU, the periodic complex K-theory spectrum, as well as connective complex or real K-theory and connective topological modular forms.

Example 2.28. Let k be a perfect field of positive characteristic p and let $\mathbb{G}$ be a 1-dimensional formal group law of height n on k. Let $E_n(\mathbb{G}, k)$ be the Lubin–Tate spectrum associated to $\mathbb{G}$ so that $\pi_* E_n(\mathbb{G}, k) \cong \mathbb{W}(k)[\![u_1, \dots, u_{n-1}]\!][u^{\pm 1}]$, where $\mathbb{W}(k)$ denotes the ring of p-typical Witt vectors of k, each u_i has degree 0, and u has degree 2. We want to show that the local Brauer group $\mathrm{LBr}(E_n(\mathbb{G}, k))$ is typically non-zero. Note that this is related to but different than the results of Hopkins and Lurie in [38], who look at the Brauer group of *$K(n)$-local* $E_n(\mathbb{G}, k)$-modules, which is different from that of $E_n(\mathbb{G}, k)$-modules. Moreover, they study the Brauer group and not just the local Brauer group.

Since $E_n(\mathbb{G}, k)$ is 2-periodic, part (iii) of Proposition 2.25 applies. To compute the groups that build $\mathrm{LBr}(E_n(\mathbb{G}, k))$, note first that $\mathrm{H}^2(\operatorname{Spec} \pi_0 E_n(\mathbb{G}, k), \mathbb{G}_m) \cong \mathrm{H}^2(\operatorname{Spec} k, \mathbb{G}_m)$ by Theorem 2.7 and $\mathrm{H}^2(\operatorname{Spec} k, \mathbb{G}_m) \cong \mathrm{Br}(k)$ is typically non-zero. Moreover, $\mathrm{H}^1(\operatorname{Spec} \pi_0 E_n(\mathbb{G}, k), \mathbb{Z}/2) \cong \mathrm{H}^1(\operatorname{Spec} k, \mathbb{Z}/2)$ by Theorem 2.8.

If $k = \mathbb{F}_{p^r}$ is finite, then the contribution from $\mathrm{H}^2(\operatorname{Spec} \mathbb{F}_{p^r}, \mathbb{G}_m) \cong \mathrm{Br}(\mathbb{F}_{p^r})$ vanishes by Wedderburn's theorem, while the group $\mathrm{H}^1(\operatorname{Spec} \mathbb{F}_{p^r}, \mathbb{Z}/2)$ equals $\mathbb{Z}/2$, as there is a unique $\mathbb{Z}/2$-Galois extension of $\mathbb{F}_{p^r}$. Thus, we obtain an exact sequence

$$0 \to \mathrm{LBr}(E_n(\mathbb{G}, \mathbb{F}_{p^r})) \to \mathbb{Z}/2 \to \mathrm{H}^3(\operatorname{Spec} \pi_0 E_n(\mathbb{G}, \mathbb{F}_{p^r}), \mathbb{G}_m),$$

which implies that $\mathrm{LBr}(E_n(\mathbb{G}, \mathbb{F}_{p^r}))$ is either zero or $\mathbb{Z}/2$.

We can be more specific if we assume that $n = 1$, since in that case we have $\pi_0 E_1(\mathbb{G}, \mathbb{F}_{p^r}) \cong \mathbb{W}(\mathbb{F}_{p^r})$ and $\mathrm{H}^3(\operatorname{Spec} \mathbb{W}(\mathbb{F}_{p^r}), \mathbb{G}_m) = 0$ by [51, 1.7a]. Thus $\mathrm{LBr}(E_1(\mathbb{G}, \mathbb{F}_{p^r}))$ must be $\mathbb{Z}/2$. Note that $E_1(\mathbb{G}, \mathbb{F}_{p^r})$ is closely related to periodic complex K-theory.

For a general height n, we claim that $\mathrm{H}^3(\operatorname{Spec} \pi_0 E_n(\mathbb{G}, \mathbb{F}_{p^r}), \mathbb{G}_m)$ is p-local. Indeed, for l prime to p, we have a short exact sequence

$$0 \to \mu_l \to \mathbb{G}_m \xrightarrow{l} \mathbb{G}_m \to 0$$

of étale sheaves. By Theorem 2.8 we have $\mathrm{H}^i(\operatorname{Spec}\pi_0 E_n(\mathbb{G},\mathbb{F}_{p^r});\mu_l)\cong \mathrm{H}^i(\operatorname{Spec}\mathbb{F}_{p^r};\mu_l)$. The latter group is zero for $i\geq 2$ since the absolute Galois group $\widehat{\mathbb{Z}}$ of $\mathbb{F}_{p^r}$ has cohomological dimension 1. Thus, multiplication by l is an isomorphism on $\mathrm{H}^3(\operatorname{Spec}\pi_0 E_n,\mathbb{G}_m)$, proving the claim. As a consequence, the map

$$\mathbb{Z}/2=\mathrm{H}^1(\operatorname{Spec}\pi_0 E_n(\mathbb{G},\mathbb{F}_{p^r}),\mathbb{Z}/2)\to \mathrm{H}^3(\operatorname{Spec}\pi_0 E_n(\mathbb{G},\mathbb{F}_{p^r}),\mathbb{G}_m)$$

must be zero in the case of odd primes p, and again we obtain that $\mathrm{LBr}(E_n(\mathbb{G},\mathbb{F}_{p^r}))\cong\mathbb{Z}/2$ at all heights if p is odd.

On the other hand, if we work over a separably closed (rather than finite) field k^{sep}, then $\mathrm{H}^2(\operatorname{Spec}k^{sep},\mathbb{G}_m)$ is again zero, but so is $\mathrm{H}^1(\operatorname{Spec}k^{sep},\mathbb{Z}/2)$. This results in the fact that $\mathrm{LBr}(E_n(\mathbb{G},k^{sep}))$ is zero, and in particular, it implies that any non-trivial Brauer class in $\mathrm{LBr}(E_n(\mathbb{G},k)$ is necessarily split by $E_n(\mathbb{G},k)\to E_n(\mathbb{G},k^{sep})$.

Remark 2.29. We do not in fact know an example where the differential

$$\mathrm{H}^1(\operatorname{Spec}\pi_0 R,\pi_1\mathbf{LBr}_{\mathcal{O}})\to \mathrm{H}^3(\operatorname{Spec}\pi_0 R,\mathbb{G}_m)$$

from Proposition 2.25(iii) is non-zero. Similarly, it would be informative to know if there is a commutative ring spectrum R such that

$$\mathrm{Pic}(R)\to \mathrm{H}^0(\operatorname{Spec}\pi_0 R,\pi_1\mathbf{LBr}_{\mathcal{O}})$$

is not surjective.

Remark 2.30. We are primarily interested in integral results as we want to understand contributions to the Brauer group for commutative ring spectra such as the various forms of topological modular forms. Nevertheless, when R is even and weakly 2-periodic and if additionally 2 is a unit on R, then there is an identifiable non-étale-locally trivial contribution to the Brauer group in general. If R is actually 2-periodic and $u\in\pi_2 R$ a unit, let A_u be the Azumaya algebra constructed in [30, Example 7.2]: it is an R-algebra with $\pi_* A_u=\pi_* R[x]$ where $|x|=1$. We let $\mathbf{BrW}_{\mathcal{O}}$ be the sheafification of the components of $\mathbf{Br}_{\mathcal{O}}$ containing 0 and the $[A_u]$ for units $u\in\pi_2 S$ for étale extensions S of R. Using that $[A_u]+[A_v]$ lies in $\mathrm{LBr}(S)$ for any units $u,v\in\pi_2 S$, there is a natural fiber sequence

$$\mathbf{LBr}_{\mathcal{O}}\to\mathbf{BrW}_{\mathcal{O}}\to\mathbb{Z}/2$$

of sheaves on $\mathrm{CAlg}_R^{\text{ét}}$. More generally, if 2 is not a unit on R (but R is still even and weakly periodic), then we can construct an extension

$$\mathbf{LBr}_{\mathcal{O}}\to\mathbf{BrW}_{\mathcal{O}}\to j_!\mathbb{Z}/2,$$

where $j:\ \operatorname{Spec}\pi_0 R[\frac{1}{2}]\to\operatorname{Spec}\pi_0 R$. An easy check using [30, Proposition 7.6] verifies that the algebraic Brauer group of R, as defined in [30], is a subgroup of $\mathrm{LBrW}(R)=\pi_0(\mathbf{BrW}_{\mathcal{O}}(R))$.

3 The Picard sheaf and local Brauer group of KO

The aim of this section is to show that the local Brauer group of KO is $\mathbb{Z}/2$. By the previous section, the key is to understand the étale Picard sheaf $\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{KO}}}$ on $\operatorname{Spec}\mathrm{KO}$. To achieve that, we essentially re-run the calculations of $\mathbf{Pic}(\mathrm{KO})$ from Gepner–Lawson and Mathew–Stojanoska, but this time in sheaves of spaces on $\operatorname{Spec}\mathrm{KO}$. As an aside we will also compute $\mathrm{Pic}(\mathrm{KO}_R)$ for any étale extension R of $\mathbb{Z}$, where KO_R denotes the étale extension of KO lifting R. (We will use similar notation for other ring spectra as well.)

Recall that $\mathrm{KO}\to\mathrm{KU}$ is a C_2-Galois extension, and consequently $\mathbf{pic}(\mathrm{KO})\simeq\tau_{\geq 0}\left(\mathbf{pic}(\mathrm{KU})^{hC_2}\right)$ by Galois descent. Similarly, if R is an étale $\mathbb{Z}$-algebra, then

$$\mathbf{pic}(\mathrm{KO}_R)\simeq\tau_{\geq 0}\left(\mathbf{pic}(\mathrm{KU}_R)^{hC_2}\right).$$

Thus, there is an equivalence

$$\mathbf{pic}_{\mathcal{O}_{\mathrm{KO}}}\simeq\tau_{\geq 0}(\mathbf{pic}_{\mathcal{O}_{\mathrm{KU}}}^{hC_2})$$

of sheaves of connective spectra on $\operatorname{Spec}\mathrm{KO}$, which results in a homotopy fixed point descent spectral sequence with signature

$$\mathscr{E}_2^{s,t}=\mathscr{H}^s(C_2,\pi_t\mathbf{pic}_{\mathcal{O}_{\mathrm{KU}}})\Rightarrow\pi_{t-s}\mathbf{pic}_{\mathcal{O}_{\mathrm{KU}}}^{hC_2}. \tag{3.1}$$

The notation $\mathscr{E}_2^{s,t} \cong \mathscr{H}^s(C_2, \pi_t \mathbf{pic}_{\mathscr{O}_{\mathrm{KU}}})$ means that the C_2-cohomology is taken in étale sheaves, and the differentials are

$$d_r^{s,t} : \mathscr{E}_r^{s,t} \to \mathscr{E}_r^{s+r,t+r-1}.$$

Note that in the figures below, we will depict this spectral sequence with the Adams indexing convention, i.e. in the $(t-s,s)$-plane.

The action of C_2 on the homotopy sheaves of $\mathbf{pic}_{\mathscr{O}_{\mathrm{KU}}}$ is as follows

$$\pi_i \mathbf{pic}_{\mathscr{O}_{\mathrm{KU}}} = \begin{cases} \mathbb{Z}/2, \text{ with trivial action, when } i = 0, \\ \mathscr{O}^\times, \text{ with trivial action, when } i = 1, \\ \mathscr{O}, \text{ with trivial action, when } i > 1,\ i \equiv 1 \mod 4, \\ \mathscr{O}, \text{ with sign action, when } i > 1,\ i \equiv 3 \mod 4. \end{cases}$$

This allows us to compute C_2-cohomology and hence the $\mathscr{E}_2$-page of (3.1).

Example 3.2. The action of C_2 on $\pi_0 \mathscr{O}^\times$ is trivial, so the cohomology sheaves are

$$\mathscr{H}^s(C_2, \mathscr{O}^\times) \cong \begin{cases} \mathscr{O}^\times & \text{if } s = 0, \\ \mu_2 & \text{if } s > 0 \text{ is odd, and} \\ \omega_2 & \text{if } s > 0 \text{ is even,} \end{cases}$$

where μ_2 and ω_2 fit into the exact sequence

$$0 \to \mu_2 \to \mathscr{O}^\times \xrightarrow{x \mapsto x^2} \mathscr{O}^\times \to \omega_2 \to 0.$$

Note that on Spec $\mathbb{Z}$, the sheaf μ_2 is isomorphic to the constant sheaf $\mathbb{Z}/2$; indeed, every étale extension of $\mathbb{Z}$ is a product of integral domains with $2 \neq 0$.

The following identification will not be necessary for our computation of LBr(KO), but we add it for completeness.

Lemma 3.3. *The sheaf ω_2 is isomorphic to $\mathscr{O}/2$ on* Spec $\mathbb{Z}$.

Proof. Since ω_2 is supported only at 2 with stalk given by $A^\times/(A^\times)^2$ where $A = \mathbb{Z}_{(2)}^{sh}$ is the strict Henselization, it is enough to compute the value of this group with its structure as a module over the absolute Galois group $\widehat{\mathbb{Z}}$ of $\mathbb{F}_2$ (cf. [53, Corollary II.3.11]). Let $\mathbb{W} = \mathbb{W}(\overline{\mathbb{F}}_2)$ be the ring of Witt vectors. There is an injection $A \hookrightarrow \mathbb{W}$ and $\mathbb{W}$ is the 2-adic completion of A. We will see that the induced map $A^\times/(A^\times)^2 \to \mathbb{W}^\times/(\mathbb{W}^\times)^2$ will turn out to be an isomorphism.

To prove that this map is injective, it suffices to show that if $u \in A^\times$ is a square in $\mathbb{W}^\times$, then it is already a square in $A^\times$. To see this, let $R = A[x]/(x^2 - u)$. This is a finite A-algebra with 2-adic completion $R_2^\wedge \cong \mathbb{W}[x]/(x^2 - u)$. By the Hensel property for A and $\mathbb{W}$, the ring R is a product of either 1 or 2 local rings (see for example [63, Tag 04GG]) and $R_2^\wedge$ is a product of the same number by looking at fraction fields. If u is a square in $\mathbb{W}$, then $R_2^\wedge$ is a product of 2 local rings, but then the same is true of R.

Next, we explicitly describe $\mathbb{W}^\times/(\mathbb{W}^\times)^2$, which will help us prove surjectivity of the above quotient map. Let $U_n = \{u \in \mathbb{W}^\times : v_2(u-1) \geq n\}$, where v_2 denotes the 2-adic valuation. One has $\mathbb{W}^\times/U_1 \cong \overline{\mathbb{F}}_2^\times$ and $U_n/U_{n+1} \cong \overline{\mathbb{F}}_2$ for $n \geq 2$. The snake lemma for the diagram

$$\begin{tikzcd} 0 \arrow[r] & U_1 \arrow[r] \arrow[d] & \mathbb{W}^\times \arrow[r] \arrow[d] & \overline{\mathbb{F}}_2^\times \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & U_1 \arrow[r] & \mathbb{W}^\times \arrow[r] & \overline{\mathbb{F}}_2^\times \arrow[r] & 0, \end{tikzcd}$$

where the vertical maps are all given by squaring, gives an isomorphism $U_1/U_1^2 \cong \mathbb{W}^\times/(\mathbb{W}^\times)^2$, as squaring is an isomorphism on $\overline{\mathbb{F}}_2^\times$. This also shows that the kernel of squaring on U_1 is isomorphic to $\mathbb{Z}/2$, identified as $\{\pm 1\} \subset \mathbb{W}^\times$.

Classical results imply that there is a logarithmic isomorphism $U_2 \cong \mathbb{W}$ (see for example the argument in [59, Sec. II.3]). Thus, there is an exact sequence

$$0 \to \mathbb{W} \cong U_2 \to U_1 \to \overline{\mathbb{F}}_2 \to 0.$$

Using the snake lemma for the squaring map for this sequence as above, gives an exact sequence

$$0 \to \mathbb{Z}/2 \to \overline{\mathbb{F}}_2 \xrightarrow{\partial} \overline{\mathbb{F}}_2 \to U_1/U_1^2 \to \overline{\mathbb{F}}_2 \to 0,$$

where we have used that $\overline{\mathbb{F}}_2$ is 2 torsion.

The boundary map ∂ is computed by lifting $x \in \overline{\mathbb{F}}_2$ to U_1 as $1 + 2\tilde{x}$ for some $\tilde{x}$ lifting x to $\mathbb{W}$ and then squaring, to find $1 + 4\tilde{x} + 4\tilde{x}^2 = 1 + 4(\tilde{x} + \tilde{x}^2)$, which is in U_2 with residue modulo squares given by $x + x^2$. We see that ∂ is surjective and that $\mathbb{W}^\times/(\mathbb{W}^\times)^2 \cong U_1/U_1^2 \cong \overline{\mathbb{F}}_2$. Explicitly, this isomorphism sends $1 + 2\tilde{x}$ to $\tilde{x} \mod 2$.

Returning to the question of surjectivity of $A^\times/(A^\times)^2 \to \mathbb{W}^\times/(\mathbb{W}^\times)^2$, since the residue fields of A and $\mathbb{W}$ agree, we can lift any element of $\overline{\mathbb{F}}_2$ in the map above to an element of the form $1 + 2\tilde{x}$ with $\tilde{x} \in A \subset \mathbb{W}$; moreover, $1 + 2\tilde{x}$ will be in $A^\times$ by the Hensel property. It follows that $A^\times/(A^\times)^2 \to \mathbb{W}^\times/(\mathbb{W}^\times)^2$ is surjective as well and that both groups are Galois-equivariantly isomorphic to $\overline{\mathbb{F}}_2$. □

Remark 3.4. The above result can also be read off from a much more sophisticated result due to Clausen, Mathew, and Morrow. They show in [20, Thm. A] that if R is p-torsion free, henselian along p, and R/p is perfect, then $\mathrm{K}(R)/p \simeq \mathrm{TC}(R)/p$. Let $A = \mathbb{Z}_{(2)}^{sh}$ and $\mathbb{W} = \mathbb{W}(\overline{\mathbb{F}}_2)$, the 2-completion of A. Applying the Clausen–Mathew–Morrow result to A and $\mathbb{W}$, one obtains

$$A^\times/(A^\times)^2 \cong \mathrm{K}_1(A)/2 \cong \mathrm{TC}_1(A)/2 \cong \mathrm{TC}_1(\mathbb{W})/2 \cong \mathrm{K}_1(\mathbb{W})/2 \cong \mathbb{W}^\times/(\mathbb{W}^\times)^2,$$

using that for any local ring R we have an isomorphism $\mathrm{K}_1(R) \cong R^\times$ and that $\mathrm{TC}(R)/p \simeq \mathrm{TC}(R_p^\wedge)/p$ for any R and any prime p, for example by [20, Lem. 5.3].

To depict the spectral sequence (3.1), we will use symbols to denote the various sheaves and Table 1 can be used as a legend.

Symbol	$\square$	$\square^\times$	$\bullet$	$\circ$
Sheaf	$\mathcal{O}$	$\mathcal{O}^\times$	$\mathcal{O}/2$	$\mathbb{Z}/2$

Table 1: An assortment of étale sheaves.

Fig. 1 and Fig. 2 show the spectral sequence (3.1). Several lemmas explain the nature of the differentials and the calculation of the $\mathscr{E}_4$-page.

Lemma 3.5. *The $\underline{\mathscr{E}}_4$-page is zero in column 0 above row 3.*

Proof. Note that our spectral sequence consists on the $\underline{\mathscr{E}}_2$-page of quasi-coherent sheaves above the antidiagonal $x + y = t = 1$. We will identify quasi-coherent sheaves on $\operatorname{Spec} \pi_0\mathrm{KO}$ with their abelian groups of global sections.

Since our spectral sequence can be seen as the sheafification of a presheaf of Picard homotopy fixed point spectral sequences, we can freely use the tools from [50]. In particular, [50, Comparison Tool 5.2.4] implies that any d_3-differential originating from above the $x + y = t = 3$ antidiagonal can be directly read off its counterpart in the homotopy fixed point spectral sequence for $\mathrm{KU}^{hC_2} \simeq \mathrm{KO}$. As in [50, Example 7.1.1], the claim follows. □

Lemma 3.6 ($d_3^{3,3}$)**.** *The differential $d_3^{3,3} : \bullet \to \bullet$ is given by $x \mapsto x + x^2$. In particular, it is a surjective map of sheaves and the kernel is $i_*\circ$, where $i : \operatorname{Spec} \mathbb{F}_2 \to \operatorname{Spec} \mathbb{Z}$ is the closed inclusion.*

Proof. The first claim follows from [50, Theorem 6.1.1], see also Example 7.1.1 in loc.cit. for the worked example in the case of the abelian group version of the spectral sequence (3.1). The map is surjective away from 2 since both sides vanish in that case. At 2, the map is surjective because $\mathcal{O}/2$ has stalks given by separably closed fields. The identification of the kernel is similar. □

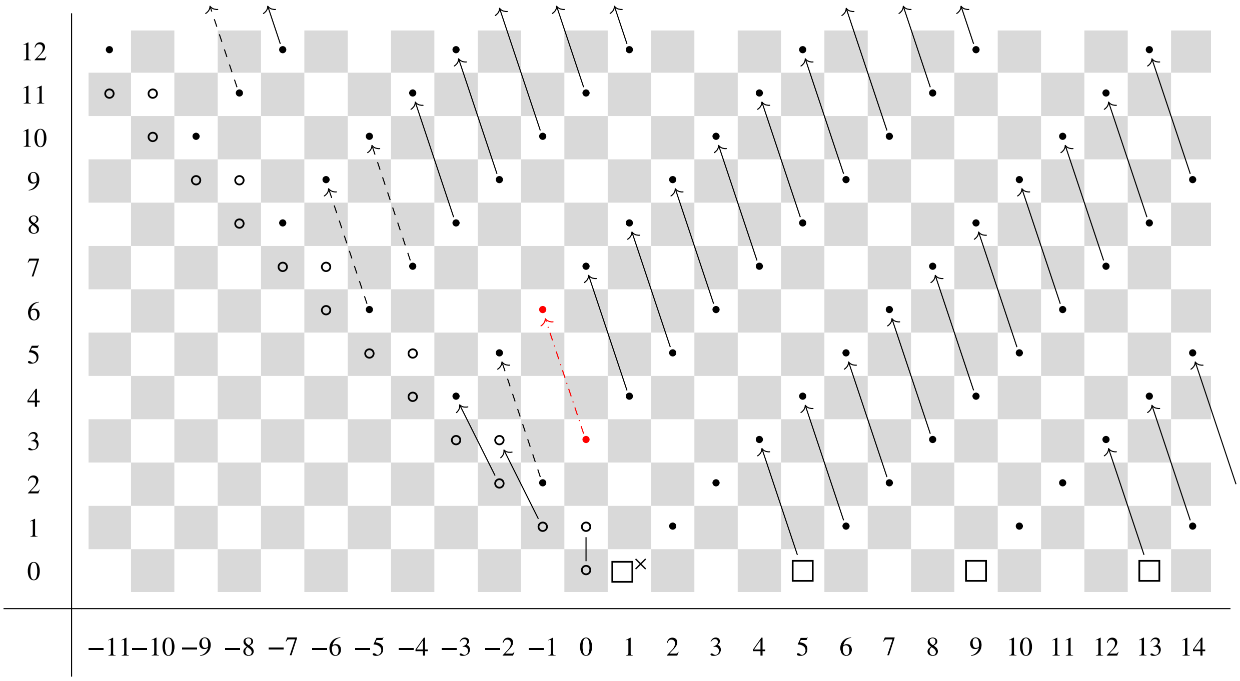

Figure 1: The $\mathscr{E}_2$-page of the spectral sequence (3.1). All differentials on all pages above the anti-diagonal line $x+y=4$ agree with their linear counterparts by [50]. Not all information is shown in degrees ≤ -2. Dashed black arrows potentially differ from their linear partners, but they do not figure into the calculation of $\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}}$. The dashed and dotted red arrow is non-linear and figures into the calculation of $\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}}$.

Remark 3.7. By [30, Proposition 7.15], the differentials $d_2^{1,0}, d_2^{2,0}$ and $d_3^{2,1}$ are nonzero on global sections (where our spectral sequence is isomorphic, at least before differentials, to the usual Picard spectral sequence for KO). The first two differentials have $\mathbb{Z}/2$ as source and are thus determined by global sections: $d_2^{1,0}$ is an isomorphism and $d_2^{2,0}$ is the unique injection $\mathbb{Z}/2 \to \mathscr{O}/2$. The differential $d_3^{2,1} : \mathscr{O}/2 \to \mathscr{O}/2$ is not determined by global sections, however, and thus remains unresolved. None of these differentials are needed for our computation of the Picard sheaf and hence of LBr(KO), though their result on global sections is used in the Gepner–Lawson computation of Br(KU|KO), which we will come back to in Remark 3.14..

These computations determine the associated graded of $\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}}$, but we can also resolve the extension problems as follows.

Proposition 3.8. *There is a filtration on $\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}}$ with associated graded pieces $\mathbb{Z}/2$, $\mathbb{Z}/2$, and $i_*\mathbb{Z}/2$, where i is the closed inclusion* Spec $\mathbb{F}_2 \to$ Spec $\mathbb{Z}$. *There is a surjective map from the constant sheaf $\mathbb{Z}/8$ to $\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}}$, resulting in a non-trivial extension*

$$0 \to i_*\mathbb{Z}/2 \to \pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}} \to \mathbb{Z}/4 \to 0. \tag{3.9}$$

Proof. The first statement was proved in the lemmas above, namely we get a filtration on the E_∞-page of the spectral sequence (3.1) with

$$\begin{tikzcd} \mathrm{F}^2 \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{F}^1 \arrow[r, hook] \arrow[d, two heads] & \pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}} \arrow[d, two heads] \\ i_*\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/2. \end{tikzcd}$$

This filtration gives an inclusion $i_*\mathbb{Z}/2 \cong \mathrm{F}^2 \to \pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}}$, and we need to identify the quotient Q with $\mathbb{Z}/4$. This quotient sits in an extension

$$0 \to \mathbb{Z}/2 \to Q \to \mathbb{Z}/2 \to 0. \tag{3.10}$$

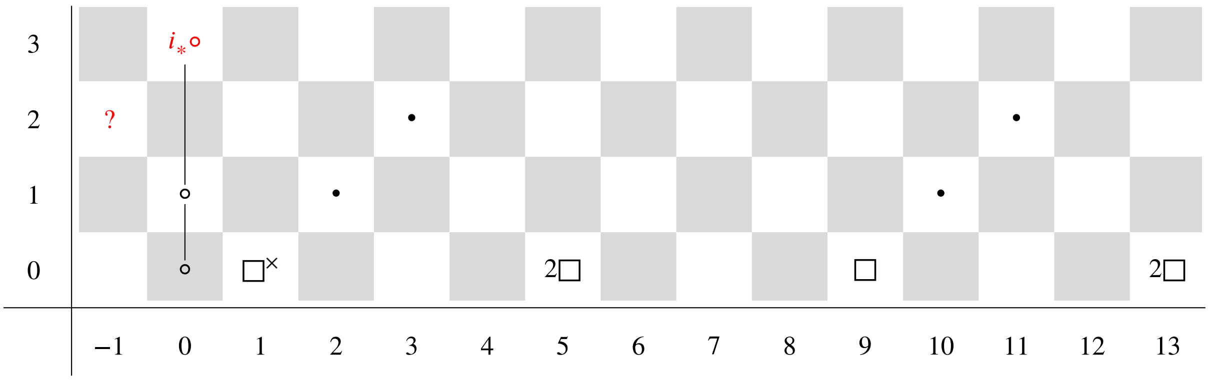

Figure 2: A part of the $\underline{\mathscr{E}}_4$-page of the spectral sequence (3.1).

The filtration implies that the group of global sections $\mathrm{H}^0(\operatorname{Spec}\mathbb{Z}, \pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})$ is a finite group of cardinality at most 8. On the other hand, note that since $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \mathbb{G}_m) = \operatorname{Pic}(\mathbb{Z}) = 0$, Proposition 2.25 implies that the homomorphism $\operatorname{Pic}(\mathrm{KO}) \to \mathrm{H}^0(\operatorname{Spec}\mathbb{Z}, \pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})$ is an injection. Composing with the isomorphism

$$\mathbb{Z}/8 \to \operatorname{Pic}(\mathrm{KO}), \qquad [1] \to \Sigma\mathrm{KO},$$

we obtain a map of sheaves $\mathbb{Z}/8 \to \pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}}$, which must be an isomorphism on global sections.

The above also gives a map $\mathbb{Z}/8 \to \underline{Q}$ that is the surjection $\mathbb{Z}/8 \to \mathbb{Z}/4$ on global sections, implying that the extension (3.10) is non-trivial. But the only non-trivial extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$ on $\operatorname{Spec}\mathbb{Z}$, which has $\mathbb{Z}/4$ as global sections, is the constant sheaf $\mathbb{Z}/4$.[3] This identifies the quotient in (3.9), and to see that this extension is also not split, we again compare with the global sections. $\square$

Corollary 3.11. *Let R be an étale extension of $\mathbb{Z}$. Then there is a short exact sequence*

$$0 \to \operatorname{Pic}(R) \to \operatorname{Pic}(\mathrm{KO}_R) \to (\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})(R) \to 0.$$

If $\operatorname{Spec} R$ *is connected, the last term sits in an extension of the form*

$$0 \to (\mathbb{Z}/2)^d \to (\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})(R) \to \mathbb{Z}/4 \to 0,$$

where d is the number of factors when decomposing $R/2$ as a product of fields.

Proof. We first show the second part. The long exact sequence in cohomology associated to the extension in Proposition 3.8 takes the form

$$0 \to (\mathbb{Z}/2)^d \to (\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})(R) \to \mathbb{Z}/4 \to \mathrm{H}^1(R; i_*\mathbb{Z}/2) \to \cdots$$

The composite $\operatorname{Pic}(\mathrm{KO}_R) \to (\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})(R) \to \mathbb{Z}/4$ is a surjection and thus we obtain the second claim.

For the first part, we can assume that $\operatorname{Spec} R$ is connected and thus R a regular integral domain. From Proposition 2.25, we have a natural exact sequence

$$0 \to \operatorname{Pic}(R) \to \operatorname{Pic}(\mathrm{KO}_R) \to (\pi_0\mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})(R) \xrightarrow{\partial_R} \operatorname{Br}(R).$$

Since $\operatorname{Pic}(\mathrm{KO}_R)$ maps surjectively onto $\mathbb{Z}/4$, the image of ∂_R is the image of the restriction $\partial_R' : (\mathbb{Z}/2)^d \to \operatorname{Br}(R)$. The

[3] Indeed, $\operatorname{Ext}_{\operatorname{Spec}\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) \cong \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}; \mathbb{Z}/2) = 0$ and thus the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

implies that $\operatorname{Ext}_{\operatorname{Spec}\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \operatorname{coker}(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2) \cong \mathbb{Z}/2$.

map $R \to R[\frac{1}{2}]$ induces a commutative square

$$\begin{tikzcd}
(\mathbb{Z}/2)^d \arrow[r] \arrow[d] & \mathrm{Br}(R) \arrow[d] \\
0 \arrow[r] & \mathrm{Br}(R[\frac{1}{2}]),
\end{tikzcd}$$

in which the horizontal arrows are the restricted boundaries ∂' for R and $R[\frac{1}{2}]$ respectively. The right-hand vertical map is an injection by Theorem 2.5 since $\operatorname{Spec} R[\frac{1}{2}] \subset \operatorname{Spec} R$ is dense. Thus $\partial'_R = 0$. □

Remark 3.12. As a consequence of the preceding corollary, we see that it is not true that for every étale extension $\mathbb{Z} \subset R$ with $\operatorname{Spec} R$ connected, we have $\mathrm{Pic}(\mathrm{KO}_R) \cong \mathrm{Pic}(R) \times \mathbb{Z}/8$ or $\mathrm{Pic}(R) \times \mathbb{Z}/4$. For example take the field $K = \mathbb{Q}(\sqrt{17})$, whose ring of integers is $\mathbb{Z}[\omega]$, where $\omega = \frac{1+\sqrt{17}}{2}$, and set $R = \mathbb{Z}[\omega][\frac{1}{17}]$. Here we have $2 = -(1+\omega)(2-\omega)$ and thus $R/2 \cong \mathbb{F}_2 \times \mathbb{F}_2$. We obtain $\mathrm{Pic}(\mathrm{KO}_R) \cong \mathbb{Z}/8 \times \mathbb{Z}/2$. In the Picard spectral sequence for KO_R, the "exotic" elements arise as the kernel of the d_3-differential

$$d_3 \colon\ R/2 \cong \mathrm{H}^3(C_2; \pi_2\mathrm{KU}_R) \to \mathrm{H}^6(C_2; \pi_4\mathrm{KU}_R) \cong R/2, \quad x \mapsto x + x^2$$

is bigger than $\mathbb{Z}/2$, namely $(\mathbb{Z}/2)^2$ in our example.

How can we understand these additional classes? Let us sketch a conjectural general picture of the filtration on $\mathrm{Pic}(A)$ from the Picard spectral sequence for a faithful G-Galois extension $A \to B$. Let $M \in \mathrm{Pic}(A)$. The 0-line detects the image $M \otimes_A B \in \mathrm{Pic}(B)$. If $M \otimes_A B \simeq B$ (and such an equivalence is chosen), the 1-line $\mathrm{H}^1(G; \pi_0 B)$ describes how the G-action on $\pi_*(M \otimes_A B)$ is twisted in comparison to that on $\pi_* B$. Thus, the E_2-term of the homotopy fixed point spectral sequence for $(M \otimes_A B)^{hC_2} \simeq M$ is isomorphic to that for $B^{hC_2} \simeq A$ if M has filtration at least 2, which we will assume now. We fix such an isomorphism. We conjecture that if M has filtration $i \geq 2$, its reduction to $\mathrm{H}^i(G, \pi_i \mathbf{pic}(B)) \cong \mathrm{H}^i(G; \pi_{i-1} B)$ equals $d_i(1)$ in the homotopy fixed point spectral sequence for $(M \otimes_A B)^{hG} \simeq M$.

Back to our example, this means that the three non-trivial classes in $\mathrm{Pic}(\mathrm{KO}_R)$ of filtration 3 correspond conjecturally to invertible KO_R-modules M such that $d_3^M(1)$ is 1, ω and $1+\omega$ respectively.

The identification of $\mathbf{Pic}_{\mathscr{O}_{\mathrm{KO}}}$ allows us to compute the local Brauer group of KO. Recall in this context that Gepner and Lawson proved in [30, Proposition 7.17] that the subgroup $\mathrm{Br}(\mathrm{KU}|\mathrm{KO}) \subseteq \mathrm{Br}(\mathrm{KO})$ of classes killed by the extension $\mathrm{KO} \to \mathrm{KU}$ is isomorphic to $\mathbb{Z}/2$. We will show that $\mathrm{LBr}(\mathrm{KO})$ is also $\mathbb{Z}/2$, and in fact it will be isomorphic to $\mathrm{Br}(\mathrm{KU}|\mathrm{KO})$.

Theorem 3.13. *There is an isomorphism* $\mathrm{LBr}(\mathrm{KO}) \cong \mathbb{Z}/2$. *The unique non-trivial class is killed by the étale cover*

$$\mathrm{KO} \to \mathrm{KO}[\tfrac{1}{2}, \zeta_4] \times \mathrm{KO}[\tfrac{1}{3}, \zeta_3].$$

Here, we use that the cyclotomic fields $\mathbb{Q}(\zeta_4)$ and $\mathbb{Q}(\zeta_3)$ are ramified only at the primes (2) and (3), respectively, to produce $\mathrm{KO}[\frac{1}{2}, \zeta_4]$ and $\mathrm{KO}[\frac{1}{3}, \zeta_3]$ as commutative ring spectra.

Proof. To use the exact sequence in Proposition 2.25, we first need to compute $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \pi_0 \mathbf{pic}_{\mathscr{O}_{\mathrm{KO}}})$, which we will do using Proposition 3.8. Since there is a unique $\mathbb{Z}/2$-Galois extension of $\operatorname{Spec}\mathbb{F}_2$, Theorem 2.9 implies

$$\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, j_*\mathbb{Z}/2) \cong \mathrm{H}^1(\operatorname{Spec}\mathbb{F}_2, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Moreover $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \mathbb{Z}/4) = 0$ as there are no unramified $\mathbb{Z}/4$-Galois extensions of $\mathbb{Q}$. Since furthermore

$$\mathrm{H}^0(\operatorname{Spec}\mathbb{Z}, \mathbf{Pic}_{\mathscr{O}_{\mathrm{KO}}}) \cong \mathrm{Pic}(\mathrm{KO}) \to \mathrm{H}^0(\operatorname{Spec}\mathbb{Z}, \mathbb{Z}/4)$$

is surjective, the long exact cohomology sequence associated with the short exact sequence of sheaves in (3.9) implies that $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathbb{Z}, \pi_0 \mathbf{Pic}_{\mathscr{O}_{\mathrm{KO}}})$ is isomorphic to $\mathbb{Z}/2$.

To conclude $\mathrm{LBr}(\mathrm{KO}) \cong \mathbb{Z}/2$ using Proposition 2.25, it remains to show the vanishing of the differential

$$\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \pi_0 \mathbf{Pic}_{\mathscr{O}_{\mathrm{KO}}}) \xrightarrow{d_2} \mathrm{H}^3_{\text{ét}}(\operatorname{Spec}\mathbb{Z}, \mathbb{G}_m).$$

We show this by comparison to KU: The map KO → KU induces a map of presheaves $\mathbf{lbr}_{\mathrm{KO}} \to \mathbf{lbr}_{\mathrm{KU}}$ on the étale site of Spec $\mathbb{Z}$, which we identify with either of the étale sites of KO and KU using the isomorphism $\pi_0\mathrm{KO} \cong \mathbb{Z} \cong \pi_0\mathrm{KU}$. Thus, we get an induced map of descent spectral sequences and in particular a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \mathbf{Pic}_{\mathscr{O}_{\mathrm{KO}}}) \arrow[r, "d_2"] \arrow[d] & \mathrm{H}^3(\operatorname{Spec}\mathbb{Z}, \mathbb{G}_m) \arrow[d] \\ \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \mathbf{Pic}_{\mathscr{O}_{\mathrm{KU}}}) \arrow[r, "d_2"] & \mathrm{H}^3(\operatorname{Spec}\mathbb{Z}, \mathbb{G}_m), \end{tikzcd}$$

where the right vertical map is an equality. Since $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \pi_0\mathbf{Pic}_{\mathscr{O}_{\mathrm{KU}}}) \cong \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \mathbb{Z}/2) = 0$, we see that the top differential must vanish. Therefore, $\mathrm{LBr}(\mathrm{KO}) \cong \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, \pi_1\mathbf{BPic}_{\mathscr{O}_{\mathrm{KO}}}) \cong \mathbb{Z}/2$.

For the second part of our claim, note first that $\mathrm{Br}(\mathbb{Z}[\frac{1}{2}, \zeta_4])$ and $\mathrm{Br}(\mathbb{Z}[\frac{1}{3}, \zeta_3])$ vanish by Example 2.4. The Brauer groups of $\mathbb{Z}[\frac{1}{2}, \zeta_4]$ and $\mathbb{Z}[\frac{1}{3}, \zeta_3]$ agree with the second étale cohomology with $\mathbb{G}_m$-coefficients since the rings are regular and noetherian. Using Proposition 2.25 again, we thus see that the non-trivial class in LBr(KO) must be killed by the extension $\mathrm{KO} \to \mathrm{KO}[\frac{1}{2}, \zeta_4] \times \mathrm{KO}[\frac{1}{3}, \zeta_3]$ if the image of the non-trivial element of $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}, j_*\mathbb{Z}/2)$ vanishes in $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[\frac{1}{2}, \zeta_4], j_*\mathbb{Z}/2)$ and $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[\frac{1}{3}, \zeta_3], j_*\mathbb{Z}/2)$. This is clear in the first case as $j_*\mathbb{Z}/2$ restricted to $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ vanishes. In the second case, we use that the extension $\mathbb{F}_2 \subset \mathbb{F}_4 \cong \mathbb{F}_2[\zeta_3]$ kills the non-trivial element of $\mathrm{H}^1(\operatorname{Spec}\mathbb{F}_2, \mathbb{Z}/2)$. □

Remark 3.14. Note that since $\mathrm{LBr}(\mathrm{KU}) = 0$, functoriality of the local Brauer group implies that the non-zero class $\alpha \in \mathrm{LBr}(\mathrm{KO}) \cong \mathbb{Z}/2$ is killed by the $\mathbb{Z}/2$-Galois extension KO → KU, i.e. lies in the relative Brauer group Br(KU|KO). By the main result of [30], LBr(KO) thus agrees with Br(KU|KO) though a priori we only get an inclusion. This gives a new proof of that the Galois-cohomological Brauer class found in [30, Proposition 7.15] is representable by an Azumaya algebra, which Gepner and Lawson prove instead with an unstable descent spectral sequence. See also Example 4.15 for another perspective.

We urge the reader to consider the analogue of the descent spectral sequence computation of Br(KU|KO) as in [30, Figure 7.2] in the case of the relative Brauer group of $\mathrm{KO}[\frac{1}{3}, \zeta_3]$ with respect to $\mathrm{KU}[\frac{1}{3}, \zeta_3]$. As the class in filtration six contributing to Br(KU|KO) has to die in $\mathrm{Br}(\mathrm{KO}[\frac{1}{3}, \zeta_3])$, there must be a new d_3 killing it. This d_3 is given by the formula in [50, Theorem 6.1.1], the point being that the image of $x \mapsto x + x^2$ on $\mathbb{Z}[\frac{1}{3}, \zeta_3]/2 \cong \mathbb{F}_4$ is $\mathbb{Z}/2 = \mathbb{F}_2 \subset \mathbb{F}_4$.

4 Brauer groups of nonconnective spectral DM stacks

In this section, we turn to Brauer groups of nonconnective spectral Deligne–Mumford (DM) stacks. A significant difference will be that the Brauer group is in general no longer π_0 of the global sections of the Brauer sheaf, yielding to a distinction between Brauer group and cohomological Brauer group, which we will explain below.

To fix notation, we recall the following definition from Lurie [47].

Definition 4.1. A **nonconnective spectral DM stack** is a spectrally ringed ∞-topos $(\mathscr{X}, \mathscr{O})$ such that there exists a covering $\coprod_{i\in I} U_i \to *$ of the final object where for each i there is an equivalence $(\mathscr{X}_{/U_i}, \mathscr{O}|_{\mathscr{X}_{U_i}}) \simeq \operatorname{Spec} R_i$ for some commutative ring spectrum R_i.[4] If $\mathscr{O}$ is connective, we say that $(\mathscr{X}, \mathscr{O})$ is a **connective spectral DM stack**; if $\mathscr{O}$ is discrete, we say that $(\mathscr{X}, \mathscr{O})$ is a **classical DM stack**.

Remark 4.2. (a) In [47], Lurie calls connective spectral DM stacks simply spectral DM stacks.

(b) Given a nonconnective spectral DM stack $(\mathscr{X}, \mathscr{O})$, there is a diagram $(\mathscr{X}, \mathscr{O}) \to (\mathscr{X}, \tau_{\geq 0}\mathscr{X}) \leftarrow (\mathscr{X}, \pi_0\mathscr{X})$ of nonconnective spectral DM stacks. The right arrow is the inclusion of the classical locus, at least if $\mathscr{X}$ arises from a 1-topos.

Construction 4.3. For a nonconnective spectral DM stack, étale sheaves on $\mathscr{X}$ are equivalent to étale sheaves on the site $\mathrm{Aff}^{\text{ét}}_{/(\mathscr{X},\mathscr{O})}$ of étale maps $\operatorname{Spec} R \to (\mathscr{X}, \mathscr{O})$ for some commutative ring spectrum R. Restricting the sheaves **Pic**, **Br**,

[4] Lurie writes Spét R for what we write as Spec R.

and $\mathbf{LBr}$ on $\mathrm{CAlg}_{\mathbb{S}}^{\mathrm{op}} \simeq \mathrm{Aff}$ from Section 2, we obtain sheaves $\mathbf{Pic}_{\mathcal{O}}$, $\mathbf{Br}_{\mathcal{O}}$, and $\mathbf{LBr}_{\mathcal{O}}$ on $\mathrm{Aff}^{\text{ét}}_{/(\mathcal{X},\mathcal{O})}$ or, equivalently, on $\mathcal{X}$.

Remark 4.4. There is a natural map $\mathbf{BPic}_{\mathcal{O}} \to \mathbf{Br}_{\mathcal{O}}$ which induces an equivalence $\mathbf{BPic}_{\mathcal{O}} \simeq \mathbf{LBr}_{\mathcal{O}}$, since again this can be checked locally. The computation of the homotopy sheaves of $\mathbf{LBr}_{\mathcal{O}}$ given in Lemma 2.18 goes through verbatim here.

Example 4.5. In general, $\pi_0 \mathbf{Br}_{\tau_{\geq 0}\mathcal{O}} = \pi_0 \mathbf{Br}_{\pi_0 \mathcal{O}} = 0$ since Brauer classes on connective commutative ring spectra are étale-locally trivial by [3, Theorem 5.11]. We also have $\pi_1 \mathbf{Br}_{\tau_{\geq 0}\mathcal{O}} \cong \pi_1 \mathbf{Br}_{\pi_0 \mathcal{O}} \cong \mathbb{Z}$ by the computation of Picard groups of connective commutative ring spectra. On the other hand, $\pi_0 \mathbf{Br}_{\mathcal{O}}$ and $\pi_1 \mathbf{Br}_{\mathcal{O}}$ are highly dependent on the nature of $\mathcal{O}$ itself.

Definition 4.6. We let $\mathrm{Br}'(\mathcal{X},\mathcal{O}) = \pi_0 \Gamma(\mathcal{X}, \mathbf{Br}_{\mathcal{O}}) = \pi_0(\mathbf{Br}_{\mathcal{O}}(\mathcal{X}))$. This is the **cohomological Brauer group** of $\mathcal{X}$. Similarly, the **cohomological local Brauer group** of $(\mathcal{X},\mathcal{O})$ is

$$\mathrm{LBr}'(\mathcal{X},\mathcal{O}) = \pi_0 \mathbf{BPic}_{\mathcal{O}}(\mathcal{X}).$$

We call the space of global sections $\mathbf{Br}_{\mathcal{O}}(\mathcal{X})$ the Brauer space and similarly for the local Brauer space $\mathbf{BPic}_{\mathcal{O}}(\mathcal{X}) \simeq \mathbf{LBr}_{\mathcal{O}}(\mathcal{X})$.

Remark 4.7. The subgroup $\mathrm{LBr}'(\mathcal{X},\mathcal{O}) \subseteq \mathrm{Br}'(\mathcal{X},\mathcal{O})$ consists of those cohomological Brauer classes which are étale locally trivial on $\mathcal{X}$. Since $(\mathcal{X},\mathcal{O})$ is a nonconnective spectral DM stack this means that for $\alpha \in \mathrm{LBr}'(\mathcal{X},\mathcal{O})$, there is a surjective family of étale maps $\{p_i : \operatorname{Spec} R_i \to (\mathcal{X},\mathcal{O})\}_{i\in I}$ such that $p_i^*\alpha = 0$ for all i.

Construction 4.8. In order to compute $\mathrm{Br}'(\mathcal{X},\mathcal{O})$ and $\mathrm{LBr}'(\mathcal{X},\mathcal{O})$, it is convenient to deloop $\mathbf{Br}_{\mathcal{O}}$ and $\mathbf{LBr}_{\mathcal{O}}$ and view them as presheaves of spectra; étale sheafification yields sheaves $\mathbf{br}_{\mathcal{O}}$ and $\mathbf{lbr}_{\mathcal{O}}$. As such we have $\pi_t \mathbf{br}_{\mathcal{O}} \cong \pi_t \mathbf{Br}_{\mathcal{O}}$ for all $t \in \mathbb{Z}$ and similarly for $\mathbf{lbr}_{\mathcal{O}}$; in particular, the homotopy sheaves vanish for $t < 0$. We have $\Omega^\infty \mathbf{br}_{\mathcal{O}}(\mathcal{X}) \simeq \mathbf{Br}_{\mathcal{O}}(\mathcal{X})$. Analogously to Proposition 2.21, we argue that $\mathbf{br}_{\mathcal{O}}$ and $\mathbf{lbr}_{\mathcal{O}}$ are Postnikov complete. Thus, we obtain a descent spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathcal{X}, \pi_t \mathbf{br}_{\mathcal{O}}) \quad \Rightarrow \quad \pi_{t-s} \mathbf{br}_{\mathcal{O}}(\mathcal{X}) \underset{t-s\geq 0}{\cong} \pi_{t-s} \mathbf{Br}_{\mathcal{O}}(\mathcal{X})$$

and similarly for $\mathbf{lbr}_{\mathcal{O}}(\mathcal{X})$.

For the following definition, recall that a quasi-coherent sheaf is perfect if it is dualizable or, equivalently, if it becomes a compact object when restricted to an affine.

Definition 4.9. A quasi-coherent sheaf $\mathcal{A}$ of $\mathcal{O}$-algebras on a nonconnective spectral DM stack $(\mathcal{X},\mathcal{O})$ is an **Azumaya algebra** if the following equivalent conditions hold:

(i) $\mathcal{A}$ is perfect, locally generates $\mathrm{QCoh}(\mathcal{X},\mathcal{O})$, and the natural map $\mathcal{A}^{\mathrm{op}} \otimes_{\mathcal{O}} \mathcal{A} \to \mathbf{End}_{\mathcal{O}}(\mathcal{A})$ is an equivalence;

(ii) there is an étale cover $\{\operatorname{Spec} R_i \xrightarrow{p_i} (\mathcal{X},\mathcal{O})\}_{i\in I}$ such that $p_i^*\mathcal{A}$ is an Azumaya R_i-algebra for all i.

Definition 4.10. Any Azumaya algebra $\mathcal{A}$ on $(\mathcal{X},\mathcal{O})$ defines a point of $\mathbf{Br}_{\mathcal{O}}$ and hence an element $[\mathcal{A}]$ of $\mathrm{Br}'(\mathcal{X},\mathcal{O})$, called the class of $\mathcal{A}$. If $\mathcal{A}$ is an Azumaya algebra, then so is the opposite algebra $\mathcal{A}^{\mathrm{op}}$ and we have $[\mathcal{A}^{\mathrm{op}}] = -[\mathcal{A}]$; if $\mathcal{B}$ is a second Azumaya algebra, then $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}$ is Azumaya and $[\mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}] = [\mathcal{A}] + [\mathcal{B}]$. These assertions may be verified locally using Definition 4.9(ii) and Definition 2.11(b). Let $\mathrm{Br}(\mathcal{X},\mathcal{O}) \subseteq \mathrm{Br}'(\mathcal{X},\mathcal{O})$ be the subgroup consisting of the classes of Azumaya algebras. Let $\mathrm{LBr}(\mathcal{X},\mathcal{O}) = \mathrm{LBr}'(\mathcal{X},\mathcal{O}) \cap \mathrm{Br}(\mathcal{X},\mathcal{O})$ inside $\mathrm{Br}'(\mathcal{X},\mathcal{O})$. We call these the **Brauer** and **local Brauer groups** of $(\mathcal{X},\mathcal{O})$.

Example 4.11. For any commutative ring spectrum R, Proposition 2.21 implies $\mathrm{Br}'(\operatorname{Spec} R) = \mathrm{Br}(\operatorname{Spec} R)$.

Definition 4.12. Let $(\mathcal{X},\mathcal{O})$ be a nonconnective spectral DM stack and let $\alpha \in \mathrm{Br}'(\mathcal{X},\mathcal{O})$. Using the inclusion $\mathbf{Br}_{\mathcal{O}} \to \mathrm{Cat}_{\mathcal{O}}$, the section $\alpha \in \mathrm{Br}'(\mathcal{X},\mathcal{O})$ defines a section of $\mathrm{Cat}_{\mathcal{O}}$ and hence a stack of stable presentable ∞-categories, $\mathrm{QCoh}_{\mathcal{O},\alpha}$. This is the stack of **α-twisted quasi-coherent sheaves** on $(\mathcal{X},\mathcal{O})$. The stable ∞-category of global sections will be denoted by $\mathrm{QCoh}(\mathcal{X},\alpha)$. An object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X},\alpha)$ is **perfect** if for every étale $p : \operatorname{Spec} R \to (\mathcal{X},\mathcal{O})$ the complex $p^*\mathcal{F}$ is a compact object of $\mathrm{QCoh}(\operatorname{Spec} R, p^*\alpha)$. Note that the latter stable ∞-category is equivalent to Mod_A where A is any Azumaya R-algebra with Brauer class $p^*\alpha$. We say that $\mathcal{F}$ is a **perfect local generator** if it is perfect and $p^*\mathcal{F}$ generates $\mathrm{QCoh}(\operatorname{Spec} R, p^*\alpha)$ for any $\operatorname{Spec} R \to (\mathcal{X},\mathcal{O})$.

Lemma 4.13. *Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral DM stack. If $\alpha \in \mathrm{Br}'(\mathcal{X}, \mathcal{O})$, then $\alpha \in \mathrm{Br}(\mathcal{X}, \mathcal{O}) \subseteq \mathrm{Br}'(\mathcal{X}, \mathcal{O})$ if and only if there exists a perfect local generator of* $\mathrm{QCoh}(\mathcal{X}, \alpha)$.

Proof. If $\mathscr{A}$ is an Azumaya algebra representing α, define $\mathrm{QCoh}(\mathcal{X}, \mathscr{A})$ as the limit of $\mathrm{Mod}_{\mathscr{A}(R)}$ over all étale maps $\mathrm{Spec}\, R \to (\mathcal{X}, \mathcal{O})$; this can be identified with a full subcategory of $\mathrm{Mod}_{\mathscr{A}}(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}))$. We have $\mathrm{QCoh}(\mathcal{X}, \mathscr{A}) \simeq \mathrm{QCoh}(\mathcal{X}, \alpha)$ and under this equivalence $\mathscr{A}$ corresponds to a perfect local generator. Conversely, given a perfect local generator $\mathscr{F}$ of $\mathrm{QCoh}(\mathcal{X}, \alpha)$, the sheaf of endomorphisms $\mathbf{End}_{\mathcal{O}}(\mathscr{F})$ is an Azumaya algebra with class α. □

Here is one example where every cohomological Brauer class is representable by an Azumaya algebra.

Proposition 4.14. *Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral DM stack. If $(\mathcal{X}, \mathcal{O})$ admits a finite étale cover $\pi :\ \mathrm{Spec}\, R \to (\mathcal{X}, \mathcal{O})$, then $\mathrm{Br}(\mathcal{X}, \mathcal{O}) = \mathrm{Br}'(\mathcal{X}, \mathcal{O})$.*

Proof. There is a compact generator $\mathscr{F}$ of $\mathrm{QCoh}(\mathrm{Spec}\, R, \pi^*\alpha)$ by Example 4.11 and Lemma 4.13. The pushforward $\pi_*\mathscr{F}$ is a perfect local generator of $\mathrm{QCoh}(\mathcal{X}, \alpha)$, as one can check étale locally. □

Example 4.15. If a finite group G acts on a commutative ring spectrum R, we obtain a finite étale map $\mathrm{Spec}\, R \to [\mathrm{Spec}\, R/G]$ to the stack quotient. In particular, $\mathrm{Br}([\mathrm{Spec}\, R/G]) \cong \mathrm{Br}'([\mathrm{Spec}\, R/G])$ by the preceding proposition. This is especially interesting if $R^{hG} \to R$ is a faithful G-Galois extension, when Galois descent implies that $\mathrm{Mod}_{R^{hG}} \simeq \mathrm{QCoh}([\mathrm{Spec}\, R/G])$. Examples include $\mathrm{KO} \to \mathrm{KU}$, $\mathrm{TMF}[\frac{1}{2}] \to \mathrm{TMF}(2)$, and $\mathrm{TMF}[\frac{1}{3}] \to \mathrm{TMF}(3)$ (see [57] and [49]).

Proposition 4.14 will not be enough to show the agreement of Br' and Br for the derived moduli stack of elliptic curve since the moduli stack of elliptic curves does not have a finite étale cover by an affine scheme [68]. This issue will be solved by Theorem 4.17 below. Before we state it, we introduce the following definition needed for its proof.

Definition 4.16. Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral DM stack. Let $\alpha \in \mathrm{Br}'(\mathcal{X}, \mathcal{O})$ be a Brauer class and let $\mathscr{F} \in \mathrm{QCoh}(\mathcal{X}, \alpha)$ be a perfect local generator. We say that $\mathscr{F}$ is a **global generator** if $\mathscr{F}$ is compact and if $\mathrm{QCoh}(\mathcal{X}, \alpha)$ is compactly generated by $\mathscr{F}$.

Theorem 4.17. *Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral DM stack and fix $\alpha \in \mathrm{Br}'(\mathcal{X}, \mathcal{O})$. If $\mathcal{X}$ admits a Zariski open cover $\{\mathcal{U}_i\}_{i=1}^n$ such that*

(a) *for each $1 \leq i, j \leq n$, the kernel of $\mathrm{QCoh}(\mathcal{U}_i, \mathcal{O}) \to \mathrm{QCoh}(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O})$ is generated by a single compact object $\mathscr{K}_{i,j}$, and*

(b) *there is a global generator $\mathscr{F}_i$ of $\mathrm{QCoh}(\mathcal{U}_i, \alpha)$ for each $i = 1, \dots, n$,*

then $\alpha \in \mathrm{Br}(\mathcal{X}, \mathcal{O})$ and there is a global generator of $\mathrm{QCoh}(\mathcal{X}, \alpha)$.

The proof follows the work of [67] and [3] which uses older arguments of Bökstedt–Neeman [14] and Bondal–van den Bergh [15] who showed that for a quasi-compact and quasi-separated scheme X, the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves admits a single compact generator, which is global in the sense above. Other important examples of $\mathrm{Br} = \mathrm{Br}'$ in the non-derived and derived context have been established in [28, 22, 35, 18].

Proof. Note first that each $\mathcal{U}_i \subseteq \mathcal{X}$ is relatively scalloped in the sense of [47, 2.5.4.1].[5]

We glue local perfect generators as in [3, Theorem 6.11] or [67, Proposition 5.9], taking care in each step to produce a global generator. Let $\mathcal{Y}_k$ be the union $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k$ in $\mathcal{X}$. It is enough to prove that there is a global generator of $\mathrm{QCoh}(\mathcal{Y}_k, \alpha)$ for each $k = 1, \dots, n$ and hence $\alpha|_{\mathcal{Y}_k}$ is in $\mathrm{Br}(\mathcal{Y}_k, \mathcal{O})$ for each k. The base case follows from assumption (b). Suppose the conclusion holds for some $1 \leq k < n$. Set $\mathcal{W} = \mathcal{Y}_k \cap \mathcal{U}_{k+1}$ and consider the pullback square

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{Y}_{k+1}, \alpha) \arrow[r] \arrow[d] & \mathrm{QCoh}(\mathcal{U}_{k+1}, \alpha) \arrow[d] \\ \mathrm{QCoh}(\mathcal{Y}_k, \alpha) \arrow[r] & \mathrm{QCoh}(\mathcal{W}, \alpha) \end{tikzcd} \tag{4.18}$$

[5] Quasi-affine morphisms are relatively scalloped; these will be enough for our applications.

of stable presentable ∞-categories. As each inclusion $\mathscr{W} \subseteq \mathscr{U}_{k+1}$, $\mathscr{W} \subseteq \mathscr{Y}_k$, $\mathscr{Y}_k \subseteq \mathscr{Y}_{k+1}$, and $\mathscr{U}_{k+1} \subseteq \mathscr{Y}_{k+1}$ is quasi-affine and hence relatively scalloped, it follows from [47, 2.5.4.3], or rather its proof,[6] that the functors in (4.18) preserve compact objects.

We want to show that the kernel of $\mathrm{QCoh}(\mathscr{U}_{k+1}, \alpha) \to \mathrm{QCoh}(\mathscr{W}, \alpha)$ is generated by the compact object

$$\mathscr{G} := \mathscr{F}_{k+1} \otimes_{\mathscr{O}} \mathscr{K}_{k+1,1} \otimes_{\mathscr{O}} \cdots \otimes_{\mathscr{O}} \mathscr{K}_{k+1,k}.$$

Compactness follows by construction. For generation, suppose that $\mathscr{M}$ is an object of $\mathrm{QCoh}(\mathscr{U}_{k+1}, \alpha)$ which restricts to zero on $\mathrm{QCoh}(\mathscr{W}, \alpha)$ and suppose additionally that the mapping spectrum $\mathrm{Map}(\mathscr{G}, \mathscr{M})$ is zero. We want to show that $\mathscr{M} \simeq 0$. But,

$$0 \simeq \mathrm{Map}(\mathscr{G}, \mathscr{M}) \simeq \mathrm{Map}(\mathscr{K}_{k+1,1} \otimes_{\mathscr{O}} \cdots \otimes_{\mathscr{O}} \mathscr{K}_{k+1,k}, \mathbf{Map}(\mathscr{F}_{k+1}, \mathscr{M}))$$

by adjunction, where $\mathbf{Map}(\mathscr{F}_{k+1}, \mathscr{M})$ denotes the internal mapping spectrum, a quasi-coherent sheaf on $\mathscr{U}_{k+1}$. Since the $\mathscr{K}_{k+1,j}$ are compact generators of the kernels of $\mathrm{QCoh}(\mathscr{U}_{k+1}, \mathscr{O}) \to \mathrm{QCoh}(\mathscr{U}_{k+1} \cap \mathscr{U}_j, \mathscr{O})$, their tensor product is a compact generator of the kernel of $\mathrm{QCoh}(\mathscr{U}_{k+1}) \to \mathrm{QCoh}(\mathscr{W})$. Denoting the inclusion $\mathscr{W} \to \mathscr{U}_{k+1}$ by i, it follows that $\mathbf{Map}(\mathscr{F}_{k+1}, \mathscr{M}) \to i_* i^* \mathbf{Map}(\mathscr{F}_{k+1}, \mathscr{M})$ is an equivalence since its fiber lies in the kernel of $\mathrm{QCoh}(\mathscr{U}_{k+1}) \to \mathrm{QCoh}(\mathscr{W})$. But, this implies that $\mathbf{Map}(\mathscr{F}_{k+1}, \mathscr{M}) \simeq \mathbf{Map}(\mathscr{F}_{k+1}|_{\mathscr{W}}, \mathscr{M}|_{\mathscr{W}})$. The latter is zero as $\mathscr{M}|_{\mathscr{W}} \simeq 0$, so $\mathbf{Map}(\mathscr{F}_{k+1}, \mathscr{M}) \simeq 0$ and hence the mapping spectrum $\mathrm{Map}(\mathscr{F}_{k+1}, \mathscr{M})$ is zero, which in turn implies that $\mathscr{M} \simeq 0$ since $\mathscr{F}_{k+1}$ is a compact generator of $\mathrm{QCoh}(\mathscr{U}_{k+1}, \alpha)$.

Using that the square (4.18) is a pullback, the vertical fibers are equivalent stable ∞-categories. Thus, $\mathscr{G}$ corresponds to a compact object of $\mathrm{QCoh}(\mathscr{Y}_{k+1}, \alpha)$ which vanishes on $\mathscr{Y}_k$. On the other hand, by induction there is a global generator $\mathscr{H}$ of $\mathrm{QCoh}(\mathscr{Y}_k, \alpha)$. Our goal will be to lift $\mathscr{H}$ to $\mathscr{Y}_{k+1}$. The fact that $\mathrm{QCoh}(\mathscr{U}_{k+1}, \alpha) \to \mathrm{QCoh}(\mathscr{W}, \alpha)$ is a localization and preserves compact objects implies that $\mathrm{QCoh}(\mathscr{W}, \alpha)$ is generated by the image of $\mathscr{F}_{k+1}$. Since the kernel is compactly generated by a compact object of $\mathrm{QCoh}(\mathscr{U}_{k+1}, \alpha)$ we are in the setting of Thomason's extension proposition [66, 5.2.2] (see [55, Corollary 0.9] for the generality needed here), which says that if $\mathscr{B} \to \mathscr{C} \to \mathscr{D}$ is a Verdier sequence of idempotent complete stable ∞-categories, then an object $\mathscr{M} \in \mathscr{D}$ lifts to $\mathscr{C}$ if and only if its class $[\mathscr{M}] \in \mathrm{K}_0(\mathscr{C})$ lifts to $\mathrm{K}_0(\mathscr{D})$. Thus, possibly by replacing $\mathscr{H}$ by $\mathscr{H} \oplus \Sigma\mathscr{H}$ (which always has vanishing class in K_0), we see that the restriction of $\mathscr{H}$ to $\mathrm{QCoh}(\mathscr{W}, \alpha)$ lifts to a compact object $\mathscr{H}_{k+1}$ of $\mathrm{QCoh}(\mathscr{U}_{k+1}, \alpha)$. Gluing $\mathscr{H}$ and $\mathscr{H}_{k+1}$ via the pullback (4.18), we obtain a compact object $\mathscr{E}$ of $\mathrm{QCoh}(X, \alpha)$. Let $\mathscr{D} = \mathscr{E} \oplus \mathscr{G}$. We claim that $\mathscr{D}$ is a global generator of $\mathrm{QCoh}(\mathscr{Y}_{k+1}, \alpha)$. Verification is standard and left to the reader. □

Corollary 4.19. *If a nonconnective spectral DM stack $(\mathscr{X}, \mathscr{O})$ satisfies the assumptions of Theorem 4.17 for every $\alpha \in S \subset \mathrm{Br}'(\mathscr{X}, \mathscr{O})$, we have*

(1) $\mathrm{Br}(\mathscr{X}, \mathscr{O}) = \mathrm{Br}'(\mathscr{X}, \mathscr{O})$ *if* $S = \mathrm{Br}'(\mathscr{X}, \mathscr{O})$,

(2) $\mathrm{LBr}(\mathscr{X}, \mathscr{O}) = \mathrm{LBr}'(\mathscr{X}, \mathscr{O})$ *if* $S = \mathrm{LBr}'(\mathscr{X}, \mathscr{O})$, *and*

(3) $\mathrm{LBrW}(\mathscr{X}, \mathscr{O}) = \mathrm{LBrW}'(\mathscr{X}, \mathscr{O})$ *if $\mathscr{O}$ is weakly 2-periodic and* $S = \mathrm{LBrW}'(\mathscr{X}, \mathscr{O})$.

This corollary will be applied in Proposition 8.1 to the derived moduli stack of elliptic curves.

5 The 0-affine case

Let $(\mathscr{X}, \mathscr{O})$ be a nonconnective spectral DM stack.

Definition 5.1. We say that $(\mathscr{X}, \mathscr{O})$ is **0-affine** if the global sections functor

$$\Gamma\colon \mathrm{QCoh}(\mathscr{X}, \mathscr{O}) \to \mathrm{Mod}_{\Gamma(\mathscr{X}, \mathscr{O})}$$

is an equivalence; equivalently, $(\mathscr{X}, \mathscr{O})$ is 0-affine if $\mathscr{O}$ is a compact generator of $\mathrm{QCoh}(\mathscr{X}, \mathscr{O})$.

In classical algebraic geometry, there are few 0-affine DM stacks. If X is a scheme, X is 0-affine if and only if it is quasi-affine, which is to say quasi-compact and can be embedded as an open subscheme of $\mathrm{Spec}\, A$ for some A. In this case, one can take $A = \mathrm{H}^0(X, \mathscr{O})$. More generally, quasi-affine connective spectral DM stacks are 0-affine.

Remarkably, in the theory nonconnective spectral DM stacks, there is an additional wealth of non-classical examples, as supplied by the following theorem of [49].

[6] One just has to repeat the proof in the twisted setting and use that a left adjoint preserves compact objects if its right adjoint preserves filtered colimits.

Theorem 5.2 ([49])**.** *Let $(\mathscr{X},\mathscr{O})$ be a nonconnective spectral DM stack such that $\mathscr{O}$ is weakly 2-periodic. Suppose that $(\mathscr{X},\pi_0\mathscr{O})$ is separated and noetherian and that the associated map $(\mathscr{X},\pi_0\mathscr{O})\to\mathscr{M}_{\mathrm{FG}}$ to the moduli of formal groups is quasi-affine and flat. Then $(\mathscr{X},\mathscr{O})$ is 0-affine.*

Our main example will be $(\mathscr{M},\mathscr{O})$, where $\mathscr{M}$ is the moduli stack of elliptic curve and $\mathscr{O}$ is the weakly 2-periodic sheaf of $\mathbb{E}_\infty$-ring spectra defined by Goerss, Hopkins and Miller [25]. Later, the nonconnective spectral DM stack $(\mathscr{M},\mathscr{O})$ was reinterpreted and reconstructed by Lurie to classify oriented spectral elliptic curves [43] and we will refer to it as the *derived moduli stack of elliptic curves*.

Corollary 5.3 ([49])**.** *The derived moduli stack $(\mathscr{M},\mathscr{O})$ of elliptic curves is 0-affine, i.e. $\Gamma\colon \mathrm{QCoh}(\mathscr{M},\mathscr{O})\xrightarrow{\simeq}\mathrm{Mod}_{\mathrm{TMF}}$ is an equivalence.*

The main point of this section is to show that for a 0-affine spectral DM stack, the canonical map $p\colon(\mathscr{X},\mathscr{O})\to\operatorname{Spec}\Gamma(\mathscr{X},\mathscr{O})$ induces an isomorphism $p^*\colon\mathrm{Br}(\operatorname{Spec}\Gamma(\mathscr{X},\mathscr{O}))\cong\mathrm{Br}(\mathscr{X},\mathscr{O})$.

Theorem 5.4. *If $(\mathscr{X},\mathscr{O})$ is a 0-affine nonconnective spectral DM stack with $R=\Gamma(\mathscr{X},\mathscr{O})$ and $p\colon(\mathscr{X},\mathscr{O})\to\operatorname{Spec}R$, then $p^*\colon\mathrm{Br}(R)\to\mathrm{Br}(\mathscr{X},\mathscr{O})$ is an isomorphism.*

Proof. By hypothesis, the functors $p^*\colon\mathrm{Mod}_R\to\mathrm{QCoh}(\mathscr{X},\mathscr{O})$ and $p_*\colon\mathrm{QCoh}(\mathscr{X},\mathscr{O})\to\mathrm{Mod}_R$ are symmetric monoidal adjoint equivalences. The functor p^* preserves Azumaya algebras. It is enough to show that p_* preserves Azumaya algebras. The condition that for an $\mathscr{O}$-algebra $\mathscr{A}$ we have $\mathscr{A}^{\mathrm{op}}\otimes_{\mathscr{O}}\mathscr{A}\simeq\mathbf{End}(\mathscr{A})$ is preserved by p_* since it is symmetric monoidal and hence also preserves internal mapping objects. We must see that if $\mathscr{A}$ is a perfect local generator of $\mathrm{QCoh}(\mathscr{X},\mathscr{O})$, then $p_*\mathscr{A}$ is a compact generator of Mod_R. However, since $\mathscr{O}$ is compact by the definition of 0-affineness, it follows that every perfect object is compact. In particular, $\mathscr{A}$ is compact in $\mathrm{QCoh}(\mathscr{X},\mathscr{O})$ and hence $p_*\mathscr{A}$ is compact in Mod_R. Now, we need to see that $p_*\mathscr{A}$ generates Mod_R. But, if $M\in\mathrm{Mod}_R$ is such that $\mathbf{Map}_R(p_*\mathscr{A},M)\simeq 0$, then $\mathbf{Map}_{\mathscr{O}}(\mathscr{A},p^*M)\simeq 0$ and hence $p^*M\simeq 0$ (since $\mathscr{A}$ is a perfect local generator). But, $M\simeq p_*(p^*M)$ so finally $M\simeq 0$. Thus, $p_*\mathscr{A}$ is a compact generator. □

Corollary 5.5. *If $(\mathscr{X},\mathscr{O})$ is a 0-affine nonconnective spectral DM stack with $R=\Gamma(\mathscr{X},\mathscr{O})$, then the isomorphism $\mathrm{Br}(R)\cong\mathrm{Br}(\mathscr{X},\mathscr{O})$ restricts to an injection $\mathrm{LBr}(R)\subseteq\mathrm{LBr}(\mathscr{X},\mathscr{O})$.*

Remark 5.6. The proof of Theorem 5.4 uses Azumaya algebras and does not say anything about cohomological Brauer classes.

Suppose that $(\mathscr{X},\mathscr{O})$ is a quasi-affine nonconnective spectral scheme. Thus, $(\mathscr{X},\mathscr{O})$ is a quasi-compact open inside $\operatorname{Spec}S$ for some commutative ring spectrum S. Let $R=\Gamma(\mathscr{X},\mathscr{O})$. Then, $(\mathscr{X},\mathscr{O})$ is 0-affine by [47, Proposition 2.4.1.4] so we see that $\mathrm{Br}(\mathscr{X},\mathscr{O})\cong\mathrm{Br}(R)$. Moreover, in this case we have $\mathrm{Br}(\mathscr{X},\mathscr{O})\cong\mathrm{Br}'(\mathscr{X},\mathscr{O})$ by Theorem 4.17, which applies because $(\mathscr{X},\mathscr{O})$ has a finite cover by affine schemes. In the next example, we use this to completely compute the Brauer group of a nonconnective $\mathbb{E}_\infty$-ring.

Example 5.7. Let $(\mathscr{X},\mathscr{O}_0)$ be the classical quasi-affine scheme given by the complement of 0 inside the affine space $\mathbb{A}^4_k=\operatorname{Spec}k[x_1,x_2,x_3,x_4]$ where k is some algebraically closed field. Let $\mathscr{O}=\mathscr{O}_0[S^1]=\mathscr{O}_0\otimes\Sigma^\infty_+S^1$ be the sheaf of $\mathbb{E}_\infty$-rings on $\mathscr{X}$ given by S^1-chains on $\mathscr{O}_0$. Thus, $\pi_0\mathscr{O}\cong\mathscr{O}_0$, $\pi_1\mathscr{O}\cong\mathscr{O}_0$, and all other homotopy sheaves vanish. Since $(\mathscr{X},\mathscr{O}_0)$ is normal, $\mathrm{H}^1(\mathscr{X},\mathbb{Z})=0$. By purity for the Brauer group, $\mathrm{H}^2(\mathscr{X},\mathbb{G}_m)\cong\mathrm{H}^2(\mathbb{A}^4_k,\mathbb{G}_m)=0$ [17]. Since $\mathscr{O}$ is connective, [3] gives that $\mathrm{Br}(\mathscr{X},\mathscr{O})\cong\mathrm{LBr}(\mathscr{X},\mathscr{O})$ (cf. Proposition 2.25). Thus, the only contribution to $\mathrm{Br}(\mathscr{X},\mathscr{O})$ in the descent spectral sequence in Construction 4.8 comes from $\mathrm{H}^3(\mathscr{X},\pi_1\mathscr{O})\cong\mathrm{H}^3(\mathscr{X},\mathscr{O}_0)\cong k[x_1^{-1},x_2^{-1},x_3^{-1},x_4^{-1}]\cdot(x_1\cdots x_4)^{-1}$.[7] All differentials out must vanish and the only thing that can hit this is $\mathrm{H}^1(\mathscr{X},\mathbb{G}_m)=\mathrm{Pic}(\mathscr{X})$, which vanishes as $\mathrm{Pic}(\mathbb{A}^4_k)=0$ and every line bundle extends (cf. e.g. the argument after (5.6) in [4]). Thus, with $R=\Gamma(\mathscr{O}_{\mathscr{X}})$ we obtain

$$\mathrm{Br}(R)\cong\mathrm{Br}(\mathscr{X},\mathscr{O})\cong\mathrm{Br}'(\mathscr{X},\mathscr{O})\cong k[x_1^{-1},x_2^{-1},x_3^{-1},x_4^{-1}]\cdot(x_1\cdots x_4)^{-1}.$$

Note that the descent sequence computing π_*R degenerates so that

$$\pi_nR\cong\begin{cases}k[x_1,x_2,x_3,x_4] & \text{if } n=0,1,\\ k[x_1^{-1},x_2^{-1},x_3^{-1},x_4^{-1}]\cdot(x_1\cdots x_4)^{-1} & \text{if } n=-3,-2,\text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

[7] Use the fact that $\mathrm{H}^3(\mathscr{X},\mathscr{O})$ is isomorphic to the local cohomology module $\mathrm{H}^4_{\{0\}}(\mathbb{A}^4_k,\mathscr{O})$; see [40, Example 7.16]. Alternatively one can use Čech cohomology as in the computation of the cohomology of $\mathbb{P}^3_k$.

Our computation gives examples of Brauer classes on a commutative ring spectrum R which are *not* killed by an étale cover. To see this, pick $\alpha \in \mathrm{Br}(R)$ and suppose that α is killed by an étale cover $R \to S$. Then, $S \otimes_R \mathcal{O}$ defines a new quasi-affine connective spectral DM stack $(\mathcal{Y}, \mathcal{O})$. (The underlying ∞-topos is naturally equivalent to $\mathcal{X} \times_{\mathbb{A}^4_k} \operatorname{Spec} \pi_0 S$.) By quasi-affineness, it follows that α restricts to 0 on $(\mathcal{Y}, \mathcal{O})$. However, the induced map on the Brauer group is

$$\mathrm{Br}(\mathcal{X}, \mathcal{O}) \cong \mathrm{H}^3(\mathcal{X}, \mathcal{O}_0) \to \mathrm{H}^3(\mathcal{Y}, \mathcal{O}_0) \cong \mathrm{Br}(\mathcal{Y}, \mathcal{O}).$$

This map is equivalent to

$$k[y_1, y_2, y_3, y_4] \to k[y_1, y_2, y_3, y_4] \otimes_{k[x_1,x_2,x_3,x_4]} \pi_0 S,$$

which is injective as $R \to S$ is faithfully flat. Thus, $\alpha = 0$ and so no nonzero class in the Brauer group can be killed by an étale cover.

6 The Picard sheaf of TMF

To compute the local Brauer group of TMF, it is necessary to first compute the Picard sheaf of TMF, which we will attack in this section. Our key tool is a sheafy version of the Picard spectral sequence used in [50], which we will introduce next.

Let $(\mathcal{M}, \mathcal{O})$ be the derived moduli stack of elliptic curves, where $\mathcal{O}$ denotes the Goerss–Hopkins–Miller–Lurie sheaf of $\mathbb{E}_\infty$-ring spectra. By Proposition 6.6, the descent spectral sequence identifies $\pi_0\mathrm{TMF}$ with $\mathrm{H}^0(\mathcal{M}, \pi_0\mathcal{O})$ and the latter one computes to be $\mathbb{Z}[j]$. Thus, the underlying classical morphism of $(\mathcal{M}, \mathcal{O}) \to \operatorname{Spec}\mathrm{TMF}$ is the map $j : \mathcal{M} \to \mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[j]$; we will denote $(\mathcal{M}, \mathcal{O}) \to \operatorname{Spec}\mathrm{TMF}$ by j as well

For every étale map $f :\ \operatorname{Spec} R \to \mathbb{A}^1$, we obtain an induced sheaf of $\mathbb{E}_\infty$-ring spectra $\mathcal{O}_R$ on the base change $\mathcal{M}_R = \mathcal{M} \times_{\mathbb{A}^1} \operatorname{Spec} R$. Let $\mathbf{Pic}_{\mathcal{O}_R}$ denote the Picard sheaf corresponding to $\mathcal{O}_R$ on $\mathcal{M}_R$ (with subscript left out if $\operatorname{Spec} R = \mathbb{A}^1$). We obtain a Picard spectral sequence $\mathrm{H}^s(\mathcal{M}_R; \pi_t\mathbf{Pic}_{\mathcal{O}_R}) \Rightarrow \pi_{t-s}\mathbf{Pic}(\mathcal{M}_R, \mathcal{O}_R)$. Sheafification thus yields a spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{R}^s j_*\pi_t\mathbf{pic}_{\mathcal{O}} \quad\Rightarrow\quad \pi_{t-s} j_*\mathbf{pic}_{\mathcal{O}} \underset{t-s\geq 0}{\cong} \pi_{t-s} j_*\mathbf{Pic}_{\mathcal{O}} \tag{6.1}$$

in the abelian category of étale sheaves of abelian groups on $\operatorname{Spec}\mathbb{Z}[j] = \mathbb{A}^1$. We note that $\mathbf{Pic}(\mathcal{M}_R, \mathcal{O}_R) \simeq \mathbf{Pic}(\mathrm{TMF}_R)$ where TMF_R is the étale extension of TMF realizing f. Indeed: the natural map

$$\mathrm{TMF}_R \to \mathcal{O}_R(\mathcal{M}_R) \simeq (\mathcal{O} \otimes_{\mathrm{TMF}} \mathrm{TMF}_R)(\mathcal{M})$$

is an equivalence since taking global sections and $\mathcal{O} \otimes_{\mathrm{TMF}} -$ are inverse equivalences between $\mathrm{QCoh}(\mathcal{M}, \mathcal{O})$ and $\mathrm{Mod}_{\mathrm{TMF}}$ by Corollary 5.3. Moreover, $(\mathcal{M}_R, \mathcal{O})$ is 0-affine by Theorem 5.2 and thus $\mathrm{QCoh}(\mathcal{M}_R, \mathcal{O}_R) \simeq \mathrm{Mod}_{\mathrm{TMF}_R}$. It follows that

$$j_*\mathbf{Pic}_{\mathcal{O}_{\mathcal{M}}} \simeq \mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}} \tag{6.2}$$

as sheaves of grouplike $\mathbb{E}_\infty$-spaces on $\operatorname{Spec}\mathbb{Z}[j]$.

As (6.1) arises as the sheafification of Picard spectral sequences, we can freely apply the tools from [50] for Picard spectral sequences. More precisely, these apply to the comparison to the analogous spectral sequence $\mathrm{R}^s j_*\pi_t\mathcal{O} \Rightarrow \pi_{t-s}\mathcal{O}_{\operatorname{Spec}\mathrm{TMF}}$. Viewing a quasi-coherent sheaf on $\operatorname{Spec}\pi_0\mathrm{TMF} \cong \operatorname{Spec}\mathbb{Z}[j]$ as a $\mathbb{Z}[j]$-module, this agrees with the usual descent spectral sequence for computing $\pi_*\mathrm{TMF}$, but remembering the $\mathbb{Z}[j]$-module structure. See in particular Proposition 6.7 for a precise statement we will be using.

Warning 6.3. In contrast to the descent spectral sequence for $\pi_*\mathrm{TMF}$, the Picard spectral sequence will in general not be $\mathbb{Z}[j]$-linear even in the range where its E_2-term agrees with a shift of the descent spectral sequence (i.e. for $t \geq 2$). We do, however, have $\mathbb{Z}[j]$-linearity in the range specified by Proposition 6.7 below. This should be seen in light of (a sheafy analogue of) [50, Corollary 5.2.3].

Remark 6.4. Alternatively, the sheafy Picard spectral sequence can be constructed as the relative descent spectral sequence for $j_*\mathbf{pic}_{\mathcal{O}}$, i.e. the spectral sequence associated to applying (sheafy) π_* to the tower $j_*\tau_{\leq\star}\mathbf{pic}_{\mathcal{O}}$. Indeed: the presheaf of Picard spectral sequence considered above is obtained by applying presheaf homotopy groups π_*^{pre} to the tower $j_*\tau_{\leq\star}\mathbf{pic}_{\mathcal{O}}$, and thus its sheafification agrees with the relative descent spectral sequence.

We will not compute the whole spectral sequence (6.1), but obtain the following result about the 0-stem, which will be crucial to our results about the local Brauer group.

Theorem 6.5. *The spectral sequence* (6.1) *induces a complete decreasing filtration* $\mathrm{F}^\star\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}}$ *on* $\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}}$ *with*

(0) $\mathrm{gr}^0\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}} \cong \mathbb{Z}/2$,

(1) $\mathrm{gr}^1\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}} \cong \mathrm{R}^1 j_*\mathbb{G}_m$, *which sits in an exact sequence*

$$0 \to (i_0)_*\mathbb{Z}/3 \oplus (i_{1728})_*\mathbb{Z}/2 \to \mathrm{R}^1 j_*\mathbb{G}_m \to \mathbb{Z}/2 \to 0$$

as established in Proposition 6.9,

(3) $\mathrm{gr}^3\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}} \cong k_* v_!\mathbb{Z}/2$, *where k and v denote the inclusions* $\operatorname{Spec}\mathbb{F}_2[j] \hookrightarrow \operatorname{Spec}\mathbb{Z}[j]$ *and* $\operatorname{Spec}\mathbb{F}_2[j^{\pm 1}] \hookrightarrow \operatorname{Spec}\mathbb{F}_2[j]$, *respectively,*

(5) $\mathrm{gr}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}}$ *is a sum of $b_*\mathbb{Z}/3$ and a subsheaf of an abelian sheaf $\mathcal{A}$, where $\mathcal{A}$ sits in a non-trivial extension*

$$0 \to \mathcal{O}/(2, j) \to \mathcal{A} \to a_*\mathbb{Z}/2 \to 0,$$

a is the closed inclusion of $\operatorname{Spec}\mathbb{F}_2$ *into* $\operatorname{Spec}\mathbb{Z}[j]$ *at* $j = 2 = 0$ *and b is the closed inclusion of* $\operatorname{Spec}\mathbb{F}_3$ *into* $\operatorname{Spec}\mathbb{Z}[j]$ *at* $j = 3 = 0$,

(7) $\mathrm{gr}^7\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}}$ *is a subsheaf of* $\mathcal{O}/(2, j)$;

all other graded pieces vanish.

In fact, in the last two items we describe the graded pieces as subsheaves of what we see on the E_6-page, but there are (at most) 2 more potential differentials originating from these spots.

The rest of this section will be devoted to the proof of the theorem. We will use Table 2 for notation for sheaves on $\operatorname{Spec}\mathbb{Z}[j]$ appearing in the spectral sequence (6.1). Fig. 3 on Page 24 shows the E_2-page of the spectral sequence (6.1). The general pattern follows from the work of Mathew–Stojanoska [50] and the computations of the homotopy groups of TMF, as in Bauer [9].

To prove Theorem 6.5, we show in the next subsection that there are no contributions in filtration degrees above 7. Then, we analyze each remaining filtration in turn.

Symbol	$\mathcal{O}$	$\mathcal{O}^\times$	$\bullet$	$\circ$	$\odot$	$\circledcirc$	$\circledast$	$\blacklozenge$	$\lozenge$
Sheaf	Structure sheaf	Units in $\mathcal{O}$	$\mathcal{O}/(2, j)$	$\mathbb{Z}/2$	$\mathcal{O}/2$	$i_{1728,*}\mathbb{Z}/2$	$k_* v_!\mathbb{Z}/2$	$\mathcal{O}/(3, j)$	$i_{0,*}\mathbb{Z}/3$

Table 2: An assortment of sheaves on $\operatorname{Spec}\mathbb{Z}[j]$.

6.1 High filtrations

In this section, we use the comparison tool of [50] to narrow down the possible filtration degrees computing to $\pi_0 j_*\mathbf{pic}_{\mathcal{O}_\mathcal{M}}$.

We use the following facts about the large-scale structure of the spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathcal{M}, \pi_t\mathcal{O}_\mathcal{M}) \Rightarrow \pi_{t-s}\mathrm{TMF},$$

which can be read off from the charts in [9] for tmf or [41] for Tmf by inverting the discriminant modular form Δ (or rather Δ^{24} since only this is a permanent cycle).

Proposition 6.6. (1) *The* E_∞*-page of the additive spectral sequence*

(a) *vanishes in columns* -1 *and* -2 *and*

(b) *vanishes above row* 0 *in column* 0.

(2) *The longest differential in the additive spectral sequence is a* d_{23}.

Here, column n always refers to $t - s = n$, i.e. to the column if drawn in Adams grading. We recall the following key tool from [50].

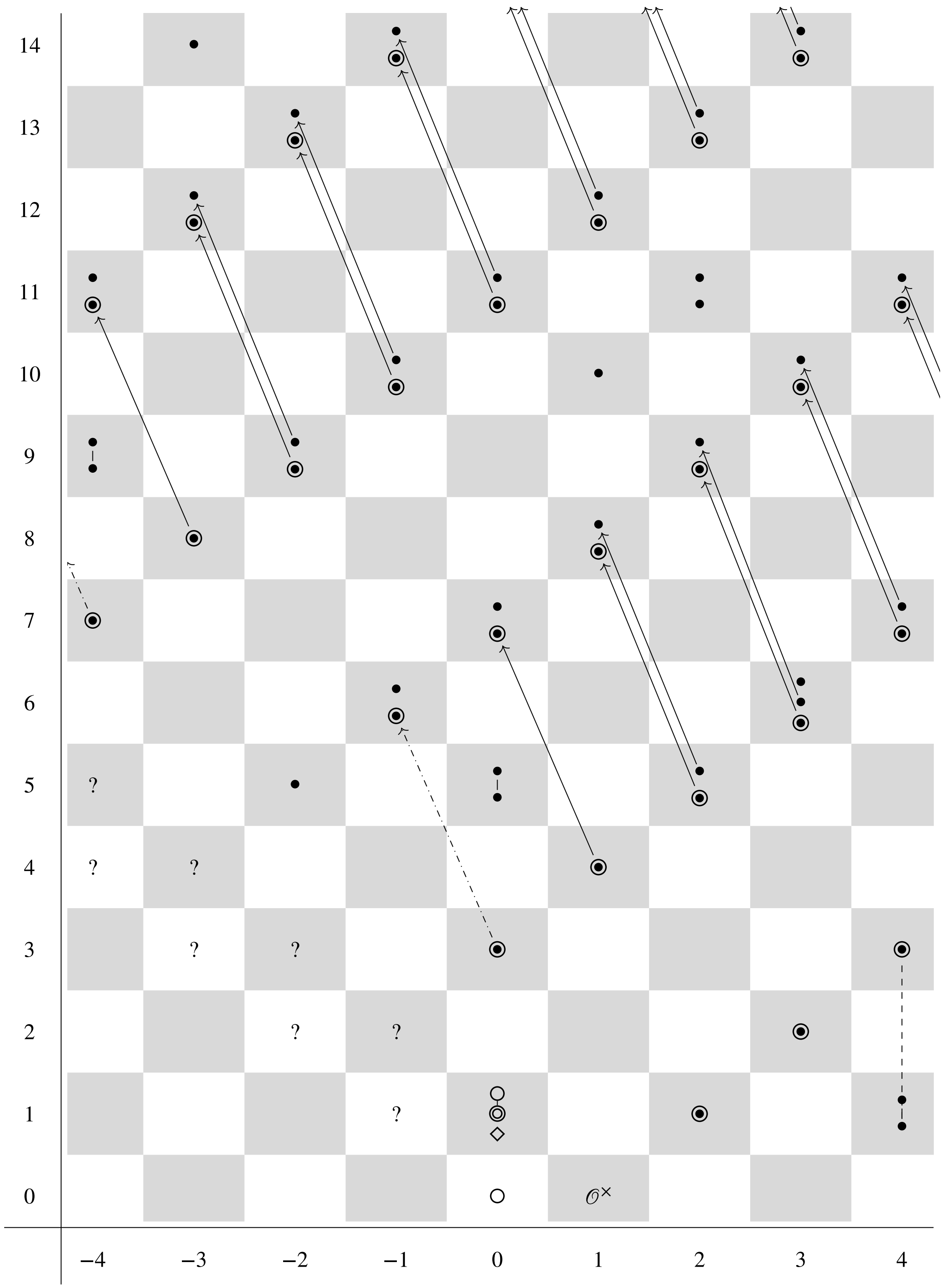

Figure 3: The E_3-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $i + j = 1$ diagonal, only 2-primary torsion information is shown.

Proposition 6.7 (Comparison Tool)**.** *For $2 \leq r \leq t-1$, $d_r^{s,t}$ in the Pic spectral sequence "is" $d_r^{s,t-1}$ in the additive spectral sequence.*

Proof. By [50, Comparison Tool 5.2.4], this is true for each term in the presheaf of Picard spectral sequence and is thus also true after sheafification. □

Using these results, we can indeed show that the Picard spectral sequence eventually vanishes in high enough degrees.

Proposition 6.8. *Everything above row 7 in column 0 vanishes in the E_∞-page of the Picard spectral sequence; likewise above row 30 in column -1. After inverting 2, the latter vanishing holds already above row 14.*

Proof. The claim about column 0 follows from the Comparison Tool (Proposition 6.7) and the further claim that in the additive spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathcal{M}, \pi_t \mathcal{O}_{\mathcal{M}}) \Rightarrow \pi_{t-s}\mathrm{TMF}$$

every spot in the (-1)-column above row 7 is killed by or supports a d_r-differential, which is an isomorphism and satisfies $r \leq t$ for $t = x + y$ being the antidiagonal of origin. Indeed, the corresponding differential also has to occur in the Picard spectral sequence and the isomorphism of groups becomes an isomorphism of quasi-coherent sheaves.

By inspection the further claim is true up to row 23 (on the E_5-page there is only one class in column -1 between row 7 and 24, namely in row 19 and this is killed by a d_9). As noted in Proposition 6.6, the longest possible differential is a d_{23} and the E_∞-term vanishes; thus everything above row 23 is killed by or supports a differential, which is at most a d_{23}. Moreover, by inspection, nothing in the additive spectral sequence in column 0 below row 23 supports a differential killing a class above row 23 in column -1.

The proof for column -1 of the Picard spectral sequence is analogous. □

The Proposition 6.8 implies that to prove Theorem 6.5 it is enough to analyze $\mathrm{gr}^n \pi_0 j_* \mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}$ for $0 \leq n \leq 7$.

6.2 Row 0

Since the geometric fibers of $j : \mathcal{M} \to \operatorname{Spec} \mathbb{Z}[j]$ are connected and $\pi_0 \mathbf{pic}_{\mathcal{O}_{\mathcal{M}}} \cong \mathbb{Z}/2$, we have $\mathrm{R}^0 j_* \mathbb{Z}/2 \cong \mathbb{Z}/2$. This term does not support any differentials since TMF[1] is a global section of the Picard sheaf which restricts to a generator of $\mathbb{Z}/2$ everywhere; this proves part (0) of Theorem 6.5.

6.3 Row 1 and the algebraic Picard sheaf

The next term to understand is $\mathrm{R}^1 j_* \mathbb{G}_m$, which appears on the E_2-page at $(s,t) = (1,1)$. This calculation is done on the classical moduli stack. The sheaf $\mathrm{R}^1 j_* \mathbb{G}_m$ is the sheafification of the presheaf that sends every étale $U \to \operatorname{Spec} \mathbb{Z}[j]$ to $\operatorname{Pic}(\mathcal{M} \times_{\operatorname{Spec} \mathbb{Z}[j]} U)$. Thus our next lemma can be seen as a sheafy analogue of the classical computation that $\operatorname{Pic}(\mathcal{M}) \cong \mathbb{Z}/12$ (see [27]), where a generator is given by the Hodge bundle λ that arises as the pushforward of the sheaf of differentials of the universal elliptic curve. We will indeed use the stronger result from [27] that the same isomorphism holds over any reduced and normal base ring R with vanishing Picard group. Moreover, we will use that $\operatorname{Pic}(\mathbb{A}^1_R) \cong \operatorname{Pic}(R)$ for any regular noetherian R, where $\mathbb{A}^1_R = \mathbb{A}^1 \times \operatorname{Spec} R$; this follows e.g. from the $\mathbb{A}^1$-invariance of the divisor class group as in [36, Proposition 6.6, Corollary 6.16].

Proposition 6.9. *Denote by $i_t : \operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \mathbb{Z}[j]$ the inclusion corresponding to the value t of the function j on $\operatorname{Spec} \mathbb{Z}[j]$ and by u_t the inclusion of its complement.*

The morphism $\operatorname{Pic}(\mathcal{M}) \cong \mathbb{Z}/12 \to \mathrm{R}^1 q_* \mathbb{G}_m$ is surjective with kernel $(u_0)_! \mathbb{Z}/3 \oplus (u_{1728})_! \mathbb{Z}/2$. Thus $\mathrm{R}^1 j_* \mathbb{G}_m$ sits in the extension

$$0 \to (i_0)_* \mathbb{Z}/3 \oplus (i_{1728})_* \mathbb{Z}/2 \to \mathrm{R}^1 j_* \mathbb{G}_m \to \mathbb{Z}/2 \to 0$$

that is pushed forward from the non-trivial extension of $\mathbb{Z}/3 \oplus \mathbb{Z}/2$ and $\mathbb{Z}/2$ of constant sheaves along the unit map $\mathbb{Z}/3 \oplus \mathbb{Z}/2 \to (i_0)_* \mathbb{Z}/3 \oplus (i_{1728})_* \mathbb{Z}/2$.

Proof. We will explain first why it suffices to show the surjectivity of $\mathbb{Z}/12 \to \mathrm{R}^1 q_* \mathbb{G}_m$ and identify its kernel. Note that there is an exact sequence

$$0 \to (u_t)_! u_t^* \mathcal{F} \to \mathcal{F} \to (i_t)_* i_t^* \mathcal{F} \to 0$$

for any t and any étale sheaf $\mathscr{F}$. Thus we obtain the claimed extension from the proposition by quotienting the first two terms of the exact sequence

$$0 \to \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to \mathbb{Z}/12 \to \mathbb{Z}/2 \to 0$$

by $(u_0)_!\mathbb{Z}/3 \oplus (u_{1728})_!\mathbb{Z}/2$ and using the snake lemma if indeed

$$0 \to (u_0)_!\mathbb{Z}/3 \oplus (u_{1728})_!\mathbb{Z}/2 \to \mathbb{Z}/12 \to \mathrm{R}^1 j_*\mathbb{G}_m \to 0$$

is exact. This exactness can be checked on the level of stalks, which is the content of the rest of the argument.

Let $\overline{x}\colon \operatorname{Spec} k \to \mathbb{A}^1$ be a geometric point (corresponding to some $j \in k$) and $x\colon \operatorname{Spec} k \to \mathscr{M}$ its unique lift. We will show that $\mathbb{Z}/12 \to (\mathrm{R}^1 j_*\mathbb{G}_m)_{\overline{x}}$ is surjective with the prescribed kernel. One can deduce from [1, Lemma 2.2.3] that the base change of $\mathscr{M}$ to the étale stalk of $\mathbb{A}^1$ at $\overline{x}$ is equivalent to the quotient stack $[\operatorname{Spec} S/\mathrm{Aut}(x)]$, where S is strictly Henselian with residue field k and $\mathrm{Aut}(x)$ is acting trivially on k (cf. [52, Proposition 8]). We can compute the stalk $(\mathrm{R}^1 j_*\mathbb{G}_m)_{\overline{x}}$ as $\mathrm{Pic}([\operatorname{Spec} S/\mathrm{Aut}(x)]) \cong \mathrm{H}^1(\mathrm{Aut}(x); \mathbb{G}_m(S))$. For the values of $\mathrm{Aut}(x)$ we refer to [62, Section III.10]. We will proceed with a case distinction based on j and the characteristic of k.

Case 1: $j \neq 0, 1728$:

If j is neither 0 nor 1728 in k, we have $\mathrm{Aut}(x) \cong C_2$, generated by $[-1]$. As the $[-1]$-automorphism is defined on all elliptic curves, $[\operatorname{Spec} S/\mathrm{Aut}(x)] \simeq BC_{2,S}$, i.e. the C_2-action on S is trivial (cf. [61, Lemma 3.2]). Hence $(\mathrm{R}^1 j_*\mathbb{G}_m)_{\overline{x}} \cong \mathrm{H}^1(C_2; \mathbb{G}_m(S)) \cong \mu_2(S) = \mathbb{Z}/2$.

Case 2: $\mathrm{char}(k) \neq 2, 3$ *and* $j = 0$ *or* 1728*:*

In general, if k is of characteristic $p \geq 0$, the group $\mathbb{G}_m(S)[\frac{1}{p}]$ is divisible (where $[\frac{1}{0}]$ is understood not to have any effect). Using the structure theory of divisible abelian groups and [63, Tag 06RR], one can show that $\mathbb{G}_m(S)[\frac{1}{p}]$ decomposes into a $\mathbb{Q}$-vector space and a torsion group, which maps isomorphically to $\mathbb{G}_m(k)[\frac{1}{p}] \cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ (cf. the proof of [52, Lemma 9]). We obtain

$$\mathrm{H}^1(\mathrm{Aut}(x); \mathbb{G}_m(S))[\tfrac{1}{p}] \cong \mathrm{H}^1(\mathrm{Aut}(x); \mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}]) \cong \mathrm{Hom}(\mathrm{Aut}(x); \mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}]).$$

If k is of characteristic not 2 or 3, we have $\mathrm{Aut}(x) \cong \mathbb{Z}/4$ if $j = 1728$ and $\mathrm{Aut}(x) \cong \mathbb{Z}/6$ if $j = 0$, which implies that the corresponding stalks of $\mathrm{R}^1 j_*\mathbb{G}_m$ are the Pontryagin duals of $\mathbb{Z}/4$ and $\mathbb{Z}/6$, i.e. isomorphic to $\mathbb{Z}/4$ and $\mathbb{Z}/6$ as well. (Note that in these cases $\mathrm{H}^1(\mathrm{Aut}(x); \mathbb{G}_m(S))$ is 12-torsion, so inverting p changes nothing.)

Concretely, the map $\mathbb{Z}/12 \cong \mathrm{Pic}(\mathscr{M}) \to \mathrm{Hom}(\mathrm{Aut}(x); \mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$ sends a line bundle $\mathscr{L}$ to the action of $\mathrm{Aut}(x)$ on $\mathscr{L}_x$ by the roots of unity $\mathbb{Q}/\mathbb{Z}[\frac{1}{p}] \cong \mu_\infty \subset \mathbb{G}_m(k)$. By the proof of [62, Theorem III.10.1], in our case a generator of $\mathrm{Aut}(x)$ acts by a fourth respectively a sixth root of unity on the invariant differential and thus on λ_x (for λ the standard generator of $\mathrm{Pic}(\mathscr{M})$ as above). Thus summarizing, we see that the map $\mathbb{Z}/12 \to (\mathrm{R}^1 j_*\mathbb{G}_m)_{\overline{x}}$ is surjective with the prescribed kernel unless $\mathrm{char}(k) = 2, 3$ and $\overline{x}$ corresponds to $j = 0 \equiv 1728$. In particular, we see that $\varphi\colon \mathbb{Z}/12 \to \mathrm{R}^1 q_*\mathbb{G}_m$ factors through $\mathscr{F} = (\mathbb{Z}/12)/(u_0)_!\mathbb{Z}/3 \oplus (u_{1728})_!\mathbb{Z}/2$.

Case 3: $\mathrm{char}(k) = 2$ *or* 3 *and* $j = 0 = 1728$*:*

From now on let $\overline{x}\colon k \to \mathbb{A}^1$ be a geometric point with $\mathrm{char}(k) = p$ for $p = 2, 3$ corresponding to $j = 0$. For a base ring R, denote by $\mathscr{M}_R$ the base change $\mathscr{M} \times \operatorname{Spec} R$. We will show that $\varphi_{\overline{x}}\colon \mathbb{Z}/12 \to (\mathrm{R}^1 j_*\mathbb{G}_m)_{\overline{x}}$ is an isomorphism by comparison with the known computation of the Picard group of $\mathrm{Pic}(\mathscr{M}_R)$ for certain R. To that purpose we will use the Leray spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathbb{A}^1_R; \mathrm{R}^t j^R_*\mathbb{G}_m) \Rightarrow \mathrm{H}^{s+t}(\mathscr{M}_R; \mathbb{G}_m)$$

for the map $j^R\colon \mathscr{M}_R \to \mathbb{A}^1_R$. Let us display the part relevant for the computation of Pic.

$$\begin{tikzcd}
\mathrm{H}^0(\mathbb{A}^1_R; \mathrm{R}^1 j^R_*\mathbb{G}_m) \arrow[drr] & \mathrm{H}^1(\mathbb{A}^1_R; \mathrm{R}^1 j^R_*\mathbb{G}_m) & \\
\mathbb{G}_m(\mathbb{A}^1_R) & \mathrm{Pic}(\mathbb{A}^1_R) & \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m)
\end{tikzcd}$$

Denoting by R the strict Henselization of the image of $\overline{x}$ in Spec $\mathbb{Z}$, the spectral sequence implies that the map $\Phi_R\colon \mathbb{Z}/12 \cong \mathrm{Pic}(\mathcal{M}_R) \to \mathrm{H}^0(\mathbb{A}^1_R; \mathrm{R}^1 j^R_* \mathbb{G}_m)$ is an isomorphism. Here we use that as $\mathbb{Z}$ is regular noetherian, R is so as well by [63, Tags 06LJ, 06LN] and thus $\mathrm{Pic}(\mathbb{A}^1_R) \cong \mathrm{Pic}(R) = 0$ and

$$\mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \cong \mathrm{H}^2(R, \mathbb{G}_m) \cong \mathrm{H}^2(\mathrm{Spec}\,\mathbb{F}_p, \mathbb{G}_m) = 0$$

by Theorem 2.5 and Theorem 2.7. The same argument shows that the map $\Phi_{\overline{\mathbb{F}}_p}\colon \mathbb{Z}/12 \cong \mathrm{Pic}(\mathcal{M}_{\overline{\mathbb{F}}_p}) \to \mathrm{H}^0(\mathbb{A}^1_{\overline{\mathbb{F}}_p}; \mathrm{R}^1 j^{\overline{\mathbb{F}}_p}_* \mathbb{G}_m)$ is an isomorphism.

If $\varphi_{\overline{x}}$ were not injective, there would be some $[m] \in \mathbb{Z}/12$ (namely [4] or [6]) such that $\varphi([m])$ is zero in every étale stalk in characteristic p and hence $\Phi_{\overline{\mathbb{F}}_p}$ would not be injective either. We see that $\varphi_{\overline{x}}$ is thus indeed injective and thus $\mathcal{F}$ agrees with the image of φ.

As above we use the notation $\mathcal{F} = (\mathbb{Z}/12)/(u_0)_!\mathbb{Z}/3 \oplus (u_{1728})_!\mathbb{Z}/2$. Denote the cokernel of $\mathbb{Z}/12 \to \mathcal{F} \to \mathrm{R}^1 j_* \mathbb{G}_m$ by $\mathcal{C}$. Its base change $\mathcal{C}_R$ agrees with the cokernel of $\mathcal{F}_R \to \mathrm{R}^1 j^R_* \mathbb{G}_m$ and it suffices to show its vanishing. By the arguments of the first paragraph, we have a short exact sequence

$$0 \to (i_0)_*\mathbb{Z}/3 \oplus (i_{1728})_*\mathbb{Z}/2 \to \mathcal{F} \to \mathbb{Z}/2 \to 0.$$

Moreover, Theorem 2.9 implies $\mathrm{H}^1(\mathbb{A}^1_R; (i_0)_*\mathbb{Z}/3 \oplus (i_{1728})_*\mathbb{Z}/2) \cong \mathrm{H}^1(R; \mathbb{Z}/3 \oplus \mathbb{Z}/2) = 0$ since the étale cohomology of anything on R vanishes. Thus $\mathbb{Z}/12 \to \mathrm{H}^0(\mathbb{A}^1_R; \mathcal{F}_R)$ is an isomorphism.

Moreover, $\mathrm{H}^1(\mathbb{A}^1_R; \mathbb{Z}/2) \cong \mathrm{H}^1(\mathbb{A}^1_R; \mu_2)$ sits in a short exact sequence with $\mathrm{H}^1(\mathbb{A}^1_R; \mathbb{G}_m)/2 = 0$ and $\mathrm{H}^2(\mathbb{A}^1_R; \mathbb{G}_m)[2] = 0$ and thus has to vanish as well. We conclude that $\mathrm{H}^1(\mathbb{A}^1_R; \mathcal{F}_R) = 0$. Summarizing, we have an exact sequence

$$0 \to \mathrm{H}^0(\mathbb{A}^1_R; \mathcal{F}_R) \to \mathrm{H}^0(\mathbb{A}^1_R; \mathrm{R}^1 j^R_* \mathbb{G}_m) \to \mathrm{H}^0(\mathbb{A}^1_R; \mathcal{C}_R) \to 0 = \mathrm{H}^1(\mathbb{A}^1_R, \mathcal{F}_R).$$

We have seen above that the natural morphisms from $\mathbb{Z}/12$ to the first two non-trivial groups are isomorphisms and thus the map between them is an isomorphism. Thus $\mathrm{H}^0(\mathbb{A}^1_R; \mathcal{C}_R)$ vanishes. As $\mathcal{C}_R$ is supported at $\overline{x}$, we see that its stalk at $\overline{x}$ vanishes and thus that $\varphi_{\overline{x}}$ is also surjective. □

We claim that there are no differentials out of $\mathrm{E}_2^{1,1} = \mathrm{R}^1 j_* \mathbb{G}_m$. Indeed, by the preceding proposition

$$\mathrm{Pic}(\mathcal{M}, \pi_0 \mathcal{O}_{\mathcal{M}}) \cong \mathbb{Z}/12 \to \mathrm{R}^1 j_* \mathbb{G}_m$$

is a surjective map of sheaves on $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{Z}[j]$. Thus, as long as the classes of $\mathbb{Z}/12$ lift to invertible sheaves on the derived moduli stack, surjectivity of the map means there cannot be differentials. But, this $\mathbb{Z}/12$ is generated by TMF[2], which gives part (1) of Theorem 6.5.

6.4 3-torsion in Row 5

For higher filtrations it is necessary to compute differentials. Differentials $d_r^{t,s}$ in the sheafified Pic spectral sequence where $r \le t-1$ (i.e., where the "length" of the differential is smaller than the coordinate $t = x + y$ of the antidiagonal of origin) can be directly read off the descent spectral sequence computing $\pi_*\mathrm{TMF}$ by Proposition 6.7. We will use this fact without further comment.

For the rest of the analysis, we will work separately with 2 and 3 inverted, to analyze the 3 and 2-torsion, respectively. The only possible contribution to 3-torsion in $\pi_0 j_* \mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}$ is the kernel of the d_9 differential on $\mathrm{E}_9^{5,5}$. We will implicitly invert 2 throughout this section.

Lemma 6.10. *The differential*

$$d_9\colon \mathrm{R}^5 j_* \pi_5 \mathbf{pic}_{\mathcal{O}_{\mathcal{M}}} \cong \mathrm{R}^5 j_* \pi_4 \mathcal{O}_{\mathcal{M}} \cong \mathcal{O}/(3,j) \to \mathrm{R}^{14} j_* \pi_{13} \mathbf{pic}_{\mathcal{O}_{\mathcal{M}}} \cong \mathrm{R}^{14} j_* \pi_{12} \mathcal{O}_{\mathcal{M}} \cong \mathcal{O}/(3,j).$$

is surjective and the kernel is $b_*\mathbb{Z}/3$, where b is the closed inclusion of Spec $\mathbb{F}_3$ *into* Spec $\mathbb{Z}[j]$ *at* $j = 3 = 0$.

Proof. The differential is the first possible outside of the exponentiable range from Proposition 6.7. In the descent spectral sequence for $\pi_*\mathrm{TMF}$ the corresponding differential is an isomorphism (see [50, (8-4)]). In contrast, [37, Theorem 7.1] implies that we can write the differential $d_9\colon \mathcal{O}/(3,j) \to \mathcal{O}/(3,j)$ in the Picard spectral sequence as

$$x \mapsto x + \zeta\beta\mathcal{P}^2(x),$$

where ζ is a unit in $\mathbb{F}_3$ and $\beta\mathscr{P}^2$ is certain power operation on $\mathbb{E}_\infty$-rings in which 2 is invertible. Moreover, $x \mapsto \zeta\beta\mathscr{P}^2(x)$ is Frobenius-semilinear in the sense that $zx \mapsto \zeta\beta\mathscr{P}^2(zx) = z^3\zeta\beta\mathscr{P}^2(x)$.

We know that our d_9 must be zero on global sections.by [50, Sec. 8.1] (as else $\mathrm{Pic}(\mathrm{TMF})_{(3)}$ could have at most 3 elements). Thus, $1 + \zeta\beta\mathscr{P}^2(1) = d_9(1)$ is zero on global sections and thus also everywhere, and hence $\zeta\beta\mathscr{P}^2(1) = -1$. By Frobenius-semilinearity, we see that $d_9(z) = z + \zeta\beta\mathscr{P}^2(z \cdot 1) = z - z^3$. It follows from Artin–Schreier theory that this differential is a surjective map of étale sheaves and that the kernel is $b_*\mathbb{Z}/3$ (cf. [53, Example 2.18c]). □

As Fig. 4 on Page 29 proves, the lemma shows that Part (5) of Theorem 6.5 holds with 2 inverted and all the other graded pieces vanish with 2 inverted. Thus it remains to analyze the 2-torsion and we will implicitly *work 2-locally everywhere*. We will compute two further differentials affecting the zeroth column of the Picard spectral sequence and then give an outlook on what remains to be done to compute all remaining differentials.

6.5 Row 3

There is a d_3-differential,

$$d_3\colon\ \mathrm{R}^3 j_*\pi_3\mathbf{pic}_{\mathscr{O}_{\mathscr{M}}} \cong \mathrm{R}^3 j_*\pi_2\mathscr{O}_{\mathscr{M}} \cong \mathscr{O}/2 \to \mathrm{R}^6 j_*\pi_5\mathbf{pic}_{\mathscr{O}_{\mathscr{M}}} \cong \mathrm{R}^6 j_*\pi_4\mathscr{O}_{\mathscr{M}} \cong \mathscr{O}/2.$$

This differential is of the form

$$x \mapsto x + jx^2 = x(1 + jx),$$

as shown in [50, Sec. 8.2]. Recall from [53, Corollary II.3.11] that k_* for $k\colon\ \operatorname{Spec}\mathbb{F}_2[j] \to \operatorname{Spec}\mathbb{Z}[j]$ induces an exact equivalence between étale sheaves on $\mathbb{F}_2[j]$ and étale sheaves on $\mathbb{Z}[j]$ supported at the prime (2); hence, we can work directly on the étale site of $\mathbb{F}_2[j]$. Under this equivalence we have $k_*\mathscr{O} \cong \mathscr{O}/2$. We claim that the $d_3\colon\ \mathscr{O} \to \mathscr{O}$ is surjective (viewed as étale sheaves on $\operatorname{Spec}\mathbb{F}_2[j]$). Indeed, given any étale morphism $\mathbb{F}_2[j] \to R$ and element $c \in R$, the extension $R \to R[x]/(jx^2 + x - c)$ is étale and surjective on geometric points: base-changing along any morphism $R \to K$ to a field, j becomes either invertible or zero and in either case $K[x]/(jx^2 + x - c)$ is nonzero. Thus, $\operatorname{Spec} R[x]/(jx^2+x-c) \to \operatorname{Spec} R$ is an étale cover and c has per construction a preimage under d_3 on $R[x]/(jx^2+x-c)$.

Any nonzero element in the stalk of the kernel must be of the form $x = \frac{1}{j}$ (since all stalks of $\mathscr{O}$ on $\mathbb{F}_2[j]$ are integral domains); thus the kernel corresponds to the étale sheaf $v_!\mathbb{Z}/2$ on $\mathbb{F}_2[j]$ (with $v\colon\ \operatorname{Spec}\mathbb{F}_2[j^{\pm 1}] \to \operatorname{Spec}\mathbb{F}_2[j]$ being the inclusion) and is $k_*v_!\mathbb{Z}/2$ as an étale sheaf on $\mathbb{Z}[j]$. There will be no further differentials from this spot because all possible further targets are supported at $(2, j)$.

6.6 2-torsion in Row 5

The next differential is the d_5-differential,

$$d_5\colon\ \mathrm{R}^5 j_*\pi_5\mathbf{pic}_{\mathscr{O}_{\mathscr{M}}} \cong \mathrm{R}^5 j_*\pi_4\mathscr{O}_{\mathscr{M}} \cong \mathscr{O}/(4, j) \to \mathrm{R}^{10} j_*\pi_9\mathbf{pic}_{\mathscr{O}_{\mathscr{M}}} \cong \mathrm{R}^{10} j_*\pi_8\mathscr{O}_{\mathscr{M}} \cong \mathscr{O}/(2, j),$$

which factors through a map $\mathscr{O}/(2, j) \to \mathscr{O}/(2, j)$. This differential is just outside of the exponentiable range and is given by [50, Thm. 6.1.1]. The map $\mathscr{O}/(2, j) \to \mathscr{O}/(2, j)$ is given by $x \mapsto x + x^2$. This is a surjective map of étale sheaves by Artin–Schreier theory: given $y \in \mathscr{O}/2$, the extension defined by $y = x + x^2$ is étale. The kernel is $a_*\mathbb{Z}/2$, where $a\colon\ \operatorname{Spec}\mathbb{F}_2 \to \operatorname{Spec}\mathbb{Z}[j]$ is the inclusion at $2 = j = 0$.

6.7 Long differentials

As already established in Proposition 6.8, in column 0 everything above row 7 must be zero on the E_∞-page. As Fig. 6 on Page 31 and the preceding discussion shows, the only remaining possible differentials are a d_{13} and d_{25} originating in row 5 and a d_{11} and a d_{23} originating in row 7. We can show the vanishing of one of these differentials.

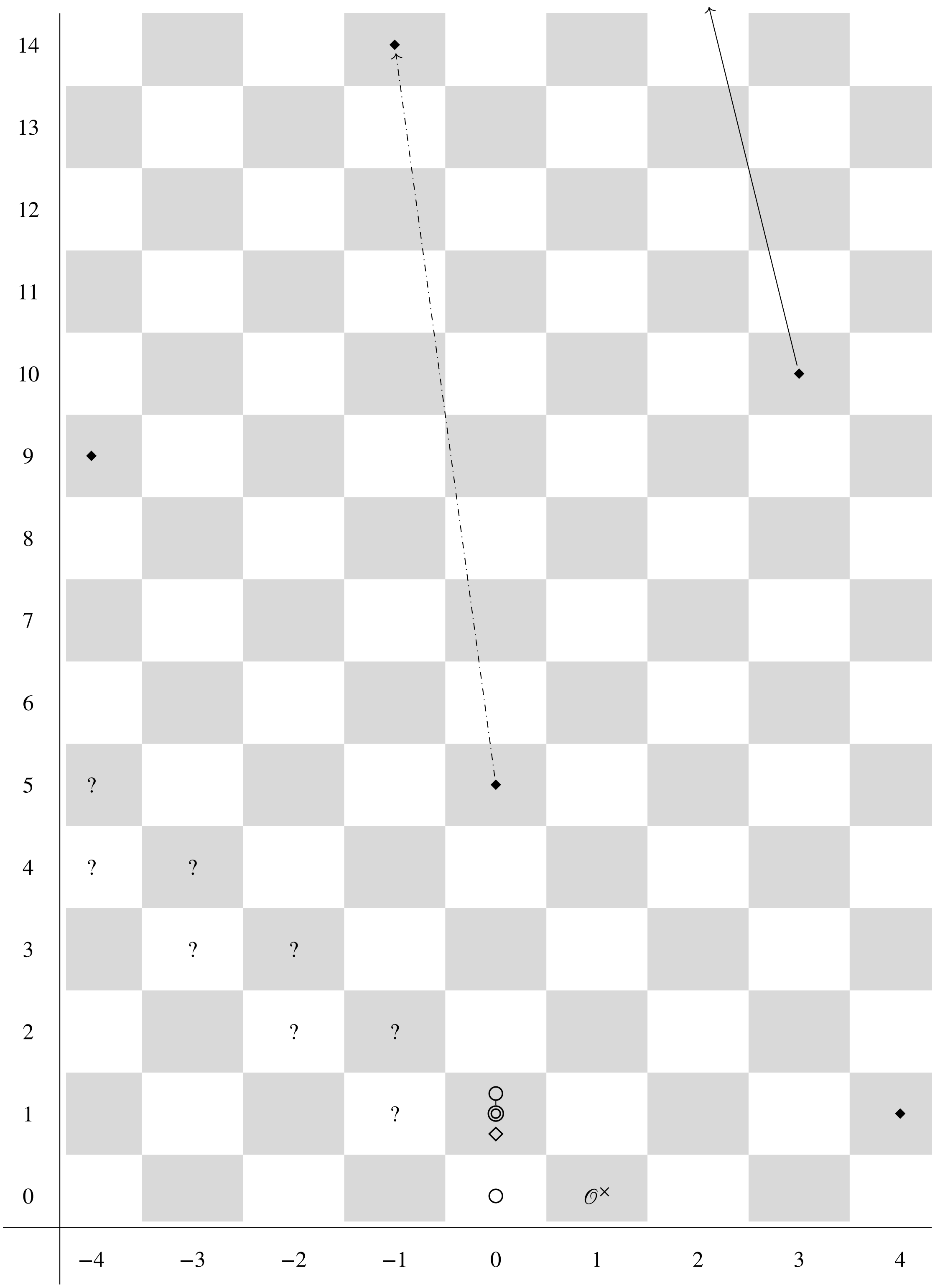

Figure 4: The E_5-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $i + j = 1$ diagonal, only 3-primary torsion information is shown.

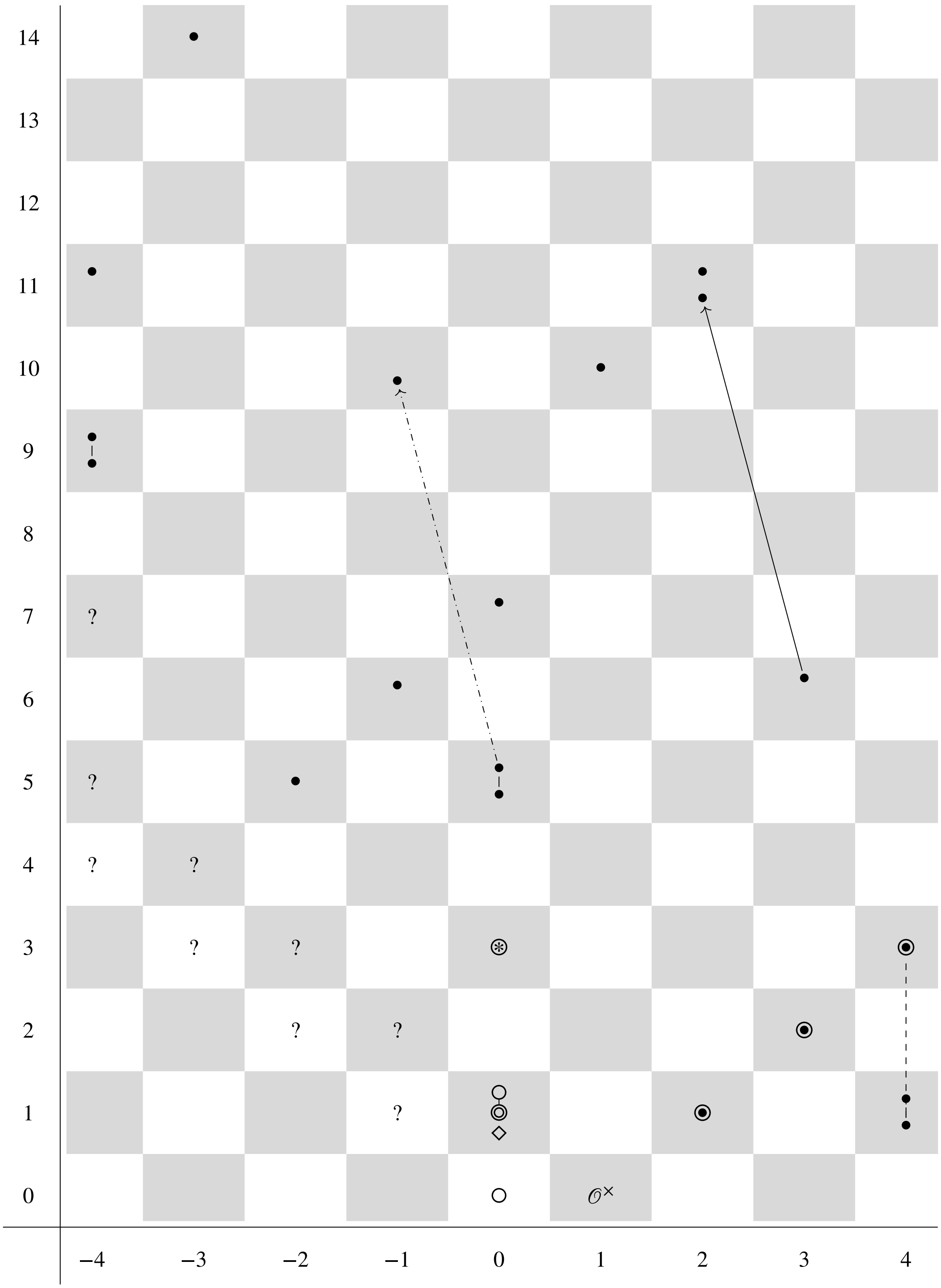

Figure 5: The E_5-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $i + j = 1$ diagonal, only 2-primary torsion information is shown.

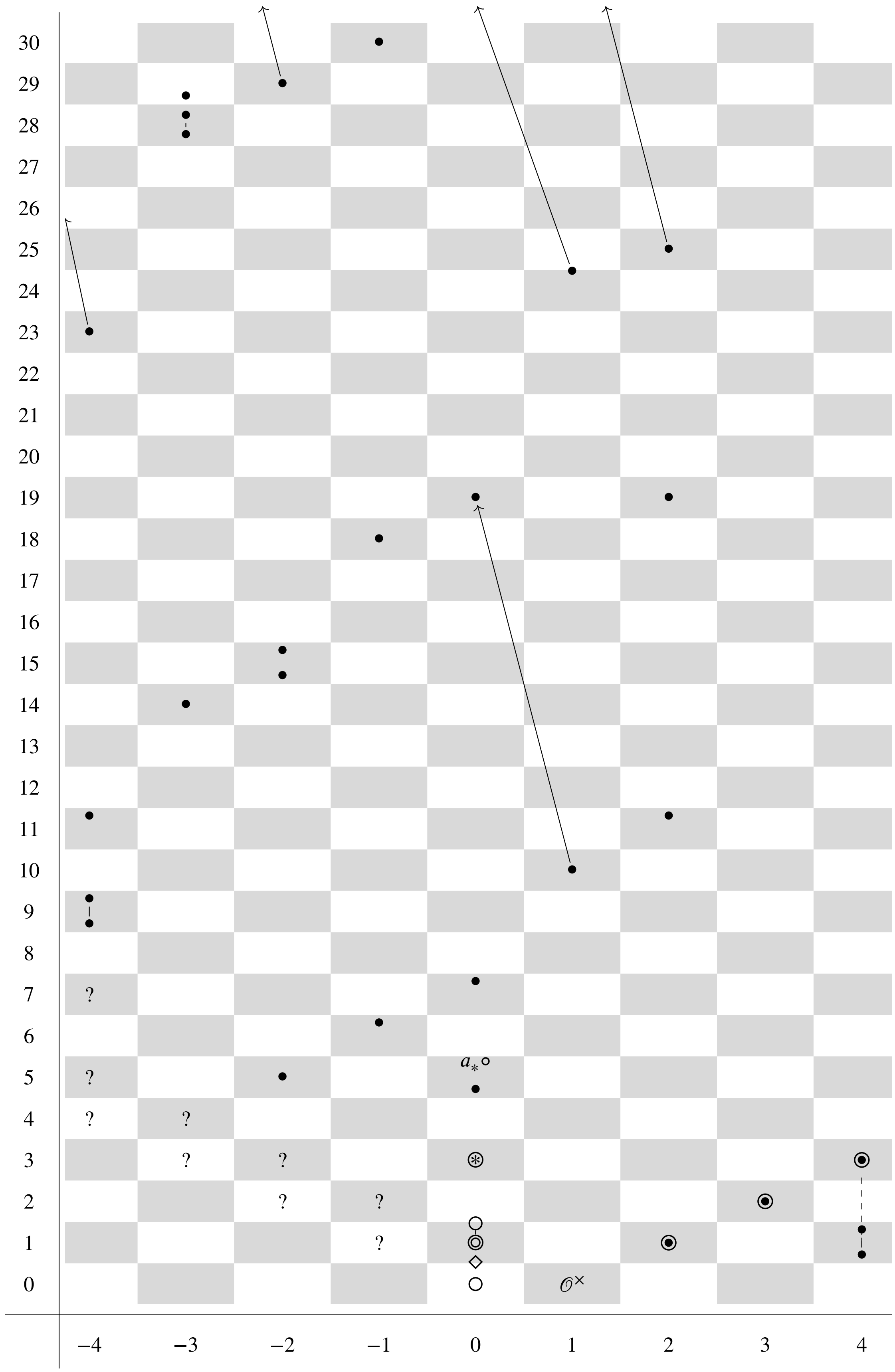

Figure 6: The E_7-page of the sheafy spectral sequence (6.1) for Pic of the moduli stack. Above the $i + j = 1$ diagonal, only 2-primary torsion information is shown.

Lemma 6.11. *The differential*

$$d_{11} : \mathrm{E}_4^{7,7} \cong \mathrm{E}_{11}^{7,7} \cong \mathscr{O}/(2,j) \to \mathrm{E}_{11}^{18,17} \cong \mathscr{O}/(2,j).$$

is zero.

Proof. We use the sequence of maps of derived stacks

$$\widehat{\mathscr{M}}(3)^{\mathrm{ss}} \to \mathscr{M}(3) \to \mathscr{M}[\tfrac{1}{3}],$$

where $\mathscr{M}(3)$ is the moduli stack of elliptic curves with full level-3-structure (see e.g. [64]) and $\widehat{\mathscr{M}}(3)^{\mathrm{ss}}$ is the completion of $\mathscr{M}(3)$ at the ideal $(2,j)$. We write j for the map from any of these stacks to Spec $\mathbb{Z}[j]$. Pushing the Picard sheaves down to Spec $\mathbb{Z}[j]$ we obtain

$$j_*\mathbf{pic}_{\mathscr{O}_{\mathscr{M}}} \to j_*\mathbf{pic}_{\mathscr{O}_{\mathscr{M}[1/3]}} \to j_*\mathbf{pic}_{\mathscr{O}_{\widehat{\mathscr{M}}(3)^{\mathrm{ss}}}}.$$

Since $\mathscr{M}(3) \to \mathscr{M}[\frac{1}{3}]$ is a Galois cover with group $\mathrm{GL}_2(\mathbb{Z}/3)$, the first of these maps induces an equivalence

$$\tau_{\geq 0} j_*\mathbf{pic}_{\mathscr{O}_{\mathscr{M}[1/3]}} \to \tau_{\geq 0}\left(j_*\mathbf{pic}^{h\mathrm{GL}_2(\mathbb{Z}/3)}_{\mathscr{O}_{\mathscr{M}(3)}}\right); \tag{6.12}$$

this kind of Galois descent follows from the sheaf property of **pic**. The 2-local Picard spectral sequence for TMF is for $t > 1$ the same as the $\mathrm{GL}_2(\mathbb{Z}/3)$-based relative descent spectral sequence for (6.12) (cf. Remark 6.4).

Completing $\mathscr{M}(3)$ at the ideal $(2,j) \subset \mathbb{Z}[j]$ results in the formal deformation space of a supersingular elliptic curve C over $\mathbb{F}_4$, which can be coordinatized as Spec $W(\mathbb{F}_4)[\![u]\!]$. Thus completing $\mathscr{M}$ itself at $(2,j)$ becomes identified with the stack quotient of Spec $W(\mathbb{F}_4)[\![u]\!]$ by $\mathrm{GL}_2(\mathbb{Z}/3)$. As source and target of the d_{11}-differential we care about are supported at $(2,j)$, completion at $(2,j)$ does not lose information; more precisely, the composition of $(2,j)$-completion and pushing forward along Spf $\mathbb{Z}_2[\![j]\!] \to$ Spec $\mathbb{Z}[j]$ is an isomorphism at the relevant spots in the sheafy Picard spectral sequence. Note that the étale topos of Spf $\mathbb{Z}_2[\![j]\!]$ agrees with that of $\mathbb{F}_2$. Thus the $(2,j)$-completed sheafy Picard spectral sequence is the sheafification of the collection of Picard spectral sequences of the higher real K-theories, assigning $E_2(\widehat{C}, \mathbb{F}_4 \otimes_{\mathbb{F}_2} k)^{h\mathrm{GL}_2(\mathbb{Z}/3)}$ to each finite extension $\mathbb{F}_2 \subset k$.

Denote the corresponding spectral sequence for $E_2(\widehat{C}, \mathbb{F}_4 \otimes_{\mathbb{F}_2} k)^{hH}$ with $H \subset \mathrm{GL}_2(\mathbb{F}_3)$ by $\mathrm{E}^{**}_{*,k,H}$, where we consider $k = \mathbb{F}_{2^n}$ or $\overline{\mathbb{F}}_2$. We will deduce from [13] that the restriction map $\mathrm{res}^{G_{48}}_{C_4} : \mathrm{E}^{s,t}_{2,\mathbb{F}_2,\mathrm{GL}_2(\mathbb{Z}/3)} \to \mathrm{E}^{s,t}_{2,\mathbb{F}_2,C_4}$ is zero for $(s,t) = (7,7)$ and an isomorphism for $(18,17)$ on the E_4-page (which agrees in this range in descent and Picard spectral sequence). Hence the same is true for all k in place of $\mathbb{F}_2$ (since the restriction maps are module maps). Moreover, we have established above that in the Picard spectral sequence $\mathrm{E}_4 = \mathrm{E}_{11}$ in these spots. As $d_{11}\,\mathrm{res}^{G_{48}}_{C_4} = \mathrm{res}^{G_{48}}_{C_4}\,d_{11}$, this implies the vanishing of our d_{11}.

To read off our claim about the restriction map from [13], we will use their notation (noting that they write G_{48} for $\mathrm{GL}_2(\mathbb{Z}/3)$). Section 2.3 of op.cit. implies that the generator of $\mathrm{E}^{7,7}_{2,\mathbb{F}_2,\mathrm{GL}_2(\mathbb{Z}/3)}$ is $\Delta^{-1}\eta^3\overline{\kappa}$. Since Δ is a d_3-cycle, it will suffice for our vanishing claim about the restriction to show that $\mathrm{res}^{G_{48}}_{C_4}(\eta^3\overline{\kappa})$ is hit by a d_3. This restriction equals $\eta^3\delta\xi^2$ by the Table above Section 2.3 of op.cit. We have the differential $d_3(\xi) = \delta^{-1}\eta\xi^2$ by [10, Proposition 2.3.1],[8] so $d_3(\delta^2\eta^2\xi) = \delta\eta^3\xi^2 = \mathrm{res}^{G_{48}}_{C_4}(\eta^3\overline{\kappa})$. Now we turn to the generator of $\mathrm{E}^{18,17}_{4,\mathbb{F}_2,\mathrm{GL}_2(\mathbb{Z}/3)}$, which is $\Delta^{-4}\kappa\overline{\kappa}^4$. Using [10, Lemma 2.2.4, Corollary 2.3.2], $\mathrm{res}^{G_{48}}_{C_4}(\Delta)$ acts like δ^3 on a torsion class like $\mathrm{res}^{G_{48}}_{C_4}(\kappa\overline{\kappa}^4) = \delta^5\nu^2\xi^8$. Thus, $\Delta^{-4}\kappa\overline{\kappa}^4$ restricts to $\delta^{17}\nu^2\xi^8$. This can be used to show that the restriction is an isomorphism on $\mathrm{E}^{18,17}_4$. Moreover, the class in $\mathrm{E}^{18,17}_{4,\mathbb{F}_2,\mathrm{GL}_2(\mathbb{Z}/3)}$ is only hit by a d_{15} and its restriction by a d_{13} (namely from $(\delta\nu^2)\delta^{20}(\delta\nu\xi)$; cf. [10, Proposition 2.3.9]). In particular, in these spots the E_4-page equals the E_{11}-page. □

We included this lemma not primarily for its intrinsic importance, but rather to demonstrate that the remaining computational mysteries of the sheafy Picard spectral sequence are purely $K(2)$-local phenomena and might thus potentially be resolved purely in the setting of Lubin–Tate spectra.

[8] As in [13], we will use [10] for information about differentials on the C_4-level, using that $\mathrm{TMF}_1(5)$ becomes a Lubin–Tate theory after $K(2)$-localization.

7 Applications to Picard groups

We can use Theorem 6.5 to compute Picard groups of various spectra related to TMF.

Example 7.1. Using Theorem 6.5, we want to compute $\mathrm{Pic}(\mathrm{TMF}[c_4^{-1}])$. Noting $\pi_0\mathrm{TMF}[c_4^{-1}] \cong \mathbb{Z}[j^{\pm 1}]$, the relevant part of the exact sequence from Proposition 2.25 is

$$0 \to \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}];\mathbb{G}_m) \to \mathrm{Pic}(\mathrm{TMF}[c_4^{-1}]) \to \mathrm{H}^0(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}],\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}) \to \mathrm{H}^2(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}];\mathbb{G}_m) \to \cdots.$$

The groups

$$\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}];\mathbb{G}_m) = \mathrm{Pic}(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}]) \subset \mathrm{Pic}(\operatorname{Spec}\mathbb{Z}[j]) \cong \mathrm{Pic}(\mathbb{Z})$$

and

$$\mathrm{H}^2(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}];\mathbb{G}_m) = \mathrm{Br}(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}])$$

vanish since $\mathbb{Z}$ is a PID and Pic is $\mathbb{A}^1$-invariant by [36, Proposition II.6.6], and by Corollary 2.6. Thus

$$\mathrm{Pic}(\mathrm{TMF}[c_4^{-1}]) \to \mathrm{H}^0(\operatorname{Spec}\mathbb{Z}[j^{\pm 1}];\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}})$$

is an isomorphism. By Theorem 6.5, the restriction of $\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}$ to $\operatorname{Spec}\mathbb{Z}[j^{\pm 1}]$ has a filtration with associated graded $\mathbb{Z}/2$, $\mathbb{Z}/2$, $(i_{1728})_*\mathbb{Z}/2$ and $k_*\mathbb{Z}/2$, where $k\colon \operatorname{Spec}\mathbb{F}_2[j^{\pm 1}] \to \operatorname{Spec}\mathbb{Z}[j^{\pm 1}]$ is the inclusion. We obtain directly that $\mathrm{Pic}(\mathrm{TMF}[c_4^{-1}])$ is 2-power torsion.

Let $\mathcal{Q}$ be the quotient of $(\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}})_{(2)}$ by everything of filtration at least 2. We obtain a short exact sequence

$$0 \to k_*\mathbb{Z}/2 \to u^*\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}} \to u^*\mathcal{Q} \to 0, \tag{7.2}$$

where $u\colon \operatorname{Spec}\mathbb{Z}[j^{\pm 1}] \to \operatorname{Spec}\mathbb{Z}[j]$ is the inclusion. Applying the long exact sequence in cohomology to this short exact sequence and to the analogous one for TMF, we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/8 \arrow[r] \arrow[d] & \mathrm{Pic}(\mathrm{TMF})_{(2)} \arrow[r] \arrow[d] & \mathrm{H}^0(\mathbb{Z}[j];\mathcal{Q}) \arrow[d] \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathrm{Pic}(\mathrm{TMF}[c_4^{-1}]) \arrow[r] & \mathrm{H}^0(\mathbb{Z}[j^{\pm 1}];u^*\mathcal{Q}).
\end{tikzcd}$$

Investigating the associated graded pieces of the Picard sheaves, we see that the first vertical map is zero. The rightmost vertical map is an injection by the four-lemma since it is an isomorphism on global sections of graded pieces. Looking at the global sections of the graded pieces, we also obtain that source and target of this map have at most 8 elements. Since $\mathrm{Pic}(\mathrm{TMF})_{(2)} \cong \mathbb{Z}/64$, we see that $\mathrm{H}^0(\mathbb{Z}[j];\mathcal{Q})$ must be $\mathbb{Z}/8$ and thus the same is true for $\mathrm{H}^0(\mathbb{Z}[j^{\pm 1}];u^*\mathcal{Q})$. As $\mathrm{TMF}[1]$ is sent to a generator of $\mathrm{H}^0(\mathbb{Z}[j];\mathcal{Q})$, we see that $\mathrm{TMF}[c_4^{-1}][1]$ is sent to a generator of $\mathrm{H}^0(\mathbb{Z}[j^{\pm 1}];u^*\mathcal{Q})$. Moreover, $\mathrm{TMF}[c_4^{-1}][1]$ generates a group of order 8 inside $\mathrm{Pic}(\mathrm{TMF}[c_4^{-1}])$. Thus, the lower exact is sequence is split short exact and $\mathrm{Pic}(\mathrm{TMF}[c_4^{-1}]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$. The extra $\mathbb{Z}/2$ provides an example of an exotic Picard group element.

The following proposition is a higher (but less explicit) analogue of Corollary 3.11.

Proposition 7.3. *Denote by $\mathcal{Q}$ the quotient of $\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}$ corresponding to (0) and (1) in Theorem 6.5 and by $\mathcal{J}$ the kernel of the quotient map. Let further R be an étale extension of $\mathbb{Z}$. Then there is a short exact sequence*

$$0 \to \mathrm{Pic}(R) \to \mathrm{Pic}(\mathrm{TMF}_R) \to \mathrm{H}^0(\mathbb{A}^1_R;\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}) \to 0.$$

If Spec R *is connected, the last term fits into a short exact sequence*

$$0 \to \mathrm{H}^0(\mathbb{A}^1_R;\mathcal{J}) \to \mathrm{H}^0(\mathbb{A}^1_R;\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}) \to \mathbb{Z}/24 \to 0.$$

Proof. We begin by proving the second claim. Applying the long exact sequence of cohomology to the extension

$$0 \to \mathcal{J} \to \pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}} \to \mathcal{Q} \to 0,$$

we obtain an exact sequence

$$0 \to \mathrm{H}^0(\mathbb{A}^1_R; \mathcal{I}) \to \mathrm{H}^0(\mathbb{A}^1_R; \pi_0 \mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}) \to \mathrm{H}^0(\mathbb{A}^1_R, \mathcal{Q}). \tag{7.4}$$

The composition

$$\begin{aligned} \mathbb{Z} \to \mathrm{Pic}(\mathrm{TMF}_R) &\to \mathrm{H}^0(\mathbb{A}^1_R; \pi_0 \mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}) \to \mathrm{H}^0(\mathbb{A}^1_R; \mathcal{Q}) \\ n \mapsto \mathrm{TMF}_R[n] & \end{aligned}$$

is a surjection by comparison to TMF (similar to the preceding example) since $\mathrm{H}^0(\mathbb{A}^1; \mathcal{Q}) \to \mathrm{H}^0(\mathbb{A}^1_R; \mathcal{Q})$ is an isomorphism (using a comparison on associated graded pieces and the five lemma). This implies that $\mathrm{H}^0(\mathbb{A}^1_R; \mathcal{Q}) \cong \mathbb{Z}/24$ and that Eq. (7.4) is short exact.

For the first claim, we recall from Proposition 2.25 the exact sequence

$$0 \to \mathrm{Pic}(\mathbb{A}^1_R) \to \mathrm{Pic}(\mathrm{TMF}_R) \to \mathrm{H}^0(\mathbb{A}^1_R; \pi_0 \mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}) \xrightarrow{\partial_R} \mathrm{Br}(\mathbb{A}^1_R).$$

Note that $\mathrm{Pic}(\mathbb{A}^1_R) \cong \mathrm{Pic}(R)$ (e.g. by [36, Proposition II.6.6]). The arguments proceeds now exactly as in Corollary 3.11, using that $\mathrm{Br}(\mathbb{A}^1_R)$ injects into $\mathrm{Br}(\mathbb{A}^1_{R[\frac{1}{6}]})$ by Theorem 2.5 and $\mathcal{I}(\mathbb{A}^1_{R[\frac{1}{6}]}) = 0$. □

8 The local Brauer groups of TMF and $(\mathcal{M}, \mathcal{O})$

The aim of this section is to show that local Brauer groups of TMF and the derived moduli stack $(\mathcal{M}, \mathcal{O})$ are infinitely generated and to compute them up to finite ambiguity. First, we observe the coincidences of various Brauer groups pertinent to this example.

Proposition 8.1. *If $(\mathcal{M}, \mathcal{O})$ is the derived moduli stack of elliptic curves, then*

(i) $\mathrm{Br}(\mathcal{M}, \mathcal{O}) \cong \mathrm{Br}'(\mathcal{M}, \mathcal{O})$,

(ii) $\mathrm{Br}(\mathcal{M}, \mathcal{O}) \cong \mathrm{Br}(\mathrm{TMF})$, *and*

(iii) $\mathrm{LBr}(\mathcal{M}, \mathcal{O}) \cong \mathrm{LBr}'(\mathcal{M}, \mathcal{O})$.

Proof. Parts (i) and (iii) follow from Corollary 4.19. Indeed, we can use the cover with opens $\mathcal{M}[\frac{1}{2}] = \mathcal{M} \times \mathrm{Spec}\, \mathbb{Z}[\frac{1}{2}]$ and $\mathcal{M}[\frac{1}{3}] = \mathcal{M} \times \mathrm{Spec}\, \mathbb{Z}[\frac{1}{3}]$. Condition (a) of Theorem 4.17 follows because the kernels of $\mathrm{QCoh}(\mathcal{M}[\frac{1}{2}]) \to \mathrm{QCoh}(\mathcal{M}[\frac{1}{6}])$ and $\mathrm{QCoh}(\mathcal{M}[\frac{1}{3}]) \to \mathrm{QCoh}(\mathcal{M}[\frac{1}{6}])$ are generated by the compact objects $\mathcal{O}/3$ and $\mathcal{O}/2$, respectively. Moreover, both $\mathcal{M}[\frac{1}{2}]$ and $\mathcal{M}[\frac{1}{3}]$ admit a finite étale cover from an affine scheme, for example the moduli stacks $\mathcal{M}(4)$ and $\mathcal{M}(3)$ of elliptic curves with full level 4 and full level 3 structures, respectively. Thus, by (the proof of) Proposition 4.14, α-twisted sheaves on $\mathcal{M}[\frac{1}{2}]$ and $\mathcal{M}[\frac{1}{3}]$ admit a local perfect generator for every Brauer class α. Condition (b) follows because $\mathcal{M}[\frac{1}{p}]$ is 0-affine by Theorem 5.2 and hence a local perfect generator is a global generator by the proof of Theorem 5.4.

Part (ii) follows from Theorem 5.4 since $(\mathcal{M}, \mathcal{O})$ is 0-affine by Corollary 5.3. □

Theorem 8.2. *The local Brauer group $\mathrm{LBr}(\mathcal{M}, \mathcal{O})$ is a torsion group. There is no p-torsion for $p > 3$. The 3-torsion is $\mathbb{Z}/3$. Moreover, there is a surjection $\mathrm{LBr}(\mathcal{M}, \mathcal{O})_{(2)} \to (\mathbb{Z}/2)^\infty$ with kernel of order 8.*

Proof. By the previous proposition, we can apply the spectral sequence from Construction 4.8 for the computation of $\mathrm{LBr}(\mathcal{M}, \mathcal{O}) = \pi_0 \mathbf{LBr}_{\mathcal{O}}(\mathcal{M})$. Up to a onefold shift, this agrees with the Picard spectral sequence for TMF from [50]. Note that $\mathrm{H}^1(\mathcal{M}; \mathbb{Z}/2) = 0$ (since $\mathcal{M}$ has no finite covers) and $\mathrm{H}^2(\mathcal{M}, \mathbb{G}_m) = \mathrm{Br}(\mathcal{M}) = 0$ by [4]. The non-sheafy version of Proposition 6.8 holds by the same arguments and thus only terms of filtration at most 30 can survive in the Picard spectral sequence in column (-1). By the results from [50], we know all differentials from the 0-column to the (-1)-column of the Picard spectral sequence: up to row 30 there are only d_3 and they are 2-local (cf. especially Figure 6 to 10 in [50]). One thus observes that the p-torsion is as stated for $p \geq 3$. In the E_∞-term, we have 2-locally the kernel of an unknown d_9-differential from $\mathrm{coker}(d_3 : \mathbb{F}_2[j] \to \mathbb{F}_2[j] \oplus \mathbb{Z}/2)$ in row 6 to a $\mathbb{Z}/2$ in row 15 (which must be abstractly isomorphic to $(\mathbb{Z}/2)^\infty$, as in the proof of Eq. (8.5) below), and further copies of $\mathbb{Z}/2$ in rows 10, 18 and 30, which cannot support differentials. Here, we use [50, Comparison Tool 5.2.4], both to show that possible targets of differentials vanish and to show the vanishing of a possible d_5 on the class in row 10. This implies the result. □

Next, we give a similar (but less precise) computation for LBr(TMF). Later, we will compare the two calculations.

Theorem 8.3. *The local Brauer group* LBr(TMF) *is a torsion group. There is no p-torsion for* $p > 3$. *The* 3*-torsion is* $\mathbb{Z}/3$. *Moreover, there is a split surjection* $\mathrm{LBr}(\mathrm{TMF})_{(2)} \to (\mathbb{Z}/2)^{\infty}$ *with finite kernel.*

Proof. By Proposition 2.25 and using $\mathrm{Br}(\pi_0\mathrm{TMF}) \cong \mathrm{Br}(\mathbb{Z}[j]) \cong \mathrm{Br}(\mathbb{Z}) = 0$ by Theorem 2.5 and Example 2.4, LBr(TMF) is isomorphic to the kernel of the differential

$$d^{\mathrm{TMF}}\colon \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j];\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}) \to \mathrm{H}^3(\operatorname{Spec}\mathbb{Z}[j];\mathbb{G}_m),$$

where we use (6.2) to identify $\pi_0\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}$ with $\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}$. We will first partially compute the source of the differential. Using Proposition 6.9, the facts that $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j];\mathbb{Z}/m) = \mathrm{H}^1(\operatorname{Spec}\mathbb{Z};\mathbb{Z}/m) = 0$ for any m, and Theorem 2.9, we deduce first that $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j];\mathrm{R}^1 j_*\mathbb{G}_m)$ vanishes. From Theorem 6.5 it thus follows that $\mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j];\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}) \cong \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j];\mathrm{F}^3\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}})$, where F^3 refers to the third filtration. The sheaf $\mathrm{F}^3\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}$ sits in an extension

$$0 \to \mathrm{F}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}} \to \mathrm{F}^3\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}} \to k_*v_!\mathbb{Z}/2 \to 0. \tag{8.4}$$

The extension must be split since $(\mathrm{F}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}})_{(2)}$ is supported at $(2, j)$, while $k_*v_!\mathbb{Z}/2$ is only nonzero on étale maps $U \to \mathbb{A}^1$ whose image does not contain $(2, j)$.

To compute $\mathrm{H}^1(\mathbb{A}^1;k_*v_!\mathbb{Z}/2)$, recall that we obtained $k_*v_!\mathbb{Z}/2$ as the kernel of a surjective differential $d_3\colon \mathcal{O}/2 \to \mathcal{O}/2$. Since $\mathcal{O}/2$ is quasi-coherent, its first cohomology vanishes and we can thus identify $\mathrm{H}^1(\mathbb{A}^1;k_*v_!\mathbb{Z}/2)$ with the cokernel of the map $d_3\colon \mathrm{H}^0(\mathbb{A}^1;\mathcal{O}/2) \cong \mathbb{F}_2[j] \to \mathbb{F}_2[j] \cong \mathrm{H}^0(\mathbb{A}^1;\mathcal{O}/2)$, which sends f to $f + jf^2$ (cf. Section 6.5). One checks that $j^2, j^4, j^6, \dots$ is a linearly independent subset in the cokernel and thus

$$\mathrm{H}^1(\mathbb{A}^1;k_*v_!\mathbb{Z}/2) \cong \mathbb{F}_2^{\infty}. \tag{8.5}$$

Next, we turn to $\mathrm{F}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}$. By Theorem 6.5, this vanishes if localized at primes bigger than 3, while 3-locally it is isomorphic to $b_*\mathbb{Z}/3$ for $b\colon \operatorname{Spec}\mathbb{F}_3 \to \mathbb{A}^1$ the inclusion at $j = 3 = 0$. Thus,

$$\mathrm{H}^1(\mathbb{A}^1;\mathrm{F}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}})_{(3)} \cong \mathrm{H}^1(\mathbb{A}^1;b_*\mathbb{Z}/3) \cong \mathrm{H}^1(\mathbb{F}_3;\mathbb{Z}/3) \cong \mathbb{Z}/3.$$

For the 2-local situation, recall from [53, Corollary II.3.11] that we can view sheaves supported at $(2, j)$ equivalently as étale sheaves on $\operatorname{Spec}\mathbb{F}_2$, whose category is equivalent to (discrete) abelian groups with a continuous action by the absolute Galois group $\mathrm{Gal}(\mathbb{F}_2) \cong \widehat{\mathbb{Z}}$; we refer to such as *discrete* $\widehat{\mathbb{Z}}$*-modules*. Let $\mathcal{F}$ be the class of discrete $\widehat{\mathbb{Z}}$-modules where $\mathrm{H}^i(\widehat{\mathbb{Z}}, -)$ is finite for all i. From the fact that for discrete $\widehat{\mathbb{Z}}$-modules, $\mathrm{H}^i(\widehat{\mathbb{Z}}, -)$ vanishes for $i > 1$, one deduces that $\mathcal{F}$ is closed under kernels, cokernels and extensions. By Proposition 6.6, we know that only finitely many discrete $\widehat{\mathbb{Z}}$-modules can contribute to $(\mathrm{F}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}})_{(2)}$ and they all lie in $\mathcal{F}$; moreover, there are only finitely many possible targets and they also lie in $\mathcal{F}$ (cf. Section 6.7). Thus, $\mathrm{H}^1(\mathbb{A}^1;\mathrm{F}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}})_{(2)}$ is finite.

It remains to study the differential

$$d^{\mathrm{TMF}}\colon \mathrm{H}^1(\operatorname{Spec}\mathbb{Z}[j];\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}}) \to \mathrm{H}^3(\operatorname{Spec}\mathbb{Z}[j];\mathbb{G}_m).$$

Let U be the complement of the image of the closed immersion $\operatorname{Spec}\mathbb{Z}/6 \to \operatorname{Spec}\mathbb{Z}[j]$ corresponding to $j = 0$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{A}^1;\mathrm{F}^5\pi_0\mathbf{BPic}_{\mathcal{O}_{\mathrm{TMF}}}) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathbb{A}^1;\pi_0\mathbf{BPic}_{\mathcal{O}_{\mathrm{TMF}}}) \arrow[r, "d^{\mathrm{TMF}}"] \arrow[d] & \mathrm{H}^3(\operatorname{Spec}\mathbb{Z}[j];\mathbb{G}_m) \arrow[d] \\
\mathrm{H}^1(\mathbb{A}^1;\mathrm{F}^5\pi_0\mathbf{BPic}_{\mathcal{O}_{\mathcal{M}\times_{\mathbb{A}^1}U}}) \arrow[r] & \mathrm{H}^1(U;\pi_0\mathbf{BPic}_{\mathcal{O}_{\mathcal{M}\times_{\mathbb{A}^1}U}}) \arrow[r, "d^{(U,\mathcal{O}_{\mathrm{TMF}}|_U)}"] & \mathrm{H}^3(U;\mathbb{G}_m)
\end{tikzcd}$$

The rightmost lower horizontal arrow is the differential in the descent spectral sequence for **BPic** on $(U, \mathcal{O}_{\mathrm{TMF}}|_U)$. The rightmost vertical map is an injection by purity [34, Théorème 6.1b]. Moreover, the leftmost vertical map is zero since $\mathrm{F}^5\pi_0\mathbf{BPic}_{\mathcal{O}_{\mathrm{TMF}}}$ is supported at $(2, j)$ and $(3, j)$. Thus, d^{TMF} vanishes when restricted to $\mathrm{H}^1(\mathbb{A}^1;\mathrm{F}^5\pi_0\mathbf{BPic}_{\mathcal{O}_{\mathrm{TMF}}})$ and the differential factors over $\mathrm{H}^1(\mathbb{A}^1;k_*v_!\mathbb{Z}/2)$.

We can cover U by $V = \operatorname{Spec} \mathbb{Z}[j^{\pm 1}, (j-1728)^{-1}]$ and $W = \operatorname{Spec} \mathbb{Z}[\frac{1}{6}, j]$. We obtain an exact sequence

$$\cdots \to \mathrm{H}^2(V \cap W; \mathbb{G}_m) \to \mathrm{H}^3(U; \mathbb{G}_m) \to \mathrm{H}^3(V; \mathbb{G}_m) \oplus \mathrm{H}^3(W; \mathbb{G}_m) \to \cdots.$$

We claim that the image of d^{TMF} in $\mathrm{H}^3(V; \mathbb{G}_m) \oplus \mathrm{H}^3(W; \mathbb{G}_m)$ is zero. Assuming this claim for the moment, we know that the image of d_3^{TMF} lies in the image of $\mathrm{H}^2(V \cap W; \mathbb{G}_m) \to \mathrm{H}^3(U; \mathbb{G}_m)$. By Theorem 2.5, we have 2-locally an isomorphism $\mathrm{H}^2(V \cap W; \mathbb{G}_m) \cong \mathrm{Br}(\mathbb{Z}[\frac{1}{6}, j^{\pm 1}, (j-1728)^{-1}]) \cong \mathrm{Br}(\mathbb{Z}[\frac{1}{6}]) \oplus \mathrm{H}^1(\mathbb{Z}[\frac{1}{6}]; \mathbb{Q}/\mathbb{Z})^{\oplus 2})$. We use the following two computations:

- $\mathrm{Br}(\mathbb{Z}[\frac{1}{6}]) \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}/2$ by Example 2.4;
- $\mathrm{H}^1(\mathbb{Z}[\frac{1}{6}]; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{Gal}(K/\mathbb{Q}), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}_2^\times \times \mathbb{Z}_3^\times, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Z}/2)^3 \oplus \mathbb{Q}_2/\mathbb{Z}_2 \oplus \mathbb{Q}_3/\mathbb{Z}_3$. Here, K is the maximal abelian extension of $\mathbb{Q}$, which is unramified at 2 and 3. We use the Kronecker–Weber theorem to identify K with the field obtained by adjoining all 2^n-th and 3^n-th roots of unity to $\mathbb{Q}$.

Thus, the image of $\mathrm{H}^2(V \cap W; \mathbb{G}_m) \to \mathrm{H}^3(U; \mathbb{G}_m)$ is 2-locally of the form (finite $\oplus$ divisible). Since the image of d_3^{TMF} must be an $\mathbb{F}_2$-vector space (as the image of an $\mathbb{F}_2$-vector space), the image of d_3^{TMF} must be finite. We deduce that the kernel of d^{TMF} consists of the finite group $\mathrm{H}^1(\mathbb{A}^1; \mathrm{F}^5\pi_0 j_* \mathbf{pic}_{\mathcal{O}_{\mathcal{M}}})$ plus an infinite-dimensional subspace of $\mathrm{H}^1(\mathbb{A}^1; k_* v_! \mathbb{Z}/2) \cong \mathbb{F}_2^\infty$, as claimed.

It remains to show that the restrictions of d^{TMF} to V and W are zero. The case of W is clear as $k_* v_! \mathbb{Z}/2$ is supported outside of W. For the case of V, recall from [61, Lemma 3.2] that the base change $\mathcal{M} \times_{\mathbb{A}^1} V$ is equivalent to $V \times \mathrm{B}C_2$, i.e. the stack quotient of V by the trivial C_2-action; this yields in particular an étale map $V \to \mathcal{M} \times_{\mathbb{A}^1} V \to \mathcal{M}$. We obtain a diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{A}^1; \mathrm{F}^3\pi_0 \mathbf{BPic}_{\mathcal{O}_{\mathcal{M}}}) \arrow[r, "\cong"] \arrow[d] & \mathrm{H}^1(\mathbb{A}^1; \pi_0 \mathbf{BPic}_{\mathcal{O}_{\mathcal{M}}}) \arrow[r, "d^{\mathrm{TMF}}"] \arrow[d] & \mathrm{H}^3(\mathbb{A}^1; \mathbb{G}_m) \arrow[d] \\
\mathrm{H}^1(V; \mathrm{F}^3\pi_0 \mathbf{BPic}_{\mathcal{O}_{\mathcal{M} \times_{\mathbb{A}^1} V}}) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathbb{A}^1; \pi_0 \mathbf{BPic}_{\mathcal{O}_{\mathcal{O}_{\mathcal{M} \times_{\mathbb{A}^1} V}}}) \arrow[r, "d^{(V, \mathcal{O}_{\mathrm{TMF}})}"] \arrow[d] & \mathrm{H}^3(V; \mathbb{G}_m) \arrow[d, "\mathrm{id}"] \\
\mathrm{H}^1(V; \mathrm{F}^3\pi_0 \mathbf{BPic}_{\mathcal{O}_V}) \arrow[r] & \mathrm{H}^1(V; \pi_0 \mathbf{BPic}_{\mathcal{O}_V}) \arrow[r, "d^{\mathcal{O}(V)}"] & \mathrm{H}^3(V; \mathbb{G}_m).
\end{tikzcd}$$

Here, $d^{\mathcal{O}(V)}$ refers to the boundary map in the long exact sequence from Proposition 2.25 for the ring spectrum $\mathcal{O}(V \to \mathcal{M})$, while $d^{(V, \mathcal{O}_{\mathrm{TMF}})}$ uses the restriction of the spectral scheme structure of $\operatorname{Spec} \mathrm{TMF}$ to V; note that both affine spectral schemes here have underlying scheme V. In particular, the rightmost vertical map is an isomorphism. Note further that $\mathrm{F}^3\pi_0 \mathbf{BPic}_{\mathcal{O}_V} = 0$ since all terms in the sheafy Picard spectral sequence of filtration 3 and higher are of the form $\mathrm{H}^{2i+1}(V; \pi_{2i}\mathcal{O})$ for $i \geq 1$, which all vanish since V is an affine scheme. Thus, we see that d^{TMF} is indeed zero after restricting to V. □

Our next goal is to compare $\mathrm{LBr}(\mathcal{M}, \mathcal{O})$ with $\mathrm{LBr}(\mathrm{TMF})$. Clearly, we have maps

$$\mathrm{LBr}(\mathrm{TMF}) \to \mathrm{LBr}(\mathcal{M}, \mathcal{O}) \to \mathrm{Br}(\mathcal{M}, \mathcal{O}).$$

Since $\mathrm{Br}(\mathrm{TMF}) \to \mathrm{Br}(\mathcal{M}, \mathcal{O})$ is an isomorphism, $\mathrm{LBr}(\mathrm{TMF}) \to \mathrm{LBr}(\mathcal{M}, \mathcal{O})$ is an injection. We want to describe how to obtain a computational handle on this injection. In conjunction with Theorem 8.2, this will also provide an alternative proof of Theorem 8.3.

Consider the sheaf $j_* \mathbf{lbr}_{\mathcal{O}}$ on $\mathbb{A}^1$. It assigns to every étale open U, the spectrum $\mathbf{lbr}(U \times_{\mathbb{A}^1} \mathcal{M}, \mathcal{O})$. The relative descent spectral sequence (cf. Remark 6.4) for $j_* \mathbf{lbr}_{\mathcal{O}}$ takes the form

$$\mathrm{E}_2^{s,t} = \mathrm{R}^s j_* \pi_t \mathbf{lbr}_{\mathcal{O}} \quad \Rightarrow \quad \pi_{t-s} j_* \mathbf{lbr}_{\mathcal{O}} \underset{t-s \geq 0}{\cong} \pi_{t-s} j_* \mathbf{LBr}_{\mathcal{O}}$$

and provides thus a method to compute $\pi_* j_* \mathbf{lbr}_{\mathcal{O}}$. But since $\mathbf{lbr}$ is just a suspension of $\mathbf{pic}$, this spectral sequence is up to a shift actually the same as the sheafy Picard spectral sequence considered in Section 6. In particular, one observes that $\pi_t j_* \mathbf{lbr}_{\mathcal{O}} \cong \pi_{t-1}\mathbf{Pic}_{\mathcal{O}_{\mathrm{TMF}}}$ for $t \geq 1$, but have additionally interesting sheaves π_t for $t \leq 0$, which are computed by the $(t-1)$-column of the sheafy Picard spectral sequence. We obtain a descent spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathbb{A}^1; \pi_t j_* \mathbf{lbr}_{\mathcal{O}}) \quad \Rightarrow \quad \pi_{t-s}\Gamma(j_*\mathbf{lbr}_{\mathcal{O}}) \underset{t-s\geq 0}{\cong} \pi_{t-s}\mathbf{lbr}(\mathcal{M}, \mathcal{O}). \tag{8.6}$$

In particular, Proposition 8.1 gives that $\pi_0\mathbf{lbr}(\mathcal{M}, \mathcal{O}) = \mathrm{LBr}'(\mathcal{M}, \mathcal{O}) = \mathrm{LBr}(\mathcal{M}, \mathcal{O})$. Thus, we are indeed computing the local Brauer group of $(\mathcal{M}, \mathcal{O})$.

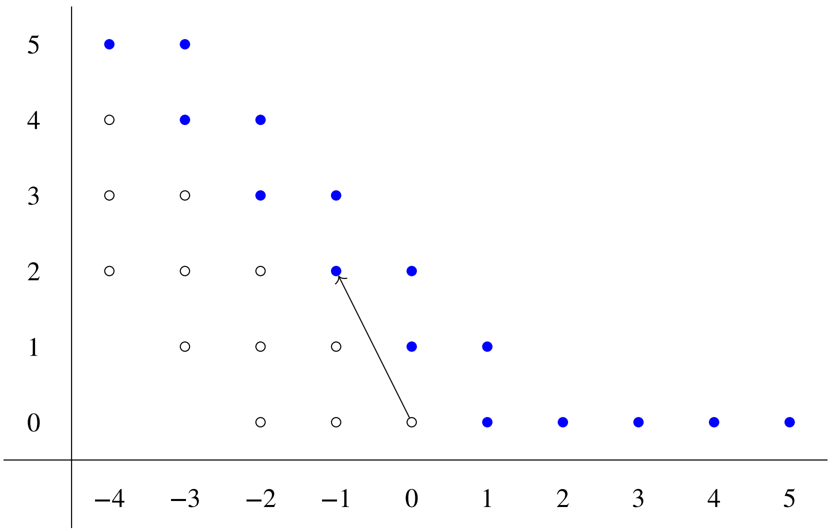

Figure 7: Schematic comparison of descent spectral sequences computing LBr(TMF) (solid, in blue) and $\mathrm{LBr}(\mathcal{M}, \mathcal{O})$

Note that the map $\mathbf{lbr}_{\mathcal{O}_{\mathrm{TMF}}} \to j_*\mathbf{lbr}_{\mathcal{O}}$ induces a map of descent spectral sequences, which is essentially the inclusion of the top two anti-diagonals. Fig. 7 gives a schematic picture of part of this map, with the image of the descent spectral sequence of $\mathbf{lbr}_{\mathcal{O}_{\mathrm{TMF}}}$ colored in blue.

Theorem 8.7. *The injection* $\mathrm{LBr}(\mathrm{TMF}) \to \mathrm{LBr}(\mathcal{M}, \mathcal{O})$ *has finite cokernel and is an isomorphism after inverting* 2.

Proof. In the spectral sequence (8.6), the only possible nonzero entries in the zeroth column are $\mathrm{H}^s(\mathbb{A}^1, \pi_s j_*\mathbf{lbr}_{\mathcal{O}})$ for $0 \leq s \leq 2$. By the above discussion, every element in $\mathrm{LBr}(\mathcal{M}, \mathcal{O})$ not coming from LBr(TMF) must be detected in $\mathrm{H}^0(\mathbb{A}^1, \pi_0 j_*\mathbf{lbr}_{\mathcal{O}})$.

The charts Fig. 4 and Fig. 6 show the possible contributions in the (-1)-column of the sheafy Picard spectral sequence to $\pi_0 j_*\mathbf{lbr}_{\mathcal{O}}$. We first note that the two question marks corresponding to $\mathrm{R}^1 j_*\mathbb{Z}/2$ and $\mathrm{R}^2 j_*\mathbb{G}_m$ cannot contribute to H^0. Indeed, by the Leray spectral sequence, $0 = \mathrm{H}^1(\mathcal{M}; \mathbb{Z}/2)$ surjects onto $\mathrm{H}^0(\mathbb{A}^1; \mathrm{R}^1 j_*\mathbb{Z}/2)$, which is thus zero as well. Likewise, from the Leray spectral sequence

$$\mathrm{H}^m(\mathbb{A}^1; \mathrm{R}^n j_*\mathbb{G}_m) \quad \Rightarrow \quad \mathrm{H}^{m+n}(\mathcal{M}; \mathbb{G}_m)$$

we see that $\mathrm{Br}(\mathcal{M}) \cong \mathrm{H}^2(\mathcal{M}; \mathbb{G}_m)$ surjects onto the cokernel of the differential $\mathrm{H}^1(\mathbb{A}^1, j_*\mathbb{G}_m) \to \mathrm{H}^0(\mathbb{A}^1; \mathrm{R}^2 j_*\mathbb{G}_m)$. But $j_*\mathbb{G}_m \cong \mathbb{G}_m$ (since every function $\mathcal{M} \to \mathbb{A}^1$ factors through j, even after étale base change) and thus $\mathrm{H}^1(\mathbb{A}^1, j_*\mathbb{G}_m) \cong \mathrm{Pic}(\mathbb{A}^1) = 0$. Moreover, $\mathrm{Br}(\mathcal{M}) = 0$ by one of the main results from [4]. Thus, $\mathrm{H}^0(\mathbb{A}^1; \mathrm{R}^2 j_*\mathbb{G}_m)$ vanishes.

Regarding the contributions in higher rows: even the E_2-term vanishes p-locally for $p > 3$. At $p = 3$, the only potential contribution is in Row 14 (see Fig. 4 and Proposition 6.6). As demonstrated in Lemma 6.10, this contribution is hit by a surjective d_9. Thus, $\mathrm{F}^3\pi_0 j_*\mathbf{lbr}_{\mathcal{O}}$ vanishes after inverting 2. We deduce $\mathrm{H}^0(\mathbb{A}^1, \pi_0 j_*\mathbf{lbr}_{\mathcal{O}})[\frac{1}{2}] = 0$ and hence that $\mathrm{LBr}(\mathrm{TMF}) \to \mathrm{LBr}(\mathcal{M}, \mathcal{O})$ is an isomorphism after inverting 2.

Regarding the 2-local picture, Fig. 6 shows that the only possible contributions are in Rows 6, 18 and 30 and each of them is an $\mathcal{O}/(2, j)$ on the E_7-page. The same argument as provided in the proof of Theorem 8.3 for the finiteness of $\mathrm{H}^1(\mathbb{A}^1; \mathrm{F}^5\pi_0 j_*\mathbf{pic}_{\mathcal{O}_{\mathcal{M}}})_{(2)}$ shows also the finiteness of $\mathrm{H}^0(\mathbb{A}^1, \pi_0 j_*\mathbf{lbr}_{\mathcal{O}})$. This in turn implies the finiteness of the cokernel of $\mathrm{LBr}(\mathrm{TMF}) \to \mathrm{LBr}(\mathcal{M}, \mathcal{O})$. □

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