DERIVED MACKEY FUNCTORS AND C_{p^n}-EQUIVARIANT COHOMOLOGY

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ABSTRACT. We establish a novel approach to computing G-equivariant cohomology for a finite group G, and demonstrate it in the case that $G = \mathsf{C}_{p^n}$.

For any commutative ring spectrum R, we prove a symmetric monoidal reconstruction theorem for genuine G-R-modules, which records them in terms of their geometric fixedpoints as well as gluing maps involving their Tate cohomologies. This reconstruction theorem follows from a symmetric monoidal stratification (in the sense of [AMGR]); here we identify the gluing functors of this stratification in terms of Tate cohomology.

Passing from genuine G-spectra to genuine G-$\mathbb{Z}$-modules (a.k.a. derived Mackey functors) provides a convenient intermediate category for calculating equivariant cohomology. Indeed, as $\mathbb{Z}$-linear Tate cohomology is far simpler than $\mathbb{S}$-linear Tate cohomology, the above reconstruction theorem gives a particularly simple algebraic description of genuine G-$\mathbb{Z}$-modules. We apply this in the case that $G = \mathsf{C}_{p^n}$ for an odd prime p, computing the Picard group of genuine G-$\mathbb{Z}$-modules (and therefore that of genuine G-spectra) as well as the $\mathsf{RO}(G)$-graded and Picard-graded G-equivariant cohomology of a point.

CONTENTS

0. Introduction

0.1. **Overview.** Let G be a finite group. Let X be a topological space equipped with an action by G. There is a natural notion of G-equivariant cohomology of X, whose output is a collection of Mackey functors of abelian groups (see e.g. [GM95a] for a motivated account of equivariant cohomology; see also Observation 1.9 below for a recollection of Mackey functors that is particularly suited to our purposes):

$$\mathsf{H}_G^V(X) \in \mathsf{Mack}_G(\mathsf{Ab}) \ ,$$

indexed by a virtual representation $V \in \mathsf{RO}(G)$.[1] (The $\mathsf{RO}(G)$-grading is of central importance. For instance, it plays a key role in equivariant duality [Wir74, LMM81, CW92].) Equivariant cohomology is quite difficult to compute, even in the case that X is a single point.

In this paper, we establish a novel approach to computing equivariant cohomology. To explain our approach, let us briefly outline a perspective on why equivariant cohomology is more difficult than non-equivariant cohomology.

Recall that ordinary cohomology of a space takes values in abelian groups; it can be computed as the cohomology of a cochain complex of abelian groups. From the point of view of ∞-categories, this can be explained by the identification of the $\mathbb{Z}$-linearization of the stable homotopy ∞-category as the derived ∞-category of its heart:

$$\mathsf{Mod}_{\mathbb{Z}} \ := \ \mathbb{Z} \otimes \mathcal{S}\mathsf{p} \ \simeq \ \mathbf{D}(\mathsf{Ab}) \ \simeq \ \mathbf{D}\big((\mathbb{Z} \otimes \mathcal{S}\mathsf{p})^{\heartsuit}\big) \ .$$

As a result, standard homological algebra tools are directly applicable for computations of ordinary cohomology.

Now, consider the $\mathbb{Z}$-linearization $\mathbb{Z} \otimes \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ of the equivariant stable homotopy ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$; following Observation 1.9, we denote this by the equivalent ∞-category

$$\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}}) \simeq \mathbb{Z} \otimes \mathcal{S}\mathsf{p}^{\mathsf{g}G} \ .$$

Now, equivariant cohomology takes values in the category $\mathsf{Mack}_G(\mathsf{Ab})$ of Mackey functors of abelian groups, which is the heart of $\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})$:

$$\mathsf{Mack}_G(\mathsf{Ab}) \ \simeq \ \mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})^{\heartsuit} \ .$$

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[1]The usual integer grading is given by multiples of the trivial representation.

However, $\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})$ is *not* the derived ∞-category of its heart:

$$\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}}) \ := \ \mathbb{Z} \otimes \mathbb{Sp}^{\mathbf{g}G} \ \not\simeq \ \mathbf{D}\big(\mathsf{Mack}_G(\mathsf{Ab})\big) \ \simeq \ \mathbf{D}\big(\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})^{\heartsuit}\big) \ .$$

In this sense, given an equivariant spectrum, it is not possible to systematically associate a cochain complex of Mackey functors that computes its equivariant cohomology. Therefore, standard homological algebra tools do not obviously apply for computations of equivariant cohomology.

The triangulated category corresponding to the stable ∞-category $\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})$ was first considered by Kaledin purely algebraically in [Kal11], under the name *derived Mackey functors*. He suggested that it should receive a functor from $\mathbb{Sp}^{\mathbf{g}G}$ compatible with both geometric and categorical fixedpoints; this is essentially the content of [Kal11, Conjectures 8.10 and 8.11]. The work in this paper may be seen as resolving and applying those conjectures.

From our point of view, it is $\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})$ that is a natural intermediate target for equivariant cohomology. While it is not the derived ∞-category of its heart, we prove as Theorem A below that it can be constructed (in a precise sense) from finitely many ∞-categories that are the derived ∞-categories of their hearts. Using this, we can import tools from standard homological algebra for computations in the equivariant stable homotopy category.

Specifically, we apply the theory of stratifications as developed in [AMGR]: for $\mathfrak{X}$ a presentable stable ∞-category, a *stratification* of it over a poset P is a collection $\{\mathcal{Z}_p \subseteq \mathfrak{X}\}_{p \in \mathsf{P}}$ of full subcategories satisfying certain natural conditions. Given a stratification of $\mathfrak{X}$, we can consider the "associated graded" stable ∞-categories $\{\mathfrak{X}_p\}_{p \in \mathsf{P}}$. We prove as [AMGR, Theorem A] that, in favorable situations such as when the poset P is finite, $\mathfrak{X}$ can be reconstructed from its associated graded ∞-categories together with, for each $p < q$ in P, a gluing functor

$$\mathfrak{X}_p \xrightarrow{\ \Gamma_q^p\ } \mathfrak{X}_q \ ,$$

as well as coherence data.

In [AMGR, Theorem E], we constructed a stratification of the ∞-category $\mathbb{Sp}^{\mathbf{g}G}$ of genuine G-spectra by the poset P_G of conjugacy classes of subgroups of G, called the ***geometric stratification***. There, we identified the associated graded ∞-categories as

$$(\mathbb{Sp}^{\mathbf{g}G})_{[H]} \ \simeq \ \mathsf{Fun}(\mathsf{BW}(H), \mathbb{Sp}) \ ,$$

where $\mathsf{W}(H) := \mathsf{N}(H)/H$ is the Weyl group (i.e. the normalizer of H in G modulo H). Moreover, the projection functors to the associated gradeds

$$\mathbb{Sp}^{\mathbf{g}G} \xrightarrow{\ \Phi^H\ } (\mathbb{Sp}^{\mathbf{g}G})_{[H]} \simeq \mathsf{Fun}(\mathsf{BW}(H), \mathbb{Sp})$$

are given by geometric H-fixedpoints. In particular, it follows that categorical fixedpoints can be expressed in terms of geometric fixedpoints and gluing data.

In fact, the stratification of $\mathbb{Sp}^{\mathbf{g}G}$ is a *symmetric monoidal* stratification, so that we are also able to describe the symmetric monoidal structure of $\mathbb{Sp}^{\mathbf{g}G}$ in terms of those of its strata and gluing functors [AMGR, Theorem C].

However, in [AMGR], we did not give an explicit general formula for the gluing functors of this stratification. Such a general formula is supplied by Theorem A below in terms of a variant of Tate cohomology.

In fact, we prove Theorem A in somewhat more generality. For $\mathcal{R}$ a presentable stable ∞-category, consider the tensor product in Pr^L

$$\mathcal{R}^{\mathsf{g}G} \;:=\; \mathcal{R} \otimes \mathcal{S}\mathsf{p}^{\mathsf{g}G} \;\simeq\; \mathsf{Mack}_G(\mathcal{R})$$

(again see Observation 1.9 for the equivalence). The stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ over P_G induces a stratification of $\mathcal{R}^{\mathsf{g}G}$ over P_G, with associated graded ∞-categories

$$\mathsf{Mack}_G(\mathcal{R})_{[H]} \;\simeq\; \mathsf{Fun}(\mathrm{BW}(H), \mathcal{R}) \;.$$

In particular, taking $\mathcal{R} = \mathsf{Mod}_{\mathbb{Z}}$, we have that the $\mathbb{Z}$-linearization of the stable equivariant homotopy ∞-category $\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})$ has a stratification by P_G, with associated graded ∞-categories given by

$$\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})_{[H]} \;\simeq\; \mathsf{Fun}(\mathrm{BW}(H), \mathsf{Mod}_{\mathbb{Z}}) \;\simeq\; \mathbf{D}(\mathsf{Rep}_{\mathrm{W}(H)}(\mathsf{Ab})) \;,$$

the derived ∞-category of abelian groups with an action by $\mathrm{W}(H)$.

Now, equivariant cohomology is by definition the cohomology of categorical fixedpoints of genuine G-spectra. By definition, the $\mathbb{Z}$-linearization functor

$$\mathcal{S}\mathsf{p}^{\mathsf{g}G} \longrightarrow \mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}}) \tag{0.1}$$

is compatible with both geometric and categorical fixedpoints. In practice, geometric fixedpoints are much easier to compute than categorical fixedpoints. In particular, the equivariant suspension spectrum functor takes fixedpoints to *geometric* fixedpoints, and moreover the geometric fixedpoints functors are symmetric monoidal.

The compatibility of the $\mathbb{Z}$-linearization functor with both types of fixedpoint functors implies that the passage from geometric fixedpoints to categorical fixedpoints can be performed in the much simpler context of derived Mackey functors. Ultimately, this yields an algebraic description of the equivariant cohomology of a genuine G-space in terms of cochains on its fixedpoint spaces.

By contrast, a standard technique in equivariant homotopy theory is to compute equivariant cohomology using resolutions of Mackey functors (see e.g. [Lew88, Gre92, Zen]). This is tantamount to performing homological algebra in the derived ∞-category $\mathbf{D}(\mathsf{Mack}_G(\mathsf{Ab}))$ of the abelian category $\mathsf{Mack}_G(\mathsf{Ab})$ of Mackey functors, which does not enjoy the favorable properties of $\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})$.

Our first main result explicitly identifies the gluing functors of this stratification of $\mathsf{Mack}_G(\mathcal{R})$ in the case that $\mathcal{R}$ is presentably symmetric monoidal and rigidly-compactly generated[2] (such as $\mathcal{R} = \mathsf{Mod}_{\mathbb{Z}}$ and $\mathcal{R} = \mathcal{S}\mathsf{p}$).

Theorem A (Theorem 6.6). *For any containment $H \subset K$ between subgroups of G, the corresponding gluing functor*

$$\mathsf{Fun}(\mathrm{BW}(H), \mathcal{R}) \xrightarrow{\;\Gamma_K^H\;} \mathsf{Fun}(\mathrm{BW}(K), \mathcal{R})$$

of the stratification of $\mathsf{Mack}_G(\mathcal{R})$ evaluates as

$$\Gamma_K^H \colon E \longmapsto \bigoplus_{[g] \in \mathrm{W}(H) \backslash C(H,K) / \mathrm{W}(K)} \mathsf{Ind}_{(\mathrm{N}(H) \cap \mathrm{N}(gKg^{-1}))/(gKg^{-1})}^{\mathrm{W}(K)} E^{\tau(gKg^{-1}/H)} \;,$$

where

- *$C(H,K)$ denotes a certain subset of G/K that carries a natural $(\mathrm{W}(H), \mathrm{W}(K))$-bimodule structure (Notation 6.4),*

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[2]That is, $\mathcal{R}$ is a compactly generated presentably symmetric monoidal stable ∞-category whose compact and dualizable objects coincide.

- $(-)^{\tau(gKg^{-1}/H)}$ *denotes a variant of Tate cohomology (Definition 4.5), and*

- *we implicitly use the isomorphism* $\mathsf{W}(gKg^{-1}) \xrightarrow{\cong} \mathsf{W}(K)$.

Remark 0.1. If the finite group G is abelian, the above gluing functors simplify as $\Gamma_K^H \colon E \mapsto E^{\tau(K/H)}$.

Note that the functors $(-)^{\tau(gKg^{-1}/H)}$ do not preserve colimits, and as a result the gluing functors of the stratification of $\mathsf{Mack}_G(\mathcal{R})$ are *not* given by tensoring those of $\mathsf{Sp}^{\mathsf{g}G}$ with $\mathcal{R}$.

The decisive advantage of working with $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G}$ is that its stratification is dramatically simpler than that of $\mathsf{Sp}^{\mathsf{g}G}$. In addition to the associated gradeds being the derived ∞-categories of their hearts, the gluing functors are substantially simpler. Indeed, the failure of the gluing functors of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G}$ to be tensored up from those of $\mathsf{Sp}^{\mathsf{g}G}$ is a feature, and not a bug: Tate cohomology in $\mathsf{Mod}_{\mathbb{Z}}$ is far simpler than Tate cohomology in Sp. In the case that $G = \mathsf{C}_{p^n}$, this simplicity is especially pronounced, due additionally to the Tate vanishing results of Nikolaus–Scholze [NS18].[3] All in all, we obtain the following.

Theorem B (Theorem 7.2). *The stratification of* $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$ *determines an equivalence between presentably symmetric monoidal stable ∞-categories:*

where $\mathsf{Mod}_{\mathbb{Z}[\mathsf{C}_{p^k}]}$ *is the derived ∞-category* $\mathbf{D}(\mathsf{Rep}_{\mathsf{C}_{p^k}}(\mathsf{Ab}))$. *In particular, a genuine* C_{p^n}-$\mathbb{Z}$-module $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$ *is equivalent to the data of*

- *its geometric* C_{p^s}-*fixedpoints*

$$E_s := \Phi^{\mathsf{C}_{p^s}}(E) \in \mathsf{Mod}_{\mathbb{Z}[\mathsf{C}_{p^{n-s}}]}$$

for all $0 \leq s \leq n$, *along with*

- *its gluing maps*

$$E_s \xrightarrow{\gamma_{s-1,s}^E} (E_{s-1})^{\mathsf{t}\mathsf{C}_p}$$

in $\mathsf{Mod}_{\mathbb{Z}[\mathsf{C}_{p^{n-s-1}}]}$ *for all* $1 \leq s \leq n$.

Moreover, this description is compatible with symmetric monoidal structures.

Theorem B is closely related to [NS18, Corollary II.4.7] (see also [NS18, Remark II.4.8]).

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[3]Whereas in general the gluing functors of a stratification only laxly compose, in that of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$ they strictly compose. More than that, all of its nontrivial composite gluing functors are zero. Neither of these facts is the case for the stratification of $\mathsf{Sp}^{\mathsf{g}\mathsf{C}_{p^n}}$. These facts support a relatively simple description of the ∞-category $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$, which we articulate as Theorem B.

0.2. **Computations.** As a demonstration of our general machinery, for $G = C_{p^n}$ with p an odd prime, we compute:

(1) the Picard group $\mathsf{Pic}(\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}}))$ and the homomorphism

$$\mathsf{RO}(G) \longrightarrow \mathsf{Pic}(\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}})) \ ,$$

as well as

(2) the $\mathsf{Pic}(\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}}))$-graded cohomology of a point (and hence in particular the $\mathsf{RO}(G)$-graded cohomology of a point).

In fact, the recent paper [Kra] proves that the homomorphism

$$\mathsf{Pic}(\mathcal{S}\mathsf{p}^{\mathsf{g}G}) \xrightarrow{\mathsf{Pic}(0.1)} \mathsf{Pic}(\mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}}))$$

is an isomorphism. Consequently, our computations explicitly identify the Picard group of $\mathcal{S}\mathsf{p}^{\mathsf{g}C_{p^n}}$. For comparison, we survey some analogous existing computations in §0.4.

Although the $\mathsf{RO}(G)$-grading of equivariant cohomology has become standard in the literature, it is more natural to grade over the Picard group $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G})$ of genuine G-$\mathbb{Z}$-modules.[4] Let X be a G-space. For a Picard element $L \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G})$ and a subgroup $H \subseteq G$, we write

$$\mathsf{C}_G^L(X)(H) := \underline{\hom}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G}}(\Sigma_G^\infty(X \times G/H)_+ \otimes \mathbb{Z}, L \otimes_{\mathbb{Z}} \underline{\mathbb{Z}}) \in \mathsf{Mod}_{\mathbb{Z}}$$

for the indicated hom-$\mathbb{Z}$-module (where we consider $\underline{\mathbb{Z}} \in \mathsf{Mack}_G(\mathsf{Ab}) \subset \mathsf{Mack}_G(\mathsf{Mod}_{\mathbb{Z}}) \simeq \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G}$). Using this, we define $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G})$-graded equivariant cohomology as

$$\mathsf{H}_G^{i+L}(X)(H) := \pi_{-i}(\mathsf{C}_G^L(X)(H)) \in \mathsf{Ab} \ .$$

This recovers $\mathsf{RO}(G)$-graded equivariant cohomology via pullback along the composite abelian group homomorphism

$$\mathsf{RO}(G) \xrightarrow{V \mapsto \mathbb{S}^V} \mathsf{Pic}(\mathcal{S}\mathsf{p}^{\mathsf{g}G}) \xrightarrow{(-) \otimes \mathbb{Z}} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G}) \ . \tag{0.2}$$

We have the following two computational results. First, we compute the Picard group of genuine C_{p^n}-$\mathbb{Z}$-modules.

Theorem C (Theorems 8.4 and 9.6). *Let p be an odd prime. There is an isomorphism between abelian groups:*

$$\mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^n (\mathbb{Z}/p^{n-s+1})^\times / \{\pm 1\} \right) \xrightarrow{\cong} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}}) \ .$$

Furthermore, the resulting homomorphism

$$\mathsf{RO}(C_{p^n}) \xrightarrow{(0.2)} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}}) \cong \mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^n (\mathbb{Z}/p^{n-s+1})^\times / \{\pm 1\} \right)$$

is given on irreducibles (which freely generate $\mathsf{RO}(C_{p^n})$) by

$$\rho_{\mathsf{triv}} \longmapsto (e_0, \vec{1}) \qquad and \qquad \rho_j \longmapsto \left(2e_0 - e_{\nu(j)+1} \ , \ \left(\frac{j}{p^{\nu(j)}} \right)_{\nu(j)+1} \right) \ ,$$

where

- ρ_{triv} *denotes the trivial (1-dimensional) representation,*

- ρ_j *denotes the 2-dimensional representation in which the generator of* C_{p^n} *acts by rotation by* $2\pi j/p^n$ *(for* $1 \leq j < p^n$*),*

- $\{e_0, \dots, e_n\} \subset \mathbb{Z}^{\oplus(n+1)}$ *is the standard basis,*

- $\vec{1} \in \bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^{\times}/\{\pm 1\}$ *is the identity element,*

- $\nu(j)$ *is the p-adic valuation of j (i.e.,* $\nu(j)$ *is the largest integer such that* $p^{\nu(j)}$ *divides j),*

- $\left(\frac{j}{p^{\nu(j)}}\right)_{\nu(j)+1} \in \bigoplus_{s=1}^{n}(\mathbb{Z}/p^{n-s+1})^{\times}/\{\pm 1\}$ *is the image of the element* $\frac{j}{p^{\nu(j)}}$ *under the canonical homomorphism* $(\mathbb{Z}/p^{n-(\nu(j)+1)+1})^{\times} \to \bigoplus_{s=1}^{n}(\mathbb{Z}/p^{n-s+1})^{\times}/\{\pm 1\}$*.*

Additionally, we compute the $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$-graded cohomology of a point.

Theorem D (Theorem 11.5). *Let p be an odd prime. Denote by* $\mathrm{D}(\mathsf{Ab})$ *the (ordinary) derived category of abelian groups. For each* $L \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$*, there is an explicit chain-level description of the* $\mathrm{D}(\mathsf{Ab})$*-valued Mackey functor*

$$\mathsf{C}_{\mathsf{C}_{p^n}}^{L}(\mathsf{pt}) \in \mathsf{Mack}_{\mathsf{C}_{p^n}}(\mathrm{D}(\mathsf{Ab}))$$

whose i^{th} *cohomology is the (ordinary) Mackey functor* $\mathsf{H}_{\mathsf{C}_{p^n}}^{i+L}(\mathsf{pt})$*.*

Even in the special case that $n = 1$ (and restricting to $\mathsf{RO}(\mathsf{C}_p)$), Theorem D gives a new proof of Stong's classical calculation of the $\mathsf{RO}(\mathsf{C}_p)$-graded cohomology of a point. We refer the reader to §0.4 for a discussion of related literature.

0.3. **Miscellaneous remarks.**

Remark 0.2. A key ingredient in the proof of Theorem D is an explicit description of the constant Mackey functor $\underline{\mathbb{Z}} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ in terms of Theorem B, i.e. in terms of its geometric C_{p^r}-fixedpoints and gluing maps (Proposition 10.4). As a consequence of this computation, we obtain an equivalence

$$\Phi^{\mathsf{C}_p}(\underline{\mathbb{Z}}) \simeq \mathsf{THH}(\mathbb{F}_p) \ ,$$

of which we would be very interested to have a conceptual description (see Corollary 10.8 and Remark 10.9).

Remark 0.3. Fix any $1 \leq s \leq n$. The summand $(\mathbb{Z}/p^{n-s+1})^{\times}/\{\pm 1\} \subseteq \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$ appearing in Theorem C arises from the homogeneous invertible elements of the ring $\pi_*(\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}}$. Indeed, for any commutative ring R, there is a commutative monoid homomorphism

$$\mathbb{Z} \times \prod_{s=1}^{n}(\pi_*(R^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}}^{\times} \longrightarrow \pi_0(\iota_0(\mathsf{Mod}_R^{\mathsf{gC}_{p^n}}))$$

given by the evident generalization of Notation 8.6 (see also Observation 8.7). In the case that $R = \mathbb{Z}$, this surjects onto the Picard group $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}) \subseteq \pi_0(\iota_0(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}))$ as a result of the fact that every Picard element of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$ is trivial up to de/suspension.

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Remark 0.4. Our methods allow for the computation of $\mathsf{Pic}(\mathsf{Mod}_R^{\mathsf{g}G})$ for more general finite groups G and commutative ring spectra $R \in \mathsf{CAlg}(\mathsf{Sp})$ (generalizing Theorem C).[5] For instance, in the case that $R = \mathbb{Q}$, we have that

$$\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Q}}^{\mathsf{g}\mathsf{C}_{p^n}}) \cong \prod_{s=0}^{n} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Q}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}) \cong \mathbb{Z}^{\oplus(n+1)}$$

as a result of the fact that the Tate construction vanishes on $\mathbb{Q}$-modules.

Likewise, our methods allow for more general computations in equivariant cohomology (generalizing Theorem D). Specifically, one can vary the (finite) group G, the G-space, and the coefficients. Moreover, we expect that one can use our techniques to describe the multiplicative structure (i.e. the Green functor) on equivariant cohomology. This would involve a more careful analysis of the multiplicative structure of Tate cohomology than is done in this paper.

We would be very interested to see any such computations along these lines.

Remark 0.5. Although the theory of stratifications developed in [AMGR] is a crucial ingredient in our work here, it does not play a major role from an expositional point of view: its main consequence that we use here is the explicit description of genuine C_{p^n}-$\mathbb{Z}$-modules (as a symmetric monoidal ∞-category) of Theorem B. So, we use the theory freely here, and refer the interested reader to [AMGR, §1] for a more detailed overview.

Remark 0.6. As explained above, our work applies to the geometric stratification of $\mathcal{R}^{\mathsf{g}G}$. However, there exist other interesting stratifications of $\mathcal{R}^{\mathsf{g}G}$. For instance, by [AMGR, Theorem D], $\mathcal{R}^{\mathsf{g}G}$ also admits an *adelic stratification* over its Balmer spectrum (which is also a symmetric monoidal stratification). The Balmer spectra of $\mathsf{Sp}^{\mathsf{g}G}$ and $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G}$ are respectively studied in [BS17] and [PSW].

0.4. **Relations with existing literature.** As mentioned in §0.1, the importance of derived Mackey functors (i.e. genuine G-$\mathbb{Z}$-modules) goes back to Kaledin [Kal11], who (in different terms) studied its geometric stratification. The idea that genuine G-objects can be expressed in terms of their geometric fixedpoints stems from the work of Greenlees and May; see in particular [Gre, GM95b]. There is also much work on similar expressions of rational G-spectra (which are simpler because the relevant Tate constructions vanish rationally), notably the reconstruction results of Greenlees–Shipley [GS18]. More recent works in this direction include [MNN17, Gla]; see also [NS18, Remark II.4.8].

The first computation of $\mathsf{RO}(G)$-graded cohomology was for $G = \mathsf{C}_p$, due to Stong (see [Lew88]). The works [HHR16, HHR17] of Hill–Hopkins–Ravenel give partial computations for $G = \mathsf{C}_{2^n}$, which play an essential role in their resolution of the Kervaire invariant one problem. Further computations include the works [Zen, Geo] of Zeng and Georgakopoulos for $G = \mathsf{C}_{p^2}$, as well as the works [HK17, HK] of Holler–Kriz for $G = (\mathsf{C}_p)^{\times n}$ (with coefficients in $\underline{\mathbb{Z}/p}$ and restricting to actual (not virtual) representations). Georgakopoulos also gives a computer program for $G = \mathsf{C}_{p^n}$.

As mentioned previously, in [Kra] Krause proves the isomorphism $\mathsf{Pic}(\mathsf{Sp}^{\mathsf{g}G}) \xrightarrow{\cong} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G})$ for any finite group G, and gives a partial computation of this Picard group in a number of examples: C_p, $(\mathsf{C}_p)^{\times 2}$, D_{2p}, and A_5.[6] Moreover, Fausk–Lewis–May [FLM01] give an algebraic description of

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[5]In the notation of [AMGR, §1.7], this may be seen as the Picard group of the fiber product $\mathscr{B}G \times_{\mathsf{Spec}(\mathbb{S})} \mathsf{Spec}(R)$.

[6]Namely, Krause computes $\mathsf{Pic}(\mathsf{Sp}^{\mathsf{g}G})$ in these cases up to unaddressed extension problems, which arise due to the inductive nature of the approach.

$\mathsf{Pic}(\mathcal{S}\mathsf{p}^{\mathsf{g}G})$ in terms of the Picard group of the Burnside ring. Using this and the results of tom Dieck–Petrie [tDP82], one can also compute the Picard group $\mathsf{Pic}(\mathcal{S}\mathsf{p}^{\mathsf{g}C_{p^n}})$, and hence (by Krause's theorem) deduce Theorem C. Our approach is more direct; in particular, it produces an explicit construction of the Picard elements of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}}$, which is needed for Theorem D.

0.5. **Outline.** This paper is divided into two parts. In Part I, we study the geometric stratification of $\mathcal{R}^{\mathsf{g}G}$, culminating with the proof of Theorem A; it is organized as follows.

§1: For a compact Lie group G and a presentable stable ∞-category $\mathcal{M}$, we introduce the ∞-category $\mathcal{M}^{\mathsf{g}G}$ of genuine G-objects in $\mathcal{M}$, and lay out some basic notations and conventions surrounding it. Taking $\mathcal{M} = \mathcal{R}$ to be a presentably symmetric monoidal stable ∞-category, the ∞-category $\mathcal{R}^{\mathsf{g}G}$ is also presentably symmetric monoidal.

§2: We introduce the geometric stratification of $\mathcal{M}^{\mathsf{g}G}$, which is inherited from that of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$. The geometric stratification of $\mathcal{R}^{\mathsf{g}G}$ is a symmetric monoidal stratification.

§3: We establish some technical results regarding the interplay between small and presentable stable ∞-categories.

§4: We introduce the proper Tate construction for objects of $\mathcal{R}^{\mathsf{h}G} := \mathsf{Fun}(\mathsf{B}G, \mathcal{R})$ (in a way making no reference to genuine equivariant homotopy theory).

§5: We establish a formula for the proper Tate construction in terms of $\mathcal{R}^{\mathsf{g}G}$. Starting here, we make the assumption that $\mathcal{R}$ is rigidly-compactly generated.

§6: We prove Theorem A, which describes the gluing functors of the geometric stratification of $\mathcal{R}^{\mathsf{g}G}$ under the further assumption that G is finite.

In Part II, we apply the results of Part I to the case that $G = C_{p^n}$ and $\mathcal{R} = \mathsf{Mod}_{\mathbb{Z}}$, proving Theorems B-D; it is organized as follows.

§7: We prove Theorem B, which gives a simple and explicit description of the geometric stratification of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}}$.

§8: We prove the first part of Theorem C, our computation of the Picard group of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}}$. Starting here, we make the assumption that the prime p is odd.

§9: We prove the second part of Theorem C, which describes the Picard elements of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}}$ that underlie (virtual) representation spheres.

§10: We study the gluing diagram of the constant Mackey functor $\underline{\mathbb{Z}} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}}$ (the coefficients for equivariant cohomology).

§11: We prove Theorem D, our computation of the $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}C_{p^n}})$-graded cohomology of a point, based on the results of §A.

§A: We record some auxiliary results in homological algebra.

Remark 0.7. Most of the work in this paper takes place at the homotopical (i.e. ∞-categorical) level. However, we work at the point-set (i.e. chain) level as well. We compartmentalize the

latter as §A, in which we produce various chain-level data (chain complexes, chain maps, and chain homotopies) and prove that they are presentations of our desired corresponding homotopical data.[7]

The primary purpose of the material in §A is as input to §11 (in which we prove Theorem D), and indeed the remainder of the paper (i.e. §§1-10) can largely be read without reference to it. However, a single straightforward computation made in §A (Lemma A.35) is used in §8, and in §9 we make use of some basic techniques in homological algebra (our conventions for which are recorded in §A.1).

0.6. **Notation and conventions.**

(1) We work within the context of ∞-categories, taking [Lur09] and [Lur] as our standard references. We work model-independently (for instance, we make no reference to the simplices of a quasicategory), and we omit all technical uses of the word "essentially" (for instance, we shorten the term "essentially surjective" to "surjective").

(2) We use the following decorations for our functors.[8]

- The arrow in the notation
$$\mathcal{C} \longhookrightarrow \mathcal{D}$$
denotes a monomorphism, i.e. the inclusion of a subcategory: a functor which is fully faithful on equivalences and induces inclusions of path components (i.e. monomorphisms) on all hom-spaces.

- The arrow in the notation
$$\mathcal{C} \xrightarrow{\text{f.f.}} \mathcal{D}$$
denotes a fully faithful functor. (However, the notation "f.f." is merely emphasis: one should not take its absence to mean that the indicated monomorphism is not fully faithful.)

- The arrow in the notation
$$\mathcal{C} \longrightarrow\!\!\!\rightarrow \mathcal{D}$$
denotes a surjection.

- The arrow in the notation
$$\mathcal{C} \downarrow \mathcal{D}$$
denotes a functor $\mathcal{C} \to \mathcal{D}$ considered as an object of the overcategory $\mathsf{Cat}_{/\mathcal{D}}$ of its target (which will often be some sort of fibration).

More generally, we use the notation $X \downarrow Y$ to denote a morphism in any ∞-category $\mathcal{C}$ that we consider as defining an object in the overcategory $\mathcal{C}_{/Y}$.

(3) Given some datum in an ∞-category (such as an object or morphism), for clarity we may use the superscript $(-)^{\circ}$ to denote the corresponding datum in the opposite ∞-category.

---

[7]In fact, these chain complexes are all quite simple: for the most part they are levelwise free of rank 0 or 1. This simplicity is ultimately afforded by certain Tate vanishing results (Observation 7.5).

[8]These are only for emphasis: the absence of such a decoration should not be taken to imply that the corresponding adjective does not apply.

(4) Given a functor F, we write F^* for pullback along it, and we respectively write $F_!$ and F_* for left and right Kan extension along it.

(5) We write Cat for the ∞-category of ∞-categories, $\mathcal{S}$ for the ∞-category of spaces, and $\mathcal{S}\mathsf{p}$ for the ∞-category of spectra. These are related by the various adjoint functors

$$\mathsf{Cat} \underset{\iota_0}{\overset{|-|}{\underset{\perp}{\overset{\perp}{\rightleftarrows}}}} \mathcal{S} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathsf{p} \ .$$

(6) Let $\mathcal{R} \in \mathsf{Alg}(\mathsf{Pr}^L_{\mathsf{st}})$ be a presentably monoidal stable ∞-category. Then, every left $\mathcal{R}$-module $\mathcal{M} \in \mathsf{LMod}_{\mathcal{R}}(\mathsf{Pr}^L_{\mathsf{st}})$ is canonically enriched over $\mathcal{R}$. In this context we simply write $\underline{\mathsf{hom}}_{\mathcal{M}}(-,-)$ for this enrichment: it is defined as a presheaf on $\mathcal{R}$ by the formula

$$\mathsf{hom}_{\mathcal{R}}(T, \underline{\mathsf{hom}}_{\mathcal{M}}(X,Y)) := \mathsf{hom}_{\mathcal{M}}(T \odot X, Y)$$

(where we write $\odot$ for the action of $\mathcal{R}$ on $\mathcal{M}$), which is representable by the adjoint functor theorem.

(7) Since we will refer to it regularly, we use the letter S to indicate a reference to [AMGR]: for example, we will refer to [AMGR, Theorem A(2)] simply as Theorem S.A(2).

Warning 0.8. In this paper, we study generalizations of a number of notions introduced in §S.5.1. We occasionally reappropriate our notation without additional decoration.

0.7. **Acknowledgments.** It is our pleasure to acknowledge our intellectual debt to Kaledin, which is clear from the discussion of §0.1; much of the material in this paper arose from thinking about the paper [Kal11]. We thank Akhil Mathew for a number of helpful conversations regarding the Tate construction, and in particular for pointing out Proposition 5.10 to us.

PART I. A SYMMETRIC MONOIDAL STRATIFICATION OF DERIVED MACKEY FUNCTORS

1. GENUINE G-OBJECTS IN PRESENTABLE STABLE ∞-CATEGORIES

In this section, we introduce the ∞-categories of genuine and homotopy G-objects in a presentable stable ∞-category as well as various basic notions surrounding them.

Notation 1.1. We assume a basic familiarity with equivariant homotopy theory; we refer the reader to §S.5.1 for a rapid review. In general, we use the notation and terminology laid out there (which is largely quite standard). Here we highlight a few conventions of particular interest in the present work.

(1) We fix an arbitrary compact Lie group G (which will sometimes be assumed to be finite).

(2) We write P_G for the poset of conjugacy classes of closed subgroups of G ordered by sub-conjugacy. We denote relation of subconjugacy by $\leq$. When we wish to indicate literal containment, we use the notation $\subseteq$.

(3) We write H and K for arbitrary closed subgroups of G. When discussing closed subgroups that are related by subconjugacy, we will always take H to be subconjugate to K.

Local Notation 1.2. In this section, we fix a presentable stable ∞-category $\mathcal{M} \in \mathsf{Pr}_{\mathsf{st}}^L$.

Definition 1.3.

(1) We define the presentable stable ∞-category of ***genuine G-objects in*** $\mathcal{M}$ to be the tensor product
$$\mathcal{M}^{\mathsf{g}G} := \mathcal{Sp}^{\mathsf{g}G} \otimes \mathcal{M}$$
in $\mathsf{Pr}_{\mathsf{st}}^L$. Given an associative ring spectrum $R \in \mathsf{Alg}(\mathcal{Sp})$, we refer to $\mathsf{Mod}_R^{\mathsf{g}G}$ as the presentable stable ∞-category of ***genuine G-R-modules***.

(2) We define the presentable stable ∞-category of ***homotopy G-objects in*** $\mathcal{M}$ to be
$$\mathcal{M}^{\mathsf{h}G} := \mathsf{Fun}(\mathsf{B}G, \mathcal{M}) \ .$$
Given an associative ring spectrum $R \in \mathsf{Alg}(\mathcal{Sp})$, we refer to $\mathsf{Mod}_R^{\mathsf{h}G}$ as the presentable stable ∞-category of ***homotopy G-R-modules***.

Observation 1.4. Suppose that $\mathcal{C} \in \mathsf{Pr}_{\mathsf{st}}^L$ is a presentable stable ∞-category. If $\mathcal{C}$ is compactly generated, then there is a canonical equivalence
$$\mathcal{C} \otimes (-) \simeq \mathsf{Fun}^{\mathsf{ex}}((\mathcal{C}^\omega)^{\mathsf{op}}, -)$$
in $\mathsf{Fun}(\mathsf{Pr}_{\mathsf{st}}^L, \mathsf{Pr}_{\mathsf{st}}^L)$. In particular, we have a canonical equivalence
$$\mathcal{C} \otimes \mathcal{M} \simeq \mathsf{Fun}^{\mathsf{ex}}((\mathcal{C}^\omega)^{\mathsf{op}}, \mathcal{M}) \ .$$

Observation 1.5. Note that $\mathcal{Sp}^{\mathsf{h}G} \simeq \mathsf{LMod}_{\Sigma_+^\infty G}(\mathcal{Sp})$ is compactly generated. Hence, by Observation 1.4, the presentable stable ∞-category of homotopy G-objects in $\mathcal{M}$ admits an identification
$$\mathcal{M}^{\mathsf{h}G} := \mathsf{Fun}(\mathsf{B}G, \mathcal{M}) \simeq \mathsf{Fun}^{\mathsf{ex}}(((\mathcal{Sp}^{\mathsf{h}G})^\omega)^{\mathsf{op}}, \mathcal{M}) \simeq \mathcal{Sp}^{\mathsf{h}G} \otimes \mathcal{M} \ .$$

We use this fact without further comment.

Notation 1.6. We simply write
$$U : \mathcal{M}^{\mathsf{g}G} := \mathcal{Sp}^{\mathsf{g}G} \otimes \mathcal{M} \xrightarrow{U \otimes \mathcal{M}} \mathcal{Sp}^{\mathsf{h}G} \otimes \mathcal{M} \simeq \mathcal{M}^{\mathsf{h}G}$$
for the tensor product with $\mathcal{M}$ of the forgetful functor $\mathcal{Sp}^{\mathsf{g}G} \xrightarrow{U} \mathcal{Sp}^{\mathsf{h}G}$ from genuine G-spectra to homotopy G-spectra (a morphism in $\mathsf{Pr}_{\mathsf{st}}^L$). Moreover, we simply write
$$\mathcal{M}^{\mathsf{g}G} \xrightarrow[\substack{\longleftarrow \\ \beta}]{U} \mathcal{M}^{\mathsf{h}G}$$
for the indicated right adjoint (which is fully faithful because the functor $(-) \otimes \mathcal{M}$ preserves colimits (in particular the quotient $\mathcal{Sp}^{\mathsf{g}G} \xrightarrow{U} \mathcal{Sp}^{\mathsf{h}G}$)).[9]

---

[9] This right adjoint may be referred to as the inclusion of the "Borel-complete" genuine G-objects in $\mathcal{M}$.

Remark 1.7. A morphism $\mathcal{A} \xrightarrow{F} \mathcal{B}$ in $\mathsf{Pr}^L_{\mathsf{st}}$ is the data of an adjunction

$$\mathcal{A} \underset{F^R}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$$

in Cat. Tensoring this morphism with any object $\mathcal{C} \in \mathsf{Pr}^L_{\mathsf{st}}$ therefore determines an adjunction

$$\mathcal{A} \otimes \mathcal{C} \underset{(F \otimes \mathcal{C})^R}{\overset{F \otimes \mathcal{C}}{\underset{\perp}{\rightleftarrows}}} \mathcal{B} \otimes \mathcal{C} \quad . \tag{1.1}$$

In general, there is no straightforward description of the right adjoint $(F \otimes \mathcal{C})^R$ (e.g. in terms of the right adjoint F^R). However, if $\mathcal{C}$ is compactly generated, then by Observation 1.4 we may identify the adjunction (1.1) as the adjunction

$$\mathcal{A} \otimes \mathcal{C} \simeq \mathsf{Fun}^{\mathsf{ex}}((\mathcal{C}^\omega)^{\mathsf{op}}, \mathcal{A}) \underset{\mathsf{Fun}^{\mathsf{ex}}((\mathcal{C}^\omega)^{\mathsf{op}}, F^R)}{\overset{\mathsf{Fun}^{\mathsf{ex}}((\mathcal{C}^\omega)^{\mathsf{op}}, F)}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}^{\mathsf{ex}}((\mathcal{C}^\omega)^{\mathsf{op}}, \mathcal{B}) \simeq \mathcal{C} \otimes \mathcal{M}$$

(because right adjoints are unique when they exist).

Notation 1.8. We simply write

$$(-)^H : \mathcal{M}^{\mathbf{g}G} := \mathsf{Sp}^{\mathbf{g}G} \otimes \mathcal{M} \xrightarrow{(-)^H \otimes \mathcal{M}} \mathsf{Sp}^{\mathbf{g}W(H)} \otimes \mathcal{M} =: \mathcal{M}^{\mathbf{g}W(H)} \ ,$$

$$(-)^H : \mathcal{M}^{\mathbf{g}G} := \mathsf{Sp}^{\mathbf{g}G} \otimes \mathcal{M} \xrightarrow{(-)^H \otimes \mathcal{M}} \mathsf{Sp}^{\mathbf{h}W(H)} \otimes \mathcal{M} \simeq \mathcal{M}^{\mathbf{h}W(H)} \ ,$$

$$\Phi^H_{\mathbf{g}} : \mathcal{M}^{\mathbf{g}G} := \mathsf{Sp}^{\mathbf{g}G} \otimes \mathcal{M} \xrightarrow{\Phi^H_{\mathbf{g}} \otimes \mathcal{M}} \mathsf{Sp}^{\mathbf{g}W(H)} \otimes \mathcal{M} =: \mathcal{M}^{\mathbf{g}W(H)} \ ,$$

and

$$\Phi^H : \mathcal{M}^{\mathbf{g}G} := \mathsf{Sp}^{\mathbf{g}G} \otimes \mathcal{M} \xrightarrow{\Phi^H \otimes \mathcal{M}} \mathsf{Sp}^{\mathbf{h}W(H)} \otimes \mathcal{M} \simeq \mathcal{M}^{\mathbf{h}W(H)}$$

for the tensor product with $\mathcal{M}$ of the various indicated H-fixedpoints functors on genuine G-spectra (all of which are morphisms in $\mathsf{Pr}^L_{\mathsf{st}}$).

Observation 1.9. Suppose that G is a finite group. Then, by [GM, Bar17] we have an equivalence

$$\mathsf{Sp}^{\mathbf{g}G} \simeq \mathsf{Mack}_G(\mathsf{Sp}) := \mathsf{Fun}^\oplus(\mathsf{Burn}_G, \mathsf{Sp})$$

where Burn_G denotes the $(2,1)$-category of spans among finite G-sets, which is preadditive and so is canonically enriched in commutative monoid spaces. It follows that the idempotent-complete stable envelope of Burn_G (i.e. that of its homwise ∞-group completion) admits a canonical identification

$$\mathsf{Env}^{\mathsf{idem}}(\mathsf{Burn}_G) \simeq ((\mathsf{Sp}^{\mathbf{g}G})^\omega)^{\mathsf{op}} \ .$$

Hence, using Observation 1.4 and the fact that $\mathsf{Sp}^{\mathbf{g}G}$ is compactly generated, we obtain a composite equivalence

$$\mathcal{M}^{\mathbf{g}G} \simeq \mathsf{Fun}^{\mathsf{ex}}(((\mathsf{Sp}^{\mathbf{g}G})^\omega)^{\mathsf{op}}, \mathcal{M}) \simeq \mathsf{Fun}^{\mathsf{ex}}(\mathsf{Env}^{\mathsf{idem}}(\mathsf{Burn}_G), \mathcal{M}) \simeq \mathsf{Fun}^\oplus(\mathsf{Burn}_G, \mathcal{M}) =: \mathsf{Mack}_G(\mathcal{M}) \ .$$

By construction, evaluating a genuine G-object $E \in \mathcal{M}^{\mathbf{g}G}$ on the finite G-set $G/H \in \mathsf{Burn}_G$ yields its categorical H-fixedpoints $E^H \in \mathcal{M}$, with the homotopy $W(H)$-action coming from its action on $G/H \in \mathsf{Burn}_G$: in other words, this equivalence extends to a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathbf{g}G} & \xrightarrow{\quad \sim \quad} & \mathsf{Mack}_G(\mathcal{M}) \\ & {\scriptstyle (-)_H} \searrow \qquad \swarrow {\scriptstyle \mathsf{ev}_{G/H}} & \\ & \mathcal{M}^{\mathbf{h}W(H)} & \end{array} \quad .$$

Definition 1.10. Suppose that G is a finite group. Given subgroups $H \leq K \leq G$, we obtain a morphism $G/H \to G/K$ between finite G-sets, which determines morphisms in both directions in Burn_G. Via Observation 1.9, evaluating a genuine G-object $E \in \mathcal{M}^{\mathsf{g}G}$ on these morphisms determines natural morphisms

$$E^H \xrightarrow{\ \mathsf{inc}_H^K(E)\ } E^K \qquad \text{and} \qquad E^K \xrightarrow{\ \mathsf{trf}_H^K(E)\ } E^H$$

in $\mathcal{M}$ in both directions between its categorical H- and K-fixedpoints, which we respectively refer to as the **inclusion** and **transfer** morphisms. As E, H, and K will always be clear from context, we will generally simply write

$$\mathsf{inc} := \mathsf{inc}_H^K(E) \qquad \text{and} \qquad \mathsf{trf} := \mathsf{trf}_H^K(E) \ .$$

Remark 1.11. In Definition 1.10, we use the term "inclusion" instead of the more familiar term "restriction" because the latter is already quite overloaded. However, note that this morphism is *not* generally a monomorphism (indeed, in a stable ∞-category, all monomorphisms are equivalences).

Observation 1.12. Suppose that G is a finite group. Given subgroups $H \leq K \leq G$, in the case that H and K are both normal in G (e.g. when G is abelian), the morphisms inc_H^K and trf_H^K admit canonical lifts from $\mathcal{M}$ to $\mathcal{M}^{\mathsf{h}(K/H)}$.[10] We use this fact without further comment.

Notation 1.13. In line with the notation introduced in Definition 1.10, we simply write inc^h for any inclusion morphism on homotopy fixedpoints.

Observation 1.14. The equivalence $\mathcal{M}^{\mathsf{g}G} \simeq \mathsf{Mack}_G(\mathcal{M})$ of Observation 1.9 is compatible with restriction, in the sense that the diagram

$$
\begin{array}{ccc}
\mathcal{M}^{\mathsf{g}G} & \xleftarrow{\ \sim\ } & \mathsf{Mack}_G(\mathcal{M}) \\
{\scriptstyle \mathsf{Res}_H^G} \downarrow & & \downarrow {\scriptstyle \left(\mathsf{Ind}_H^G\right)^*} \\
\mathcal{M}^{\mathsf{g}H} & \xleftarrow[\ \sim\ ]{} & \mathsf{Mack}_H(\mathcal{M})
\end{array}
$$

commutes. It is also compatible with categorical fixedpoints, in the sense that the diagram

$$
\begin{array}{ccc}
\mathcal{M}^{\mathsf{g}G} & \xleftarrow{\ \sim\ } & \mathsf{Mack}_G(\mathcal{M}) \\
{\scriptstyle \mathsf{Res}_{N(H)}^G} \downarrow & & \downarrow {\scriptstyle \left(\mathsf{Ind}_{N(H)}^G\right)^*} \\
\mathcal{M}^{\mathsf{g}N(H)} & \xleftarrow[\ \sim\ ]{} & \mathsf{Mack}_{N(H)}(\mathcal{M}) \\
{\scriptstyle (-)^H} \downarrow & & \downarrow {\scriptstyle \left(\mathsf{Res}_{N(H)}^{W(H)}\right)^*} \\
\mathcal{M}^{\mathsf{g}W(H)} & \xleftarrow[\ \sim\ ]{} & \mathsf{Mack}_{W(H)}(\mathcal{M})
\end{array}
$$

with the outer left arrow $(-)^H$ from $\mathcal{M}^{\mathsf{g}G}$ to $\mathcal{M}^{\mathsf{g}W(H)}$.

commutes. We use these facts without further comment.

Notation 1.15. We denote by $\odot$ the action on $\mathcal{M}^{\mathsf{g}G}$ of $\mathcal{S}_*^{\mathsf{g}G}$ (via that of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$).

---

[10]More generally, these morphisms are equivariant for the *relative Weyl group* (Definition S.5.4.8).

Observation 1.16. It follows directly from the definitions that the diagram

$$
\begin{array}{ccc}
\mathcal{M}^{\mathrm{g}G} & \xrightarrow{\widetilde{\mathsf{E}}\mathscr{F}_{\not\geq_H}\odot(-)} & \mathcal{M}^{\mathrm{g}G} \\
& \searrow{\scriptstyle\Phi_{\mathrm{g}}^{H}} & \downarrow{\scriptstyle(-)^{H}} \\
& & \mathcal{M}^{\mathrm{g}\mathsf{W}(H)}
\end{array}
$$

commutes (see Definition S.5.1.12).

2. The geometric stratification of genuine G-objects

In this brief section, we introduce the **geometric stratification** of genuine G-objects (Definition 2.4). Throughout it, we refer freely to the notions introduced in [AMGR] (recall Remark 0.5).

Observation 2.1. Let $\mathcal{X}$ be a presentable stable ∞-category. A closed subcategory $\mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ is precisely the data of an adjunction

$$
\mathcal{Z} \underset{y}{\overset{i_L}{\underset{\perp}{\rightleftarrows}}} \mathcal{X}
$$

in $\mathsf{Pr}_{\mathsf{st}}^{L}$ whose unit is an equivalence. It follows that taking the tensor product in $\mathsf{Pr}_{\mathsf{st}}^{L}$ with any presentable stable ∞-category $\mathcal{M}$ determines a functor

$$
\begin{array}{ccc}
\mathbf{Cls}_{\mathcal{X}} & \xrightarrow{\;-\otimes\mathcal{M}\;} & \mathbf{Cls}_{\mathcal{X}\otimes\mathcal{M}} \\
\cup & & \cup \\
\mathcal{Z} & \longmapsto & \mathcal{Z}\otimes\mathcal{M}
\end{array} \qquad . \tag{2.1}
$$

Observe further that the functor (2.1) preserves colimits and finite products. It follows that for any stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_{\bullet}} \mathbf{Cls}_{\mathcal{X}}$, postcomposition determines a stratification

$$
\begin{array}{ccccc}
\mathsf{P} & \xrightarrow{\;\mathcal{Z}_{\bullet}\;} & \mathbf{Cls}_{\mathcal{X}} & \xrightarrow{\;-\otimes\mathcal{M}\;} & \mathbf{Cls}_{\mathcal{X}\otimes\mathcal{M}} \\
\cup & & & & \cup \\
p & & \longmapsto & & \mathcal{Z}_p\otimes\mathcal{M}
\end{array} \tag{2.2}
$$

of $\mathcal{X}\otimes\mathcal{M}$: the factorizations guaranteed by the stratification condition define commutative squares in $\mathsf{Pr}_{\mathsf{st}}^{L}$, and so persist upon tensoring with $\mathcal{M}$. Moreover, because the functor $\mathsf{Pr}_{\mathsf{st}}^{L} \xrightarrow{-\otimes\mathcal{M}} \mathsf{Pr}_{\mathsf{st}}^{L}$ preserves colimits, for each $p \in \mathsf{P}$ we may identify the pth stratum of the stratification (2.2) as

$$
(\mathcal{X} \otimes \mathcal{M})_p \simeq \mathcal{X}_p \otimes \mathcal{M} \; .
$$

It follows immediately that for each $p \in \mathsf{P}$ we may identify the pth geometric localization functor of the stratification (2.2) as

$$
\mathcal{X} \otimes \mathcal{M} \xrightarrow{\;\Phi_p\otimes\mathcal{M}\;} \mathcal{X}_p \otimes \mathcal{M} \; .
$$

Remark 2.2. In the situation of Observation 2.1, if $\mathcal{M}$ is compactly generated then by Observation 1.4 we may identify the pth localization adjunction of the stratification (2.2) as

$$
\mathcal{X} \otimes \mathcal{M} \simeq \mathsf{Fun}((\mathcal{M}^{\omega})^{\mathsf{op}},\mathcal{X}) \underset{\mathsf{Fun}^{\mathsf{ex}}((\mathcal{M}^{\omega})^{\mathsf{op}},\rho^P)}{\overset{\mathsf{Fun}^{\mathsf{ex}}((\mathcal{M}^{\omega})^{\mathsf{op}},\Phi_p)}{\underset{\perp}{\rightleftarrows}}} \mathsf{Fun}((\mathcal{M}^{\omega})^{\mathsf{op}},\mathcal{X}_p) \simeq \mathcal{X}_p \otimes \mathcal{M} \quad .
$$

In fact, in this case the entire gluing diagram of $\mathfrak{X} \otimes \mathfrak{M}$ is tensored up from that of $\mathfrak{X}$, in the sense that it is given by the composite

$$\mathscr{G}(\mathfrak{X} \otimes \mathfrak{M}) : \mathsf{P} \xrightarrow{\;\mathscr{G}(\mathfrak{X})\;}_{\text{l.lax}} \mathsf{Pr}_{\mathsf{st}} \xrightarrow{\;\mathsf{Fun}^{\mathsf{ex}}((\mathfrak{M}^\omega)^{\mathsf{op}},-)\;} \mathsf{Pr}_{\mathsf{st}} \quad .$$

Observation 2.3. Let $\mathfrak{X}$ and $\mathfrak{M}$ be presentably symmetric monoidal stable ∞-categories. Then, $\mathfrak{X} \otimes \mathfrak{M}$ is canonically a presentably symmetric monoidal stable ∞-category as well, and the functor (2.1) admits a refinement

$$
\begin{array}{ccc}
\mathbf{Cls}_{\mathfrak{X}} & \xrightarrow{\;-\otimes\mathfrak{M}\;} & \mathbf{Cls}_{\mathfrak{X}\otimes\mathfrak{M}} \\
\uparrow & & \uparrow \\
\mathbf{Idl}_{\mathfrak{X}} & \dashrightarrow{\;-\otimes\mathfrak{M}\;} & \mathbf{Idl}_{\mathfrak{X}\otimes\mathfrak{M}}
\end{array}
\quad .
$$

Definition 2.4. The **geometric stratification** of the presentable stable ∞-category $\mathfrak{M}^{\mathsf{g}G}$ of genuine G-objects in $\mathfrak{M}$ is the composite functor

$$
\begin{array}{ccc}
\mathsf{P}_G & \xrightarrow{\;\mathcal{S}\mathsf{p}^{\mathsf{g}G}_{\leq\bullet}\;} \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}} & \xrightarrow{\;-\otimes\mathfrak{M}\;} \mathbf{Cls}_{\mathfrak{M}^{\mathsf{g}G}} \\
\mathrel{\rotatebox{90}{$\in$}} & & \mathrel{\rotatebox{90}{$\in$}} \\
H & \longmapsto & \mathcal{S}\mathsf{p}^{\mathsf{g}G}_{\leq H} \otimes \mathfrak{M}
\end{array}
\quad ,
$$

where the first functor is the symmetric monoidal geometric stratification of genuine G-spectra of Theorem S.E (cf. Definition S.5.1.8); the fact that this composite functor is a stratification follows from Observation 2.1. In the case that $\mathfrak{M}$ is a presentably symmetric monoidal stable ∞-category, we use the same name to refer to the symmetric monoidal stratification

$$\mathsf{P}_G \xrightarrow{\;\mathcal{S}\mathsf{p}^{\mathsf{g}G}_{\leq\bullet}\;} \mathbf{Idl}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}} \xrightarrow{\;-\otimes\mathfrak{M}\;} \mathbf{Idl}_{\mathfrak{M}^{\mathsf{g}G}} \tag{2.3}$$

guaranteed by Observation 2.3.

3. Some algebra and analysis of stable ∞-categories

In this section, we establish a number of technical results regarding the interplay between small and presentable stable ∞-categories, which we use in our proof that the gluing functors for the geometric stratification of genuine G-objects are proper Tate constructions (Proposition 5.9).

Definition 3.1. The **stable quotient** of an exact functor

$$\mathcal{A} \xrightarrow{\;F\;} \mathcal{B}$$

between (small or large) stable ∞-categories is the cofiber

$$\mathcal{B}/^{\mathsf{St}}\mathcal{A} := \mathsf{cofib}\left(\mathcal{A} \xrightarrow{\;F\;} \mathcal{B}\right) \ ,$$

considered among (resp. small or large) stable ∞-categories. (Stable quotients always exist, by Observations 3.3 and 3.4.)

Warning 3.2. We will be taking stable quotients of functors between *presentable* (and in particular, large) stable ∞-categories.

Observation 3.3. The stable quotient of an exact functor between stable ∞-categories only depends on its image, because it is merely a condition (as opposed to additional data) for an exact functor among stable ∞-categories to be zero. In other words, to understand stable quotients it suffices to understand stable quotients by full stable subcategories. Likewise, it suffices to consider stable quotients by full stable subcategories that are closed under retracts.

Observation 3.4. Suppose that

$$\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B}$$

is a fully faithful exact functor between small stable ∞-categories. Applying the functor

$$\mathsf{St} \xrightarrow{\mathsf{Ind}} \mathsf{Pr}^L_{\mathsf{st}} \,,$$

we obtain a fully faithful functor

$$\mathsf{Ind}(\mathcal{A}) \xhookrightarrow{\mathsf{Ind}(i)=i_!} \mathsf{Ind}(\mathcal{B})$$

between presentable stable ∞-categories. In fact, this is the inclusion of a closed subcategory: the functor $i_!$ preserves colimits, as does its right adjoint i^*. Hence, we obtain a recollement

(see e.g. Definition S.1.1.1). Observe that the functor p_L preserves compact objects (because the functor i^* preserves filtered colimits). From here, it is straightforward to see that the composite

$$\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B} \overset{p}{\longrightarrow} (\mathsf{Ind}(\mathcal{B})/\mathsf{Ind}(\mathcal{A}))^\omega$$

is a cofiber sequence among small stable ∞-categories after idempotent completion, where p denotes the restriction of p_L to $\mathcal{B} \subseteq \mathsf{Ind}(\mathcal{B})$, and thereafter that the stable quotient of i itself is the full stable subcategory

$$\mathcal{B}/^{\mathsf{St}}\mathcal{A} = p(\mathcal{B}) \subseteq (\mathsf{Ind}(\mathcal{B})/\mathsf{Ind}(\mathcal{A}))^\omega \,,$$

the image of p (which is automatically stable).[11] Moreover, by enlarging our Grothendieck universe, we can apply this same construction to exact functors between not-necessarily-small stable ∞-categories.

Remark 3.5. Let $\mathfrak{X}$ be a presentable stable ∞-category and let $\mathfrak{Z} \subseteq \mathfrak{X}$ be a full presentable stable subcategory. Then, the stable and presentable quotients of $\mathfrak{X}$ by $\mathfrak{Z}$ coincide: the canonical morphism

$$\mathfrak{X}/^{\mathsf{St}}\mathfrak{Z} \longrightarrow \mathfrak{X}/\mathfrak{Z}$$

is an equivalence. Indeed, the presentable quotient satisfies the universal property of the stable quotient: writing

$$\mathfrak{Z} \xrightleftharpoons[i^R]{i} \mathfrak{X} \xrightleftharpoons[j]{j^L} \mathfrak{X}/\mathfrak{Z}$$

for the resulting diagram in Cat (see Definition S.2.3.6 and Observation S.2.3.5), given any stable ∞-category $\mathcal{C}$ and any exact functor $\mathfrak{X} \overset{F}{\longrightarrow} \mathcal{C}$ such that $Fi \simeq 0$, the morphism

$$F \longrightarrow Fjj^L$$

---

[11]In particular, $(\mathsf{Ind}(\mathcal{B})/\mathsf{Ind}(\mathcal{A}))^\omega$ is the idempotent completion of $\mathcal{B}/^{\mathsf{St}}\mathcal{A}$.

is an equivalence (because for each $X \in \mathcal{X}$ the cofiber sequence $ii^R X \to X \to jj^L X$ is carried by F to a cofiber sequence).

Definition 3.6. Let $\mathcal{C}$ be a stably symmetric monoidal ∞-category. A full stable subcategory $\mathcal{I} \subseteq \mathcal{C}$ is a ***thick ideal*** if it is closed under retracts and contagious under the symmetric monoidal structure.

Notation 3.7. Let $\mathcal{C}$ be a stably symmetric monoidal ∞-category. Given a set $\{K_s \in \mathcal{C}\}_{s \in S}$ of objects of $\mathcal{C}$, we write $\langle K_s \rangle_{s \in S}^{\mathsf{St}, \otimes} \subseteq \mathcal{C}$ for the thick ideal that they generate.

Observation 3.8. Suppose that $\mathcal{C} \in \mathsf{CAlg}(\mathsf{St})$ is a stably symmetric monoidal ∞-category and $\mathcal{I} \subseteq \mathcal{C}$ is a thick ideal. Then, $\mathsf{Ind}(\mathcal{I}) \subseteq \mathsf{Ind}(\mathcal{C})$ is an ideal (see Definition S.4.2.5). Moreover, for any set of objects $\{K_s \in \mathcal{C}\}_{s \in S}$ we have

$$\mathsf{Ind}(\langle K_s \rangle_{s \in S}^{\mathsf{St}, \otimes}) = \langle K_s \rangle_{s \in S}^{\otimes}$$

(see Notation S.4.2.6).

4. The proper Tate construction

In this section, we introduce the proper Tate construction (Definition 4.5) and study its basic features. In fact, we introduce a generalization of the proper Tate construction, which makes reference to a family of closed subgroups; this is no more difficult to study, and has the added benefit of recovering the ordinary Tate construction as a special case (Observation 4.8 and Remark 4.9).

Local Notation 4.1. In this section, we fix a family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$ of closed subgroups of G (i.e. a collection of closed subgroups of G that is closed under subconjugacy), and we fix a presentably symmetric monoidal stable ∞-category $\mathcal{R}$.

Notation 4.2. Given an object $E \in \mathcal{R}$ and a homotopy G-space $X \in \mathcal{S}^{hG}$, we write

$$E^{\mathsf{h}} \langle X \rangle := (\Sigma^\infty X_+ \otimes E) \in \mathcal{S}\mathsf{p}^{hG} \otimes \mathcal{R} \simeq \mathcal{R}^{hG} \ .$$

Warning 4.3. We write G/H both for the object of $\mathcal{S}^{\mathsf{g}G}$ and its image under the forgetful functor $\mathcal{S}^{\mathsf{g}G} \xrightarrow{U} \mathcal{S}^{hG}$. However, it will always be clear from context which object is being referred to.

Notation 4.4. We write

$$\mathcal{I}_{\mathscr{F}}^{\mathsf{h}} := \left\langle \mathbb{1}_{\mathcal{R}}^{\mathsf{h}} \langle G/H \rangle \right\rangle_{H \in \mathscr{F}}^{\mathsf{St}, \otimes} \subseteq \mathcal{R}^{hG}$$

for the thick ideal of $\mathcal{R}^{hG}$ generated by the objects $\{\mathbb{1}_{\mathcal{R}}^{\mathsf{h}} \langle G/H \rangle \in \mathcal{R}^{hG}\}_{H \in \mathscr{F}}$.

Definition 4.5. For any closed subgroup $H \in \mathsf{P}_G$, the ***$\mathscr{F}$-H-Tate construction*** is the composite functor

$$(-)^{\tau_{\mathscr{F}} H} : \mathcal{R}^{hG} \xrightarrow{p} \mathcal{R}^{hG} /^{\mathsf{St}} \mathcal{I}_{\mathscr{F}}^{\mathsf{h}} \xrightarrow{\underline{\mathsf{hom}}(p(\mathbb{1}_{\mathcal{R}}^{\mathsf{h}} \langle G/H \rangle), -)} \mathcal{R}^{h\mathsf{W}(H)} \ ,$$

where p denotes the canonical functor to the stable quotient. In the special case where $\mathscr{F} = (\not\geq H)$, we simply write

$$(-)^{\tau H} := (-)^{\tau_{(\not\geq H)} H}$$

and refer to this functor as the ***proper H-Tate construction***.

Remark 4.6. We refer the reader to Proposition 5.9 for the relationship between the $\mathscr{F}$-H-Tate construction and genuine G-objects, and to Proposition 5.10 for an explicit formula for the proper G-Tate construction.

Remark 4.7. If G is a finite group that is not of prime-power order, then $(-)^{\tau G}$ is zero. This follows from the argument of [Kal11, Lemma 7.15(i)] (see also [NS18, Lemma II.6.7]).

Observation 4.8. Suppose that G is a finite group and that $\mathcal{R} = \mathsf{Mod}_R$ for some commutative ring spectrum $R \in \mathsf{CAlg}(\mathcal{Sp})$. Then, the $\{e\}$-G-Tate construction recovers the usual G-Tate construction

$$(-)^{\tau_{\{e\}}G} \simeq (-)^{\mathsf{t}G} := \mathsf{cofib}\left((-)_{\mathsf{h}G} \xrightarrow{\mathsf{Nm}_G} (-)^{\mathsf{h}G}\right) .$$

This is proved as [NS18, Lemma I.3.8(iii)] in the case that $R = \mathbb{S}$ (so that $\mathcal{R} = \mathsf{Mod}_{\mathbb{S}}(\mathcal{Sp}) \simeq \mathcal{Sp}$), and the same proof applies verbatim in general.

Remark 4.9. In fact, for a finite group G, the $\{e\}$-G-Tate construction recovers the usual G-Tate construction for any presentably symmetric monoidal stable ∞-category $\mathcal{R}$.[12] To verify this, it suffices to prove the analog of the first equivalence of [NS18, Lemma I.3.8(iii)], namely that for any object $X \in \mathcal{R}^{\mathsf{h}G}$ the morphism

$$\left(\mathsf{colim}_{Y \in (\mathcal{R}^{\mathsf{h}G}_{\mathsf{Ind}})_{/X}} Y\right) \longrightarrow X$$

is an equivalence, where we write $\mathcal{R}^{\mathsf{h}G}_{\mathsf{Ind}} \subseteq \mathcal{R}^{\mathsf{h}G}$ for the stable subcategory generated by the image of the induction functor $\mathcal{R} \xrightarrow{\mathsf{Ind}^G_e} \mathcal{R}^{\mathsf{h}G}$. To see this, observe first that this subcategory is the stable envelope of the Kleisli ∞-category associated to the monadic adjunction

$$\mathcal{R} \underset{\mathsf{Res}^G_e}{\overset{\mathsf{Ind}^G_e}{\underset{\perp}{\rightleftarrows}}} \mathcal{R}^{\mathsf{h}G} \quad .$$

Hence, the claim follows from the spectrally-enriched analog of Lemma 4.10 and the fact that for any stable ∞-category $\mathcal{C}$ and any spectrally-enriched ∞-category $\mathcal{I}$ the diagram

$$\mathsf{Fun}^{\mathsf{ex}}(\mathsf{Env}(\mathcal{I}), \mathcal{C}) \xrightarrow{\ \sim\ } \underline{\mathsf{Fun}}(\mathcal{I}, \mathcal{C})$$
$$\underset{\mathsf{colim}_{\mathsf{Env}(\mathcal{I})}}{\searrow} \quad \mathcal{C} \quad \underset{\mathsf{colim}_{\mathcal{I}}}{\swarrow}$$

commutes (where $\mathsf{Env}(\mathcal{I})$ denotes the stable envelope of $\mathcal{I}$ and the underlines signify the analogous enriched notions).

Lemma 4.10. *Let $\mathcal{C}$ be an ∞-category and let $T \in \mathsf{Alg}(\mathsf{Fun}(\mathcal{C}, \mathcal{C}))$ be a monad on $\mathcal{C}$. Let us write*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathsf{Alg}_T(\mathcal{C})$$

for the corresponding free/forget adjunction (so that $T \simeq UF$), and let us write

$$\mathsf{Kl}_T(\mathcal{C}) \subseteq \mathsf{Alg}_T(\mathcal{C})$$

for the Kleisli ∞-category of T (i.e. the full subcategory on the free T-algebras). Then, for any T-algebra $X \in \mathsf{Alg}_T(\mathcal{C})$ the canonical morphism

$$\mathsf{colim}_{FY \in \mathsf{Kl}_T(\mathcal{C})_{/X}} FY \longrightarrow X \tag{4.1}$$

in $\mathsf{Alg}_T(\mathcal{C})$ is an equivalence.

---

[12]We do not need this fact, and so we do not prove it. Nevertheless, we include the present discussion in order to motivate Definition 4.11.

Proof. It suffices to observe that the morphism (4.1) extends to a commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_{FY \in \mathsf{Kl}_T(\mathcal{C})_{/X}} |(FU)^{\bullet+1}(FY)| & \longrightarrow & |(FU)^{\bullet+1}X| \\ \Big\downarrow \wr & & \Big\downarrow \wr \\ \operatorname{colim}_{FY \in \mathsf{Kl}_T(\mathcal{C})_{/X}} FY & \longrightarrow & X \end{array}$$

in $\mathsf{Alg}_T(\mathcal{C})$. $\qquad\square$

Definition 4.11. In view of Observation 4.8 and Remark 4.9, in the special case where $\mathscr{F} = \{e\}$, we simply write

$$(-)^{\mathsf{t}H} := (-)^{\tau_{\{e\}}H}$$

and refer to this functor as the *H-Tate construction*.

Notation 4.12. Suppose that G is a finite group. We write

$$(-)_{\mathsf{h}G} \xrightarrow{\mathsf{Nm}_G} (-)^{\mathsf{h}G} \xrightarrow{\mathsf{Q}_G} (-)^{\mathsf{t}G}$$

for the cofiber sequence in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{R}^{\mathsf{h}G}, \mathcal{R})$ that defines the G-Tate construction.[13]

Observation 4.13. Let G be a finite group and let $H \leq G$ be a normal subgroup. We record the following facts for future use.

(1) Parametrizing the norm map $(-)_{\mathsf{h}H} \xrightarrow{\mathsf{Nm}_H} (-)^{\mathsf{h}H}$ in $\mathsf{Fun}(\mathcal{R}^{\mathsf{h}H}, \mathcal{R})$ over the $\mathsf{B}H$-bundle $\mathsf{B}G \to \mathsf{B}(G/H)$, we obtain a homotopy (G/H)-equivariant norm map, i.e. a morphism

$$(-)_{\mathsf{h}H} \xrightarrow{\mathsf{Nm}_H} (-)^{\mathsf{h}H}$$

in $\mathsf{Fun}(\mathcal{R}^{\mathsf{h}G}, \mathcal{R}^{\mathsf{h}(G/H)})$. Hence, we obtain a cofiber sequence

$$(-)_{\mathsf{h}H} \xrightarrow{\mathsf{Nm}_H} (-)^{\mathsf{h}H} \xrightarrow{\mathsf{Q}_H} (-)^{\mathsf{t}H} \tag{4.2}$$

in $\mathsf{Fun}(\mathcal{R}^{\mathsf{h}G}, \mathcal{R}^{\mathsf{h}(G/H)})$ lifting the analogous cofiber sequence in $\mathsf{Fun}(\mathcal{R}^{\mathsf{h}G}, \mathcal{R})$ (because the forgetful functor $\mathcal{R}^{\mathsf{h}(G/H)} \xrightarrow{\mathsf{fgt}} \mathcal{R}$ is exact). In particular, we obtain a residual homotopy (G/H)-equivariant structure on the H-Tate construction. We use these facts without further comment.

(2) Consider the morphism

$$(-)^{\mathsf{h}H} \xrightarrow{\mathsf{Q}_H} (-)^{\mathsf{t}H} \tag{4.3}$$

in $\mathsf{Fun}(\mathcal{R}^{\mathsf{h}G}, \mathcal{R}^{\mathsf{h}(G/H)})$, the second morphism in the cofiber sequence (4.2). Evidently, the source of the morphism (4.3) is canonically right-laxly symmetric monoidal. Thereafter, there exists a canonical enhancement of the morphism (4.3) to one of right-laxly symmetric monoidal functors, obtained by working fiberwise over $\mathsf{B}(G/H)$ and applying [NS18, Theorem I.3.1] (which guarantees that there is in fact a unique such enhancement when $H = G$).

---

[13]We have chosen the notation "Q" to invoke the idea that this the map to the quotient (of homotopy G-fixedpoints by homotopy G-orbits).

(3) Suppose that $H \leq K \leq G$. Then, for any $E \in \mathcal{R}^{hG}$, by [HL, Proposition 4.2.2] we have the following natural commutative diagram in $\mathcal{R}^{h(G/K)}$:

5. The proper Tate construction from genuine G-objects

In §4, we introduced the $\mathscr{F}$-H-Tate construction for homotopy G-objects (Definition 4.5). Our main goal in this section is to show that this functor can also be described in terms of genuine G-objects (Proposition 5.9); this description will be a key ingredient of our proof of Theorem A in §6. We also use it to give an explicit formula for the proper G-Tate construction (Proposition 5.10), which we state and prove directly after stating Proposition 5.9. We then make a number of preliminary observations before proving Proposition 5.9 at the end of the section.

Local Notation 5.1. In this section, we fix a compact Lie group G, a family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$, and a rigidly-compactly generated presentably symmetric monoidal stable ∞-category $\mathcal{R}$.

Notation 5.2. We define the subcategory
$$\mathcal{I}_{\mathscr{F}} := \mathsf{Sp}_{\mathscr{F}}^{\mathsf{g}G} \otimes \mathcal{R} := \langle \Sigma_G^{\infty}(G/H)_+ \rangle_{H \in \mathscr{F}} \otimes \mathcal{R} \subseteq \mathsf{Sp}^{\mathsf{g}G} \otimes \mathcal{R} =: \mathcal{R}^{\mathsf{g}G} .$$

Observation 5.3. By Observations S.5.1.15 and 2.3, the subcategory $\mathcal{I}_{\mathscr{F}} \subseteq \mathcal{R}^{\mathsf{g}G}$ is a closed ideal.

Notation 5.4. Given an object $E \in \mathcal{R}$ and a genuine G-space $X \in \mathcal{S}^{\mathsf{g}G}$, we write
$$E \langle X \rangle := (\Sigma_G^{\infty} X_+ \otimes E) \in \mathsf{Sp}^{\mathsf{g}G} \otimes \mathcal{R} =: \mathcal{R}^{\mathsf{g}G} .$$

Observation 5.5. Notation 5.4 is compatible with Notation 4.2, in the sense that for any genuine G-space $X \in \mathcal{S}^{\mathsf{g}G}$ and any object $E \in \mathcal{R}$ we have an equivalence
$$U(E \langle X \rangle) \simeq E^{\mathsf{h}} \langle U(X) \rangle$$
in $\mathcal{R}^{hG}$ (due to the equivalence $U(\Sigma_G^{\infty} X_+) \simeq \Sigma^{\infty} U(X)_+$ in Sp^{hG}).

Observation 5.6. There is an identification
$$\mathcal{I}_{\mathscr{F}} = \langle \mathbb{1}_{\mathcal{R}} \langle G/H \rangle \rangle_{H \in \mathscr{F}}^{\otimes}$$
among ideals of $\mathcal{R}^{\mathsf{g}G}$.

Definition 5.7. For any closed subgroup $H \in \mathsf{P}_G$, the $\mathscr{F}$-H-**geometric fixedpoints** functor is

$$\Phi_{\mathscr{F}}^H : \mathcal{R}^{\mathsf{g}G} \xrightarrow{p_L} \mathcal{R}^{\mathsf{g}G}/\mathcal{I}_{\mathscr{F}} \xrightarrow{\underline{\hom}_{\mathcal{R}^{\mathsf{g}G}/\mathcal{I}_{\mathscr{F}}}(p_L(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle),-)} \mathcal{R}^{\mathsf{hW}(H)} .$$

Observation 5.8. Definition 5.7 generalizes the functor $\mathcal{R}^{\mathsf{g}G} \xrightarrow{\Phi^H} \mathcal{R}^{\mathsf{hW}(H)}$ of Notation 1.8, as we now explain. Given a presentable stable ∞-category $\mathcal{C} \in \mathsf{Pr}_{\mathsf{st}}^L$ and a compact object $X \in \mathcal{C}^\omega$, we obtain an adjunction

$$\mathsf{Sp} \underset{\underline{\hom}_{\mathcal{C}}(X,-)}{\overset{(-)\otimes X}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}$$

in $\mathsf{Pr}_{\mathsf{st}}^L$. When tensored with $\mathcal{R}$, this gives an adjunction

$$\mathcal{R} \underset{\underline{\hom}_{\mathcal{C}}(X,-)\otimes\mathcal{R}}{\overset{X\otimes(-)}{\underset{\perp}{\rightleftarrows}}} \mathcal{C} \otimes \mathcal{R}$$

in $\mathsf{Mod}_{\mathcal{R}}(\mathsf{Pr}_{\mathsf{st}}^L)$, which yields a commutative triangle

$$
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{R} & \xrightarrow{\underline{\hom}_{\mathcal{C}}(X,-)\otimes\mathcal{R}} & \mathsf{Sp} \otimes \mathcal{R} \\
 & \underline{\hom}_{\mathcal{C}\otimes\mathcal{R}}(X\otimes\mathbb{1}_{\mathcal{R}},-) \searrow & \downarrow \wr \\
 & & \mathcal{R}
\end{array}
\quad .
$$

We apply this in the case that $\mathcal{C} = \mathsf{Sp}^{\mathsf{g}G}/\mathsf{Sp}_{\mathscr{F}}^{\mathsf{g}G}$ and $X = p_L(\Sigma_G^\infty(G/H)_+)$ (which is compact because p_L preserves compact objects since ν preserves colimits). Using the fact that the functor $(-)\otimes\mathcal{R}$ preserves colimits, we find that the $\mathscr{F}$-H-geometric fixedpoints functor may be described as the tensor product

$$\left(\mathsf{Sp}^{\mathsf{g}G} \xrightarrow{p_L} \mathsf{Sp}^{\mathsf{g}G}/\mathsf{Sp}_{\mathscr{F}}^{\mathsf{g}G} \xrightarrow{\underline{\hom}_{\mathsf{Sp}^{\mathsf{g}G}/\mathsf{Sp}_{\mathscr{F}}^{\mathsf{g}G}}(p_L(\Sigma_G^\infty(G/H)_+),-)} \mathsf{Sp}^{\mathsf{hW}(H)} \right) \otimes \mathcal{R} .$$

In particular, in the case that $\mathscr{F} = (\not\geq H)$, we obtain an equivalence

$$\Phi_{(\not\geq H)}^H \simeq \Phi^H$$

in $\mathsf{Fun}(\mathcal{R}^{\mathsf{g}G}, \mathcal{R}^{\mathsf{hW}(H)})$. We use this fact without further comment.

Proposition 5.9. *For any closed subgroup $H \leq G$, we have a canonical commutative triangle*

$$
\begin{array}{ccc}
\mathcal{R}^{\mathsf{h}G} & \xrightarrow{\quad\beta\quad} & \mathcal{R}^{\mathsf{g}G} \\
(-)^{\tau_{\mathscr{F}}H} \searrow & & \swarrow \Phi_{\mathscr{F}}^H \\
 & \mathcal{R}^{\mathsf{hW}(H)} &
\end{array}
\quad .
$$

We learned the following result from Akhil Mathew.

Proposition 5.10. *Assume that G is a finite group, and fix any homotopy G-object $E \in \mathcal{R}^{\mathsf{h}G}$ in $\mathcal{R}$.*

(1) *There is a canonical equivalence*

$$E^{\tau G} \xleftarrow{\sim} \mathsf{colim}_{n\in\mathbb{N}} \left((S^{n\tilde{\rho}} \odot E)^{\mathsf{h}G} \right)$$

in $\mathcal{R}$, in which

- *$\odot$ denotes the action of $\mathsf{S}_*^{\mathsf{h}G}$ on $\mathcal{R}^{\mathsf{h}G}$,*

- *$\tilde{\rho}$ denotes the reduced (real) regular representation of G,*

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- S^V denotes the representation sphere corresponding to a representation V, and

- the morphisms in the colimit are induced by the inclusion $\{0\} \to \tilde{\rho}$.

(2) Suppose that $\mathcal{R} = \mathsf{Mod}_R$ for some commutative ring spectrum $R \in \mathsf{CAlg}(\mathcal{Sp})$. Then, a complex orientation of R (e.g. a $\mathbb{Z}$-algebra structure) determines an element

$$e \in \pi_{-2(|G|-1)}(R^{\mathrm{h}G}) \ ,$$

the Euler class of the reduced complex regular representation (via the complex orientation), with respect to which the canonical morphism $E^{\tau G} \leftarrow E^{\mathrm{h}G}$ in $\mathcal{R}$ exhibits $E^{\tau G}$ as the localization at e; that is,

$$E^{\tau G} \simeq E^{\mathrm{h}G}[e^{-1}] \ .$$

Proof. We begin with part (1). Observe the equivalences

$$E^{\tau G} \simeq \Phi^G \beta(E) \simeq (\widetilde{\mathsf{E}}\mathscr{F}_{\not\geq G} \odot \beta(E))^G \ , \tag{5.1}$$

the first by Proposition 5.9. By [MNN19, Proposition 2.7], we have an equivalence

$$\widetilde{\mathsf{E}}\mathscr{F}_{\not\geq G} \simeq \mathsf{colim}_{n\in\mathbb{N}}(S^{n\tilde{\rho}}) \tag{5.2}$$

in $\mathcal{S}_*^{\mathsf{g}G}$. Combining the equivalences (5.1) and (5.2), we obtain the first equivalence in the composite equivalence

$$E^{\tau G} \simeq ((\mathsf{colim}_{n\in\mathbb{N}}(S^{n\tilde{\rho}})) \odot \beta(E))^G \xleftarrow{\sim} \mathsf{colim}_{n\in\mathbb{N}}((S^{n\tilde{\rho}} \odot \beta(E))^G) \ . \tag{5.3}$$

On the other hand, since $\Sigma_G^\infty S^{n\tilde{\rho}} \in \mathcal{Sp}^{\mathsf{g}G}$ is dualizable, we have an equivalence

$$S^{n\tilde{\rho}} \odot \beta(E) \xrightarrow{\sim} \beta(S^{n\tilde{\rho}} \odot E) \tag{5.4}$$

in $\mathcal{R}^{\mathsf{g}G}$. Therefore, combining equivalences (5.3) and (5.4), we obtain an equivalence

$$E^{\tau G} \simeq \mathsf{colim}_{n\in\mathbb{N}}(\beta(S^{n\tilde{\rho}} \odot E)^G) \simeq \mathsf{colim}_{n\in\mathbb{N}}((S^{n\tilde{\rho}} \odot E)^{\mathrm{h}G}) \ ,$$

as asserted.

Now, part (2) follows from part (1) along with the observation that $2\tilde{\rho}$ is precisely the reduced complex regular representation. $\qquad\square$

Notation 5.11. We write

$$\mathsf{Ind}(\mathcal{R}^{\mathrm{h}G}) \underset{i}{\overset{\mathsf{colim}}{\underset{\longleftarrow}{\rightleftarrows}}} \mathcal{R}^{\mathrm{h}G}$$

for the canonical adjunction resulting from the fact that $\mathcal{R}^{\mathrm{h}G}$ admits filtered colimits: its right adjoint is the canonical fully faithful inclusion, and its left adjoint is given by taking colimits of filtered diagrams.

Observation 5.12. Because the functor

$$\mathcal{R}^{\mathsf{g}G} \xrightarrow{U} \mathcal{R}^{\mathrm{h}G}$$

preserves colimits (being a left adjoint) and the presentable stable ∞-category $\mathcal{R}^{\mathsf{g}G}$ is compactly generated (because both $\mathcal{Sp}^{\mathsf{g}G}$ and $\mathcal{R}$ are), there exists a canonical factorization

$$\begin{array}{ccc} \mathcal{R}^{\mathsf{g}G} & \xrightarrow{\quad U \quad} & \mathcal{R}^{\mathrm{h}G} \\ & \underset{\tilde{U}}{\searrow} \quad \nearrow_{\mathsf{colim}} & \\ & \mathsf{Ind}(\mathcal{R}^{\mathrm{h}G}) & \end{array} \ , \tag{5.5}$$

namely the functor

$$\tilde{U} := \mathsf{Ind}\left((\mathcal{R}^{\mathfrak{g}G})^\omega \longrightarrow \mathcal{R}^{\mathfrak{g}G} \xrightarrow{\ U\ } \mathcal{R}^{\mathsf{h}G}\right) \ .$$

Observation 5.13. By construction, the functor $\mathcal{R}^{\mathfrak{g}G} \xrightarrow{\tilde{U}} \mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})$ preserves compact objects: given a compact object $E \in (\mathcal{R}^{\mathfrak{g}G})^\omega \subseteq \mathcal{R}^{\mathfrak{g}G}$, there is a canonical equivalence

$$\tilde{U}(E) \xrightarrow{\ \sim\ } i(U(E)) \ .$$

Observation 5.14. Both the source and target of the functor $\mathcal{R}^{\mathfrak{g}G} \xrightarrow{\tilde{U}} \mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})$ are compactly generated. Combining this fact with Observation 5.13, we see that it admits a colimit-preserving right adjoint, which we denote by

$$\mathcal{R}^{\mathfrak{g}G} \ \underset{\tilde{\beta}}{\overset{\tilde{U}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \ \mathsf{Ind}(\mathcal{R}^{\mathsf{h}G}) \quad .$$

Observation 5.15. Passing to right adjoints in the commutative diagram (5.5), we obtain a commutative diagram

$$\mathcal{R}^{\mathfrak{g}G} \xleftarrow{\hspace{1.5cm} \beta \hspace{1.5cm}} \mathcal{R}^{\mathsf{h}G}$$
$$\tilde{\beta} \searrow \qquad \swarrow i$$
$$\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})$$

.

Observation 5.16. The presentably symmetric monoidal stable ∞-category $\mathcal{R}^{\mathfrak{g}G} := \mathcal{Sp}^{\mathfrak{g}G} \otimes \mathcal{R}$ is rigidly-compactly generated (since the functor $\mathsf{St} \xrightarrow{\mathsf{Ind}} \mathsf{Pr}^L_{\mathsf{st}}$ is symmetric monoidal).

Observation 5.17. The functor $\mathcal{R}^{\mathfrak{g}G} \xrightarrow{\tilde{U}} \mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})$ is canonically symmetric monoidal, because the composite $(\mathcal{R}^{\mathfrak{g}G})^\omega \hookrightarrow \mathcal{R}^{\mathfrak{g}G} \xrightarrow{\ U\ } \mathcal{R}^{\mathsf{h}G}$ is symmetric monoidal by Observation 5.16.

Observation 5.18. Because $\mathcal{R}^{\mathfrak{g}G}$ is rigidly-compactly generated by Observation 5.16 and $\tilde{U}$ is symmetric monoidal by Observation 5.17, by [GR17, Chapter 1, Lemma 9.3.6] the right adjoint $\tilde{\beta}$ is $\mathcal{R}^{\mathfrak{g}G}$-linear: in other words, for any $E \in \mathcal{R}^{\mathfrak{g}G}$ and $F \in \mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})$ we have the projection formula

$$\tilde{\beta}(\tilde{U}(E) \otimes F) \simeq E \otimes \tilde{\beta}(F) \ .$$

Observation 5.19. By Observation 5.3 and Corollary S.4.2.15, the ideal

$$\left\langle \tilde{U}(\mathcal{I}_{\mathscr{F}}) \right\rangle^{\otimes} \subseteq \mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})$$

is a closed ideal and moreover

$$\nu\left(\mathbb{1}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})/\langle\tilde{U}(\mathcal{I}_{\mathscr{F}})\rangle^{\otimes}}\right) \simeq \tilde{U}(\nu(\mathbb{1}_{\mathcal{Sp}^{\mathfrak{g}G}/\mathcal{I}_{\mathscr{F}}})) \ .$$

Observation 5.20. We have the string of identifications

$$\left\langle \tilde{U}(\mathcal{I}_{\mathscr{F}}) \right\rangle^{\otimes} = \left\langle \tilde{U}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle) \right\rangle^{\otimes}_{H\in\mathscr{F}} \tag{5.6}$$

$$= \langle iU(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle)\rangle^{\otimes}_{H\in\mathscr{F}} \tag{5.7}$$

$$= \left\langle i(\mathbb{1}^{\mathsf{h}}_{\mathcal{R}}\langle G/H\rangle) \right\rangle^{\otimes}_{H\in\mathscr{F}} \tag{5.8}$$

$$= \mathsf{Ind}(\mathcal{I}^{\mathsf{h}}_{\mathscr{F}}) \tag{5.9}$$

among subcategories of $\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})$, where identification (5.6) follows from Observation 5.6, identification (5.7) follows from Observation 5.13, identification (5.8) follows from Observation 5.5, and identification (5.9) follows from Observation 3.8.

Proof of Proposition 5.9. Our proof takes place within the context of the diagram in Figure 1, in
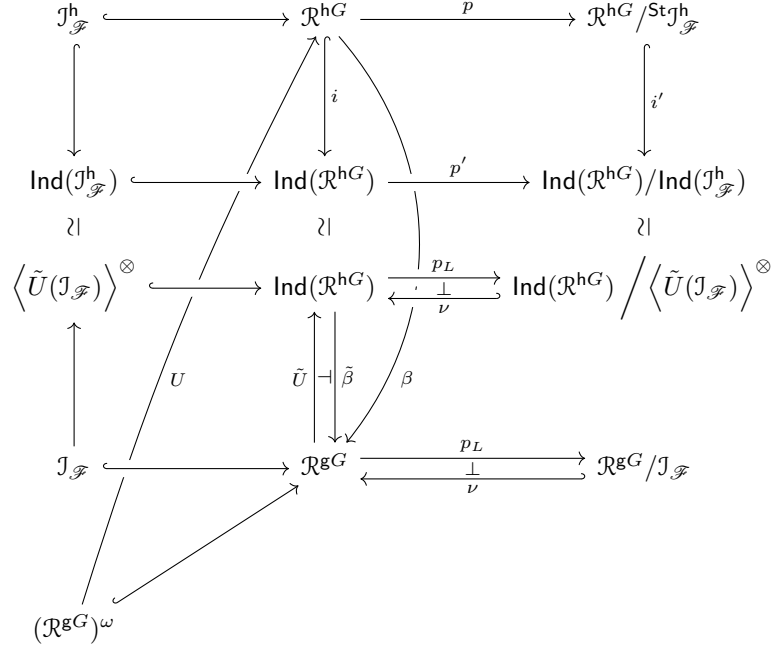

FIGURE 1. The proof of Proposition 5.9 takes places within this diagram.

which

- the upper row is a stable quotient sequence,

- the functor i' is the fully faithful inclusion into $\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G}/{}^{\mathsf{St}}\mathcal{J}^{\mathsf{h}}_{\mathscr{F}}) \simeq \mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})/\mathsf{Ind}(\mathcal{J}^{\mathsf{h}}_{\mathscr{F}})$ (note that the functor $\mathsf{St} \xrightarrow{\mathsf{Ind}} \mathsf{Pr}^L_{\mathsf{st}}$ preserves colimits),[14]

- the lower two rows are presentable quotient sequences in which the kernels are closed ideals by Observations 5.19 and 5.3,

- the back triangles commute by Observations 5.15 and 5.13, and

- the equivalences in the middle row follow from Observation 5.20.

Namely, for any $E \in \mathcal{R}^{\mathsf{h}G}$, we have natural equivalences

$$E^{\tau_{\mathscr{F}}H} := \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{h}G}/{}^{\mathsf{St}}\mathcal{J}^{\mathsf{h}}_{\mathscr{F}}}(p(\mathbb{1}^{\mathsf{h}}_{\mathcal{R}}\langle G/H\rangle), pE)$$

$$\simeq \underline{\mathsf{hom}}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})/\mathsf{Ind}(\mathcal{J}^{\mathsf{h}}_{\mathscr{F}})}(i'p(\mathbb{1}^{\mathsf{h}}_{\mathcal{R}}\langle G/H\rangle), i'pE)$$

$$\simeq \underline{\mathsf{hom}}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})/\mathsf{Ind}(\mathcal{J}^{\mathsf{h}}_{\mathscr{F}})}((p'i(\mathbb{1}^{\mathsf{h}}_{\mathcal{R}}\langle G/H\rangle)), p'iE)$$

$$\simeq \underline{\mathsf{hom}}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})}(i(\mathbb{1}^{\mathsf{h}}_{\mathcal{R}}\langle G/H\rangle), \nu p_L iE)$$

$$\simeq \underline{\mathsf{hom}}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})}(iU(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle), \nu p_L iE) \tag{5.10}$$

---

[14]By Remark 3.5, the functor i' can also be seen as the induced functor on stable quotients.

$$\simeq \underline{\hom}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})}(\tilde{U}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle), \nu p_L i E) \tag{5.11}$$

$$\simeq \underline{\hom}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})}(\tilde{U}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle), (\nu p_L \, \mathbb{1}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})}) \otimes (iE))$$

$$\simeq \underline{\hom}_{\mathsf{Ind}(\mathcal{R}^{\mathsf{h}G})}(\tilde{U}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle), (\tilde{U}\nu p_L \, \mathbb{1}_{\mathcal{R}^{\mathsf{g}G}}) \otimes (iE)) \tag{5.12}$$

$$\simeq \underline{\hom}_{\mathcal{R}^{\mathsf{g}G}}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle, \tilde{\beta}((\tilde{U}\nu p_L \, \mathbb{1}_{\mathcal{R}^{\mathsf{g}G}}) \otimes (iE)))$$

$$\simeq \underline{\hom}_{\mathcal{R}^{\mathsf{g}G}}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle, (\nu p_L \, \mathbb{1}_{\mathcal{R}^{\mathsf{g}G}}) \otimes (\tilde{\beta}iE)) \tag{5.13}$$

$$\simeq \underline{\hom}_{\mathcal{R}^{\mathsf{g}G}}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle, (\nu p_L \, \mathbb{1}_{\mathcal{R}^{\mathsf{g}G}}) \otimes (\beta E))$$

$$\simeq \underline{\hom}_{\mathcal{R}^{\mathsf{g}G}}(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle, \nu p_L \beta E)$$

$$\simeq \underline{\hom}_{\mathcal{R}^{\mathsf{g}G}/\mathcal{I}_{\mathscr{F}}}(p_L(\mathbb{1}_{\mathcal{R}}\langle G/H\rangle), p_L \beta E)$$

$$=: \Phi^H_{\mathscr{F}} \beta E$$

in $\mathcal{R}^{\mathsf{h}\mathsf{W}(H)}$, in which

- equivalence (5.10) follows from Observation 5.5,

- equivalence (5.11) follows from the fact that $\mathbb{1}_{\mathcal{R}}\langle G/H\rangle \in \mathcal{R}^{\mathsf{g}G}$ is compact (because $\mathbb{1}_{\mathcal{R}} \in \mathcal{R}$ and $\Sigma^\infty_G(G/H)_+ \in \mathsf{Sp}^{\mathsf{g}G}$ are compact),

- equivalence (5.12) follows from Observation 5.19, and

- equivalence (5.13) follows from Observation 5.18. $\qquad\square$

6. Gluing functors for the geometric stratification of genuine G-objects

In this section, we prove Theorem A as Theorem 6.6.

Local Notation 6.1. In this section, we fix a finite group G and a rigidly-compactly generated presentably symmetric monoidal stable ∞-category $\mathcal{R}$.

Notation 6.2. Given subgroups $K, H \subseteq G$, we define the subset

$$\tilde{C}(H, K) := \{g \in G : H \subseteq gKg^{-1} \subseteq \mathsf{N}(H)\} \subseteq G$$

of those elements of G that conjugate K to lie between H and the normalizer of H.

Observation 6.3. Considering G as a (G, G)-bimodule set by left and right multiplication, the subset $\tilde{C}(H, K) \subseteq G$ inherits a $(\mathsf{N}(H), \mathsf{N}(K))$-bimodule structure: that is, it is carried into itself by left multiplication by elements of $\mathsf{N}(H)$ and by right multiplication by elements of $\mathsf{N}(K)$. We use this fact without further comment.

Notation 6.4. We write

$$C(H, K) := \tilde{C}(H, K)/K$$

for the set obtained by quotienting $\tilde{C}(H, K)$ by its right K-action.

Observation 6.5. Of course, $C(H, K)$ naturally inherits a $(\mathsf{N}(H), \mathsf{W}(K))$-bimodule structure. Moreover, its left H-action is trivial, so that this descends to a $(\mathsf{W}(H), \mathsf{W}(K))$-bimodule structure.[15] We use these facts without further comment.

---

[15]See Lemma 6.16 for a more conceptual description of the double quotient $\mathsf{W}(H)\backslash C(H, K)/\mathsf{W}(K)$.

Theorem 6.6. *For any morphism $H \leq K$ in P_G, the gluing functor*

$$\mathcal{R}^{\mathsf{hW}(H)} \xrightarrow{\Gamma_K^H} \mathcal{R}^{\mathsf{hW}(K)}$$

of the geometric stratification (2.3) of $\mathcal{R}^{\mathsf{g}G}$ is given by the formula

$$E \longmapsto \bigoplus_{[g] \in \mathsf{W}(H) \backslash C(H,K) / \mathsf{W}(K)} \mathsf{Ind}_{(\mathsf{N}(H) \cap \mathsf{N}(gKg^{-1}))/(gKg^{-1})}^{\mathsf{W}(K)} E^{\tau(gKg^{-1})/H} \tag{6.1}$$

(where we implicitly use the isomorphism $\mathsf{W}(gKg^{-1}) \xrightarrow{\cong} \mathsf{W}(K)$ given by conjugation by g^{-1}).

Remark 6.7. The set $C(H,K)$ is empty unless we have $H \leq K \leq \mathsf{N}(H)$ in P_G. At the other extreme, if we have $H \subseteq K \subseteq G$ with H normal in K and K normal in G, then $C(H,K) = G/K$. In this case, it is not hard to see that the formula (6.1) reduces to the formula

$$E \longmapsto \mathsf{Ind}_{\mathsf{N}(H)/K}^{G/K} E^{\tau(K/H)} \ .$$

If furthermore H is normal in G, then the formula (6.1) reduces to the even simpler formula

$$E \longmapsto E^{\tau(K/H)} \ .$$

Remark 6.8. While the description of the functor Γ_K^H given by Theorem 6.6 is fairly explicit, it has the drawback of making reference to specific representatives of objects of P_G (i.e. conjugacy classes of subgroups of G). Here are two alternative descriptions that are more invariant.

(1) If we replace the double quotient appearing in formula (6.1) with the *groupoid* double quotient, we obtain a span

$$\mathsf{BW}(H) \longleftarrow C(H,K)_{\mathsf{h}(\mathsf{W}(H) \times \mathsf{W}(K))} \longrightarrow \mathsf{BW}(K) \ ,$$

from which Γ_K^H may be obtained as a sort of pull-push operation: ordinary pullback along the leftwards functor followed by the fiberwise proper Tate construction and the indicated induction.

(2) A variant of the argument used to prove Proposition 5.9 shows that Γ_K^H can also be described as the composite

$$\mathcal{R}^{\mathsf{hW}(H)} \xrightarrow{p} \mathcal{R}^{\mathsf{hW}(H)}/^{\mathsf{St}}\mathfrak{I}_{\mathscr{F}}^{\mathsf{h}} \xrightarrow{\underline{\mathsf{hom}}\left(p\left(\mathbb{1}_{\mathcal{R}}\left\langle (G/H)^K \right\rangle\right), (-)\right)} \mathcal{R}^{\mathsf{hW}(K)} \ ,$$

where $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_{\mathsf{W}(H)}}$ denotes the family defined in Local Notation 6.11(3). (Lemmas 6.14 and 6.19 below can be applied to give an explicit description of the object $p(\mathbb{1}_{\mathcal{R}}\left\langle (G/H)^K \right\rangle) \in \mathcal{R}^{\mathsf{hW}(H)}/^{\mathsf{St}}\mathfrak{I}_{\mathscr{F}}^{\mathsf{h}}$.)

Example 6.9 (genuine A_4-spectra)**.** Let A_4 denote the alternating group on four letters. We describe the reconstruction theorem for genuine A_4-spectra that results from combining Theorem 6.6

with Theorems S.A(2) and S.E.[16] The poset of conjugacy classes of closed subgroups of A_4 is

$$
\mathsf{P}_{\mathsf{A}_4} = \left\{
\begin{array}{ccc}
e & \longrightarrow & \mathsf{C}_2 \\
\downarrow & & \downarrow \\
\mathsf{C}_3 & & \mathsf{V}_4 \\
& \searrow & \downarrow \\
& & \mathsf{A}_4
\end{array}
\right\}
=: \left\{
\begin{array}{ccc}
1 & \longrightarrow & 2 \\
\downarrow & & \downarrow \\
3 & & 4 \\
& \searrow & \downarrow \\
& & 12
\end{array}
\right\} ,
$$

where $\mathsf{V}_4 \cong \mathsf{C}_2 \times \mathsf{C}_2$ denotes the Klein four-group and we label strata by cardinality. By Observation S.5.2.6, the gluing functors $\Gamma_{\mathsf{A}_4}^e$ and $\Gamma_{\mathsf{A}_4}^{\mathsf{C}_2}$ vanish. Moreover, by Observation S.5.2.2, the gluing functor $\Gamma_{\mathsf{A}_4}^{\mathsf{C}_3}$ also vanishes. Altogether, we obtain an equivalence

$$
\mathcal{S}\mathsf{p}^{g\mathsf{A}_4} \xrightarrow[\sim]{g} \lim{}^{\mathrm{r.lax}} \left(
\begin{array}{ccc}
\mathcal{S}\mathsf{p}^{h\mathsf{A}_4} & \xrightarrow{(-)^{t\mathsf{C}_2}} & \mathcal{S}\mathsf{p}^{h\mathsf{C}_2} \\
\downarrow{\scriptstyle (-)^{t\mathsf{C}_3}} & \searrow{\scriptstyle (-)_{\tau\mathsf{V}_4}} \nearrow & \downarrow{\scriptstyle (-)^{t\mathsf{C}_2}} \\
\mathcal{S}\mathsf{p} & & \mathcal{S}\mathsf{p}^{h\mathsf{C}_3} \\
& & \downarrow{\scriptstyle (-)^{t\mathsf{C}_3}} \\
& & \mathcal{S}\mathsf{p}
\end{array}
\right) .
$$

So, a genuine A_4-spectrum $E \in \mathcal{S}\mathsf{p}^{g\mathsf{A}_4}$ is equivalently specified by the data of

- the objects

$$E_1 := \Phi^e E \in \mathcal{S}\mathsf{p}^{h\mathsf{A}_4} , \qquad E_2 := \Phi^{\mathsf{C}_2} E \in \mathcal{S}\mathsf{p}^{h\mathsf{C}_2} , \qquad E_3 := \Phi^{\mathsf{C}_3} E \in \mathcal{S}\mathsf{p} , \qquad E_4 := \Phi^{\mathsf{V}_4} E \in \mathcal{S}\mathsf{p}^{h\mathsf{C}_3} , \qquad \text{and} \qquad E_{12} := \Phi^{\mathsf{A}_4} E \in \mathcal{S}\mathsf{p} ,$$

- the morphisms

$$E_2 \xrightarrow{\gamma_{1,2}} E_1^{t\mathsf{C}_2} , \qquad E_3 \xrightarrow{\gamma_{1,3}} E_1^{t\mathsf{C}_3} , \qquad E_1^{\tau\mathsf{V}_4} \xleftarrow{\gamma_{1,4}} E_4 \xrightarrow{\gamma_{2,4}} E_2^{t\mathsf{C}_2} , \qquad \text{and} \qquad E_{12} \xrightarrow{\gamma_{4,12}} E_4^{t\mathsf{C}_3} ,$$

and

- a homotopy making the square

$$
\begin{array}{ccc}
E_4 & \xrightarrow{\gamma_{2,4}} & E_2^{t\mathsf{C}_2} \\
\downarrow{\scriptstyle \gamma_{1,4}} & & \downarrow{\scriptstyle \gamma_{1,2}^{t\mathsf{C}_2}} \\
E_1^{\tau\mathsf{V}_4} & \longrightarrow & (E_1^{t\mathsf{C}_2})^{t\mathsf{C}_2}
\end{array}
$$

commute (in which the lower horizontal morphism is the canonical one).

Remark 6.10. We prove Theorem 6.6 at the end of this section, after some preliminary work. These preliminaries decompose into three main logically distinct pieces, which are organized into environments as follows.

---

[16]For brevity, we assume familiarity with the latter results. We refer the reader to §S.5.3 for a number of simpler examples. (In particular, the present example has a number of similarities with Example S.5.3.10 of genuine S_3-spectra.)

6.11-6.14: The proof of Theorem 6.6 will begin by applying a projection formula to replace the family $\mathscr{F}_{\not\geq K} \in \mathsf{Down}_{\mathsf{P}_G}$ with a certain family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_{\mathsf{W}(H)}}$. The relevance of the set $C(H, K)$ is that it is clearly a subset of $(G/K)^H$, and it turns out to consist of precisely those orbits that aren't annihilated by the functor $\widetilde{\mathsf{E}}\mathscr{F} \wedge (-)_+$.

6.15-6.19: We decompose $C(H, K)$ into $(\mathsf{W}(H), \mathsf{W}(K))$-orbits by identifying $\mathsf{W}(H)\backslash C(H, K)/\mathsf{W}(K)$ with conjugacy classes of stabilizer subgroups in $\mathsf{W}(H)$ (with respect to the left action). We then describe these orbits explicitly, in terms of elements $g \in G$.

6.20-6.21: We exhibit each $(\mathsf{W}(H), \mathsf{W}(K))$-orbit in $C(H, K)$ as being itself induced from an orbit: namely, it is induced up from $\mathsf{W}_{\mathsf{N}(H)}(gKg^{-1}) = (\mathsf{N}(H) \cap \mathsf{N}(gKg^{-1}))/(gKg^{-1})$.

Local Notation 6.11. We use the following notation through the remainder of this section.

(1) We fix subgroups $H \subseteq K \subseteq G$.[17]

(2) We write $\mathsf{N}(H) \xrightarrow{p} \mathsf{W}(H)$ for the quotient homomorphism.

(3) We write

$$\mathscr{F} := \lim \left(\begin{array}{c} \mathscr{F}_{\not\geq K} \\ \downarrow \\ \mathsf{P}_{\mathsf{W}(H)} \xrightarrow[p^{-1}]{} \mathsf{P}_{\mathsf{N}(H)} \longrightarrow \mathsf{P}_G \end{array} \right) \cong \left\{ L \subseteq \mathsf{W}(H) : \begin{array}{c} K \text{ is not subconjugate} \\ \text{to } p^{-1}(L) \text{ in } G \end{array} \right\} \in \mathsf{Down}_{\mathsf{P}_{\mathsf{W}(H)}}$$

for the indicated family of subgroups of $\mathsf{W}(H)$.

Warning 6.12. The functor $\mathsf{P}_{\mathsf{N}(H)} \to \mathsf{P}_G$ is not generally fully faithful.[18] (Whenever it is fully faithful, the set $\mathsf{W}(H)\backslash C(H, K)/\mathsf{W}(K)$ is a singleton; see Lemma 6.16.)

Observation 6.13. The defining inclusion $\tilde{C}(H, K) \hookrightarrow G$ extends to a diagram

of $(\mathsf{N}(H), \mathsf{N}(K))$-bimodule sets: the dashed factorization arises from the defining fact that if $gK \in C(H, K)$ then $H \subseteq gKg^{-1}$. In particular, we obtain a canonical inclusion

$$C(H, K) \longrightarrow (G/K)^H \tag{6.2}$$

of $(\mathsf{W}(H), \mathsf{W}(K))$-sets. We regard this as a morphism of homotopy $\mathsf{W}(K)$-objects in genuine $\mathsf{W}(H)$-spaces.

---

[17]It is clear that in proving Theorem 6.6 we may assume without real loss of generality that H is contained in K, and doing so leads to some notational simplification.

[18]For example, take $G = \mathsf{S}_4$ to be the symmetric group on four letters, define elements $h, j \in G$ by $h = (12)(34)$ and $j = (13)(24)$ (using cycle notation), and define subgroups $H = \langle h \rangle$ and $J = \langle j \rangle$ of G. Then we have $J \subseteq \mathsf{N}(H)$, and moreover $J \not\cong H$ in $\mathsf{P}_{\mathsf{N}(H)}$ but $J \cong H$ in P_G.

Lemma 6.14. *The morphism*

$$\widetilde{\mathsf{E}}\mathscr{F} \wedge (6.2)_+ := \widetilde{\mathsf{E}}\mathscr{F} \wedge \big(C(H,K) \longrightarrow (G/K)^H\big)_+$$

in $(\mathcal{S}_*^{\mathsf{gW}(H)})^{\mathsf{hW}(K)}$ *is an equivalence.*

Proof. We check that the composite functor

$$\mathscr{O}_{\mathsf{W}(H)} \longrightarrow \mathcal{S}^{\mathsf{gW}(H)} \xrightarrow{\ \widetilde{\mathsf{E}}\mathscr{F} \wedge (-)_+\ } \mathcal{S}_*^{\mathsf{gW}(H)}$$

annihilates (i.e. carries to the zero object) all $\mathsf{W}(H)$-orbits in $(G/K)^H$ that do not lie in the image of the inclusion (6.2).

Consider an arbitrary element $gK \in G/K$. Observe that its stabilizer under the G-action is gKg^{-1}. Moreover, it is H-fixed if and only if $H \subseteq gKg^{-1}$. In this case, the stabilizer of the element $gK \in (G/K)^H$ under the $\mathsf{W}(H)$-action is

$$(gKg^{-1} \cap \mathsf{N}(H))/H \ .$$

So, the orbit $\mathsf{W}(H) \cdot gK \subseteq (G/K)^H$ is annihilated by the functor $\widetilde{\mathsf{E}}\mathscr{F} \wedge (-)_+$ if (and only if) we have

$$(gKg^{-1} \cap \mathsf{N}(H))/H \in \mathscr{F} \in \mathsf{Down}_{\mathsf{P}_{\mathsf{W}(H)}} \ ,$$

which by definition is equivalent to the assertion that

$$(gKg^{-1} \cap \mathsf{N}(H)) \in \mathscr{F}_{\not\geq K} \in \mathsf{Down}_{\mathsf{P}_G} \ ,$$

i.e. that K is not subconjugate to $(gKg^{-1} \cap \mathsf{N}(H))$ in G. So, it suffices to show that $gKg^{-1} \not\subseteq \mathsf{N}(H)$ implies that $K \not\leq (gKg^{-1} \cap \mathsf{N}(H))$. We show the contrapositive, namely that $K \leq (gKg^{-1} \cap \mathsf{N}(H))$ implies that $gKg^{-1} \subseteq \mathsf{N}(H)$.

Suppose there exists some $x \in G$ such that $xKx^{-1} \subseteq (gKg^{-1} \cap \mathsf{N}(H))$. On the one hand, this implies that $xKx^{-1} \subseteq gKg^{-1}$, which because K is finite implies that $xKx^{-1} = gKg^{-1}$. On the other hand, this implies that $xKx^{-1} \subseteq \mathsf{N}(H)$. Combining these two implications yields the desired conclusion that $gKg^{-1} = xKx^{-1} \subseteq \mathsf{N}(H)$. $\qquad\square$

Observation 6.15. Carrying each element of $C(H,K)$ to its stabilizer under the left $\mathsf{W}(H)$-action determines a function

$$
\begin{array}{ccc}
C(H,K) & \longrightarrow & \mathsf{P}_{\mathsf{W}(H)} \\
\cup & & \cup \\
gK & \longmapsto & (gKg^{-1})/H
\end{array}
\tag{6.3}
$$

on underlying sets. We note the following two properties of the function (6.3).

(1) It takes values in those conjugacy classes of subgroups of $\mathsf{W}(H)$ whose preimages in $\mathsf{N}(H)$ are conjugate in G to K.

(2) It is invariant with respect to both the left $\mathsf{W}(H)$-action and the right $\mathsf{W}(K)$-action on $C(H,K)$.

The function (6.3) therefore admits a factorization

$$
\begin{array}{ccc}
C(H,K) & \longrightarrow & \mathsf{P}_{\mathsf{W}(H)} \\
\downarrow & & \uparrow \\
\mathsf{W}(H)\backslash C(H,K)/\mathsf{W}(K) & \dashrightarrow & \mathsf{P}_{\mathsf{W}(H)} \times_{\mathsf{P}_G} \{K\}
\end{array}
\qquad (6.4)
$$

We use these facts without further comment.

Lemma 6.16. *The factorization* (6.4) *is an isomorphism*

$$
\mathsf{W}(H)\backslash C(H,K)/\mathsf{W}(K) \xrightarrow{\ \cong\ } \mathsf{P}_{\mathsf{W}(H)} \times_{\mathsf{P}_G} \{K\}
$$

among sets.

Proof. We first verify that the function (6.4) is surjective. For this, choose any subgroup $J \subseteq \mathsf{W}(H)$ such that $gKg^{-1} = p^{-1}(J)$ for some $g \in G$. Then, the function (6.4) carries the equivalence class of the element $gK \in C(H,K)$ to the equivalence class of J in $\mathsf{P}_{\mathsf{W}(H)} \times_{\mathsf{P}_G} \{K\}$. So indeed, the function (6.4) is surjective.

We now verify that the function (6.4) is injective. For this, consider a pair of elements $g_1, g_2 \in \tilde{C}(H,K)$. Then, observe the equivalence of the conditions

(1) $(g_1 K g_1^{-1})/H$ and $(g_2 K g_2^{-1})/H$ are conjugate in $\mathsf{W}(H)$,

(2) $g_1 K g_1^{-1}$ and $g_2 K g_2^{-1}$ are conjugate in $\mathsf{N}(H)$,

(3) there exists some $x \in \mathsf{N}(H)$ such that $x g_1 K g_1^{-1} x^{-1} = g_2 K g_2^{-1}$,

(4) there exists some $x \in \mathsf{N}(H)$ such that $g_2^{-1} x g_1 \in \mathsf{N}(K)$,

(5) there exist some $x \in \mathsf{N}(H)$ and $y \in \mathsf{N}(K)$ such that $g_2 = x g_1 y$,

and

(6) $[g_1] = [g_2]$ in $\mathsf{W}(H)\backslash C(H,K)/\mathsf{W}(K)$.

The equivalence (1) $\Leftrightarrow$ (6) implies that the function (6.4) is injective. $\qquad\square$

Observation 6.17. For each element $[g] \in \mathsf{W}(H)\backslash C(H,K)/\mathsf{W}(K)$, the fiber

$$
\begin{array}{ccc}
C(H,K)_{[g]} & \lhook\joinrel\longrightarrow & C(H,K) \\
\downarrow & & \downarrow \\
\{[g]\} & \lhook\joinrel\longrightarrow & \mathsf{W}(H)\backslash C(H,K)/\mathsf{W}(K)
\end{array}
$$

is a $(\mathsf{W}(H), \mathsf{W}(K))$-subbimodule of $C(H,K)$.

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Observation 6.18. Fix an element $g \in \tilde{C}(H, K)$. We have a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\;\;z \longmapsto zg\;\;} & G \\[1mm]
\big\uparrow & & \big\uparrow \\[1mm]
\mathsf{N}(H) \cdot \mathsf{N}(gKg^{-1}) & \xleftarrow{\;\;xy \longmapsto xyg\;\;} & \tilde{C}(H, K) \\[1mm]
\big\downarrow & & \big\downarrow \\[1mm]
(\mathsf{N}(H) \cdot \mathsf{N}(gKg^{-1}))/(gKg^{-1}) & \xleftarrow{\;xy(gKg^{-1}) \longmapsto xygK\;} & C(H, K) \\[1mm]
\big\vdots & & \big\downarrow \\[1mm]
\{[g]\} & \longleftarrow & \mathsf{W}(H)\backslash C(H, K)/\mathsf{W}(K)
\end{array}
\tag{6.5}
$$

of sets, in which

- all functions are left $\mathsf{N}(H)$-equivariant,

- the left column is right $\mathsf{N}(gKg^{-1})$-equivariant,

- the right column is right $\mathsf{N}(K)$-equivariant, and

- the horizontal functions are globally right-equivariant with respect to the conjugation iso-
  morphism $\mathsf{N}(gKg^{-1}) \xrightarrow{g^{-1}(-)g} \mathsf{N}(K)$.

Hence, by considering the sets in the left column of diagram (6.5) as right $\mathsf{N}(K)$-sets via the conjugation isomorphism $\mathsf{N}(K) \xrightarrow{g(-)g^{-1}} \mathsf{N}(gKg^{-1})$, we may consider the entire diagram (6.5) as one of $(\mathsf{N}(H), \mathsf{N}(K))$-bimodule sets. In particular, we obtain an injective function

$$
(\mathsf{N}(H) \cdot \mathsf{N}(gKg^{-1}))/(gKg^{-1}) \lhook\joinrel\longrightarrow C(H, K)_{[g]}
\tag{6.6}
$$

of $(\mathsf{N}(H), \mathsf{N}(K))$-bimodule sets, which using Observation 6.17 we consider as an injective function of $(\mathsf{W}(H), \mathsf{W}(K))$-bimodule sets.

Lemma 6.19. *For every $g \in \tilde{C}(H, K)$, the function (6.6) is an isomorphism among $(\mathsf{N}(H), \mathsf{N}(K))$-bimodule sets.*

Proof. It remains to show that the function (6.6) is surjective. For this, we fix an element $zK \in C(H, K)$, and study the condition

(1) $zK \in C(H, K)_{[g]}$.

By definition, condition (1) is equivalent to the condition

(2) $(zKz^{-1})/H$ is conjugate to $(gKg^{-1})/H$ in $\mathsf{W}(H)$,

which is equivalent to the condition

(3) zKz^{-1} is conjugate to gKg^{-1} in $\mathsf{N}(H)$,

which is equivalent to the condition

(4) there exists some $x \in \mathsf{N}(H)$ such that $xzKz^{-1}x^{-1} = gKg^{-1}$.

Because $xzKz^{-1}x^{-1} = (xzg^{-1})(gKg^{-1})(xzg^{-1})^{-1}$, condition (4) is equivalent to the condition

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(5) there exists some $x \in \mathsf{N}(H)$ such that $xzg^{-1} \in \mathsf{N}(gKg^{-1})$.

Now, we claim that condition (5) is equivalent to the condition

(6) $zg^{-1} \in \mathsf{N}(H) \cdot \mathsf{N}(gKg^{-1})$.

On the one hand, we have that (5) $\Rightarrow$ (6) via left multiplication by $x^{-1} \in \mathsf{N}(H)$. On the other hand, if $zg^{-1} \in \mathsf{N}(H) \cdot \mathsf{N}(gKg^{-1})$, then there exist elements $x^{-1} \in \mathsf{N}(H)$ and $y \in \mathsf{N}(gKg^{-1})$ such that $zg^{-1} = x^{-1}y$, which implies that $xzg^{-1} = y \in \mathsf{N}(gKg^{-1})$. So indeed, (6) $\Rightarrow$ (5). Finally, condition (6) is clearly equivalent to the condition

(7) $z \in \mathsf{N}(H) \cdot \mathsf{N}(gKg^{-1}) \cdot g \subseteq \tilde{C}(H, K)$.

So, via the composite equivalence (1) $\Leftrightarrow$ (7) and by inspection of diagram (6.5), we find that every element $zK \in C(H, K)_{[g]}$ is indeed in the image of the function (6.6). $\qquad\square$

Observation 6.20. Fix a subgroup $L \subseteq G$ such that $H \subseteq L \subseteq \mathsf{N}(H)$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathsf{N}(H) & \xhookrightarrow{\ x \mapsto x \cdot e = x\ } & \mathsf{N}(H) \cdot \mathsf{N}(L) \\
\downarrow & & \downarrow \\
\mathsf{N}(H)/L & \xdashrightarrow[\ xL \mapsto (x \cdot e)L = xL\ ]{} & (\mathsf{N}(H) \cdot \mathsf{N}(L))/L
\end{array}
\tag{6.7}
$$

of $(\mathsf{N}(H), \mathsf{N}(H) \cap \mathsf{N}(L))$-bimodule sets. Furthermore, the left H-actions on both sets in the lower row of diagram (6.7) are trivial. So, we may consider the lower morphism of diagram (6.7) as a morphism

$$
\mathsf{N}(H)/L \longrightarrow (\mathsf{N}(H) \cdot \mathsf{N}(L))/L
\tag{6.8}
$$

of $(\mathsf{W}(H), (\mathsf{N}(H) \cap \mathsf{N}(L))/L)$-bimodule sets. Moreover, the $(\mathsf{W}(H), (\mathsf{N}(H) \cap \mathsf{N}(L))/L)$-bimodule structure on $(\mathsf{N}(H) \cdot \mathsf{N}(L))/L$ extends to a $(\mathsf{W}(H), \mathsf{W}(L))$-bimodule structure (via the inclusion $(\mathsf{N}(H) \cap \mathsf{N}(L))/L \subseteq \mathsf{N}(L)/L =: \mathsf{W}(L)$). Hence, via the adjunction

$$
\mathsf{BiMod}_{(\mathsf{W}(H),(\mathsf{N}(H)\cap\mathsf{N}(L))/L)} \overset{\mathsf{Ind}^{\mathsf{W}(L)}_{(\mathsf{N}(H)\cap\mathsf{N}(L))/L}}{\underset{\mathsf{Res}^{\mathsf{W}(L)}_{(\mathsf{N}(H)\cap\mathsf{N}(L))/L}}{\overset{\perp}{\rightleftarrows}}} \mathsf{BiMod}_{(\mathsf{W}(H),\mathsf{W}(L))} \quad,
$$

the morphism (6.8) upgrades to a morphism

$$
\mathsf{N}(H)/L \longrightarrow \mathsf{Res}^{\mathsf{W}(L)}_{(\mathsf{N}(H)\cap\mathsf{N}(L))/L}((\mathsf{N}(H) \cdot \mathsf{N}(L))/L)
$$

in $\mathsf{BiMod}_{(\mathsf{W}(H),(\mathsf{N}(H)\cap\mathsf{N}(L))/L)}$, which corresponds to a morphism

$$
\mathsf{Ind}^{\mathsf{W}(L)}_{(\mathsf{N}(H)\cap\mathsf{N}(L))/L}(\mathsf{N}(H)/L) \longrightarrow (\mathsf{N}(H) \cdot \mathsf{N}(L))/L
\tag{6.9}
$$

in $\mathsf{BiMod}_{(\mathsf{W}(H),\mathsf{W}(L))}$.

Lemma 6.21. *For any subgroup $L \subseteq G$ such that $H \subseteq L \subseteq \mathsf{N}(H)$, the morphism (6.9) is an isomorphism.*

Proof. As the forgetful functor $\mathsf{BiMod}_{(\mathsf{W}(H),\mathsf{W}(L))} \to \mathsf{RMod}_{\mathsf{W}(L)}$ is conservative, it suffices to show that the morphism (6.9) becomes an isomorphism in $\mathsf{RMod}_{\mathsf{W}(L)}$. Now, note that we have a commutative square

$$
\begin{array}{ccc}
\mathsf{RMod}_{\mathsf{N}(H) \cap \mathsf{N}(L)} & \xrightarrow{\ \mathsf{Ind}_{\mathsf{N}(H) \cap \mathsf{N}(L)}^{\mathsf{N}(L)}\ } & \mathsf{RMod}_{\mathsf{N}(L)} \\
\downarrow {\scriptstyle (-)/L} & & \downarrow {\scriptstyle (-)/L} \\
\mathsf{RMod}_{(\mathsf{N}(H) \cap \mathsf{N}(L))/L} & \xrightarrow[\ \mathsf{Ind}_{(\mathsf{N}(H) \cap \mathsf{N}(L))/L}^{\mathsf{W}(L)}\ ]{} & \mathsf{RMod}_{\mathsf{W}(L)}
\end{array}
$$

(which commutes because it clearly commutes upon passing to right adjoints). Therefore, it suffices to observe the isomorphism

$$
\mathsf{Ind}_{\mathsf{N}(H) \cap \mathsf{N}(L)}^{\mathsf{N}(L)} \mathsf{N}(H) \cong (\mathsf{N}(H) \times \mathsf{N}(L))/(\mathsf{N}(H) \cap \mathsf{N}(L)) \cong \mathsf{N}(H) \cdot \mathsf{N}(L)
$$

in $\mathsf{RMod}_{\mathsf{N}(L)}$. $\qquad\square$

Observation 6.22. Consider the composite adjunction

$$
\mathcal{R}\mathsf{g} G \underset{\mathsf{coInd}_{\mathsf{N}(H)}^G}{\overset{\mathsf{Res}_{\mathsf{N}(H)}^G}{\rightleftarrows}} \mathcal{R}\mathsf{g}\mathsf{N}(H) \underset{\rho_{\mathsf{g}}^H}{\overset{\Phi_{\mathsf{g}}^H}{\rightleftarrows}} \mathcal{R}\mathsf{g}\mathsf{W}(H) \tag{6.10}
$$

(compare with Observation S.5.1.22). Because $\mathcal{R}\mathsf{g} G$ is rigidly-compactly generated by Observation 5.16 and the composite left adjoint $\Phi_{\mathsf{g}}^H \circ \mathsf{Res}_{\mathsf{N}(H)}^G$ is symmetric monoidal, by [GR17, Chapter 1, Lemma 9.3.6] the composite right adjoint $\mathsf{coInd}_{\mathsf{N}(H)}^G \circ \rho_{\mathsf{g}}^H$ is $\mathcal{R}\mathsf{g} G$-linear: in other words, for any $E \in \mathcal{R}\mathsf{g} G$ and $F \in \mathcal{R}\mathsf{g}\mathsf{W}(H)$ we have the projection formula

$$
\mathsf{coInd}_{\mathsf{N}(H)}^G \left(\rho_{\mathsf{g}}^H \left(\Phi_{\mathsf{g}}^H \left(\mathsf{Res}_{\mathsf{N}(H)}^G(E) \right) \otimes F \right) \right) \simeq E \otimes \mathsf{coInd}_{\mathsf{N}(H)}^G \left(\rho_{\mathsf{g}}^H(F) \right) \ .
$$

Proof of Theorem 6.6. It is clear that we have an identification

$$
\Phi^K \simeq \underline{\mathsf{hom}}_{\mathcal{R}\mathsf{g} G} \left(\mathbb{1}_{\mathcal{R}} \langle G/K \rangle , (-) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F}_{\ngeq K} \right\rangle \right)
$$

in $\mathsf{Fun}(\mathcal{R}\mathsf{g} G, \mathcal{R}^{\mathsf{hW}(K)})$ (see Definition S.5.1.20). It then follows from Observation S.5.2.1 that we may identify the gluing functor Γ_K^H as the composite

$$
\mathcal{R}^{\mathsf{hW}(H)} \xrightarrow{\beta_{\mathsf{W}(H)}} \mathcal{R}\mathsf{g}\mathsf{W}(H) \xrightarrow{\rho_{\mathsf{g}}^H} \mathcal{R}\mathsf{g}\mathsf{N}(H) \xrightarrow{\mathsf{coInd}_{\mathsf{N}(H)}^G} \mathcal{R}\mathsf{g} G \xrightarrow{\underline{\mathsf{hom}}_{\mathcal{R}\mathsf{g} G}\left(\mathbb{1}_{\mathcal{R}} \langle G/K \rangle , (-) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F}_{\ngeq K} \right\rangle \right)} \mathcal{R}^{\mathsf{hW}(K)} \ .
$$

In other words, for any $E \in \mathcal{R}^{\mathsf{hW}(H)}$ we have a natural equivalence

$$
\Gamma_K^H(E) \simeq \underline{\mathsf{hom}}_{\mathcal{R}\mathsf{g} G} \left(\mathbb{1}_{\mathcal{R}} \langle G/K \rangle , \mathsf{coInd}_{\mathsf{N}(H)}^G(\rho_{\mathsf{g}}^H(\beta_{\mathsf{W}(H)}(E))) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F}_{\ngeq K} \right\rangle \right) \tag{6.11}
$$

in $\mathcal{R}^{\mathsf{hW}(K)}$. By Observation 6.22, we have an equivalence

$$
\mathsf{coInd}_{\mathsf{N}(H)}^G(\rho_{\mathsf{g}}^H(\beta_{\mathsf{W}(H)}(E))) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F}_{\ngeq K} \right\rangle \simeq \mathsf{coInd}_{\mathsf{N}(H)}^G \left(\rho_{\mathsf{g}}^H \left(\beta_{\mathsf{W}(H)}(E) \otimes \Phi_{\mathsf{g}}^H \left(\mathsf{Res}_{\mathsf{N}(H)}^G \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F}_{\ngeq K} \right\rangle \right) \right) \right) \right)
$$

in $\mathcal{R}\mathsf{g} G$. Therefore, by the composite adjunction (6.10) we find that

$$
(6.11) \simeq \underline{\mathsf{hom}}_{\mathcal{R}\mathsf{g}\mathsf{W}(H)} \left(\Phi_{\mathsf{g}}^H \left(\mathsf{Res}_{\mathsf{N}(H)}^G \left(\mathbb{1}_{\mathcal{R}} \langle G/K \rangle \right) \right) , \beta_{\mathsf{W}(H)}(E) \otimes \Phi_{\mathsf{g}}^H \left(\mathsf{Res}_{\mathsf{N}(H)}^G \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F}_{\ngeq K} \right\rangle \right) \right) \right)
$$

$$
\simeq \underline{\mathsf{hom}}_{\mathcal{R}\mathsf{g}\mathsf{W}(H)} \left(\mathbb{1}_{\mathcal{R}} \left\langle (G/K)^H \right\rangle , \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F} \right\rangle \right) \tag{6.12}
$$

(recall Local Notation 6.11(3)). Thereafter, we find that

$$
(6.12) \simeq \underline{\mathsf{hom}}_{\mathcal{R}\mathsf{g}\mathsf{W}(H)} \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F} \right\rangle \otimes \mathbb{1}_{\mathcal{R}} \left\langle (G/K)^H \right\rangle , \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}} \mathscr{F} \right\rangle \right) \tag{6.13}
$$

$$\simeq \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \otimes \mathbb{1}_\mathcal{R} \left\langle C(H,K) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right) \tag{6.14}$$

$$\simeq \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle C(H,K) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right), \tag{6.15}$$

where equivalences (6.13) and (6.15) follow from the fact that $\widetilde{\mathsf{E}}\mathscr{F} \in \mathcal{S}_*^{\mathsf{g}\mathsf{W}(H)}$ is idempotent and equivalence (6.14) follows from Lemma 6.14. Then, using Lemma 6.19 we obtain an equivalence

$$(6.15) \simeq \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle \coprod_{[g] \in \mathsf{W}(H) \backslash C(H,K)/\mathsf{W}(K)} C(H,K)_{[g]} \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right)$$

$$\simeq \bigoplus_{[g] \in \mathsf{W}(H) \backslash C(H,K)/\mathsf{W}(K)} \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle C(H,K)_{[g]} \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right). \tag{6.16}$$

To simplify our notation, we fix an element $[g] \in \mathsf{W}(H) \backslash C(H,K)/\mathsf{W}(K)$ and study the corresponding summand

$$\underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle C(H,K)_{[g]} \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right) \tag{6.17}$$

of (6.16). Then, we obtain equivalences

$$(6.17) \simeq \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle (\mathsf{N}(H) \cdot \mathsf{N}(gKg^{-1}))/(gKg^{-1}) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right) \tag{6.18}$$

$$\simeq \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle \mathsf{Ind}_{(\mathsf{N}(H) \cap \mathsf{N}(gKg^{-1}))/(gKg^{-1})}^{\mathsf{W}(gKg^{-1})} (\mathsf{N}(H)/(gKg^{-1})) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right) \tag{6.19}$$

$$\simeq \mathsf{coInd}_{(\mathsf{N}(H) \cap \mathsf{N}(gKg^{-1}))/(gKg^{-1})}^{\mathsf{W}(gKg^{-1})} \left(\underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle \mathsf{N}(H)/(gKg^{-1}) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right) \right)$$

$$\simeq \mathsf{Ind}_{(\mathsf{N}(H) \cap \mathsf{N}(gKg^{-1}))/(gKg^{-1})}^{\mathsf{W}(gKg^{-1})} \left(\underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle \mathsf{N}(H)/(gKg^{-1}) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right) \right), \tag{6.20}$$

where equivalence (6.18) follows from Lemma 6.19 and equivalence (6.19) follows from Lemma 6.21. To simplify our notation, we study the object

$$\underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle \mathsf{N}(H)/(gKg^{-1}) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right) \tag{6.21}$$

of $\mathcal{R}^{\mathsf{h}(\mathsf{N}(H) \cap \mathsf{N}(gKg^{-1}))/(gKg^{-1})}$ (whose induction to $\mathsf{W}(gKg^{-1})$ is (6.20)). Specifically, we compute that

$$(6.21) \simeq \underline{\mathsf{hom}}_{\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}} \left(\mathbb{1}_\mathcal{R} \left\langle \mathsf{W}(H)/((gKg^{-1})/H) \right\rangle, \beta_{\mathsf{W}(H)}(E) \otimes \mathbb{1}_\mathcal{R} \left\langle \widetilde{\mathsf{E}}\mathscr{F} \right\rangle \right)$$

$$=: \Phi_\mathscr{F}^{(gKg^{-1})/H} \beta_{\mathsf{W}(H)}(E). \tag{6.22}$$

Finally, observe that $(gKg^{-1})/H \in \mathsf{P}_{\mathsf{W}(H)} \backslash \mathscr{F}$ is a minimal element. Hence, by Observation S.5.1.22 we have an equivalence

$$\Phi_\mathscr{F}^{(gKg^{-1})/H} \simeq \Phi_{\not\geq (gKg^{-1})/H}^{(gKg^{-1})/H} =: \Phi^{(gKg^{-1})/H}$$

in $\mathsf{Fun}(\mathcal{R}^{\mathsf{g}\mathsf{W}(H)}, \mathcal{R}^{\mathsf{h}(\mathsf{N}(H) \cap \mathsf{N}(gKg^{-1}))/(gKg^{-1})})$. Therefore, by Proposition 5.9 we have equivalences

$$(6.22) \simeq \Phi^{(gKg^{-1})/H} \beta_{\mathsf{W}(H)} E \simeq E^{\tau(gKg^{-1})/H},$$

completing the proof. $\qquad \square$

7. THE GEOMETRIC STRATIFICATION OF GENUINE C_{p^n}-$\mathbb{Z}$-MODULES

In this section, we apply the material of Part I to describe the symmetric monoidal *geometric stratification* (see Definition 2.4) of the presentably symmetric monoidal stable ∞-category

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}} := \mathsf{Sp}^{\mathsf{gC}_{p^n}} \otimes \mathsf{Mod}_{\mathbb{Z}} \simeq \mathsf{Mack}_{\mathsf{C}_{p^n}}(\mathsf{Mod}_{\mathbb{Z}})$$

of genuine C_{p^n}-$\mathbb{Z}$-modules (see Definition 1.3(1) and Observation 1.9), where p is prime. This description (originally stated as Theorem B) is recorded as Theorem 7.2; its consequences are unpacked more explicitly in Observation 7.10 (after its proof). We also describe categorical fixedpoints in these terms as Proposition 7.19.

Notation 7.1.

(1) We fix a prime p and a nonnegative integer $n \geq 0$; these determine a finite cyclic group C_{p^n} of order p^n.

(2) To ease our notation, we use the identification $\mathsf{P}_{\mathsf{C}_{p^n}} \cong [n]$ for the poset of subgroups of C_{p^n}, and for any $s \in [n]$ we may use the identification $\mathsf{C}_{p^n}/\mathsf{C}_{p^s} \cong \mathsf{C}_{p^{n-s}}$.[19]

(3) Given a genuine C_{p^n}-$\mathbb{Z}$-module $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$, for any $s \in [n]$ we may simply write

$$E_s := \Phi^{\mathsf{C}_{p^s}}(E) \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$$

for its C_{p^s}-geometric fixedpoints.

(4) We fix a generator $\sigma \in \mathsf{C}_{p^n}$. For any $s \in [n]$, we also denote by $\sigma \in \mathsf{C}_{p^{n-s}}$ its image under the quotient homomorphism $\mathsf{C}_{p^n} \to \mathsf{C}_{p^{n-s}}$.

(5) In the interest of brevity, we simply write $\otimes := \otimes_{\mathbb{Z}}$ for the symmetric monoidal structure of $\mathsf{Mod}_{\mathbb{Z}}$.

(6) For any $i \geq 0$, we simply write $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^i}}$ for the trivial C_{p^i}-action on $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}$.

Theorem 7.2. *The symmetric monoidal geometric stratification*

$$[n] \xrightarrow{\mathsf{Sp}_{\leq \bullet}^{\mathsf{gC}_{p^n}}} \mathbf{Idl}_{\mathsf{Sp}^{\mathsf{gC}_{p^n}}} \xrightarrow{-\otimes \mathsf{Mod}_{\mathbb{Z}}} \mathbf{Idl}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}} \tag{7.1}$$

of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ has the following features.

(1) Its underlying stratification is strict.[20]

(2) Its symmetric monoidal gluing diagram is the functor

$$[n] \xrightarrow{\mathscr{G}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})} \mathsf{CAlg}^{\mathrm{r.lax}}(\mathsf{Cat})$$

---

[19]We use the letter "s" because it stands for the word "stratum".

[20]That is, the stratification is convergent (as guaranteed by Theorem S.A because the poset $[n]$ is finite) and moreover the gluing functors compose strictly (as opposed to left-laxly). See §S.6.3 for more discussion of this notion.

selecting the diagram

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^n}} \xrightarrow{(-)^{\mathsf{tC}_p}} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-1}}} \xrightarrow{(-)^{\mathsf{tC}_p}} \cdots \xrightarrow{(-)^{\mathsf{tC}_p}} \mathsf{Mod}_{\mathbb{Z}} \; . \tag{7.2}$$

(3) *All nontrivial composite gluing functors in its symmetric monoidal gluing diagram (7.2) are zero.*

Lemma 7.3. *Let $\mathcal{R}$ be a rigidly-compactly generated presentably symmetric monoidal stable ∞-category, let $G = \mathsf{C}_{p^n}$, and let $K < H \leq G$ be any strict containment among subgroups of G. Then, the composite natural transformation in the diagram*

(in which the bottom triangle commutes) is an equivalence in $\mathsf{Fun}(\mathcal{R}^{\mathsf{g}G}, \mathcal{R}^{\mathsf{g}G/H})$: in other words, for any $E \in \mathcal{R}^{\mathsf{g}G}$ the canonical morphism

$$\Phi_{\mathsf{g}}^{H/K}\left(E^K\right) \longrightarrow \Phi_{\mathsf{g}}^{H/K}\left(\Phi_{\mathsf{g}}^K(E)\right) \simeq \Phi_{\mathsf{g}}^H(E) \tag{7.3}$$

is an equivalence.

Remark 7.4. In fact, Lemma 7.3 applies to the rigidly-compactly generated presentably symmetric monoidal stable ∞-categories

$$\mathsf{Sp}^{\mathsf{g}\mathsf{C}_{p^\infty}} := \lim\left(\cdots \xrightarrow{\mathsf{Res}_{\mathsf{C}_{p^2}}^{\mathsf{C}_{p^3}}} \mathsf{Sp}^{\mathsf{g}\mathsf{C}_{p^2}} \xrightarrow{\mathsf{Res}_{\mathsf{C}_p}^{\mathsf{C}_{p^2}}} \mathsf{Sp}^{\mathsf{g}\mathsf{C}_p} \xrightarrow{\mathsf{Res}_e^{\mathsf{C}_p}} \mathsf{Sp}\right) \qquad \text{and} \qquad \mathsf{Sp}^{\mathsf{g}^{<p}\mathbb{T}} := \left\langle \Sigma_{\mathbb{T}}^\infty (\mathbb{T}/\mathsf{C}_{p^i})_+ \right\rangle_{i \geq 0} \subseteq \mathsf{Sp}^{\mathsf{g}\mathbb{T}},$$

both of which are stratified over the poset $\mathbb{Z}_{\geq 0}$ (under the requirement that the subgroups $K < H \leq G$ be closed): the key point is that this poset is totally ordered. Alternatively, these statements can be readily deduced from Lemma 7.3 itself: for $\mathsf{Sp}^{\mathsf{g}\mathsf{C}_{p^\infty}}$ this is immediate, and for $\mathsf{Sp}^{\mathsf{g}^{<p}\mathbb{T}}$ it suffices to observe that the forgetful functor $\mathsf{Sp}^{\mathsf{g}^{<p}\mathbb{T}} \to \mathsf{Sp}^{\mathsf{g}\mathsf{C}_{p^\infty}}$ is conservative.

Proof of Lemma 7.3. We begin by observing the commutative diagram

in $\mathsf{CAlg}(\mathsf{Pr}^L)$. We observe too that in the adjunction

because $\mathcal{R}^{\mathfrak{g}(G/K)}$ is rigidly-compactly generated by Observation 5.16 and the left adjoint $\mathsf{Res}_G^{G/K}$ is symmetric monoidal, by [GR17, Chapter 1, Lemma 9.3.6] the right adjoint $(-)^K$ is $\mathcal{R}^{\mathfrak{g}(G/K)}$-linear: in other words, for any $F \in \mathcal{R}^{\mathfrak{g}(G/K)}$ and $E \in \mathcal{R}^{\mathfrak{g}G}$ we have the projection formula

$$F \otimes E^K \simeq \left(\mathsf{Res}_G^{G/K}(F) \otimes E \right)^K .$$

Using these two observations, we respectively identify the source and target of the morphism (7.3) as

$$\Phi_{\mathfrak{g}}^{H/K} \left(E^K \right) \simeq \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \odot E^K \right)^{H/K} \tag{7.4}$$

$$\simeq \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right\rangle \otimes E^K \right)^{H/K}$$

$$\simeq \left(\left(\mathsf{Res}_G^{G/K} \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right\rangle \right) \otimes E \right)^K \right)^{H/K}$$

$$\simeq \left(\mathsf{Res}_G^{G/K} \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right\rangle \right) \otimes E \right)^{H}$$

$$\simeq \left(\mathsf{Res}_G^{G/K} \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right) \odot E \right)^{H}$$

and

$$\Phi_{\mathfrak{g}}^{H/K} \left(\Phi_{\mathfrak{g}}^{K}(E) \right) \simeq \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \odot \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq K} \odot E \right)^K \right)^{H/K} \tag{7.5}$$

$$\simeq \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right\rangle \otimes \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq K} \right\rangle \otimes E \right)^K \right)^{H/K}$$

$$\simeq \left(\left(\mathsf{Res}_G^{G/K} \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right\rangle \right) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq K} \right\rangle \otimes E \right)^K \right)^{H/K}$$

$$\simeq \left(\mathsf{Res}_G^{G/K} \left(\mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right\rangle \right) \otimes \mathbb{1}_{\mathcal{R}} \left\langle \widetilde{\mathsf{E}}\mathscr{F}_{\geq K} \right\rangle \otimes E \right)^{H}$$

$$\simeq \left(\left(\mathsf{Res}_G^{G/K} \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right) \wedge \widetilde{\mathsf{E}}\mathscr{F}_{\geq K} \right) \odot E \right)^{H} ,$$

where equivalences (7.4) and (7.5) follow from Observation 1.16. It is now clear that the morphism (7.3) itself may be obtained by applying the composite functor

$$\mathcal{S}_*^{\mathfrak{g}G} \xrightarrow{(-)\odot E} \mathcal{R}^{\mathfrak{g}G} \xrightarrow{(-)^H} \mathcal{R}^{\mathfrak{g}(G/H)}$$

to the evident morphism

$$\mathsf{Res}_G^{G/K} \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right) \simeq \mathsf{Res}_G^{G/K} \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right) \wedge S^0 \longrightarrow \mathsf{Res}_G^{G/K} \left(\widetilde{\mathsf{E}}\mathscr{F}_{\geq H/K} \right) \wedge \widetilde{\mathsf{E}}\mathscr{F}_{\geq K} \tag{7.6}$$

in $\mathcal{S}_*^{\mathfrak{g}G}$, which it therefore suffices to show is an equivalence. For this, fix an arbitrary closed subgroup $J \leq G$. Now, for any genuine G/K-space $X \in \mathcal{S}_*^{\mathfrak{g}(G/K)}$ we have that

$$\mathsf{Res}_G^{G/K}(X)^J \simeq X^{J/(K \cap J)}$$

(note that $J/(K \cap J)$ is the image of the composite $J \hookrightarrow G \twoheadrightarrow G/K$). Hence, it suffices to observe that if $J/(J \cap K) \geq H/K$ in $\mathsf{P}_{G/K}$ then $J \geq K$, which follows from the fact that the poset P_G is totally ordered. $\qquad\square$

Observation 7.5. For any $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^2}}$, all three terms in the cofiber sequence

$$\left(E_{\mathsf{hC}_p} \xrightarrow{\mathsf{Nm}_{\mathsf{C}_p}(E)} E^{\mathsf{hC}_p} \xrightarrow{\mathsf{Q}_{\mathsf{C}_p}(E)} E^{\mathsf{tC}_p} \right)^{\mathsf{tC}_p}$$

in $\mathsf{Mod}_{\mathbb{Z}}$ are zero; this follows from [NS18, Footnote 9] (see also [NS18, Lemma I.2.7]). Equivalently, the norm maps

$$(E_{\mathsf{hC}_p})_{\mathsf{hC}_p} \xrightarrow{\mathsf{Nm}_{\mathsf{C}_p}(E_{\mathsf{hC}_p})} (E_{\mathsf{hC}_p})^{\mathsf{hC}_p} \,, \qquad (E^{\mathsf{hC}_p})_{\mathsf{hC}_p} \xrightarrow{\mathsf{Nm}_{\mathsf{C}_p}(E^{\mathsf{hC}_p})} (E^{\mathsf{hC}_p})^{\mathsf{hC}_p} \,, \qquad \text{and} \qquad (E^{\mathsf{tC}_p})_{\mathsf{hC}_p} \xrightarrow{\mathsf{Nm}_{\mathsf{C}_p}(E^{\mathsf{tC}_p})} (E^{\mathsf{tC}_p})^{\mathsf{hC}_p}$$

are equivalences.

Observation 7.6. For any $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^n}}$ and any $1 \leq s \leq n$, if $E^{\mathsf{tC}_{p^s}} \simeq 0$ then $E^{\mathsf{tC}_{p^i}} \simeq 0$ for all $s \leq i \leq n$. To see this, by induction it suffices to verify the case that $i = s + 1$ (assuming that $s < n$, otherwise the assertion is vacuously true). And indeed, we have that $E^{\mathsf{tC}_{p^{s+1}}} \simeq 0$ as a result of the diagram

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s+1}}}$, which commutes by Observation 4.13(3) and in which the lower left morphism is an equivalence by Observation 7.5 (and the equivalence $E_{\mathsf{hC}_{p^s}} \xrightarrow{\sim} (E_{\mathsf{hC}_{p^{s-1}}})_{\mathsf{hC}_p}$) and the lower right morphism is an equivalence by assumption.

Proof of Theorem 7.2. We note that the assertions only make reference to the underlying stratification of the symmetric monoidal stratification (7.1), so it suffices to verify them at that level. For each $s \in [n]$, its sth stratum is given by

$$(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})_s := (\mathsf{Sp}^{\mathsf{gC}_{p^n}} \otimes \mathsf{Mod}_{\mathbb{Z}})_s \simeq (\mathsf{Sp}^{\mathsf{gC}_{p^n}})_s \otimes \mathsf{Mod}_{\mathbb{Z}} \simeq \mathsf{Sp}^{\mathsf{hC}_{p^{n-s}}} \otimes \mathsf{Mod}_{\mathbb{Z}} \simeq \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}} \,,$$

where the first equivalence follows from Observation 2.1 and the second equivalence follows from Theorem S.5.1.27. Thereafter, it follows from Proposition 5.9 and Observation 1.4 (and the fact that $\mathsf{Mod}_{\mathbb{Z}}$ is compactly generated) that for each morphism $i \to j$ in $[n]$ its corresponding gluing functor

$$(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})_i \xrightarrow{\Gamma_j^i} (\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})_j$$

is given by the proper $\mathsf{C}_{p^{j-i}}$-Tate construction

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-i}}} \xrightarrow{(-)^{\tau\mathsf{C}_{p^{j-i}}}} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-j}}} \,.$$

Now, assuming $j - i \geq 1$, then for any $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-i}}}$ we have that

$$E^{\tau\mathsf{C}_{p^{j-i}}} \simeq \Phi^{\mathsf{C}_{p^{j-i}}}(\beta E) \tag{7.7}$$

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$$\simeq \Phi^{\mathsf{C}_p}\left((\beta E)^{\mathsf{C}_{p^{j-i-1}}}\right) \tag{7.8}$$

$$\simeq \Phi^{\mathsf{C}_p}\left(\beta\left(E^{\mathsf{hC}_{p^{j-i-1}}}\right)\right)$$

$$\simeq \left(E^{\mathsf{hC}_{p^{j-i-1}}}\right)^{\tau\mathsf{C}_p} \tag{7.9}$$

$$\simeq \left(E^{\mathsf{hC}_{p^{j-i-1}}}\right)^{\mathsf{tC}_p}, \tag{7.10}$$

where

- equivalences (7.7) and (7.9) follow from Proposition 5.9,

- equivalence (7.8) follows from Lemma 7.3 when $j-i>1$ and is trivially true when $j-i=1$, and

- equivalence (7.10) follows from Observation 4.8.

All three claims now follow from Observation 7.5. $\qquad\qquad\square$

Notation 7.7. We define the full subcategory

$$\mathsf{Zig}_n := \{(i \to j) \in \mathsf{TwAr}([n]) : j-i \le 1\} \subseteq \mathsf{TwAr}([n]) \,,$$

which we depict as

$$\mathsf{Zig}_n = \left(\begin{array}{cccccc} (0\to 0) & (1\to 1) & \cdots & \cdots & (n\to n) & \\ & \searrow\ \downarrow\ \nearrow & \downarrow & & \searrow\ \downarrow & \\ & (0\to 1) & (1\to 2) & \cdots & ((n-1)\to n) & \end{array}\right).$$

Remark 7.8. When possible, we use the "sawtooth" depiction of Zig_n (as in Notation 7.7) and of diagrams indexed thereover: the columns will correspond to the strata of the geometric stratification of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ of Theorem 7.2. However, in order to depict natural transformations between diagrams indexed over Zig_n (such as in Proposition 7.19), we will use a more symmetric depiction.

Observation 7.9. The inclusion

$$\mathsf{Zig}_n \hookrightarrow \mathsf{TwAr}([n])$$

is initial. We use this fact without further comment.

Observation 7.10. We unpack the following consequences of Theorem 7.2.

(1) At the macrocosm level, applying Theorems S.2.5.14 and S.4.5.1 along with Observation S.6.3.5 to the strict symmetric monoidal stratification (7.1), we obtain an identification

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}} \simeq \mathsf{Glue}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}) := \lim_{[n]}^{\mathsf{r.lax}}\left(\mathscr{G}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})\right) \simeq \Gamma_{[n]^{\mathsf{op}}}\left(\mathscr{G}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})^{\overset{\mathsf{cocart}}{\vee}}\right)$$

$$\simeq \lim\left(\begin{array}{cccccc} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^n}} & \mathsf{Ar}\left(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-1}}}\right) & \cdots & \cdots & \mathsf{Ar}\left(\mathsf{Mod}_{\mathbb{Z}}\right) \\ & {\scriptstyle(-)^{\mathsf{tC}_p}}\searrow\ \downarrow {\scriptstyle t}\ {\scriptstyle(-)^{\mathsf{tC}_{p_{\mathsf{os}}}}}\searrow & \downarrow {\scriptstyle t} & & {\scriptstyle(-)^{\mathsf{tC}_{p_{\mathsf{os}}}}}\searrow\ \downarrow {\scriptstyle t} \\ & \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-1}}} & \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-2}}} & \cdots & \mathsf{Mod}_{\mathbb{Z}} \end{array}\right),$$

where the limit is taken in $\mathsf{CAlg}^{\mathsf{r.lax}}(\mathsf{Cat})$.

(2) We now proceed to the microcosm level.

    (a) By Observations S.6.3.2 and S.2.7.4, the fact that the stratification (7.1) is strict implies that for each object $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$ we have a natural diagram of equivalences

$$E \xrightarrow{\sim} \lim_{\mathsf{TwAr}([n])}(g(E)) \xrightarrow{\sim} \lim_{\mathsf{Zig}_n}(g(E)) \ .$$

In particular, the object $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$ is recorded by the data of its gluing diagram

$$\left(\begin{array}{ccccc} E_0 & E_1 & \cdots & \cdots & E_n \\ & \downarrow{\scriptstyle\gamma^E_{0,1}} & \downarrow{\scriptstyle\gamma^E_{1,2}} & & \downarrow{\scriptstyle\gamma^E_{n-1,n}} \\ & (E_0)^{\mathsf{t}\mathsf{C}_p} & (E_1)^{\mathsf{t}\mathsf{C}_p} & \cdots & (E_{n-1})^{\mathsf{t}\mathsf{C}_p} \end{array}\right) .$$

    (b) The fact that the stratification (7.1) is symmetric monoidal implies that for any objects $E, F \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$, their tensor product $(E \otimes F) \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$ has $(E \otimes F)_s \simeq E_s \otimes F_s$ for all $s \in [n]$ and gluing morphisms the composites

$$\gamma^{E \otimes F}_{s-1,s} : E_s \otimes F_s \xrightarrow{\gamma^E_{s-1,s} \otimes \gamma^F_{s-1,s}} (E_{s-1})^{\mathsf{t}\mathsf{C}_p} \otimes (F_{s-1})^{\mathsf{t}\mathsf{C}_p} \longrightarrow (E_{s-1} \otimes F_{s-1})^{\mathsf{t}\mathsf{C}_p} \ ,$$

where the second morphism arises from the fact that the functor

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s+1}}} \xrightarrow{(-)^{\mathsf{t}\mathsf{C}_p}} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}$$

is right-laxly symmetric monoidal by Observation 4.13(2).

(3) At the nanocosm level, again by Observation S.2.7.4, for each $E, F \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$ we have an equivalence

$$\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}}(F, E) \xrightarrow{\sim} \lim_{(i \to j) \in \mathsf{Zig}_n} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-j}}}}(F_j, (E_i)^{\mathsf{t}\mathsf{C}_{p^{j-i}}}) \tag{7.11}$$

in $\mathsf{Mod}_{\mathbb{Z}}$, where (by definition of Zig_n) we have that $j - i$ is either 0 or 1. More explicitly, the diagram $\mathsf{Zig}_n \to \mathsf{Mod}_{\mathbb{Z}}$ whose limit is the target of the equivalence (7.11) is

$$\left(\begin{array}{ccccc} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^n}}}(F_0, E_0) & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-1}}}}(F_1, E_1) & \cdots & \cdots & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}}(F_n, E_n) \\ & \downarrow & & & \downarrow \\ & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-1}}}}(F_1, (E_0)^{\mathsf{t}\mathsf{C}_p}) & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-2}}}}(F_2, (E_1)^{\mathsf{t}\mathsf{C}_p}) & \cdots & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}}(F_n, (E_{n-1})^{\mathsf{t}\mathsf{C}_p}) \end{array}\right),$$

where for all $1 \le s \le n$ the sth diagonal morphism is the composite

$$\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s+1}}}}(F_{s-1}, E_{s-1}) \xrightarrow{(-)^{\mathsf{t}\mathsf{C}_p}} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}}((F_{s-1})^{\mathsf{t}\mathsf{C}_p}, (E_{s-1})^{\mathsf{t}\mathsf{C}_p}) \xrightarrow{\gamma^F_{s-1,s}} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}}(F_s, (E_{s-1})^{\mathsf{t}\mathsf{C}_p})$$

and the sth vertical morphism is

$$\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}}(F_s, E_s) \xrightarrow{\gamma^E_{s-1,s}} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}}(F_s, (E_{s-1})^{\mathsf{t}\mathsf{C}_p}) \ .$$

Remark 7.11. In what follows, we record a number of basic facts about categorical fixedpoints of genuine C_{p^n}-$\mathbb{Z}$-modules. In effect, we obtain these by applying the discussion of §S.5.4 (regarding categorical fixedpoints of genuine G-spectra for an arbitrary finite group G). The present situation is simpler both because C_{p^n} is abelian and because we work $\mathbb{Z}$-linearly.

Observation 7.12. For any $0 \leq a < n$, the inclusion morphisms

$$E^{\mathsf{C}_{p^{a+1}}} \xrightarrow{\ \mathsf{inc}\ } E^{\mathsf{C}_{p^a}}$$

for $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ define a natural transformation

$$
\begin{array}{ccc}
 & & \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a-1}}} \\
 & \nearrow^{(-)^{\mathsf{C}_{p^{a+1}}}} & \downarrow^{\mathsf{triv}} \\
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}} & \xrightarrow{\ \swarrow\ } & \\
 & \searrow_{(\sim)^{\mathsf{C}_{p^a}}} & \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}
\end{array}
\qquad . \tag{7.12}
$$

Indeed, the natural transformation in diagram (7.12) is corepresented by the morphism

$$\mathbb{Z}\left\langle \mathsf{C}_{p^n}/\mathsf{C}_{p^a} \longrightarrow \mathsf{C}_{p^n}/\mathsf{C}_{p^{a+1}} \right\rangle$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$.

Observation 7.13. For any $0 \leq a < n$, the transfer morphisms

$$E^{\mathsf{C}_{p^a}} \xrightarrow{\ \mathsf{trf}\ } E^{\mathsf{C}_{p^{a+1}}}$$

for $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ define a natural transformation

$$
\begin{array}{ccc}
 & & \mathsf{Mod}^{\mathsf{hC}_{p^{n-a-1}}} \\
 & \nearrow^{(-)^{\mathsf{C}_{p^{a+1}}}} & \downarrow^{\mathsf{triv}} \\
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}} & \xrightarrow{\ \nearrow\ } & \\
 & \searrow_{(\sim)^{\mathsf{C}_{p^a}}} & \mathsf{Mod}^{\mathsf{hC}_{p^{n-a}}}
\end{array}
\qquad . \tag{7.13}
$$

Indeed, the natural transformation in diagram (7.13) is corepresented by the morphism

$$\mathbb{Z}\left\langle \mathsf{C}_{p^n}/\mathsf{C}_{p^{a+1}} \right\rangle \longrightarrow \mathbb{Z}\left\langle \mathsf{C}_{p^n}/\mathsf{C}_{p^a} \right\rangle \tag{7.14}$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ obtained by applying the functor $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^{a+1}}} \xrightarrow{\ \mathsf{Ind}_{\mathsf{C}_{p^{a+1}}}^{\mathsf{C}_{p^n}}\ } \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ to the morphism

$$\mathbb{Z} \simeq \mathbb{Z}\left\langle \mathsf{C}_{p^{a+1}}/\mathsf{C}_{p^{a+1}} \right\rangle \longrightarrow \mathbb{Z}\left\langle \mathsf{C}_{p^{a+1}}/\mathsf{C}_{p^a} \right\rangle \simeq \mathsf{coInd}_{\mathsf{C}_{p^a}}^{\mathsf{C}_{p^{a+1}}}\left(\mathbb{Z}\left\langle \mathsf{C}_{p^a}/\mathsf{C}_{p^a} \right\rangle \right)$$

corresponding to the identity morphism

$$\mathsf{Res}_{\mathsf{C}_{p^a}}^{\mathsf{C}_{p^{a+1}}}\left(\mathbb{Z}\left\langle \mathsf{C}_{p^{a+1}}/\mathsf{C}_{p^{a+1}} \right\rangle \right) \longrightarrow \mathbb{Z}\left\langle \mathsf{C}_{p^a}/\mathsf{C}_{p^a} \right\rangle$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^a}}$.

Definition 7.14. The ***homotopy transfer*** (***for*** C_p) is the morphism

$$\mathsf{id}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_p}} \xrightarrow{\ \mathsf{trf}_{\mathsf{C}_p}^{\mathsf{h}}\ } \mathsf{triv} \circ (-)^{\mathsf{hC}_p}$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_p}, \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_p})$ that is corepresented by the morphism

$$\mathbb{Z} \longrightarrow \mathbb{Z}^{\mathsf{h}}\left\langle \mathsf{C}_p/e \right\rangle \ , \tag{7.15}$$

i.e. the morphism U (7.14) in the case that $n = 1$ and $a = 0$. We may simply write

$$\mathsf{trf}^\mathsf{h} := \mathsf{trf}^\mathsf{h}_{\mathsf{C}_p}(E)$$

for its component at an object $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_p}$ (in line with the notation introduced in Definition 1.10).

Observation 7.15. The morphism (7.15) is between discrete objects in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_p}$ (i.e. abelian groups with C_p-action) which are cyclic as discrete $\mathbb{Z}[\mathsf{C}_p]$-modules, and it is characterized by the fact that it carries $1 \in \mathbb{Z}$ to the element

$$N := (1 + \sigma + \cdots + \sigma^{p-1}) \in \mathbb{Z}^\mathsf{h} \langle \mathsf{C}_p/e \rangle$$

(see Definition A.20). So, for any object $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_p}$, the morphism $\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_p}}((7.15), E)$ is the composite

$$E \longrightarrow E_{\mathsf{hC}_p} \xrightarrow{\ \mathsf{Nm}_{\mathsf{C}_p}(E)\ } E^{\mathsf{hC}_p} \ .$$

Observation 7.16. Fix any $0 \le a < n$. We claim that for any $s \in [n]$, applying the functor

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}} \xrightarrow{\ \Phi^{\mathsf{C}_{p^s}}\ } \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$$

to the morphism (7.14) gives

- the morphism

$$0 \longrightarrow 0$$

  when $s > a + 1$;

- the morphism

$$\mathbb{Z}^\mathsf{h} \langle \mathsf{C}_{p^{n-s}}/\mathsf{C}_{p^{a+1-s}} \rangle \longrightarrow 0$$

  when $s = a + 1$; and

- the morphism

$$\mathbb{Z}^\mathsf{h} \langle \mathsf{C}_{p^{n-s}}/\mathsf{C}_{p^{a+1-s}} \rangle \longrightarrow \mathbb{Z}^\mathsf{h} \langle \mathsf{C}_{p^{n-s}}/\mathsf{C}_{p^{a-s}} \rangle \tag{7.16}$$

  when $s \le a$.

The only nontrivial case to verify is when $s \le a$. Then, the morphism (7.16) is between discrete objects in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$ (i.e. abelian groups with $\mathsf{C}_{p^{n-s}}$-action) which are cyclic as discrete $\mathbb{Z}[\mathsf{C}_{p^{n-s}}]$-modules, and it is characterized by the fact that it carries the element $1 \in \mathbb{Z}^\mathsf{h} \langle \mathsf{C}_{p^{n-s}}/\mathsf{C}_{p^{a+1-s}} \rangle$ to the element

$$N := (1 + \sigma + \cdots + \sigma^{p-1}) \in \mathbb{Z}^\mathsf{h} \langle \mathsf{C}_{p^{n-s}}/\mathsf{C}_{p^{a-s}} \rangle \ .$$

So by Observation 7.15, the morphism (7.16) corepresents the natural morphism

$$E^{\mathsf{hC}_{p^{a-s}}} \xrightarrow{\ \mathsf{trf}^\mathsf{h}_{\mathsf{C}_p}(E)^{\mathsf{hC}_{p^{a-s}}}\ } E^{\mathsf{hC}_{p^{a+1-s}}} \tag{7.17}$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ for any object $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$.

Notation 7.17. For any $0 \le s \le a < n$ and any $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$, we may simply write

$$\mathsf{trf}^\mathsf{h} := \mathsf{trf}^\mathsf{h}_{\mathsf{C}_p}(E)^{\mathsf{hC}_{p^{a-s}}}$$

for the morphism (7.17).

Observation 7.18. For any $i \geq 0$ and any $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{i+1}}}$, we have a canonical commutative diagram

$$
\begin{array}{ccc}
E & & \\
\downarrow & \searrow^{\mathsf{trf}_{\mathsf{C}_p}^h(E)} & \\
E_{\mathsf{hC}_p} & \xrightarrow[\mathsf{Nm}_{\mathsf{C}_p}(E)]{} & E^{\mathsf{hC}_p} \\
\downarrow & & \downarrow^{\mathsf{Q}_{\mathsf{C}_p}(E)} \\
0 & \longrightarrow & E^{\mathsf{tC}_p}
\end{array}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^i}}$, in which the upper triangle commutes by Observation 7.15 and the square is the defining pushout.

Proposition 7.19. *Fix any genuine C_{p^n}-$\mathbb{Z}$-module $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$.*

(1) For any $0 \leq a \leq n$, its C_{p^a}-fixedpoints $E^{\mathsf{C}_{p^a}} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ is the limit of the diagram

$$
\mathsf{Zig}_a \xrightarrow{\mathsf{D}_a(E)} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}
$$

given by

$$
\tag{7.18}
$$

Moreover, for any $0 \leq a < n$ the functors $\mathsf{D}_a(E)$ and $\mathsf{D}_{a+1}(E)$ participate in a commutative square

$$
\begin{array}{ccc}
\mathsf{Zig}_a & \overset{\mathsf{Zig}_{i \mapsto i}}{\hookrightarrow} & \mathsf{Zig}_{a+1} \\
{\scriptstyle \mathsf{D}_a(E)}\downarrow & & \downarrow{\scriptstyle \mathsf{D}_{a+1}(E)} \\
\mathsf{Mod}^{\mathsf{hC}_{p^{n-a}}} & \xrightarrow[(-)^{\mathsf{hC}_p}]{} & \mathsf{Mod}^{\mathsf{hC}_{p^{n-a-1}}}
\end{array}
\tag{7.19}
$$

(in which the functor $\mathsf{Zig}_{i \mapsto i}$ is the evident fully faithful inclusion).

(2) For any $0 \leq a < n$, the inclusion morphism

$$
E^{\mathsf{C}_{p^{a+1}}} \xrightarrow{\mathsf{inc}} E^{\mathsf{C}_{p^a}}
$$

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in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ is the limit of the morphism

$$
\begin{array}{cccccc}
(E_0)^{\mathsf{hC}_{p^{a+1}}} & \cdots & & (E_a)^{\mathsf{hC}_{p^1}} & & (E_{a+1})^{\mathsf{hC}_{p^0}} \\[2em]
& \cdots & ((E_{a-1})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^1}} & & ((E_a)^{\mathsf{tC}_p})^{\mathsf{hC}_{p^0}} & \\[2em]
(E_0)^{\mathsf{hC}_{p^a}} & \cdots & & (E_a)^{\mathsf{hC}_{p^0}} & & 0 \\[2em]
& \cdots & ((E_{a-1})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^0}} & & 0 &
\end{array}
\tag{7.20}
$$

in $\mathsf{Fun}(\mathsf{Zig}_{a+1}, \mathsf{Mod}^{\mathsf{hC}_{p^{n-a}}})$ in which the nontrivial non-vertical morphisms are as in diagram (7.18).[21,22]

(3) For any $0 \le a < n$, the transfer morphism

$$
E^{\mathsf{C}_{p^a}} \xrightarrow{\ \mathsf{trf}\ } E^{\mathsf{C}_{p^{a+1}}}
$$

---

[21] Recall from Notation 1.13 that $\mathsf{inc}^{\mathsf{h}}$ denotes an inclusion morphism among homotopy fixedpoints.

[22] That is, it arises from the functoriality of limits for the diagram

$$
\begin{array}{ccc}
\mathsf{Zig}_a & \xrightarrow{\ \mathsf{Zig}_{i \mapsto i}\ } & \mathsf{Zig}_{a+1} \\[1em]
{\scriptstyle \mathsf{D}_a(E)} \downarrow & & \downarrow {\scriptstyle \mathsf{D}_{a+1}(E)} \\[1em]
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}} & \xrightarrow{\ (-)^{\mathsf{hC}_p}\ } & \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a-1}}} \\[1em]
{\scriptstyle \mathsf{id}} \downarrow & \swarrow {\scriptstyle \mathsf{triv}} & \\[1em]
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}} & &
\end{array}
$$

consisting of the commutative square (7.19) and the counit of the adjunction $\mathsf{triv} \dashv (-)^{\mathsf{hC}_p}$.

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ is the limit of the morphism

$$
\begin{array}{ccccccc}
(E_0)^{\mathsf{hC}_{p^{a+1}}} & \cdots & & (E_a)^{\mathsf{hC}_{p^1}} & & (E_{a+1})^{\mathsf{hC}_{p^0}} \\[2ex]
& \cdots & ((E_{a-1})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^1}} & & ((E_a)^{\mathsf{tC}_p})^{\mathsf{hC}_{p^0}} & \\[2ex]
\mathsf{trf}^{\mathsf{h}} & & \mathsf{trf}^{\mathsf{h}} & \mathsf{trf}^{\mathsf{h}} & & (7.21)\\[2ex]
(E_0)^{\mathsf{hC}_{p^a}} & \mathsf{trf}^{\mathsf{h}} \quad \cdots \quad \mathsf{trf}^{\mathsf{h}} & & (E_a)^{\mathsf{hC}_{p^0}} & & 0 \\[2ex]
& \cdots & ((E_{a-1})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^0}} & & 0 &
\end{array}
$$

in $\mathsf{Fun}(\mathsf{Zig}_{a+1}, \mathsf{Mod}^{\mathsf{hC}_{p^{n-a}}})$, in which the nontrivial non-vertical morphisms are as in diagram (7.18) and in which the second to rightmost square commutes by Observation 7.18.[23]

Proof. Part (1) is a special case of Observation 7.10(3) (with the commutative square (7.19) following from its naturality in $F \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$). Part (2) then follows by applying Observation S.5.4.9, and part (3) follows by applying Observations S.5.4.10, 7.13, and 7.16. $\qquad\square$

8. The Picard group of genuine C_{p^n}-$\mathbb{Z}$-modules

In this section, we prove the first part of Theorem C: namely, we use our symmetric monoidal stratification of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ (as described in §7) to compute its Picard group (under the assumption that p is odd). In order to help the reader appreciate the flow of the computation, we state its output as quickly as possible as Theorem 8.4, and then proceed to unpack it; its proof appears at the end of the section.

Local Notation 8.1. Through §11 (i.e. for the remainder of the paper except for §A), we assume that the prime p that was fixed in Notation 7.1(1) is odd.

Notation 8.2. Given a symmetric monoidal ∞-category $\mathcal{C}$, we write $\pi_0(\iota_0(\mathcal{C}))$ for the commutative monoid of equivalence classes of objects of $\mathcal{C}$. We make no notational distinction between an object of $\mathcal{C}$ and its equivalence class in $\pi_0(\iota_0(\mathcal{C}))$.

Definition 8.3. The ***Picard group*** of a symmetric monoidal ∞-category $\mathcal{C}$ is the maximal subgroup

$$\mathsf{Pic}(\mathcal{C}) \subseteq \pi_0(\iota_0(\mathcal{C})) \ ,$$

---

[23]That is, it arises from the functoriality of limits for a diagram

$$
\begin{array}{ccc}
\mathsf{Zig}_a & \xrightarrow{\ \mathsf{Zig}_{i\mapsto i}\ } & \mathsf{Zig}_{a+1} \\[1ex]
{\scriptstyle \mathsf{D}_a(E)}\Big\downarrow & \raisebox{0.5ex}{$\scriptstyle (\mathsf{Zig}_{i\mapsto i})_*(\mathsf{D}_a(E))$} \Downarrow & \Big\downarrow{\scriptstyle \mathsf{D}_{a+1}(E)} \\[1ex]
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}} & \xleftarrow{\ \mathsf{triv}\ } & \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a-1}}}
\end{array}
$$

in which upper triangle is a right Kan extension (i.e. extension by zero).

i.e. the abelian group of those equivalence classes of objects $c \in \mathcal{C}$ such that there exist an object $c' \in \mathcal{C}$ and an equivalence $c \otimes c' \simeq \mathbb{1}_{\mathcal{C}}$. We refer to an object of $\mathcal{C}$ whose equivalence class lies in $\mathsf{Pic}(\mathcal{C})$ as a ***Picard element***.

Theorem 8.4. *The abelian group homomorphism*

$$\mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^{\times} \right) \xrightarrow{L^{\bullet}} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}) \tag{8.1}$$

of Notation 8.13 is surjective, and descends to an isomorphism

$$\mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^{\times}/\{\pm 1\} \right) \xrightarrow{\cong} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}) \ .$$

Observation 8.5. For any $s \in [n]$, the functors

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}} \xrightarrow{\Phi^{\mathsf{C}_{p^s}}} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}} \xrightarrow{\mathsf{fgt}} \mathsf{Mod}_{\mathbb{Z}}$$

are symmetric monoidal, and therefore carry Picard elements to Picard elements. We use this to make some basic deductions regarding Picard elements of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$.

First of all, every Picard element of $\mathsf{Mod}_{\mathbb{Z}}$ is equivalent to $\Sigma^{\alpha}\mathbb{Z}$ for some $\alpha \in \mathbb{Z}$. Moreover, as a result of the equivalences $\mathsf{aut}_{\mathsf{Mod}_{\mathbb{Z}}}(\Sigma^{\alpha}\mathbb{Z}) \simeq \mathsf{aut}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}) \simeq \{\pm 1\}$ and the fact that p is odd, there are no nontrivial $\mathsf{C}_{p^{n-s}}$-actions on $\Sigma^{\alpha}\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}$ for any $\alpha \in \mathbb{Z}$. So, every Picard element of $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}$ is equivalent to $\Sigma^{\alpha}\mathbb{Z}$ for some $\alpha \in \mathbb{Z}$, where we simply write $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}$ for the trivial $\mathsf{C}_{p^{n-s}}$-action on $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}$ (as specified by Notation 7.1(6)).

So, given any Picard element $E \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}$, for each $s \in [n]$ there must exist an equivalence $\Phi^{\mathsf{C}_s}E \simeq \Sigma^{\alpha_s}\mathbb{Z}$ for some $\alpha_s \in \mathbb{Z}$. Choosing such equivalences, by Theorem 7.2 (and Observation 7.10(2)(a)), this Picard element is recorded by the data of a gluing diagram

$$\left(\begin{array}{ccccc} \Sigma^{\alpha_0}\mathbb{Z} & \Sigma^{\alpha_0+\alpha_1}\mathbb{Z} & \cdots & \cdots & \Sigma^{\alpha_0+\alpha_1+\cdots+\alpha_n}\mathbb{Z} \\ {\scriptstyle\swarrow} & {\scriptstyle\downarrow\gamma_1} {\scriptstyle\swarrow} & {\scriptstyle\downarrow\gamma_2} & & {\scriptstyle\swarrow} {\scriptstyle\downarrow\gamma_n} \\ \Sigma^{\alpha_0}\mathbb{Z}^{\mathsf{t}\mathsf{C}_p} & \Sigma^{\alpha_0+\alpha_1}\mathbb{Z}^{\mathsf{t}\mathsf{C}_p} & \cdots & \Sigma^{\alpha_0+\alpha_1+\cdots+\alpha_{n-1}}\mathbb{Z}^{\mathsf{t}\mathsf{C}_p} \end{array} \right) \ . \tag{8.2}$$

For all $1 \le s \le n$, we consider the sth gluing morphism of the gluing diagram (8.2) as an element

$$\gamma_s \in \pi_0 \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}} \left(\Sigma^{\alpha_0+\alpha_1+\cdots+\alpha_s}\mathbb{Z}, \Sigma^{\alpha_0+\alpha_1+\cdots+\alpha_{s-1}}\mathbb{Z}^{\mathsf{t}\mathsf{C}_p} \right)$$

$$\cong \pi_0 \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}\mathsf{C}_{p^{n-s}}}} \left(\Sigma^{\alpha_s}\mathbb{Z}, \mathbb{Z}^{\mathsf{t}\mathsf{C}_p} \right)$$

$$\cong \pi_0 \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}} \left(\Sigma^{\alpha_s}\mathbb{Z}, (\mathbb{Z}^{\mathsf{t}\mathsf{C}_p})^{\mathsf{h}\mathsf{C}_{p^{n-s}}} \right)$$

$$\cong \pi_{\alpha_s} \left((\mathbb{Z}^{\mathsf{t}\mathsf{C}_p})^{\mathsf{h}\mathsf{C}_{p^{n-s}}} \right) \ .$$

Notation 8.6. Given a pair of elements

$$\vec{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}^{\oplus(n+1)} \qquad \text{and} \qquad \vec{\gamma} = \left(\gamma_s \in \pi_{\alpha_s}((\mathbb{Z}^{\mathsf{t}\mathsf{C}_p})^{\mathsf{h}\mathsf{C}_{p^{n-s}}}) \right)_{1 \le s \le n} \ ,$$

we write

$$K^{(\vec{\alpha},\vec{\gamma})} \in \pi_0(\iota_0(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}\mathsf{C}_{p^n}}))$$

for the equivalence class of genuine C_{p^n}-$\mathbb{Z}$-module corresponding to the gluing diagram (8.2) via Theorem 7.2 (and Observation 7.10(2)(a)). Evidently, such pairs $(\vec{\alpha}, \vec{\gamma})$ may be equivalently considered as elements of the product

$$\left(\alpha_0 \in \mathbb{Z}, (\gamma_s \in \pi_{\alpha_s}((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}}))_{1 \leq s \leq n}\right) \in \mathbb{Z} \times \prod_{s=1}^{n} \pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}}$$

(where the subscript denotes the restriction to homogeneous elements), through which identification this construction assembles as a function

$$\mathbb{Z} \times \prod_{s=1}^{n} \pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}} \xrightarrow{K^\bullet} \pi_0(\iota_0(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}))$$

between sets.

Observation 8.7. By Observation 4.13(2), for each $1 \leq s \leq n$ the object $(\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}} \in \mathsf{Mod}_{\mathbb{Z}}$ admits a canonical lift to an object of $\mathsf{CAlg}(\mathsf{Mod}_{\mathbb{Z}})$, so that the graded abelian group $\pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})$ acquires the structure of a graded-commutative ring. Thereafter, the source of the function

$$\mathbb{Z} \times \prod_{s=1}^{n} \pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}} \xrightarrow{K^\bullet} \pi_0(\iota_0(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}))$$

obtained in Notation 8.6 naturally inherits the structure of a monoid. By Theorem 7.2 (and Observation 7.10(2)(b)), with respect to this structure the function $K^\bullet$ is a monoid homomorphism. We use these facts without further comment.

Observation 8.8. By Lemma A.35, for each $1 \leq s \leq n$ we have an isomorphism

$$\pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}}) \cong (\mathbb{Z}/p^{n-s+1})[(c_{n-s+1})^{\pm}]$$

of graded-commutative rings, where $|c_{n-s+1}| = -2$.

Notation 8.9. For each $1 \leq s \leq n$, we define the commutative submonoid

$$\mathbb{M}_s := \{(\alpha, \gamma) \in \mathbb{Z} \oplus \mathbb{Z}/p^{n-s+1} : \text{if } \alpha \text{ is odd then } \gamma = 0\} \subseteq (\mathbb{Z} \oplus \mathbb{Z}/p^{n-s+1}) := ((\mathbb{Z}, +) \oplus (\mathbb{Z}/p^{n-s+1}, \cdot))$$

of the indicated commutative monoid (namely the direct sum of $\mathbb{Z}$ (considered as a commutative monoid via addition) and $\mathbb{Z}/p^{n-s+1}$ (considered as a commutative monoid via multiplication)). Projection to the first summand gives the commutative monoid $\mathbb{M}_s$ the structure of a graded-commutative monoid. We also define the commutative monoid

$$\mathbb{M} := \mathbb{Z} \oplus \left(\bigoplus_{s=1}^{n} \mathbb{M}_s\right).$$

Via the injection

$$\mathbb{M} := \mathbb{Z} \oplus (\bigoplus_{s=1}^{n} \mathbb{M}_s) \subseteq \mathbb{Z} \oplus (\bigoplus_{s=1}^{n} \mathbb{Z} \oplus (\mathbb{Z}/p^{n-s+1})^{\times}) \cong \mathbb{Z}^{\oplus(n+1)} \oplus (\bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^{\times})$$

$$\cup \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup$$

$$(\alpha_0, (\alpha_1, \gamma_1), \ldots, (\alpha_n, \gamma_n)) \longmapsto ((\alpha_0, \ldots, \alpha_n), (\gamma_1, \ldots, \gamma_n))$$

we denote an element in $\mathbb{M}$ as a pair of vectors $(\vec{\alpha}, \vec{\gamma})$. We denote the identity element in this group as $(\vec{0}, \vec{1})$.

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Observation 8.10. For each $1 \leq s \leq n$, by Observation 8.8 we have an isomorphism

$$\mathbb{M}_s \xrightarrow{\ \cong\ } \pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}}$$
$$\Cup \qquad\qquad\qquad \Cup$$
$$(\alpha, \gamma) \longmapsto (c_{n-s+1})^{-\frac{\alpha}{2}}\gamma$$

of graded-commutative monoids, where we take the convention that $(c_{n-s+1})^{-\frac{\alpha}{2}} := 0$ whenever α is odd. Thereafter, we obtain a likewise isomorphism

$$\mathbb{M} \xrightarrow{\ \cong\ } \mathbb{Z} \times \prod_{s=1}^{n} \pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}}$$
$$\Cup \qquad\qquad\qquad\qquad \Cup$$
$$(\vec{\alpha}, \vec{\gamma}) \longmapsto \left(\alpha_0, (c_n)^{-\frac{\alpha_1}{2}}\gamma_1, \ldots, (c_1)^{-\frac{\alpha_n}{2}}\gamma_n\right)$$

of monoids. We employ both of these isomorphisms – and in particular the fact that all monoids under consideration are in fact commutative – without further comment.

Observation 8.11. For any $1 \leq s \leq n$, the inclusion

$$\mathbb{Z} \oplus (\mathbb{Z}/p^{n-s+1})^{\times} \lhook\joinrel\longrightarrow \mathbb{M}_s \xrightarrow{\ \cong\ } \pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}}$$
$$\Cup \qquad\qquad\qquad \Cup \qquad\qquad\qquad \Cup$$
$$(\beta, \gamma) \longmapsto (2\beta, \gamma) \longmapsto (c_{n-s+1})^{-\beta}\gamma$$

is that of the subgroup of invertible elements. Thereafter, the inclusion

$$\mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n}(\mathbb{Z}/p^{n-s+1})^{\times}\right) \lhook\joinrel\longrightarrow \mathbb{M} \xrightarrow{\ \cong\ } \mathbb{Z} \times \prod_{s=1}^{n}\pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}}$$
$$\Cup \qquad\qquad\qquad\qquad \Cup \qquad\qquad\qquad\qquad \Cup$$
$$(\vec{\beta}, \vec{\gamma}) := ((\beta_0, \ldots, \beta_n), (\gamma_1, \ldots, \gamma_n)) \longmapsto ((\beta_0, 2\beta_1, \ldots, 2\beta_n), (\gamma_1, \ldots, \gamma_n)) \longmapsto \left(\beta_0, (c_n)^{-\beta_1}\gamma_1, \ldots, (c_1)^{-\beta_n}\gamma_n\right)$$

is likewise that of the subgroup of invertible elements.

Notation 8.12. We define the abelian groups $\mathbb{K}$, $\tilde{\mathbb{P}}$, and $\mathbb{P}$ according to the commutative diagram

$$\mathbb{K} := 0^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n}\{\pm 1\}^{\times}\right) \xrightarrow{\ \cong\ } \left\{\left(0 \in \mathbb{Z}, \left(\pm 1 \in \pi_0((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})\right)\right)\right\}_{1 \leq s \leq n}$$

$$\tilde{\mathbb{P}} := \mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n}(\mathbb{Z}/p^{n-s+1})^{\times}\right) \lhook\joinrel\longrightarrow \mathbb{M} \xrightarrow{\ \cong\ } \mathbb{Z} \times \prod_{s=1}^{n}\pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})_{\mathsf{homog}} \qquad \text{[24]}$$

$$\mathbb{P} := \mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n}(\mathbb{Z}/p^{n-s+1})^{\times}/\{\pm 1\}\right)$$

in which the middle horizontal inclusion is that of the subgroup of invertible elements by Observation 8.11 and the left vertical composite is an exact sequence among abelian groups.

---

[24]Actually, $\tilde{\mathbb{P}}$ and $\mathbb{P}$ were already implicitly introduced in the statement of Theorem 8.4.

Notation 8.13. We write

$$\begin{array}{ccc} \tilde{\mathbb{P}} & \xrightarrow{\;\; L^{\bullet} := ((\vec{\beta},\vec{\gamma}) \longmapsto L^{(\vec{\beta},\vec{\gamma})}) \;\;} & \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}) \\ \downarrow & & \downarrow \\ \mathbb{M} & \xrightarrow[\;\; K^{\bullet} := ((\vec{\alpha},\vec{\gamma}) \longmapsto K^{(\vec{\alpha},\vec{\gamma})}) \;\;]{} & \pi_0(\iota_0(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})) \end{array} \qquad (8.3)$$

for the induced homomorphism on subgroups of invertible elements.

Lemma 8.14. *The kernel of the commutative monoid homomorphism*

$$\mathbb{M} \xrightarrow{\; K^{\bullet} \;} \pi_0(\iota_0(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}))$$

is the subgroup $\mathbb{K} \subseteq \mathbb{M}$.

Proof. First of all, it is clear that we have an equivalence

$$K^{(\vec{0},\vec{1})} \simeq \mathbb{1}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}}$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$. To prove the claim, we will show that for any $(\vec{\alpha}, \vec{\gamma}) \in \mathbb{M}$ there exists an equivalence

$$K^{(\vec{\alpha},\vec{\gamma})} \longrightarrow K^{(\vec{0},\vec{1})} \qquad (8.4)$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ if and only if

$$\alpha_s = 0 \in \mathbb{Z}$$

for all $0 \leq s \leq n$ and

$$\gamma_s = \pm 1 \in \pi_{\alpha_s}((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}}) = \pi_0((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}})$$

for all $1 \leq s \leq n$.

First of all, observe that for each $s \in [n]$ there exists an equivalence

$$\Phi^{\mathsf{C}_{p^s}} K^{(\vec{\alpha},\vec{\gamma})} \simeq \Sigma^{\alpha_0 + \cdots + \alpha_s} \mathbb{Z} \longrightarrow \mathbb{Z} \simeq \Phi^{\mathsf{C}_{p^s}} K^{(\vec{0},\vec{1})}$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$ if and only if $\alpha_0 + \cdots + \alpha_s = 0$. Hence, in order for there to exist an equivalence (8.4), it must indeed be the case that $\alpha_s = 0$ for all $0 \leq s \leq n$. So we assume this, and proceed.

Now, by Theorem 7.2 (and Observation 7.10(3)), we have an equivalence

$$\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}}(K^{(\vec{0},\vec{\gamma})}, K^{(\vec{0},\vec{1})}) \xrightarrow{\;\sim\;} \lim_{(i \to j) \in \mathsf{Zig}_n} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-j}}}}(\mathbb{Z}, \mathbb{Z}^{\mathsf{tC}_{p^{j-i}}})$$

in $\mathsf{Mod}_{\mathbb{Z}}$; more explicitly, the limit is of the diagram $\mathsf{Zig}_n \to \mathsf{Mod}_{\mathbb{Z}}$ given by

$$\left(\begin{array}{ccccc} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^n}}}(\mathbb{Z}, \mathbb{Z}) & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-1}}}}(\mathbb{Z}, \mathbb{Z}) & \cdots & \cdots & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}) \\ & \searrow^{\gamma_1} \quad \downarrow^1 & \searrow^{\gamma_2} \quad \downarrow^1 & & \searrow^{\gamma_n} \quad \downarrow^1 \\ & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-1}}}}(\mathbb{Z}, \mathbb{Z}^{\mathsf{tC}_p}) & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-2}}}}(\mathbb{Z}, \mathbb{Z}^{\mathsf{tC}_p}) & \cdots & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}^{\mathsf{tC}_p}) \end{array} \right),$$
$$(8.5)$$

where for all $1 \leq s \leq n$ we slightly abuse notation by writing

$$\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s+1}}}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\gamma_s} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}}(\mathbb{Z}, \mathbb{Z}^{\mathsf{tC}_p})$$

for the composite

$$\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s+1}}}}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{(-)^{\mathsf{tC}_p}} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}}(\mathbb{Z}^{\mathsf{tC}_p}, \mathbb{Z}^{\mathsf{tC}_p}) \xrightarrow{\gamma_s} \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}}(\mathbb{Z}, \mathbb{Z}^{\mathsf{tC}_p}) .$$

Evidently, we may rewrite the diagram (8.5) more simply as

$$
\left(
\begin{array}{ccccccc}
\mathbb{Z}^{\mathsf{hC}_{p^n}} & & \mathbb{Z}^{\mathsf{hC}_{p^{n-1}}} & & \cdots & & \mathbb{Z} \\
& \searrow_{\gamma_1} & \downarrow 1 & \searrow_{\gamma_2} & \downarrow 1 & \searrow_{\gamma_n} & \downarrow 1 \\
& & \left(\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^{n-1}}} & & \left(\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^{n-2}}} & \cdots & \mathbb{Z}^{\mathsf{tC}_p}
\end{array}
\right) . \qquad (8.6)
$$

Hence, we obtain the composite isomorphism

$$
\pi_0(\underline{\hom}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}}(K^{(\vec{0},\vec{\gamma})}, K^{(\vec{0},\vec{1})})) \xrightarrow{\cong} \pi_0(\lim_{\mathsf{Zig}_n}(8.6))
$$

$$
\xrightarrow{\cong} \lim_{\mathsf{Zig}_n}(\pi_0(8.6)) \qquad (8.7)
$$

$$
\cong \lim \left(
\begin{array}{ccccccc}
\mathbb{Z} & & \mathbb{Z} & & \cdots & & \mathbb{Z} \\
& \searrow_{\gamma_1} & \downarrow 1 & \searrow_{\gamma_2} & \downarrow 1 & \searrow_{\gamma_n} & \downarrow 1 \\
& & \mathbb{Z}/p^n & & \mathbb{Z}/p^{n-1} & \cdots & \mathbb{Z}/p
\end{array}
\right)
$$

$$
(8.8)
$$

among abelian groups, in which

- isomorphism (8.7) follows from the fact that all of the $\mathbb{Z}$-modules on the lower row of diagram (8.6) have vanishing π_1 by Observation 8.8, and

- isomorphism (8.8) follows from the evident isomorphisms $\pi_0(\mathbb{Z}^{\mathsf{hC}_{p^{n-s}}}) \cong \mathbb{Z}$ for all $s \in [n]$ as well as the isomorphisms $\pi_0((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{n-s}}}) \cong \mathbb{Z}/p^{n-s+1}$ for all $1 \le s \le n$ of Observation 8.8.

By Theorem 7.2 (and Observation 7.10(1)), the functor

$$
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}} \xrightarrow{\prod_{s \in [n]} \Phi^{\mathsf{C}_{p^s}}} \prod_{s \in [n]} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}
$$

is conservative. Hence, an element of $\pi_0(\underline{\hom}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}}(K^{(\vec{0},\vec{\gamma})}, K^{(\vec{0},\vec{1})}))$ is an isomorphism in $\mathsf{ho}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$ if and only if its image under the composite homomorphism

$$
\pi_0(\underline{\hom}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}}(K^{(\vec{0},\vec{\gamma})}, K^{(\vec{0},\vec{1})})) \xrightarrow{\cong} \pi_0(\lim_{\mathsf{Zig}_n}((8.6))) \longrightarrow \prod_{s \in [n]} \mathbb{Z}
$$

(in which the last morphism is the projection to the upper factors in the diagram appearing in isomorphism (8.8)) lies in the subset $\prod_{s \in [n]} \mathbb{Z}^{\times} \subseteq \prod_{s \in [n]} \mathbb{Z}$. From here, we see inductively that there exists an element of $\pi_0(\underline{\hom}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}}(K^{(\vec{0},\vec{\gamma})}, K^{(\vec{0},\vec{1})}))$ that is an isomorphism in $\mathsf{ho}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$ if and only if $\gamma_s = \pm 1$ for all $1 \le s \le n$. □

Proof of Theorem 8.4. By Observation 8.5, the image of the commutative monoid homomorphism $K^{\bullet}$ contains the Picard group. Moreover, by Lemma 8.14 its kernel is a subgroup of $\mathbb{M}$ (i.e. it only contains invertible elements). Together, these two facts imply that the abelian group homomorphism $L^{\bullet}$ is surjective (and in fact that the commutative square (8.3) is a pullback). Appealing again to Lemma 8.14 completes the proof. □

9. FROM VIRTUAL C_{p^n}-REPRESENTATIONS TO PICARD GENUINE C_{p^n}-$\mathbb{Z}$-MODULES

In this section, we describe the composite abelian group homomorphism

$$\mathsf{RO}(\mathsf{C}_{p^n}) \xrightarrow{V \longmapsto \mathbb{S}^V} \mathsf{Pic}(\mathsf{Sp}^{\mathbf{g}\mathsf{C}_{p^n}}) \xrightarrow{(-) \otimes \mathbb{Z}} \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathbf{g}\mathsf{C}_{p^n}})$$

in terms of the identification of the target $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathbf{g}\mathsf{C}_{p^n}})$ given by Theorem 8.4. More precisely, in Theorem 9.6 (which is proved at the end of the section, and which was originally stated as the second part of Theorem C) we describe in these terms its values on irreducible C_{p^n}-representations, which freely generate $\mathsf{RO}(\mathsf{C}_{p^n})$ (as an abelian group).

Definition 9.1. We use the term C_{p^n}-*representation* to mean a finite-dimensional real orthogonal C_{p^n}-representation (or equivalently a homomorphism $\mathsf{C}_{p^n} \to \mathsf{O}(d)$ for some $d \geq 0$). Isomorphism classes of C_{p^n}-representations form a commutative monoid under direct sum, whose group completion we denote by $\mathsf{RO}(\mathsf{C}_{p^n})$. We refer to the elements of $\mathsf{RO}(\mathsf{C}_{p^n})$ as *virtual representations*.

Notation 9.2. For any virtual C_{p^n}-representation $V \in \mathsf{RO}(\mathsf{C}_{p^n})$ we write

$$\mathbb{Z}^V := \mathbb{S}^V \otimes \mathbb{Z} \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathbf{g}\mathsf{C}_{p^n}})$$

for the corresponding Picard element of $\mathsf{Mod}_{\mathbb{Z}}^{\mathbf{g}\mathsf{C}_{p^n}}$. Moreover, for any representation $\mathsf{C}_{p^n} \xrightarrow{\rho} \mathsf{O}(d)$ we write

$$\mathbb{Z}^\rho := \mathbb{Z}^{(\mathbb{R}^d, \rho)} \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathbf{g}\mathsf{C}_{p^n}}) \ .$$

Notation 9.3. Recall the abelian group

$$\tilde{\mathbb{P}} := \mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^{\times} \right)$$

of Notation 8.12.

(1) We express elements of $\tilde{\mathbb{P}}$ as ordered pairs, according to the indicated direct sum decomposition.

(2) We write

$$\{\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_n\} \subseteq \mathbb{Z}^{\oplus(n+1)} \subseteq \tilde{\mathbb{P}}$$

for the standard basis of the torsionfree subgroup of $\tilde{\mathbb{P}}$.

(3) For any $1 \leq s \leq n$, we implicitly consider the inclusion

$$(\mathbb{Z}/p^{n-s+1})^{\times} \subseteq \left(\bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^{\times} \right) \subseteq \tilde{\mathbb{P}} \ ,$$

and in doing so, for $\gamma \in (\mathbb{Z}/p^{n-s+1})^{\times}$ we simply denote by γ the element $(1, \ldots, 1, \gamma, 1, \ldots, 1) \in \tilde{\mathbb{P}}$ in which the sth coordinate is γ and every other coordinate is 1.

(4) We write

$$\vec{1} \in \left(\bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^{\times} \right) \subseteq \tilde{\mathbb{P}}$$

for the identity element of the torsion subgroup of $\tilde{\mathbb{P}}$.

So for instance, given an element $\gamma \in (\mathbb{Z}/p^n)^\times$, we write

$$(3\vec{e}_0 - 5\vec{e}_2, \gamma) := ((3,0,-5,0,\ldots,0),(\gamma,1,\ldots,1)) \in \mathbb{Z}^{\oplus(n+1)} \oplus \left(\bigoplus_{s=1}^{n} (\mathbb{Z}/p^{n-s+1})^\times \right) =: \tilde{\mathbb{P}} \ .$$

Observation 9.4. Because p is odd, every C_{p^n}-representation is orientable, and moreover every irreducible C_{p^n}-representation has dimension 1 or 2. We use these facts without further comment.

Notation 9.5.

(1) We write

$$\mathsf{C}_{p^n} \xrightarrow{\rho_{\mathrm{triv}}} \mathsf{SO}(1)$$

for the unique 1-dimensional C_{p^n}-representation.

(2) For brevity, we use the identification

$$\mathsf{SO}(2) \cong \mathsf{U}(1) \ .$$

(3) For each $j \in \{1,\ldots,p^n-1\}$, we write

$$\mathsf{C}_{p^n} \xrightarrow{\rho_j} \mathsf{U}(1)$$

for the 2-dimensional C_{p^n}-representation characterized by the fact that $\rho_j(\sigma) = e^{2\pi i j/p^n}$, where $\sigma \in \mathsf{C}_{p^n}$ is the standard generator.

(4) For each $j \in \{1,\ldots,p^n-1\}$, we write $\nu(j) := \nu_p(j)$ for the p-adic valuation of j, and

$$\gamma(j) := \frac{j}{p^{\nu(j)}} \ .^{25}$$

We note that $0 \leq \nu(j) < n$, and we often implicitly consider $\gamma(j) \in (\mathbb{Z}/p^{n-\nu(j)})^\times$.

We prove the following result at the end of this subsection.

Theorem 9.6. *The Picard elements of* $\mathrm{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ *determined by the irreducible* C_{p^n}-*representations are given by the formulas*

$$\mathbb{Z}^{\rho_{\mathrm{triv}}} = L^{(\vec{e}_0,\vec{1})} \qquad and \qquad \mathbb{Z}^{\rho_j} = L^{(2\vec{e}_0-\vec{e}_{\nu(j)+1},\gamma(j))} \ .$$

Definition 9.7. We define the ∞-categories of ***naive*** C_{p^n}-***spectra*** and ***naive*** C_{p^n}-$\mathbb{Z}$-***modules*** to be

$$\mathsf{Sp}^{n\mathsf{C}_{p^n}} := \mathsf{Fun}(\mathcal{O}_{\mathsf{C}_{p^n}}^{\mathrm{op}}, \mathsf{Sp}) \qquad and \qquad \mathrm{Mod}_{\mathbb{Z}}^{n\mathsf{C}_{p^n}} := \mathsf{Fun}(\mathcal{O}_{\mathsf{C}_{p^n}}^{\mathrm{op}}, \mathrm{Mod}_{\mathbb{Z}}) \ .$$

Observation 9.8. We have equivalences

$$\mathsf{Sp}^{n\mathsf{C}_{p^n}} \simeq \mathcal{S}_*^{\mathsf{gC}_{p^n}} \otimes \mathsf{Sp} \qquad and \qquad \mathrm{Mod}_{\mathbb{Z}}^{n\mathsf{C}_{p^n}} \simeq \mathcal{S}_*^{\mathsf{gC}_{p^n}} \otimes \mathrm{Mod}_{\mathbb{Z}}$$

in Pr^L (in fact in $\mathsf{CAlg}(\mathsf{Pr}^L)$).

---

[25]In other words, the positive integers $\nu(j)$ and $\gamma(j)$ are characterized by the fact that $j = p^{\nu(j)} \cdot \gamma(j)$ where $\gamma(j)$ is coprime to p.

Notation 9.9. We define the functors $\Psi_{\mathbb{S}}$, $\Psi_{\mathbb{Z}}$, $\tilde{\mathsf{C}}(-)$, and $\tilde{\mathbb{Z}}\langle-\rangle$ as those appearing in the commutative diagram

$$
\begin{array}{c}
\xrightarrow{\quad\tilde{\mathsf{C}}(-)\quad} \\
\mathcal{S}^{\mathsf{gC}_{p^n}}_{*} \xrightarrow{\;\Sigma^{\infty}\;} \mathsf{Sp}^{\mathsf{nC}_{p^n}} \xrightarrow{\;(-)\otimes\mathbb{Z}\;} \mathsf{Mod}^{\mathsf{nC}_{p^n}}_{\mathbb{Z}} \\
\Sigma^{\infty}_{\mathsf{C}_{p^n}} \quad\Big\downarrow{\Psi_{\mathbb{S}}} \qquad\qquad \Big\downarrow{\Psi_{\mathbb{Z}}} \\
\mathsf{Sp}^{\mathsf{gC}_{p^n}} \xrightarrow{\;(-)\otimes\mathbb{Z}\;} \mathsf{Mod}^{\mathsf{gC}_{p^n}}_{\mathbb{Z}} \\
\xrightarrow{\quad\tilde{\mathbb{Z}}\langle-\rangle\quad}
\end{array}
$$

in Pr^L, in which the functors $\Psi_{\mathbb{S}}$ and $\Psi_{\mathbb{Z}}$ respectively arise from the adjunctions

$$
\mathsf{Pr}^L \underset{\mathsf{fgt}}{\overset{(-)\otimes\mathsf{Sp}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mod}_{\mathsf{Sp}}(\mathsf{Pr}^L) \qquad \text{and} \qquad \mathsf{Pr}^L \underset{\mathsf{fgt}}{\overset{(-)\otimes\mathsf{Mod}_{\mathbb{Z}}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mod}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathsf{Pr}^L)
$$

using Observation 9.8.

Notation 9.10.

(1) For any ∞-category $\mathcal{C}$, we depict an object $X \in \mathsf{Fun}(\mathcal{O}^{\mathsf{op}}_{\mathsf{C}_{p^n}}, \mathcal{C})$ by the diagram

$$
\begin{array}{ccccccccc}
X_n & \xrightarrow{\delta_n} & X_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots & \xrightarrow{\delta_2} & X_1 & \xrightarrow{\delta_1} & X_0 \\
\circlearrowleft & & \circlearrowleft & & & & \circlearrowleft & & \circlearrowleft \\
e & & \mathsf{C}_p & & & & \mathsf{C}_{p^{n-1}} & & \mathsf{C}_{p^n}
\end{array} , \tag{9.1}
$$

in which

- for every $s \in [n]$, we write $X_s := X((\mathsf{C}_{p^n}/\mathsf{C}_{p^s})^{\circ}) \in \mathcal{C}$,

- the curved arrows schematically depict actions of Weyl groups, and

- for every $1 \le s \le n$, the morphism δ_s denotes the value of X on the distinguished morphism

$$
(\mathsf{C}_{p^n}/\mathsf{C}_{p^s})^{\circ} \longrightarrow (\mathsf{C}_{p^n}/\mathsf{C}_{p^{s-1}})^{\circ} \tag{9.2}
$$

in $\mathcal{O}^{\mathsf{op}}_{\mathsf{C}_{p^n}}$ opposite to the evident quotient homomorphism among quotient groups of C_{p^n}.

(2) We take the convention that if the sth horizontal morphism (counting from the left) in a diagram such as (9.1) is unlabeled, then it corresponds to the identity morphism on underlying objects and moreover the C_{p^s}-action on its target is pulled back from the $\mathsf{C}_{p^{s-1}}$-action on its source via the quotient homomorphism.

(3) Given an object $X \in \mathsf{Fun}(\mathcal{O}^{\mathsf{op}}_{\mathsf{C}_{p^n}}, \mathcal{C})$ as depicted in diagram (9.1), for any $1 \le s \le n$ we denote by

$$
\begin{array}{ccc}
 & (X_{s-1})^{\mathsf{hC}_p} & \\
\overset{\tilde{\delta}_s}{\nearrow} & & \searrow \\
X_s & \xrightarrow{\quad\delta_s\quad} & X_{s-1}
\end{array}
$$

the canonical factorization (determined by the fact that the morphism (9.2) is C_p-equivariant with respect to the trivial C_p-action on its source), a morphism in $\mathcal{C}^{\mathsf{hC}_{p^{n-s}}} := \mathsf{Fun}(B\mathsf{C}_{p^{n-s}}, \mathcal{C})$.

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Observation 9.11. The functor

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{nC}_{p^n}} \xrightarrow{\Psi_{\mathbb{Z}}} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$$

carries the object (9.1) (with $\mathcal{C} = \mathsf{Mod}_{\mathbb{Z}}$) to the genuine C_{p^n}-$\mathbb{Z}$-module whose gluing diagram via Theorem 7.2 (and Observation 7.10(2)(a)) is

$$\left(\begin{array}{ccccc}
X_0 & X_1 & \cdots & \cdots & X_n \\
 & \downarrow{\scriptstyle\tilde{\delta}_1} & & & \downarrow{\scriptstyle\tilde{\delta}_n} \\
(X_0)^{\mathsf{hC}_p} & & \cdots & \cdots & (X_{n-1})^{\mathsf{hC}_p} \\
 & \downarrow{\scriptstyle\mathsf{Q}_{\mathsf{C}_p}(X_0)} & & & \downarrow{\scriptstyle\mathsf{Q}_{\mathsf{C}_p}(X_{n-1})} \\
(X_0)^{\mathsf{tC}_p} & (X_1)^{\mathsf{tC}_p} & \cdots & (X_{n-1})^{\mathsf{tC}_p}
\end{array}\right) .$$

Indeed, this follows from tom Dieck splitting along with the defining equivalence $\Psi_{\mathbb{Z}} \circ \tilde{\mathsf{C}}(-) \simeq \tilde{\mathbb{Z}}\langle -\rangle$.

Proof of Theorem 9.6. It is clear that

$$\mathbb{Z}^{\rho_{\mathsf{triv}}} := \mathbb{S}^{\rho_{\mathsf{triv}}} \otimes \mathbb{Z} = \Sigma\mathbb{Z} = L^{(\vec{e}_0, \bar{1})} \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}) .$$

So, we fix an element $j \in \{1, \ldots, p^n - 1\}$ and turn our attention to the corresponding Picard element

$$\mathbb{Z}^{\rho_j} \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}) .$$

For convenience, we introduce the following notation.

- We simply write $\rho := \rho_j$, $\nu := \nu(j)$, and $\gamma := \gamma(j)$. So by definition, $\mathbb{Z}^{\rho} := \mathbb{Z}^{\rho_j}$.

- We respectively write $S^{\rho} \in \mathbb{S}_*^{\mathsf{gC}_{p^n}}$ and $S(\rho) \in \mathbb{S}^{\mathsf{gC}_{p^n}}$ for the representation sphere and unit sphere of ρ.[26] So by definition, $\mathbb{Z}^{\rho} := \tilde{\mathbb{Z}}\langle S^{\rho}\rangle$.

- We define $\tau \in \mathsf{C}_{p^{n-\nu}}$ to be the unique element such that $\tau^{\gamma} = \sigma$.[27]

Now, we begin by observing the equivalence

$$S^{\rho} \simeq \Sigma S(\rho) \tag{9.3}$$

in $\mathbb{S}_*^{\mathsf{gC}_{p^n}}$,[28] as well as the equivalence

$$S(\rho) \simeq \mathsf{colim}\left(\mathsf{C}_{p^n}/\mathsf{C}_{p^{\nu}} \underset{\tau}{\overset{\mathsf{id}}{\rightrightarrows}} \mathsf{C}_{p^n}/\mathsf{C}_{p^{\nu}} \right) \tag{9.4}$$

---

[26]That is, S^{ρ} denotes the one-point compactification of $\mathbb{R}^2$ and $S(\rho)$ denotes its unit circle, both considered as genuine C_{p^n}-spaces (the former pointed).

[27]Explicitly, we set $\tau := \sigma^{\gamma^{-1}}$, where γ^{-1} denotes the multiplicative inverse of the unit γ in the commutative ring $\mathbb{Z}/p^{n-\nu}$.

[28]This follows from the fact that at the level of topological spaces with C_{p^n}-action, the representation sphere is the *unreduced* suspension of the unit sphere.

in $\mathcal{S}^{g \mathsf{C}_{p^n}}$.[29] We use these to give an explicit presentation of the object $\tilde{\mathsf{C}}(S^\rho) \in \mathsf{Mod}_{\mathbb{Z}}^{n\mathsf{C}_{p^n}}$. Namely, we define an object $\tilde{\mathsf{c}}(S^\rho) \in \mathsf{Fun}(\mathscr{O}_{\mathsf{C}_{p^n}}^{\mathsf{op}}, \mathsf{Ch}_{\mathbb{Z}})$ as follows. First of all, we define the object

$$\mathtt{Y} := \left(\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}[\mathsf{C}_{p^{n-\nu}}] \xrightarrow{1-\tau} \mathbb{Z}[\mathsf{C}_{p^{n-\nu}}] \xrightarrow{\sigma \mapsto 1} \underset{\sim}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \right) \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-\nu}}]} \simeq \mathsf{Fun}(\mathsf{BC}_{p^{n-\nu}}, \mathsf{Ch}_{\mathbb{Z}})$$

(where in degree 0 we endow $\mathbb{Z}$ with the trivial $\mathsf{C}_{p^{n-\nu}}$-action). Moreover, we simply write

$$\underset{\sim}{\mathbb{Z}} := \left(\cdots \longrightarrow 0 \longrightarrow \underset{\sim}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \right) \in \mathsf{Ch}_{\mathbb{Z}} \ .$$

Now, consider the morphism

$$\underset{\sim}{\mathbb{Z}} \xrightarrow{\ z\ } \mathtt{Y}$$

in $\mathsf{Ch}_{\mathbb{Z}}$ that is the identity in degree 0, which is evidently $\mathsf{C}_{p^{n-\nu}}$-equivariant with respect to the trivial $\mathsf{C}_{p^{n-\nu}}$-action on the source. We then define the functor

$$\tilde{\mathsf{c}}(S^\rho) : \mathscr{O}_{\mathsf{C}_{p^n}}^{\mathsf{op}} \longrightarrow (\mathsf{BC}_{p^{n-\nu}})^{\triangleleft} \xrightarrow{\ z\ } \mathsf{Ch}_{\mathbb{Z}} \ ,$$

where the first morphism is the right adjoint retraction onto the full subcategory on the objects

$$(\mathsf{C}_{p^n}/\mathsf{C}_{p^n})^\circ, (\mathsf{C}_{p^n}/\mathsf{C}_{p^\nu})^\circ \in \mathscr{O}_{\mathsf{C}_{p^n}}^{\mathsf{op}} \ .[30]$$

It is clear from the equivalences (9.3) and (9.4) that $\tilde{\mathsf{c}}(S^\rho)$ is indeed a presentation of $\tilde{\mathsf{C}}(S^\rho)$, i.e. that we have the assignment

$$\mathsf{Fun}(\mathscr{O}_{\mathsf{C}_{p^n}}^{\mathsf{op}}, \mathsf{Ch}_{\mathbb{Z}}) \longrightarrow \mathsf{Fun}(\mathscr{O}_{\mathsf{C}_{p^n}}^{\mathsf{op}}, \mathsf{Mod}_{\mathbb{Z}})$$
$$\cup\!\!\!| \qquad\qquad\qquad\qquad \cup\!\!\!| \qquad\qquad .$$
$$\tilde{\mathsf{c}}(S^\rho) \longmapsto\!\!\!\!\!\!\!\longmapsto \tilde{\mathsf{C}}(S^\rho)$$

In order to proceed, we introduce the following additional notation.

- We write $Y := \Pi_\infty(\mathtt{Y}) \in \mathsf{Mod}_{\mathbb{Z}}^{h\mathsf{C}_{p^{n-\nu}}}$ for the underlying object of $\mathtt{Y} \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-\nu}}]}$.

- We write $\mathbb{Z} \xrightarrow{z} Y$ for the underlying morphism in $\mathsf{Mod}_{\mathbb{Z}}$ of the morphism $\underset{\sim}{\mathbb{Z}} \xrightarrow{z} \mathtt{Y}$ in $\mathsf{Ch}_{\mathbb{Z}}$, which is likewise $\mathsf{C}_{p^{n-\nu}}$-equivariant with respect to the trivial $\mathsf{C}_{p^{n-\nu}}$-action on the source.[31]

---

[29]Indeed, the homomorphism $\mathsf{C}_{p^n} \xrightarrow{\rho} \mathsf{SO}(2) \cong \mathsf{U}(1)$ factors through the subgroup $\mu_{p^{n-\nu}} \subseteq \mathsf{U}(1)$ of $(p^{n-\nu})^{\text{th}}$ roots of unity according to the formula $\sigma \mapsto e^{2\pi i \gamma / p^{n-\nu}}$, and the unit circle has an evident $\mu_{p^{n-\nu}}$-CW complex structure that affords its description as the coequalizer

$$\mathsf{colim} \left(\mu_{p^{n-\nu}}/e \xrightarrow[e^{2\pi i / p^{n-\nu}}]{\mathsf{id}} \mu_{p^{n-\nu}}/e \right) \in \mathcal{S}^{g\mu_{p^{n-\nu}}} \ .$$

[30]So we may depict $\tilde{\mathsf{c}}(S^\rho) \in \mathsf{Fun}(\mathscr{O}_{\mathsf{C}_{p^n}}^{\mathsf{op}}, \mathsf{Ch}_{\mathbb{Z}})$ as the diagram

$$\underset{\sim}{\mathbb{Z}} \longrightarrow \underset{\sim}{\mathbb{Z}} \longrightarrow \cdots \longrightarrow \underset{\sim}{\mathbb{Z}} \xrightarrow{z} \mathtt{Y} \longrightarrow \cdots \longrightarrow \mathtt{Y} \quad .$$

with vertical markers $e, \ \mathsf{C}_p, \ \mathsf{C}_{p^{n-\nu-1}}, \ \mathsf{C}_{p^{n-\nu}}, \ \mathsf{C}_{p^n}$

[31]So, we may depict $\tilde{\mathsf{C}}(S^\rho) \in \mathsf{Mod}_{\mathbb{Z}}^{n\mathsf{C}_{p^n}}$ as the diagram

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{z} Y \longrightarrow \cdots \longrightarrow Y \quad .$$

with vertical markers $e, \ \mathsf{C}_p, \ \mathsf{C}_{p^{n-\nu-1}}, \ \mathsf{C}_{p^{n-\nu}}, \ \mathsf{C}_{p^n}$

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Now, applying Observation 9.11, we find that the genuine C_{p^n}-$\mathbb{Z}$-module

$$\mathbb{Z}^\rho := \tilde{\mathbb{Z}}\langle S^\rho \rangle \simeq \Psi_{\mathbb{Z}}(\tilde{\mathsf{C}}(S^\rho)) \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$$

has gluing diagram

$$\left(\begin{array}{ccccccccc} Y & Y & \cdots & Y & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} & \mathbb{Z} \\ & \downarrow{\scriptstyle\tilde{\mathsf{id}}} & & & \downarrow{\scriptstyle\tilde{z}} & \downarrow{\scriptstyle\tilde{\mathsf{id}}} & & & \downarrow{\scriptstyle\tilde{\mathsf{id}}} \\ & Y^{\mathsf{hC}_p} & \cdots & \cdots & Y^{\mathsf{hC}_p} & \mathbb{Z}^{\mathsf{hC}_p} & \cdots & \cdots & \mathbb{Z}^{\mathsf{hC}_p} \\ & \downarrow{\scriptstyle\mathsf{Q}_{\mathsf{C}_p}(Y)} & & & \downarrow{\scriptstyle\mathsf{Q}_{\mathsf{C}_p}(Y)} & \downarrow{\scriptstyle\mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z})} & & & \downarrow{\scriptstyle\mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z})} \\ Y^{\mathsf{tC}_p} & Y^{\mathsf{tC}_p} & \cdots & Y^{\mathsf{tC}_p} & \mathbb{Z}^{\mathsf{tC}_p} & \mathbb{Z}^{\mathsf{tC}_p} & \cdots & & \mathbb{Z}^{\mathsf{tC}_p} \end{array}\right).$$

Noting the equivalence $Y \simeq \Sigma^2\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-\nu}}}$, we find that

$$\mathbb{Z}^\rho = L^{(2\vec{e}_0 - \vec{e}_{\nu+1}, \mathsf{Q}_{\mathsf{C}_p}(Y)\circ\tilde{z})} \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}) .$$

So it remains to show that

$$\mathsf{Q}_{\mathsf{C}_p}(Y) \circ \tilde{z} = \gamma \in (\mathbb{Z}/p^{n-\nu})^\times .$$

For this, we define a quasi-isomorphism

$$\mathrm{Y} \xrightarrow{\approx} \mathrm{Y}'$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-\nu}}]}$ as

$$\begin{array}{ccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[\mathsf{C}_{p^{n-\nu}}] & \xrightarrow{1-\tau} & \mathbb{Z}[\mathsf{C}_{p^{n-\nu}}] & \xrightarrow{\sigma\mapsto 1} & \underset{\sim}{\mathbb{Z}} & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1+\tau+\cdots+\tau^{\gamma-1}} & & \downarrow{\scriptstyle\gamma N} & & \downarrow & \\ \cdots \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[\mathsf{C}_{p^{n-\nu}}] & \xrightarrow{1-\sigma} & \mathbb{Z}[\mathsf{C}_{p^{n-\nu}}] & \xrightarrow{N} & \underset{\sim\sim}{\mathbb{Z}[\mathsf{C}_{p^{n-\nu}}]} & \xrightarrow{1-\sigma} & \mathbb{Z}[\mathsf{C}_{p^{n-\nu}}] & \xrightarrow{N} \cdots \end{array}.$$

From this, we see that the composite morphism

$$\underset{\sim}{\mathbb{Z}} \xrightarrow{z} \mathrm{Y} \xrightarrow{\approx} \mathrm{Y}'$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-\nu}}]}$ (where the source is endowed with the trivial $\mathsf{C}_{p^{n-\nu}}$-action) selects the element

$$\gamma \cdot c_{n-\nu} \in \pi_{-2}(\mathbb{Z}^{\mathsf{hC}_{p^{n-\nu}}}) \cong \pi_0 \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Z}, \Sigma^2\mathbb{Z}^{\mathsf{hC}_{p^{n-\nu}}}) \cong \pi_0 \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-\nu}}}}(\mathbb{Z}, \Sigma^2\mathbb{Z}) ,$$

which proves the claim. $\qquad\square$

10. The constant Mackey functor at $\mathbb{Z}$

In this section, we describe the constant Mackey functor at $\mathbb{Z}$ (i.e. the coefficients for equivariant cohomology) in terms of our stratification; this is given as Proposition 10.4. We also record a resulting identification of $\mathsf{THH}(\mathbb{F}_p)$ as Corollary 10.8.

Notation 10.1. We write

$$\underline{\mathbb{Z}} \in \mathsf{Mack}_{\mathsf{C}_{p^n}}(\mathsf{Ab}) \subseteq \mathsf{Mack}_{\mathsf{C}_{p^n}}(\mathsf{Mod}_{\mathbb{Z}}) \simeq \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$$

for the constant Mackey functor at $\mathbb{Z}$, considered as a genuine C_{p^n}-$\mathbb{Z}$-module (using Observation 1.9): its categorical fixedpoints for any subgroup $\mathsf{C}_{p^s} \leq \mathsf{C}_{p^n}$ are

$$\underline{\mathbb{Z}}^{\mathsf{C}_{p^s}} := \mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$$

(i.e. the abelian group $\mathbb{Z} \in \mathsf{Ab} \subseteq \mathsf{Mod}_{\mathbb{Z}}$ equipped with the trivial $\mathsf{C}_{p^{n-s}}$-action), and for any morphism

$$\mathsf{C}_{p^n}/\mathsf{C}_{p^s} \longrightarrow \mathsf{C}_{p^n}/\mathsf{C}_{p^t}$$

in $\mathscr{O}_{\mathsf{C}_{p^n}}$ its corresponding restriction and transfer maps are respectively

$$\underline{\mathbb{Z}}^{\mathsf{C}_{p^t}} := \mathbb{Z} \xrightarrow{1} \mathbb{Z} =: \underline{\mathbb{Z}}^{\mathsf{C}_{p^s}} \qquad \text{and} \qquad \underline{\mathbb{Z}}^{\mathsf{C}_{p^s}} := \mathbb{Z} \xrightarrow{p^{t-s}} \mathbb{Z} =: \underline{\mathbb{Z}}^{\mathsf{C}_{p^t}}.^{[32]}$$

For convenience, more generally for any $i \geq 0$ we also simply write

$$\underline{\mathbb{Z}} \in \mathsf{Mack}_{\mathsf{C}_{p^i}}(\mathsf{Ab}) \subseteq \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^i}}$$

for the constant Mackey functor at $\mathbb{Z}$.

Observation 10.2. For every $s \in [n]$, there is an evident identification

$$
\begin{array}{ccc}
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}} & \xrightarrow{(-)^{\mathsf{C}_{p^s}}} & \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^{n-s}}} \\
\cup & & \cup \\
\underline{\mathbb{Z}} & \longmapsto & \underline{\mathbb{Z}}
\end{array}
$$
.

We use this fact without further comment.

Notation 10.3. Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. We write $\mathcal{C} \xrightarrow{\tau_{\geq 0}} \mathcal{C}$ for the connective cover functor, and for any object $E \in \mathcal{C}$ we write

$$\tau_{\geq 0} E \xrightarrow{\varepsilon_{\geq 0}(E)} E$$

for the canonical morphism to it from its connective cover. For simplicity, we write $\varepsilon_{\geq 0} := \varepsilon_{\geq 0}(E)$ or even $\varepsilon := \varepsilon_{\geq 0}$ when the meaning is clear from context.

Proposition 10.4. *Via the geometric stratification of* $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ *of Theorem 7.2 (and Observation 7.10(2)(a)), the object* $\underline{\mathbb{Z}} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ *is recorded by the data of the gluing diagram*

Observation 10.5. Fix any $i \geq 1$. Consider the unit morphism

$$\underline{\mathbb{Z}} \longrightarrow \beta U \underline{\mathbb{Z}} \simeq \beta \mathbb{Z} \tag{10.1}$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^i}}$. Taking categorical C_p-fixedpoints of the morphism (10.1), we obtain the morphism

$$\underline{\mathbb{Z}}^{\mathsf{C}_p} := \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}^{\mathsf{hC}_p} \simeq (\beta \mathbb{Z})^{\mathsf{C}_p}$$

---

[32] The factor p^{t-s} arises as the index $|\mathsf{C}_{p^t} : \mathsf{C}_{p^r}|$.

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{i-1}}}$. Hence, on isotropy separation sequences the morphism (10.1) determines a commutative diagram

$$
\begin{array}{ccc}
\underline{\mathbb{Z}}^{\mathsf{C}_p} & \longrightarrow & \Phi^{\mathsf{C}_p}\underline{\mathbb{Z}} \\[4pt]
\| \| & & \wr| \\[4pt]
\mathbb{Z}_{\mathsf{hC}_p} \longrightarrow \mathbb{Z} & \longrightarrow & \tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p} \\[4pt]
\| & \downarrow{\varepsilon} & \downarrow{\varepsilon} \\[4pt]
\mathbb{Z}_{\mathsf{hC}_p} \xrightarrow{\mathsf{Nm}_{\mathsf{C}_p}(\mathbb{Z})} \mathbb{Z}^{\mathsf{hC}_p} \xrightarrow{\mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z})} \mathbb{Z}^{\mathsf{tC}_p} \\[4pt]
\wr| & & \wr| \\[4pt]
(\beta\mathbb{Z})^{\mathsf{C}_p} & \longrightarrow & \Phi^{\mathsf{C}_p}\beta\mathbb{Z}
\end{array}
\tag{10.2}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{i-1}}}$ (and in particular the equivalence $\Phi^{\mathsf{C}_p}\underline{\mathbb{Z}} \simeq \tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}$), in which both horizontal composites are cofiber sequences and hence the middle right square is a pullback.

Notation 10.6. Fix any $i \geq 1$. We write

$$
\mathbb{Z}_{\mathsf{hC}_p} \xrightarrow{\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})} \mathbb{Z} \xrightarrow{\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})} \tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}
$$

for the morphisms in the upper horizontal composite in diagram (10.2) in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{i-1}}}$.

Observation 10.7. Fix any $i \geq 2$. By Observations 10.5 and 7.5, applying the C_p-Tate construction to the morphism

$$
\mathbb{Z} \xrightarrow{\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})} \tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{i-1}}}$ yields an equivalence

$$
\mathbb{Z}^{\mathsf{tC}_p} \xrightarrow[\sim]{\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{tC}_p}} (\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{tC}_p}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{i-2}}}$.

Proof of Proposition 10.4. We give the proof assuming that $n \geq 3$, proving the cases $0 \leq n \leq 2$ implicitly along the way. It is clear that the gluing diagram of $\underline{\mathbb{Z}} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}}$ restricts as asserted over $0 \in [n]$. The fact that it restricts as asserted over $1 \in [n]$ follows from Observation 10.5 (in the case that $i = n$). Thereafter, the fact that it restricts as asserted over $2 \in [n]$ follows from the commutative diagram

$$
\begin{array}{ccc}
\Phi^{\mathsf{C}_p}\underline{\mathbb{Z}} =: \Phi^{\mathsf{C}_p}(\underline{\mathbb{Z}}^{\mathsf{C}_p}) & \xrightarrow{\sim} & \Phi^{\mathsf{C}_p}(\Phi_{\mathsf{g}}^{\mathsf{C}_p}\underline{\mathbb{Z}}) \\[6pt]
\downarrow & & \downarrow \\[6pt]
\mathbb{Z}^{\mathsf{tC}_p} =: (\underline{\mathbb{Z}}^{\mathsf{C}_p})^{\mathsf{tC}_p} & \xrightarrow{\sim} & (\Phi^{\mathsf{C}_p}\underline{\mathbb{Z}})^{\mathsf{tC}_p}
\end{array}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-2}}}$, in which the upper horizontal morphism is an equivalence by Lemma 7.3 and the lower horizontal morphism is an equivalence by Observations 10.5 and 10.7 (in the case that $i = n$). Thereafter, the fact that it restricts as asserted over $s \in [n]$ for any $s \geq 3$ follows from the commutative diagram

$$
\begin{array}{ccc}
\Phi^{\mathsf{C}_{p^2}}(\underline{\mathbb{Z}}) =: \Phi^{\mathsf{C}_p}(\Phi_{\mathsf{g}}^{\mathsf{C}_p}(\underline{\mathbb{Z}}^{\mathsf{C}_{p^{s-2}}})) & \xrightarrow{\sim} & \Phi^{\mathsf{C}_{p^s}}\underline{\mathbb{Z}} \\[6pt]
\downarrow & & \downarrow \\[6pt]
(\Phi^{\mathsf{C}_p}\underline{\mathbb{Z}})^{\mathsf{tC}_p} =: (\Phi^{\mathsf{C}_p}(\underline{\mathbb{Z}}^{\mathsf{C}_{p^{s-2}}}))^{\mathsf{tC}_p} & \xrightarrow{\sim} & (\Phi^{\mathsf{C}_{p^{s-1}}}\underline{\mathbb{Z}})^{\mathsf{tC}_p}
\end{array}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$, in which both horizontal morphisms are equivalences by Lemma 7.3. $\qquad\square$

Corollary 10.8. *Considering $\underline{\mathbb{Z}}$ as an object of the ∞-category*

$$\mathsf{Sp}^{\mathsf{gC}_{p^\infty}} := \lim\left(\cdots \xrightarrow{\mathsf{Res}^{\mathsf{C}_{p^3}}_{\mathsf{C}_{p^2}}} \mathsf{Sp}^{\mathsf{gC}_{p^2}} \xrightarrow{\mathsf{Res}^{\mathsf{C}_{p^2}}_{\mathsf{C}_p}} \mathsf{Sp}^{\mathsf{gC}_p} \xrightarrow{\mathsf{Res}^{\mathsf{C}_p}_e} \mathsf{Sp} \right)$$

in the evident way, there is an equivalence

$$\Phi^{\mathsf{C}_p}(\underline{\mathbb{Z}}) \simeq \mathsf{THH}(\mathbb{F}_p)$$

in $\mathsf{Sp}^{\mathsf{gC}_{p^\infty}}$.

Proof. This follows by combining Proposition 10.4 with [NS18, Corollary IV.4.16 et seq.]. $\qquad\square$

Remark 10.9. In fact, the object $\underline{\mathbb{Z}} \in \mathsf{Sp}^{\mathsf{gC}_{p^\infty}}$ naturally carries the structure of a p-typical topological Cartier module in the sense of [AN]; thereafter, Corollary 10.8 gives an equivalence of p-cyclotomic spectra. We would be very interested in a conceptual explanation of this equivalence, which would give a conceptual explanation for Bökstedt's computation [Bök85] of $\pi_*\mathsf{THH}(\mathbb{F}_p)$.

11. The $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$-graded cohomology of a point

In this section, we establish our explicit chain-level description of equivariant cohomology (Theorem D) as Theorem 11.5. We begin by recalling the notation established in §0.2 as Notation 11.2.

Remark 11.1. In this section, we freely use the material of §A. We refer the reader to Notations A.22 and A.37 for Lemmas 11.7 and 11.8, and additionally to Notations A.25, A.29, and A.32 for Lemma 11.9. Furthermore, we refer the reader to Remark A.18 for some mnemonics regarding these various notations.

Notation 11.2. Given a finite group G, a genuine G-space $X \in \mathcal{S}^{\mathsf{g}G}$, a subgroup $H \leq G$, and a Picard element $L \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G})$, we write

$$\mathsf{C}_G^L(X)(H) := \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{g}G}}(\Sigma_G^\infty(X \times G/H)_+ \otimes \mathbb{Z}, L \otimes \underline{\mathbb{Z}}) \in \mathsf{Mod}_{\mathbb{Z}} \ .$$

This assembles as a $\mathsf{Mod}_{\mathbb{Z}}$-valued Mackey functor for G, i.e. an additive functor

$$\mathsf{Burn}_G \xrightarrow{\mathsf{C}_G^L(X)} \mathsf{Mod}_{\mathbb{Z}}$$

carrying $G/H \in \mathsf{Burn}_G$ to $\mathsf{C}_G^L(X)(H) \in \mathsf{Mod}_{\mathbb{Z}}$. Thereafter, for any integer $i \in \mathbb{Z}$ we write

$$\mathsf{H}_G^{i+L}(X)(H) := \pi_{-i}(\mathsf{C}_G^L(X)(H)) \in \mathsf{Ab} \ .$$

This assembles as an Ab-valued Mackey functor for G, i.e. an additive functor

$$\mathsf{Burn}_G \xrightarrow{\mathsf{H}_G^{i+L}(X)} \mathsf{Ab}$$

carrying $G/H \in \mathsf{Burn}_G$ to $\mathsf{H}_G^{i+L}(X)(H) \in \mathsf{Ab}$.

Observation 11.3. Let $\mathcal{A}$ be an additive category. Then, $\mathcal{A}$-valued Mackey functors for G are equivalent data to their values on G/H along with inclusion and transfer morphisms satisfying a double coset formula. We use this fact without further comment.

Local Notation 11.4. We fix a Picard element $L \in \mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$, as well as a presentation $L = L^{(\vec{\beta}, \vec{\gamma})}$ for some $(\vec{\beta}, \vec{\gamma}) \in \tilde{\mathbb{P}}$ as afforded by Theorem 8.4 (see Notations 8.12, 8.13, and 9.3). We furthermore fix arbitrary lifts $\tilde{\gamma}_s \in \mathbb{Z}$ of the elements $\gamma_s \in (\mathbb{Z}/p^{n-s+1})^\times$.

Theorem 11.5. *The* $\mathsf{ho}(\mathsf{Mod}_{\mathbb{Z}})$-*valued Mackey functor*

$$\mathsf{Burn}_{\mathsf{C}_{p^n}} \xrightarrow{\mathsf{C}^L_{\mathsf{C}_{p^n}}(\mathsf{pt})} \mathsf{Mod}_{\mathbb{Z}} \longrightarrow \mathsf{ho}(\mathsf{Mod}_{\mathbb{Z}})$$

admits an explicit chain-level description as follows: its values, restriction maps, and transfer maps are respectively as described in Lemmas 11.7, 11.8, and 11.9.

Proof. This follows from Lemmas 11.7, 11.8, and 11.9. $\qquad\square$

Notation 11.6. For any $1 \le i \le n$, we write

$$\beta_{\le i} := \beta_0 + 2(\beta_1 + \cdots + \beta_i) \ ,$$

for simplicity.

Lemma 11.7. *For any $0 \le a \le n$, diagram (7.18) in* $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ *in the case that $E = L^{(\vec{\beta}, \vec{\gamma})} \otimes \mathbb{Z}$ is presented by the diagram*

$$(11.1)$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-a}}]}$, *in which for all $1 \le s \le a$ the sth vertical morphism is the composite*

$$\Sigma^{\beta_{\le s}} \mathsf{C}_{n-a}^{a-s} \xrightarrow{\mathsf{e}_{n-a}^{a-s}} \Sigma^{\beta_{\le s}} \mathsf{T}_{n-a}^{a-s} \xrightarrow{\tilde{\gamma}_s(\mathsf{c}_{n-a}^{a-s})^{\beta_s}} \Sigma^{\beta_{\le(s-1)}} \mathsf{T}_{n-a}^{a-s} \ . \qquad (11.2)$$

In particular, by Proposition 7.19(1) the object

$$\mathsf{C}^L_{\mathsf{C}_{p^n}}(\mathsf{pt})(\mathsf{C}_{p^a}) := (L^{(\vec{\beta}, \vec{\gamma})} \otimes \underline{\mathbb{Z}})^{\mathsf{C}_{p^a}} \in \mathsf{ho}(\mathsf{Mod}_{\mathbb{Z}})$$

is the homotopy limit of the diagram (11.1).

Proof. The fact that the constituents and the diagonal morphisms of diagram (11.1) in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-a}}]}$ present those of diagram (7.18) in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ follows from Proposition 10.4 and Lemma A.24. So, it remains to show that for any $1 \le s \le a$ the morphism (11.2) in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-a}}]}$ is a presentation of the morphism

$$\Sigma^{\beta_{\le(s-1)}} \left(\Sigma^{2\beta_s} \mathbb{Z} \otimes \tau_{\ge 0} \mathbb{Z}^{\mathsf{tC}_p} \xrightarrow{\gamma_s c_s^{\beta_s} \otimes \varepsilon} \mathbb{Z}^{\mathsf{tC}_p} \otimes \mathbb{Z}^{\mathsf{tC}_p} \xrightarrow{\mu} \mathbb{Z}^{\mathsf{tC}_p} \right)^{\mathsf{hC}_{a-s}} \qquad (11.3)$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$, in which the parenthesized morphism lies in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$ and μ denotes the multiplication morphism resulting from Observation 4.13(2). For this, observe the commutative diagram

$$
\begin{array}{ccc}
\Sigma^{2\beta_s}\mathbb{Z}\otimes\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p} & \xrightarrow{\ \gamma_s c_s^{\beta_s}\otimes\varepsilon\ } & \mathbb{Z}^{\mathsf{tC}_p}\otimes\mathbb{Z}^{\mathsf{tC}_p}\xrightarrow{\ \mu\ }\mathbb{Z}^{\mathsf{tC}_p}
\end{array}
\tag{11.4}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-s}}}$ (in which we use Notation A.36 for the dashed morphism). It follows from Lemmas A.24 and A.39 that the lower composite in diagram (11.4) is presented by the morphism

$$
\Sigma^{2\beta_s}\mathrm{C}_{n-s}^0 \xrightarrow{\ \tilde{\gamma}_s e_{n-s}^0\ } \Sigma^{2\beta_s}\mathrm{T}_{n-s}^0 \xrightarrow{\ (c_{n-s}^0)^{\beta_s}\ } \mathrm{T}_{n-s}^0
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-s}}]}$. Hence, it follows from Lemma A.28 and Observations A.27, A.38, A.9(2)(a), and A.14 that the morphism (11.3) in $\mathsf{Mod}^{\mathsf{hC}_{p^{n-a}}}$ is indeed presented by the morphism (11.2) in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-a}}]}$. $\square$

Lemma 11.8. *For any $0 \leq a < n$, the morphism of diagrams (7.20) in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ in the case that $E = L^{(\vec{\beta},\vec{\gamma})} \otimes \underline{\mathbb{Z}}$ is presented by the morphism of diagrams*

$$\tag{11.5}$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-a}}]}$, in which the nontrivial non-vertical maps are as in diagram (11.1) and we implicitly apply Observation A.27. In particular, by Lemma 11.7 and Proposition 7.19(2), the inclusion morphism

$$
\mathsf{C}_{\mathsf{C}_{p^n}}^L(\mathsf{pt})(\mathsf{C}_{p^{a+1}}) \xrightarrow{\ \mathsf{inc}\ } \mathsf{C}_{\mathsf{C}_{p^n}}^L(\mathsf{pt})(\mathsf{C}_{p^a})
$$

in $\mathsf{Mod}_{\mathbb{Z}}$ is the induced map on homotopy limits.

Proof. Note that all terms in the lower zigzag of diagram (11.5) are adapted to homotopy C_p-fixedpoints by Lemma A.28 and Observation A.9(2)(a). Hence, the claim follows from Observation A.17. $\square$

Lemma 11.9. *For any $0 \le a < n$, the morphism of diagrams (7.21) in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ in the case that $E = L^{(\vec{\beta},\vec{\gamma})} \otimes \underline{\mathbb{Z}}$ is presented by the homotopy-coherent morphism of diagrams*

$$(11.6)$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-a}}]}$, in which

- *in the lower zigzag we replace C_{n-a}^0 with $\widetilde{\mathsf{C}}_{n-a}$ via the quasi-isomorphism k_{n-a} of Observation A.31(1), so that the nontrivial non-vertical morphism out of it is the composite*

$$\Sigma^{\beta_{\le a}}\widetilde{\mathsf{C}}_{n-a} \xrightarrow[\approx]{\mathsf{k}_{n-a}} \Sigma^{\beta_{\le a}}\mathsf{C}_{n-a}^0 \xrightarrow{\mathsf{e}_{n-a}^0} \Sigma^{\beta_{\le a}}\mathsf{T}_{n-a}^0 \xrightarrow{\tilde{\gamma}_a(\mathsf{c}_{n-a}^0)^{\beta_a}} \Sigma^{\beta_{\le (a-1)}}\mathsf{T}_{n-a}^0 \ ,$$

- *the remaining nontrivial non-vertical morphisms are as in diagram (11.1), and*

- *we implicitly apply Observation A.27.*

In particular, by Lemma 11.7 and Proposition 7.19(3), the transfer morphism

$$\mathsf{C}_{\mathsf{C}_{p^n}}^L(\mathsf{pt})(\mathsf{C}_{p^a}) \xrightarrow{\mathsf{trf}} \mathsf{C}_{\mathsf{C}_{p^n}}^L(\mathsf{pt})(\mathsf{C}_{p^{a+1}})$$

in $\mathsf{Mod}_{\mathbb{Z}}$ is the induced map on homotopy limits.

Proof. Note that all terms in the lower zigzag of diagram (11.6) are adapted to homotopy C_p-fixedpoints by Lemma A.28 and Observation A.9(2)(a). Hence, the claim follows from Observation A.17 and Lemma A.33. $\qquad\square$

Appendix A. Some homological algebra

In this appendix, we collect the various results in homological algebra that underlie the work carried out in the main body of the paper – primarily our cohomology computation of the $\mathsf{Pic}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{gC}_{p^n}})$-graded cohomology of a point (Theorem D). It is organized as follows.

§A.1: We lay out our notations and conventions regarding homological algebra.

§A.2: We study the notion of *adaptedness* of chain complexes with respect to various homotopical operations (a close analog of co/fibrancy in a model category).

§A.3: We introduce most of the chain complexes and chain maps that participate in our computation of equivariant cohomology, and prove that they present the correct homotopical data.

§A.4: We study the chain complexes introduced in §A.3 in more depth, towards establishing chain-level data that presents the inclusion and transfer maps on equivariant cohomology.

§A.5: We compute a certain Tate cohomology ring, and we prove a related result identifying a chain-level automorphism as a presentation of a resulting homotopical automorphism.

A.1. **Basic notation and conventions.** In this subsection, we establish the basic notation and conventions that we use for chain complexes in this paper. (These are only used in §§A and 11.)

Warning A.1. In this section, we discuss chain complexes of discrete modules over discrete rings. We simply use the word "module" to refer both to discrete modules and to module spectra; our meaning will always be clear from context.

Notation A.2. Fix an associative ring R.

(1) We write Ch_R for the category of chain complexes of R-modules.

(2) We freely identify elements of R with endomorphisms of R-modules.

(3) In depicting a chain complex, we indicate its degree-0 term with a squiggled underline.

(4) We generally denote point-set objects (e.g. chain complexes and morphisms between them) using typewriter text font.

(5) For typographical reasons, we depict chain complexes horizontally and morphisms of chain complexes vertically.

(6) When indexing a $\mathbb{Z}$-indexed family of morphisms (e.g. the differentials in a chain complex, the constituent morphisms in a chain map, or the constituent morphisms in a chain homotopy), we always number a morphism according to the degree of its *source*. So for example, a complex $\mathtt{M} \in \mathsf{Ch}_R$ may be depicted as a diagram

$$\cdots \xrightarrow{\partial_2^{\mathtt{M}}} \mathtt{M}_1 \xrightarrow{\partial_1^{\mathtt{M}}} \underset{\sim}{\mathtt{M}_0} \xrightarrow{\partial_0^{\mathtt{M}}} \mathtt{M}_{-1} \xrightarrow{\partial_{-1}^{\mathtt{M}}} \cdots .$$

(7) Given a morphism $\mathtt{M} \xrightarrow{\mathtt{f}} \mathtt{N}$ in Ch_R, we take the convention that its cone is the complex $\mathsf{cone}(\mathtt{f}) \in \mathsf{Ch}_R$ such that

$$\mathsf{cone}(\mathtt{f})_n := \begin{pmatrix} \mathtt{M}_{n-1} \\ \oplus \\ \mathtt{N}_n \end{pmatrix} \qquad \text{and} \qquad \partial_n^{\mathsf{cone}(\mathtt{f})} := \begin{pmatrix} \partial_{n-1}^{\mathtt{M}} & 0 \\ (-1)^{n-1}\mathtt{f}_{n-1} & \partial_n^{\mathtt{N}} \end{pmatrix}.$$

(8) We write $\mathsf{Ch}_R \xrightarrow{\Sigma} \mathsf{Ch}_R$ for the autoequivalence given by shifting (even though it is not literally the suspension functor on the category Ch_R).

(9) We do not notationally distinguish between a chain map and its shifts, since (given the source and target of the shifted morphism) the meaning is unambiguous.

Remark A.3. Note that parts (7) and (8) of Notation A.2 are consistent: there is an evident isomorphism $\Sigma \cong \mathsf{cone}(\mathsf{id}_R \to 0)$ in $\mathsf{Fun}(\mathsf{Ch}_R, \mathsf{Ch}_R)$.

Notation A.4. Given a dg-algebra $R \in \mathsf{Alg}(\mathsf{Ch}_\mathbb{Z})$, we write $\mathsf{Ch}_R := \mathsf{LMod}_R(\mathsf{Ch}_\mathbb{Z})$ for the category of (left) dg-modules over it.

Observation A.5. By [DS07, Theorem 1.4], for any dg-algebra $R \in \mathsf{Alg}(\mathsf{Ch}_\mathbb{Z})$, freely inverting the quasi-isomorphisms in the category Ch_R yields the ∞-category $\mathsf{Mod}_{\Pi_\infty(R)} := \mathsf{Mod}_{\Pi_\infty(R)}(\mathsf{Mod}_\mathbb{Z})$. In particular, for any finite group G, freely inverting the quasi-isomorphisms in the category $\mathsf{Ch}_{\mathbb{Z}[G]}$ yields the ∞-category $\mathsf{Mod}_{\mathbb{Z}[G]} \simeq \mathsf{Mod}_\mathbb{Z}^{hG}$. We use these facts without further comment.

Notation A.6. For any dg-algebra $R \in \mathsf{Alg}(\mathsf{Ch}_\mathbb{Z})$, we write $\mathsf{Ch}_R \xrightarrow{\Pi_\infty} \mathsf{Mod}_{\Pi_\infty(R)}$ for the localization functor.

A.2. **Adaptedness to homotopy fixedpoints and homotopy orbits.** In this subsection, we study the notion of *adaptedness* of chain complexes to homotopy fixedpoints and homotopy orbits. The primary output is Observation A.17, which establishes that for a chain complex that is adapted to homotopy fixedpoints, its point-set inclusion and transfer morphisms are presentations of its homotopical inclusion and transfer morphisms.

Local Notation A.7. In this subsection, we fix a finite group G and a subgroup $H \leq G$.

Definition A.8. Observe the canonical natural transformation

$$
\begin{array}{ccc}
\mathsf{Ch}_{\mathbb{Z}[G]} & \xrightarrow{\ (-)^G\ } & \mathsf{Ch}_\mathbb{Z} \\
{\scriptstyle \Pi_\infty}\downarrow & \swarrow & \downarrow{\scriptstyle \Pi_\infty} \\
\mathsf{Mod}_\mathbb{Z}^{hG} & \xrightarrow{\ (-)^{hG}\ } & \mathsf{Mod}_\mathbb{Z}
\end{array} \ .
$$

We say that a chain complex $M \in \mathsf{Ch}_{\mathbb{Z}[G]}$ is ***adapted to homotopy G-fixedpoints*** if the morphism

$$ \Pi_\infty(M^G) \longrightarrow \Pi_\infty(M)^{hG} $$

is an equivalence. More generally, we say that $M \in \mathsf{Ch}_{\mathbb{Z}[G]}$ is adapted to homotopy H-fixedpoints if its image under the forgetful functor $\mathsf{Ch}_{\mathbb{Z}[G]} \xrightarrow{\mathsf{fgt}} \mathsf{Ch}_{\mathbb{Z}[H]}$ is so.

Observation A.9.

(1) A free $\mathbb{Z}[G]$-module concentrated in degree zero is adapted to homotopy G-fixedpoints.

(2) Complexes that are adapted to homotopy G-fixedpoints are stable under taking

   (a) shifts,

   (b) cones, and

   (c) homotopy limits.

(3) Combining parts (1) and (2), it follows that a bounded-above levelwise-free complex of $\mathbb{Z}[G]$-modules is adapted to homotopy G-fixedpoints. Indeed, this follows by examining its presentation as a homotopy limit of its truncations from below.

Definition A.10. Observe the canonical natural transformation

$$
\begin{array}{ccc}
\mathsf{Ch}_{\mathbb{Z}[G]} & \xrightarrow{\ (-)_G\ } & \mathsf{Ch}_{\mathbb{Z}} \\
{\scriptstyle \Pi_\infty}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle \Pi_\infty} \\
\mathsf{Mod}_{\mathbb{Z}}^{\mathrm{h}G} & \xrightarrow{\ (-)_{\mathrm{h}G}\ } & \mathsf{Mod}_{\mathbb{Z}}
\end{array} \quad .
$$

We say that a chain complex $\mathsf{M} \in \mathsf{Ch}_{\mathbb{Z}[G]}$ is **adapted to homotopy G-orbits** if the morphism

$$
\Pi_\infty(\mathsf{M})_{\mathrm{h}G} \longrightarrow \Pi_\infty(\mathsf{M}_G)
$$

is an equivalence. More generally, we say that $\mathsf{M} \in \mathsf{Ch}_{\mathbb{Z}[G]}$ is adapted to homotopy H-orbits if its image under the forgetful functor $\mathsf{Ch}_{\mathbb{Z}[G]} \xrightarrow{\ \mathsf{fgt}\ } \mathsf{Ch}_{\mathbb{Z}[H]}$ is so.

Observation A.11.

(1) A free $\mathbb{Z}[G]$-module concentrated in degree zero is adapted to homotopy G-orbits.

(2) Complexes that are adapted to homotopy G-orbits are stable under taking

    (a) shifts,

    (b) cones, and

    (c) homotopy colimits.

(3) Combining parts (1) and (2), it follows that a bounded-below levelwise-free complex of $\mathbb{Z}[G]$-modules is adapted to homotopy G-orbits. Indeed, this follows by examining its presentation as a homotopy colimit of its truncations from above.

Lemma A.12. *Consider a commutative square*

$$
\begin{array}{ccc}
\tilde{\mathcal{D}} & \xleftarrow{\ \tilde{F}\ } & \tilde{\mathcal{C}} \\
{\scriptstyle M}\big\downarrow & & \big\downarrow{\scriptstyle L} \\
\mathcal{D} & \xleftarrow{\ F\ } & \mathcal{C}
\end{array}
$$

in Cat *in which the functor L is a localization and the functors $\tilde{F}$ and F admit right adjoints. Then, the resulting diagram*

$$
\begin{array}{ccc}
\tilde{\mathcal{D}} \xrightarrow{\ \tilde{F}^R\ } \tilde{\mathcal{C}} \xrightarrow{\ L\ } \mathcal{C} & & \tilde{\mathcal{D}} \xrightarrow{\ \tilde{F}^R\ } \tilde{\mathcal{C}} \\
{\scriptstyle M}\big\downarrow \quad {\scriptstyle \nwarrow} \quad \nearrow & := & {\scriptstyle M}\big\downarrow \quad {\scriptstyle \nwarrow} \quad \big\downarrow{\scriptstyle L} \\
\mathcal{D} \qquad\ {\scriptstyle F^R} & & \mathcal{D} \xrightarrow{\ F^R\ } \mathcal{C}
\end{array}
$$

is a left Kan extension diagram.

Proof. Consider the diagram

$$
\begin{array}{ccc}
\mathsf{Fun}(\tilde{\mathcal{C}}, \mathcal{C}) & \xleftarrow[\ \ L^*\ \ ]{\ \ \top\ \ } & \mathsf{Fun}(\mathcal{C}, \mathcal{C}) \\
{\scriptstyle (\tilde{F}^R)^*}\big\uparrow\big\downarrow{\scriptstyle \tilde{F}^*} & & {\scriptstyle (F^R)^*}\big\uparrow\big\downarrow{\scriptstyle F^*} \\
\mathsf{Fun}(\tilde{\mathcal{D}}, \mathcal{C}) & \xleftarrow[\ \ M^*\ \ ]{\ \ \top\ \ } & \mathsf{Fun}(\mathcal{D}, \mathcal{C})
\end{array}
$$

in Cat, in which the right adjoints commute and the dashed horizontal left adjoints may only be partially defined. It suffices to observe that we have assignments

$$
\begin{array}{ccc}
L & \longmapsto & \mathrm{id}_{\mathcal{C}} \\
\downarrow & & \downarrow \\
L \circ \tilde{F}^R & & F^R
\end{array}
\ ,
$$

by the uniqueness of (partially defined) left adjoints. $\qquad\square$

Corollary A.13. *Let* $\mathrm{R}, \mathrm{S} \in \mathsf{Alg}(\mathsf{Ch}_{\mathbb{Z}})$ *be dg-algebras, let* $\mathrm{M} \in \mathsf{BiMod}_{(\mathrm{S},\mathrm{R})}(\mathsf{Ch}_{\mathbb{Z}})$ *be a dg-bimodule, and suppose that the functor*

$$
\mathsf{Ch}_{\mathbb{Z}} \xrightarrow{\ \mathrm{M} \otimes_{\mathbb{Z}} (-)\ } \mathsf{Ch}_{\mathbb{Z}}
$$

preserves quasi-isomorphisms. Then, the canonical diagram

$$
\begin{array}{ccccc}
\mathsf{Ch}_{\mathrm{S}} & \xrightarrow{\ \underline{\mathrm{hom}}_{\mathsf{Ch}_{\mathrm{S}}}(\mathrm{M},-)\ } & \mathsf{Ch}_{\mathrm{R}} & \xrightarrow{\ \Pi_{\infty}\ } & \mathsf{Mod}_{\Pi_{\infty}(\mathrm{R})} \\
\Pi_{\infty}\downarrow & \overset{\nearrow}{\nLeftarrow} & & & \\
\mathsf{Mod}_{\Pi_{\infty}(\mathrm{S})} & \xrightarrow[\ \underline{\mathrm{hom}}_{\mathsf{Mod}_{\Pi_{\infty}(\mathrm{S})}}(\Pi_{\infty}(\mathrm{M}),-)\ ]{} & & &
\end{array}
$$

is a left Kan extension.

Proof. To simplify our notation, we write $R := \Pi_{\infty}(\mathrm{R})$, $S := \Pi_{\infty}(\mathrm{S})$, and $M := \Pi_{\infty}(\mathrm{M})$.

Because the functor $\mathsf{Mod}_R \xrightarrow{\ \mathsf{fgt}\ } \mathsf{Mod}_{\mathbb{Z}}$ is colimit-preserving and conservative, it suffices to show that the diagram

$$
\begin{array}{ccccccc}
\mathsf{Ch}_{\mathrm{S}} & \xrightarrow{\ \underline{\mathrm{hom}}_{\mathsf{Ch}_{\mathrm{S}}}(\mathrm{M},-)\ } & \mathsf{Ch}_{\mathrm{R}} & \xrightarrow{\ \Pi_{\infty}\ } & \mathsf{Mod}_R & \xrightarrow{\ \mathsf{fgt}\ } & \mathsf{Mod}_{\mathbb{Z}} \\
\Pi_{\infty}\downarrow & \nLeftarrow & & & & & \\
\mathsf{Mod}_S & \xrightarrow[\ \underline{\mathrm{hom}}_{\mathsf{Mod}_S}(M,-)\ ]{} & & & & &
\end{array}
$$

is a left Kan extension, which is equivalent to the condition that the diagram

$$
\begin{array}{ccccc}
\mathsf{Ch}_{\mathrm{S}} & \xrightarrow{\ \underline{\mathrm{hom}}_{\mathsf{Ch}_{\mathrm{S}}}(\mathrm{M},-)\ } & \mathsf{Ch}_{\mathbb{Z}} & \xrightarrow{\ \Pi_{\infty}\ } & \mathsf{Mod}_{\mathbb{Z}} \\
\Pi_{\infty}\downarrow & \nLeftarrow & & & \\
\mathsf{Mod}_S & \xrightarrow[\ \underline{\mathrm{hom}}_{\mathsf{Mod}_S}(M,-)\ ]{} & & &
\end{array}
\qquad (\mathrm{A.1})
$$

is a left Kan extension. Now, the canonical lax-commutative square

$$
\begin{array}{ccc}
\mathsf{Ch}_{\mathrm{S}} & \xleftarrow{\ \mathrm{M} \otimes_{\mathbb{Z}} (-)\ } & \mathsf{Ch}_{\mathbb{Z}} \\
\Pi_{\infty}\downarrow & \nwarrow & \downarrow \Pi_{\infty} \\
\mathsf{Mod}_S & \xleftarrow[\ M \otimes_{\mathbb{Z}}(-)\ ]{} & \mathsf{Mod}_{\mathbb{Z}}
\end{array}
$$

commutes by our assumption (e.g. by taking a projective resolution of $\mathrm{M} \in \mathsf{Ch}_{\mathbb{Z}}$ to compute the derived tensor product (i.e. the tensor product in $\mathsf{Mod}_{\mathbb{Z}}$)). Hence, the fact that the diagram (A.1) is a left Kan extension follows from Lemma A.12. $\qquad\square$

Observation A.14. We apply Corollary A.13 in the case that $\mathrm{S} = \mathbb{Z}[G], \mathrm{R} = \mathbb{Z}[\mathsf{W}(H)] \in \mathsf{Alg}(\mathsf{Ab}) \subset \mathsf{Alg}(\mathsf{Ch}_{\mathbb{Z}})$ and $\mathrm{M} = \mathbb{Z}[G/H]$. Then, we have identifications

$$
\underline{\mathrm{hom}}_{\mathsf{Ch}_{\mathbb{Z}[G]}}(\mathbb{Z}[G/H], -) \simeq (-)^H \qquad \text{and} \qquad \underline{\mathrm{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}G}}(\mathbb{Z}[G/H], -) \simeq (-)^{\mathsf{h}H} \ .
$$

As $\mathbb{Z}[G/H] \in \mathsf{Ab} \subset \mathsf{Ch}_{\mathbb{Z}}$ is a free abelian group, we find that the canonical diagram

$$
\begin{array}{ccc}
\mathsf{Ch}_{\mathbb{Z}[G]} & \xrightarrow{\ (-)^H\ } \mathsf{Ch}_{\mathbb{Z}[\mathsf{W}(H)]} \xrightarrow{\ \Pi_\infty\ } \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hW}(H)} \\
{\scriptstyle \Pi_\infty}\Big\downarrow \quad\quad {\scriptstyle \nearrow} & \\
\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}G} & \xrightarrow[\ \ (-)^{\mathsf{hC}_j}\ \ ]{}
\end{array}
\tag{A.2}
$$

is a left Kan extension. In particular, it follows that for any morphism $\mathtt{M} \xrightarrow{\mathtt{f}} \mathtt{N}$ in $\mathsf{Ch}_{\mathbb{Z}[G]}$, if $\mathtt{M}$ and $\mathtt{N}$ are adapted to homotopy H-fixedpoints then the morphism $\mathtt{f}^H$ in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{W}(H)]}$ is a presentation of the morphism $(\Pi_\infty(\mathtt{f}))^{\mathsf{h}H}$ in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hW}(H)}$.

Notation A.15. Consider the morphisms

$$
\mathbb{Z}[G/e] \xrightarrow{\ 1 \longmapsto 1\ } \mathbb{Z}[G/H]
\tag{A.3}
$$

and

$$
\mathbb{Z}[G/H] \xrightarrow{\ 1 \longmapsto \sum_{h \in H} h\ } \mathbb{Z}[G/e]
\tag{A.4}
$$

in $\mathsf{Mod}_{\mathbb{Z}[G]}(\mathsf{Ab}) \subset \mathsf{Mod}_{\mathbb{Z}[G]}(\mathsf{Ch}_{\mathbb{Z}})$. For any $\mathtt{M} \in \mathsf{Ch}_{\mathbb{Z}[G]}$, we write

$$
\left(\mathtt{M} \xleftarrow{\ \mathtt{inc}^{\mathsf{h}}\ } \mathtt{M}^H \right) := \underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}[G]}}((\mathrm{A}.3), \mathtt{M}) \qquad \text{and} \qquad \left(\mathtt{M}^H \xleftarrow{\ \mathtt{trf}^{\mathsf{h}}\ } \mathtt{M} \right) := \underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}[G]}}((\mathrm{A}.4), \mathtt{M})
$$

for the indicated morphisms in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{N}(H)]}$.[33]

Observation A.16. Corollary A.13 (and in particular Observation A.14) is functorial in the dg-bimodule $\mathtt{M}$. That is, there is a natural transformation

$$
(\Pi_\infty)_!(\Pi_\infty(\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathsf{S}}}(-,=))) \longrightarrow \underline{\mathsf{hom}}_{\mathsf{Mod}_{\Pi_\infty(\mathsf{S})}}(\Pi_\infty(-),=)
$$

in $\mathsf{Fun}(\mathsf{BiMod}_{(\mathsf{S},\mathsf{R})}(\mathsf{Ch}_{\mathbb{Z}})^{\mathsf{op}} \times \mathsf{Ch}_{\mathsf{S}}, \mathsf{Mod}_{\Pi_\infty(\mathsf{R})})$, which is an equivalence when restricting to those dg-bimodules $\mathtt{M}$ such that the functor $\mathsf{Ch}_{\mathbb{Z}} \xrightarrow{\mathtt{M} \otimes_{\mathbb{Z}} (-)} \mathsf{Ch}_{\mathbb{Z}}$ preserves quasi-isomorphisms.

Observation A.17. Fix a subgroup $H \leq G$. Using Observation A.16, we apply Observation A.14 to the morphisms (A.3) and (A.4) to obtain for each $\mathtt{M} \in \mathsf{Ch}_{\mathbb{Z}[G]}$ natural commutative squares

$$
\begin{array}{ccc}
\Pi_\infty(\mathtt{M}^H) & \xrightarrow{\ \Pi_\infty(\mathtt{inc}^{\mathsf{h}})\ } & \Pi_\infty(\mathtt{M}^e) \\
\downarrow & & \downarrow{\scriptstyle \wr} \\
\Pi_\infty(\mathtt{M})^{\mathsf{h}H} & \xrightarrow[\ \mathsf{inc}^{\mathsf{h}}\ ]{} & \Pi_\infty(\mathtt{M})^{\mathsf{h}e}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\Pi_\infty(\mathtt{M}^e) & \xrightarrow{\ \Pi_\infty(\mathtt{trf}^{\mathsf{h}})\ } & \Pi_\infty(\mathtt{M}^H) \\
{\scriptstyle \wr}\downarrow & & \downarrow \\
\Pi_\infty(\mathtt{M})^{\mathsf{h}e} & \xrightarrow[\ \mathsf{trf}^{\mathsf{h}}\ ]{} & \Pi_\infty(\mathtt{M})^{\mathsf{h}H}
\end{array}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{h}G}$, in which all vertical morphisms are components of natural transformations of the form (A.2). In particular, if $\mathtt{M}$ is adapted to homotopy H-fixedpoints, then we obtain canonical identifications $\mathsf{inc}^{\mathsf{h}} \simeq \Pi_\infty(\mathtt{inc}^{\mathsf{h}})$ and $\mathsf{trf}^{\mathsf{h}} \simeq \Pi_\infty(\mathtt{trf}^{\mathsf{h}})$ in $\mathsf{Ar}(\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hW}(H)})$.

---

[33]Recall from Notations 1.13 and 7.17 that in the case that $G = \mathsf{C}_{p^i}$ and $H = \mathsf{C}_p$, for any $M \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^i}}$ we likewise simply write

$$
\left(M \xleftarrow{\ \mathsf{inc}^{\mathsf{h}}\ } M^{\mathsf{hC}_p} \right) := \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^i}}}((\mathrm{A}.3), M) \qquad \text{and} \qquad \left(M^{\mathsf{hC}_p} \xleftarrow{\ \mathsf{trf}^{\mathsf{h}}\ } M \right) := \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^i}}}((\mathrm{A}.4), M)
$$

for the indicated morphisms.

A.3. **Chain-level data for cohomology groups.** In this subsection, we introduce and study the specific chain complexes and most of the chain maps that participate in our computation of the values of equivariant cohomology (Notation A.22). The main output is Lemma A.24, which establishes the underlying homotopical content of these chain-level data.

Remark A.18. We have tried to make the notation that is introduced in the remainder of the paper mnemonical, as we now describe.

- The number a is that appearing in §§7 and 11, where it participates in the categorical fixedpoints functor $(-)^{\mathsf{C}_{p^a}}$.

- The letter r stands for "residual equivariance". This is generally recorded as a subscript.

- Superscripts generally record "the number of times that homotopy C_p-fixedpoints has been taken". Correspondingly, we use subscripts on the left to record "the number of times that homotopy C_p-orbits has been taken".

- The letter Z stands for "integers", and the letter S is chosen because it is similar to Z.

- The letter T stands for "Tate".

- The letter C stands for "connective cover of Tate".

- The letter q stands for "quotient", and the letter g is chosen because it is similar to q.

- The letter e is chosen because it is similar to ε.

- The letter c stands for "Chern class".

Notation A.19. We fix nonnegative integers $a, r \geq 0$.

Definition A.20. The **norm element** for the group C_{p^r} is the element

$$N := N_r := \left(\sum_{i=1}^{p^r} \sigma^i \right) = \left(1 + \sigma + \cdots + \sigma^{p^r - 1} \right) \in \mathbb{Z}[\mathsf{C}_{p^r}]$$

of its group ring.

Remark A.21. Notations A.22 and A.23 are formatted in a way that makes them well-suited for application in §11. However, from a certain point of view this formatting is suboptimal: most of the data defined therein is independent of the choice of a.

Notation A.22. We define a diagram

$$\text{(A.5)}$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ (i.e. a functor $\mathsf{Zig}_a \to \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$) as follows.

(1) We define the objects in diagram (A.5) as follows; all are levelwise free $\mathbb{Z}[\mathsf{C}_{p^r}]$-modules of rank 0 or 1.

(a) We define the object

$$Z^a_r := \left(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \underset{\sim}{\mathbb{Z}[\mathsf{C}_{p^r}]} \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^a N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^a N} \cdots \right).$$

(b) For any $i \geq 0$, we define the object

$$C^i_r := \left(\cdots \xrightarrow{p^{i+1} N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+1} N} \underset{\sim}{\mathbb{Z}[\mathsf{C}_{p^r}]} \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^i N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^i N} \cdots \right). [34]$$

(c) For any $i \geq 0$, we define the object

$$T^i_r := \left(\cdots \xrightarrow{p^{i+1} N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+1} N} \underset{\sim}{\mathbb{Z}[\mathsf{C}_{p^r}]} \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+1} N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \cdots \right).$$

(2) We define the morphisms in diagram (A.5) as follows.

(a) Assuming that $a \geq 1$, we define the morphism

$$Z^a_r \xrightarrow{\mathsf{q}^{a-1}_r} T^{a-1}_r$$

as

i.e. it is the identity in all nonpositive degrees.

(b) For any $i \geq 0$, we define the morphism

$$C^i_r \xrightarrow{\mathsf{e}^i_r} T^i_r$$

as

i.e. it is the identity in each positive degree and, for all $j \geq 0$, in degree $-j$ it is multiplication by $p^{\lfloor \frac{j}{2} \rfloor}$.[35]

(c) For any $i \geq 0$, we define the morphism

$$C^{i+1}_r \xrightarrow{\mathsf{g}^i_r} T^i_r$$

[34]Note that the differentials in nonnegative degrees differ from those in negative degrees.

[35]Here the exponent is the floor of $\frac{j}{2}$, i.e. the largest integer that is at most $\frac{j}{2}$.

as

$$\cdots \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+2}N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+2}N} \underwave{\mathbb{Z}[\mathsf{C}_{p^r}]} \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+1}N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \cdots$$

$$\Big\downarrow{p^2} \qquad \Big\downarrow{p} \qquad \Big\downarrow{p} \qquad \Big\downarrow{1} \qquad \Big\downarrow{1} \qquad \Big\downarrow{1}$$

$$\cdots \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+1}N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+1}N} \underwave{\mathbb{Z}[\mathsf{C}_{p^r}]} \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^{i+1}N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \cdots$$

,

i.e. it is the identity in each negative degree and, for all $j \geq 0$, in degree j it is multiplication by $p^{\lceil \frac{j}{2} \rceil}$.[36]

Notation A.23. We define a diagram

(A.6)

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ (i.e. a functor $\mathsf{Zig}_a \to \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$) as follows.

(a) Assuming that $a \geq 1$, we define the morphism q_r^{a-1} as

$$\mathbb{Z}^{\mathsf{hC}_{p^a}} \xleftarrow{\sim} \left(\mathbb{Z}^{\mathsf{hC}_p}\right)^{\mathsf{hC}_{p^{a-1}}} \xrightarrow{\mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{hC}_{p^{a-1}}}} \left(\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^{a-1}}} ,$$

the homotopy $\mathsf{C}_{p^{a-1}}$-fixedpoints of the canonical morphism from the homotopy C_p-fixedpoints to the C_p-Tate construction for the object $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^n}}$.

(b) For any $i \geq 0$, we define the morphism e_r^i as

$$\left(\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^i}} \xrightarrow{\varepsilon_{\geq 0}\left(\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^i}}} \left(\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^i}} ,$$

the homotopy C_{p^i}-fixedpoints of the canonical morphism to the C_p-Tate construction on the object $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+i+1}}}$ from its connective cover.

(c) For any $i \geq 0$, we define the morphism g_r^i as

$$\left(\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^{i+1}}} \xleftarrow{\sim} \left(\left(\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_p}\right)^{\mathsf{hC}_{p^i}} \xrightarrow{\mathsf{Q}_{\mathsf{C}_p}\left(\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^i}}} \left(\left(\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^i}} \xleftarrow[\sim]{\left(\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^i}}} \left(\mathbb{Z}^{\mathsf{tC}_p}\right)^{\mathsf{hC}_{p^i}} ,$$

the composite of

- the homotopy C_{p^i}-fixedpoints of the canonical map from the homotopy C_p-fixedpoints to the C_p-Tate construction for the connective cover of the C_p-Tate construction on the object $\mathbb{Z} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+i+2}}}$ and

- the homotopy C_{p^i}-fixedpoints of the inverse of the equivalence $\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{tC}_p}$ in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+i}}}$ of Observation 10.7.

Lemma A.24. *The diagram* (A.5) *in* $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ *presents the diagram* (A.6) *in* $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$.

---

[36]Here the exponent is the ceiling of $\frac{j}{2}$, i.e. the smallest integer that is at least $\frac{j}{2}$.

Proof. The case where $a = 0$ is trivial to verify, so let us assume that $a \geq 1$. We also fix any $i \geq 0$. Then, we prove that the morphisms q_r^{a-1}, g_r^i, and e_r^i in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ are equivalent to other morphisms that are manifestly presented by the morphisms q_r^{a-1}, g_r^i, and e_r^i in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$, respectively. In particular, this shows that the objects

$$\mathsf{Z}_r^a, \mathsf{C}_r^i, \mathsf{T}_r^i \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$$

are indeed presentations of the objects

$$\mathbb{Z}^{\mathsf{hC}_{p^a}}, (\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}}, (\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}} ,$$

respectively.

We begin with the morphisms q_r^{a-1} and e_r^i. For these, note the commutative diagram in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$

(A.7)

FIGURE 2. This commutative diagram in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{n-a}}}$ contains the morphisms q_r^{a-1} and e_r^i. All three horizontal composites are cofiber sequences.

of Figure 2, obtained as follows.

- The two upper commutative squares are obtained by applying the functor

$$\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+i}}} \xrightarrow{(-)^{\mathsf{hC}_{p^i}}} \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+i}}}$$

to (the middle two rows of) the commutative diagram (10.2), and so are cofiber sequences by Observation 10.5.

- The lower left pentagon commutes by Observation 4.13(3).

- The morphism $\mathsf{Nm}_{\mathsf{C}_{p^i}}(\mathbb{Z}_{\mathsf{hC}_p})$ is an equivalence

  - trivially if $i = 0$,

  - by Observation 7.5 if $i = 1$, and

  - by Observations 7.5 and 7.6 if $i \geq 2$.

- The lower right vertical equivalence is the induced equivalence between cofibers.

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We now argue as follows.

(a) In the case that $i = a - 1$, the morphism q_r^{a-1} is evidently a presentation of the morphism $\mathsf{Q}_{\mathsf{C}_{p^a}}(\mathbb{Z})$, which proves that it is indeed a presentation of the morphism q_r^{a-1} by diagram (A.7).

(b) Consider the morphism of cofiber sequences

$$
\begin{array}{ccccc}
\mathbb{Z}_{\mathsf{hC}_{p^{i+1}}} & \xrightarrow{\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{hC}_{p^i}} \circ \mathsf{Nm}_{\mathsf{C}_{p^i}}(\mathbb{Z}_{\mathsf{hC}_p})} & \mathbb{Z}^{\mathsf{hC}_{p^i}} & \xrightarrow{\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{hC}_{p^i}}} & (\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}} \\
\| & & \downarrow{\scriptstyle \varepsilon_{\geq 0}(\mathbb{Z}^{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}}} & & \downarrow{\scriptstyle e_r^i} \\
\mathbb{Z}_{\mathsf{hC}_{p^{i+1}}} & \xrightarrow[\mathsf{Nm}_{\mathsf{C}_{p^{i+1}}}(\mathbb{Z})]{} & \mathbb{Z}^{\mathsf{hC}_{p^{i+1}}} & \xrightarrow[\mathsf{Q}_{\mathsf{C}_{p^{i+1}}}(\mathbb{Z})]{} & \mathbb{Z}^{\mathsf{tC}_{p^{i+1}}}
\end{array}
\tag{A.8}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ extracted from diagram (A.7). Noting that the middle vertical morphism in diagram (A.8) induces an equivalence on connective covers (and that $\mathbb{Z}_{\mathsf{hC}_{p^{i+1}}} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ is connective), we see that the morphism e_r^i is indeed a presentation of the morphism e_r^i.

We now proceed to the morphism g_r^i. For this, note the commutative diagram in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ of

$$
\begin{array}{ccccc}
((\mathbb{Z}_{\mathsf{hC}_p})_{\mathsf{hC}_p})_{\mathsf{hC}_{p^i}} & \xrightarrow{(\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})_{\mathsf{hC}_p})_{\mathsf{hC}_{p^i}}} & (\mathbb{Z}_{\mathsf{hC}_p})_{\mathsf{hC}_{p^i}} & & \\
\downarrow{\scriptstyle \mathsf{Nm}_{\mathsf{C}_{p^i}}((\mathbb{Z}_{\mathsf{hC}_p})_{\mathsf{hC}_p})}\wr & & \downarrow\wr{\scriptstyle \mathsf{Nm}_{\mathsf{C}_{p^i}}(\mathbb{Z}_{\mathsf{hC}_p})} & & \\
((\mathbb{Z}_{\mathsf{hC}_p})_{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}} & \xrightarrow{(\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})_{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}}} & (\mathbb{Z}_{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}} & \xrightarrow{(\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})_{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}}} & ((\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})_{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}} \\
\downarrow{\scriptstyle \mathsf{Nm}_{\mathsf{C}_p}(\mathbb{Z}_{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}}}\wr & & \downarrow{\scriptstyle \mathsf{Nm}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{hC}_{p^i}}} & & \downarrow{\scriptstyle \mathsf{Nm}_{\mathsf{C}_p}(\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}}} \\
((\mathbb{Z}_{\mathsf{hC}_p})^{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}} & \xrightarrow{(\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}}} & (\mathbb{Z}^{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}} & \xrightarrow{(\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}}} & ((\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}} \\
\downarrow{\scriptstyle \mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z}_{\mathsf{hC}_p})^{\mathsf{hC}_{p^i}}} & & \downarrow{\scriptstyle \mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{hC}_{p^i}}} & & \downarrow{\scriptstyle \mathsf{Q}_{\mathsf{C}_p}(\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}}} \\
0 \simeq ((\mathbb{Z}_{\mathsf{hC}_p})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}} & \xrightarrow{(\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}}} & (\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}} & \xleftarrow[\quad(\widetilde{\mathsf{Q}}_{\mathsf{C}_p}(\mathbb{Z})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}}\quad]{\sim} & ((\tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{tC}_p})^{\mathsf{hC}_{p^i}}
\end{array}
$$

$$
\tag{A.9}
$$

FIGURE 3. This commutative diagram in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ contains the morphism g_r^i. Omitting the top row, all three horizontal composites and all three vertical composites are cofiber sequences.

Figure 3, obtained as follows.

- The bottom three rows are obtained from the second row of diagram (10.2) by applying the cofiber sequence

$$(-)_{hC_p} \xrightarrow{\mathsf{Nm}_{C_p}} (-)^{hC_p} \xrightarrow{\mathsf{Q}_{C_p}} (-)^{tC_p}$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathsf{Mod}_{\mathbb{Z}}^{hC_{p^{r+1}}}, \mathsf{Mod}_{\mathbb{Z}}^{hC_{p^r}})$ followed by the functor

$$\mathsf{Mod}_{\mathbb{Z}}^{hC_{p^{r+i}}} \xrightarrow{(-)^{hC_{p^i}}} \mathsf{Mod}_{\mathbb{Z}}^{hC_{p^r}} .$$

So indeed, the three lower vertical composites are indeed cofiber sequences, and moreover the lower three rows are cofiber sequences by Observation 10.5.

- The equivalence $0 \simeq ((\mathbb{Z}_{hC_p})^{tC_p})^{hC_{p^i}}$ follows from Observation 7.5, and it implies that the morphisms $\mathsf{Nm}_{C_p}(\mathbb{Z}_{hC_p})^{hC_{p^i}}$ and $(\widetilde{\mathsf{Q}}_{C_p}(\mathbb{Z})^{tC_p})^{hC_{p^i}}$ are both equivalences.

- The morphisms $\mathsf{Nm}_{C_{p^i}}((\mathbb{Z}_{hC_p})_{hC_p})$ and $\mathsf{Nm}_{C_{p^i}}(\mathbb{Z}_{hC_p})$ are equivalences

  - trivially if $i = 0$,

  - by Observation 7.5 if $i = 1$, and

  - by Observations 7.5 and 7.6 if $i \geq 2$.

We now argue as follows.

(c) Consider the morphism of cofiber sequences

$$(A.10)$$

in $\mathsf{Mod}_{\mathbb{Z}}^{hC_{p^r}}$ extracted from diagram (A.9), where we have used Observation 4.13(3) to reidentify the left two horizontal morphisms. Noting that the left vertical morphism in diagram (A.10) induces an equivalence after coconnective truncation (and that $\mathbb{Z}^{hC_{p^{i+1}}} \in \mathsf{Mod}_{\mathbb{Z}}^{hC_{p^r}}$ is coconnective), we see that the morphism g_r^i is indeed a presentation of the morphism g_r^i. $\qquad\square$

A.4. **Chain-level data for inclusion and transfer.** In this subsection, we establish the results that support our identification of the inclusion and transfer morphisms in equivariant cohomology. These involve some new auxiliary chain complexes and chain maps introduced in Notations A.25 and A.29, as well as a chain homotopy introduced in Notation A.32. The main results are Lemma A.28 (which establishes the relevant adaptedness) and Lemma A.33 (which proves that the chain homotopy of Notation A.29 presents the desired ∞-categorical homotopy).

Notation A.25. For any $i \geq 0$, we define the object $_i S_r \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ as

$$_i S_r := \left(\cdots \xrightarrow{p^i N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{p^i N} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \underwave{\mathbb{Z}[\mathsf{C}_{p^r}]} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \right) .$$

Local Notation A.26. In this subsection, we henceforth assume that $r \geq 1$.

Observation A.27. For any $i \geq 0$, there are evident isomorphisms

$$(\mathbb{Z}_r^a)^{\mathsf{C}_p} \cong \mathbb{Z}_{r-1}^{a+1} , \qquad (\mathsf{C}_r^i)^{\mathsf{C}_p} \cong \mathsf{C}_{r-1}^{i+1} , \qquad \text{and} \qquad (\mathsf{T}_r^i)^{\mathsf{C}_p} \cong \mathsf{T}_{r-1}^{i+1}$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r-1}}]}$, as well as evident isomorphisms

$$(\mathsf{q}_r^{a-1})^{\mathsf{C}_p} \cong \mathsf{q}_{r-1}^a , \qquad (\mathsf{g}_r^i)^{\mathsf{C}_p} \cong \mathsf{g}_{r-1}^{i+1} , \qquad \text{and} \qquad (\mathsf{e}_r^i)^{\mathsf{C}_p} \cong \mathsf{e}_{r-1}^{i+1}$$

in $\mathsf{Ar}(\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r-1}}]})$.

Lemma A.28. *For any $i \geq 0$, the complexes $\mathbb{Z}_r^a, \mathsf{C}_r^i, \mathsf{T}_r^i \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ are adapted to homotopy C_p-fixedpoints.*

Proof. It follows from Observation A.9(3) that $\mathbb{Z}_r^a$ is adapted to homotopy C_p-fixedpoints.

There is an evident diagram $\mathbb{Z}_r^i \leftarrow {}_{i+1}S_r \rightarrow \mathbb{Z}_r^{i+1}$ in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ given by

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \underwave{\mathbb{Z}[\mathsf{C}_{p^r}]} & \xrightarrow{1-\sigma} & \mathbb{Z}[\mathsf{C}_{p^r}] & \xrightarrow{p^i N} & \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \cdots \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle p^{i+1}N} & & \uparrow & & \uparrow \\
\cdots \xrightarrow{1-\sigma} & \mathbb{Z}[\mathsf{C}_{p^r}] & \xrightarrow{p^{i+1}N} & \mathbb{Z}[\mathsf{C}_{p^r}] & \xrightarrow{1-\sigma} & & \underwave{\mathbb{Z}[\mathsf{C}_{p^r}]} & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p^{i+1}N} & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \underwave{\mathbb{Z}[\mathsf{C}_{p^r}]} & \xrightarrow{1-\sigma} & \mathbb{Z}[\mathsf{C}_{p^r}] & \xrightarrow{p^{i+1}N} & \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1-\sigma} \cdots
\end{array}
$$

(reading from top to bottom) such that we have isomorphisms

$$\mathsf{C}_r^i \cong \mathsf{cone}({}_{i+1}S_r \longrightarrow \mathbb{Z}_r^i) \qquad \text{and} \qquad \mathsf{T}_r^i \cong \mathsf{cone}({}_{i+1}S_r \longrightarrow \mathbb{Z}_r^{i+1}) .$$

So by Observation A.9(2)(b), to show that C_r^i and $\mathbb{Z}_r^i$ are adapted to homotopy C_p-fixedpoints it suffices to show that ${}_{i+1}S_r$, $\mathbb{Z}_r^i$, and $\mathbb{Z}_r^{i+1}$ are. We have just seen that the latter two are adapted to homotopy C_p-fixedpoints, so it remains to prove that $\mathsf{S} := {}_{i+1}S_r$ is as well. For this, let us write

$$N_1 := \left(\sum_{j=1}^{p} (\sigma^{p^{r-1}})^j \right) = \left(1 + \sigma^{p^{r-1}} + \cdots + \sigma^{(p-1)\cdot p^{r-1}} \right) \in \mathbb{Z}[\mathsf{C}_{p^r}]$$

for the image of the norm element for C_p under the ring homomorphism $\mathbb{Z}[\mathsf{C}_p \hookrightarrow \mathsf{C}_{p^{n-a}}]$. Then, consider the evident factorization

$$
\begin{array}{ccc}
\mathsf{S} & \xrightarrow{N_1} & \mathsf{S} \\
\downarrow & & \uparrow \\
\mathsf{S}_{\mathsf{C}_p} & \dashrightarrow[N_1] & \mathsf{S}^{\mathsf{C}_p}
\end{array}
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$, which determines a commutative diagram

$$
\begin{array}{ccc}
\Pi_\infty(\mathsf{S})_{\mathsf{hC}_p} & \xrightarrow{\;\mathsf{Nm}_{\mathsf{C}_p}(\Pi_\infty(\mathsf{S}))\;} & \Pi_\infty(\mathsf{S})^{\mathsf{hC}_p} \\
\downarrow & & \uparrow \\
\Pi_\infty(\mathsf{S}_{\mathsf{C}_p}) & \xrightarrow[\;\Pi_\infty(N_1)\;]{} & \Pi_\infty(\mathsf{S}^{\mathsf{C}_p})
\end{array}
\tag{A.11}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$. We claim that the three morphisms in diagram (A.11) aside from the right vertical morphism are equivalences.

- Its upper horizontal morphism is an equivalence due to the evident equivalence $\Pi_\infty(\mathsf{S}) \simeq \mathbb{Z}_{\mathsf{hC}_{p^{i+1}}}$ in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ and using Observation 7.5.

- Its left vertical morphism is an equivalence because $\mathsf{S} \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ is adapted to homotopy C_p-orbits by Observation A.11(3).

- Its lower horizontal morphism is an equivalence because the morphism $\mathsf{S}_{\mathsf{C}_p} \xrightarrow{N_1} \mathsf{S}^{\mathsf{C}_p}$ in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ is evidently an isomorphism.

Hence, the right vertical morphism in diagram (A.11) is an equivalence, i.e. S is adapted to homotopy C_p-fixedpoints. $\qquad\square$

Notation A.29. We define the morphisms

$$
{}_1\mathsf{S}_r \xrightarrow{\;\widetilde{\mathsf{Nm}}\;} {}_0\mathsf{S}_r \xrightarrow{\;\mathsf{i}\;} \mathsf{Z}_r^0 \xrightarrow{\;\mathsf{j}\;} \mathsf{Z}_{r-1}^1
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{n-a}}]}$ as

Moreover, we define the chain complex

$$
\widetilde{\mathsf{C}}_r := \mathsf{cone}\left(\widetilde{\mathsf{Nm}}\right) \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]} ,
$$

and we write

$$
\begin{array}{ccc}
{}_1\mathsf{S}_r & \xrightarrow{\;\widetilde{\mathsf{Nm}}\;} & {}_0\mathsf{S}_r & \longrightarrow & \widetilde{\mathsf{C}}_r \\
\| & & \downarrow{\scriptstyle \mathsf{i}} & & \vdots{\scriptstyle \mathsf{k}_r} \\
{}_1\mathsf{S}_r & \xrightarrow[\;\mathsf{io}\widetilde{\mathsf{Nm}}\;]{} & \mathsf{Z}_r^0 & \longrightarrow & \mathsf{C}_r^0
\end{array}
$$

for the induced morphism on cones.

Local Notation A.30. For simplicity, we often omit the functor $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]} \xleftarrow{\mathsf{triv}} \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r-1}}]}$ from our notation. Moreover, we use the notation $(-)^\dagger$ to denote passage to adjunct morphisms in the adjunction

$$\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]} \underset{\mathsf{triv}}{\overset{(-)_{\mathsf{C}_p}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r-1}}]} \quad .$$

Observation A.31.

(1) There is a canonical isomorphism

$$\mathsf{cone}\left(\mathtt{i} \circ \widetilde{\mathsf{Nm}}\right) \cong \mathsf{C}_r^0$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$. Moreover, the morphism

$$\widetilde{\mathsf{C}}_r := \mathsf{cone}\left(\widetilde{\mathsf{Nm}}\right) \xrightarrow{\mathtt{k}_r} \mathsf{cone}\left(\mathtt{i} \circ \widetilde{\mathsf{Nm}}\right) \cong \mathsf{C}_r^0$$

is a quasi-isomorphism, because the morphism $_0\mathsf{S}_r \xrightarrow{\mathtt{i}} \mathsf{Z}_r^0$ is a quasi-isomorphism.

(2) We have a canonical commutative diagram

$$
\begin{array}{ccccc}
(_1\mathsf{S}_r)_{\mathsf{C}_p} & \xrightarrow{(\mathtt{j}\circ\mathtt{i}\circ\widetilde{\mathsf{Nm}})^\dagger} & \mathsf{Z}_{r-1}^1 & \longrightarrow & \mathsf{C}_{r-1}^1 \\
{\scriptstyle\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}}\Big\downarrow & & \Big\| & & \Big\downarrow{\scriptstyle\mathsf{g}_{r-1}^0} \\
(_0\mathsf{S}_r)_{\mathsf{C}_p} & \xrightarrow[(\mathtt{j}\circ\mathtt{i})^\dagger]{} & \mathsf{Z}_{r-1}^1 & \longrightarrow & \mathsf{T}_{r-1}^0
\end{array}
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r-1}}]}$ in which both rows are cone sequences and g_{r-1}^0 is the induced morphism on cones.

Notation A.32. We write h' for the canonical nullhomotopy of the composite

$$(\widetilde{\mathsf{C}}_r)_{\mathsf{C}_p} \longrightarrow \mathsf{C}_{r-1}^1 \xrightarrow{\mathsf{g}_{r-1}^0} \mathsf{T}_{r-1}^0$$

determined by the commutative triangle

$$
\begin{array}{ccc}
(_1\mathsf{S}_r)_{\mathsf{C}_p} & \xrightarrow{\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}} & (_0\mathsf{S}_r)_{\mathsf{C}_p} \\
{\scriptstyle(\mathtt{j}\circ\mathtt{i}\circ\widetilde{\mathsf{Nm}})^\dagger}\searrow & & \swarrow{\scriptstyle(\mathtt{j}\circ\mathtt{i})^\dagger} \\
& \mathsf{Z}_{r-1}^1 &
\end{array}
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ (using Observation A.31(2)), and we write h for its precomposition with the quotient morphism $\widetilde{\mathsf{C}}_r \to (\widetilde{\mathsf{C}}_r)_{\mathsf{C}_p}$. So explicitly, h is a sequence

$$\mathsf{h} := \left((\widetilde{\mathsf{C}}_r)_n \xrightarrow{\mathsf{h}_n} (\mathsf{T}_{r-1}^0)_{n+1} \right)_{n\in\mathbb{Z}}$$

of maps of $\mathbb{Z}[\mathsf{C}_{p^r}]$-modules given as follows:

- for $n < 0$, h_n is the zero map

$$(\widetilde{\mathsf{C}}_r)_n := 0 \longrightarrow \mathbb{Z}[\mathsf{C}_{p^{r-1}}] =: (\mathsf{T}_{r-1}^0)_{n+1} \; ;$$

- h_0 is the quotient map

$$(\widetilde{\mathsf{C}}_r)_0 := \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1\mapsto 1} \mathbb{Z}[\mathsf{C}_{p^{r-1}}] =: (\mathsf{T}_{r-1}^0)_{n+1} \; ;$$

- for $n > 0$, h_n is the composite

$$(\widetilde{\mathsf{C}}_r)_n := \begin{pmatrix} \mathbb{Z}[\mathsf{C}_{p^r}] \\ \oplus \\ \mathbb{Z}[\mathsf{C}_{p^r}] \end{pmatrix} \xrightarrow{(0\ 1)} \mathbb{Z}[\mathsf{C}_{p^r}] \xrightarrow{1 \longmapsto 1} \mathbb{Z}[\mathsf{C}_{p^{r-1}}] =: (\mathtt{T}^0_{r-1})_{n+1}$$

of the projection onto the second factor followed by the quotient map.

Lemma A.33. *The homotopy-commutative diagram*

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ *is a presentation of the commutative diagram*

$$(\mathrm{A.12})$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ *of Observation 7.18 (applied to* $E = \tau_{\geq 0}\mathbb{Z}^{\mathsf{tC}_p} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ *). In particular, the homotopy-commutative square*

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ *is a presentation of the commutative square*

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}_{\mathsf{hC}_p} & \xrightarrow{\ \widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})\ } & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}_{\mathsf{hC}_{p^2}} & \xrightarrow[\ \widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})_{\mathsf{hC}_p}\ ]{} & \mathbb{Z}_{\mathsf{hC}_p} & \xrightarrow[\ \mathsf{Nm}_{\mathsf{C}_p}(\mathbb{Z})\ ]{} & \mathbb{Z}^{\mathsf{hC}_p}
\end{array}
\tag{A.13}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$. Passing to cofibers as indicated, this yields a commutative diagram

$$
\begin{array}{ccc}
\mathrm{cofib}(\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})) & & \\
\downarrow & & \\
\mathrm{cofib}(\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})_{\mathsf{hC}_p}) & \longrightarrow & \mathrm{cofib}(\mathsf{Nm}_{\mathsf{C}_p}(\mathbb{Z}) \circ \widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})_{\mathsf{hC}_p}) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{cofib}(\mathsf{Nm}_{\mathsf{C}_p}(\mathbb{Z}))
\end{array}
\tag{A.14}
$$

in $\mathsf{Mod}^{\mathsf{hC}_{p^r}}$ in which the square is a pushout. Moreover, it follows from the proof of Lemma A.24 that the commutative diagram (A.14) is precisely the commutative diagram (A.12). So to conclude, it suffices to show that the commutative diagram (A.13) in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ is presented by the commutative diagram

$$
\begin{array}{ccc}
{}_1 S_r & \xrightarrow{\ \widetilde{\mathsf{Nm}}\ } & {}_0 S_r \\
\downarrow & & \downarrow \quad \searrow^{\mathtt{joi}} \\
({}_1 S_r)_{\mathsf{C}_p} & \xrightarrow{\ \widetilde{\mathsf{Nm}}_{\mathsf{C}_p}\ } & ({}_0 S_r)_{\mathsf{C}_p} \xrightarrow{\ (\mathtt{joi})^{\dagger}\ } Z^1_{r-1} \\
& \searrow_{(\mathtt{joi}\circ\widetilde{\mathsf{Nm}})^{\dagger}} &
\end{array}
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$. For this, we note that it follows from Observation A.31(1) that the morphism $\widetilde{\mathsf{Nm}}$ presents the morphism $\widetilde{\mathsf{Nm}}_{\mathsf{C}_p}(\mathbb{Z})$ and moreover that ${}_1 S_r$ and ${}_0 S_r$ are adapted to homotopy C_p-orbits by Observation A.11(3). $\qquad\square$

A.5. **Multiplicative structure of Tate cohomology.** In this subsection, we study the multiplicative structure on Tate cohomology. Namely, in Lemma A.35 we compute the ring structure on the homotopy groups of $(\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{a-1}}} \in \mathsf{Mod}_{\mathbb{Z}}$, and in Lemma A.39 we give a chain-level presentation of an endomorphism of $\mathbb{Z}^{\mathsf{tC}_p} \in \mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ given by multiplying by a C_{p^r}-equivariant homotopy element.

Notation A.34. We respectively write

$$
c'_a \in \pi_{-2}(\mathbb{Z}^{\mathsf{hC}_{p^a}}) \qquad \text{and} \qquad c_a \in \pi_{-2}((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{a-1}}})
$$

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for the elements represented by the cycles

$$1 \in \mathbb{Z} =: (\mathbb{Z}_0^a)_{-2} \qquad \text{and} \qquad 1 \in \mathbb{Z} =: (\mathsf{T}_0^{a-1})_{-2}$$

(using Lemma A.24).[37]

Lemma A.35. *The homomorphism*

$$\pi_*(\mathbb{Z}^{\mathsf{hC}_{p^a}}) \xrightarrow{\pi_*(q_0^{a-1})} \pi_*((\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^{a-1}}}) \tag{A.15}$$

of graded-commutative rings guaranteed by Observation 4.13(2) (which lifts q_0^{a-1} from a morphism in $\mathsf{Mod}_\mathbb{Z}$ to a morphism in $\mathsf{CAlg}(\mathsf{Mod}_\mathbb{Z})$) is the homomorphism

$$\mathbb{Z}[c_a']/(p^a c_a') \longrightarrow (\mathbb{Z}/p^a)[c_a^\pm] \tag{A.16}$$

of graded-commutative rings characterized by the fact that it carries c_a' to c_a.

Proof. By Lemma A.24, the morphism q_0^{a-1} in $\mathsf{Ch}_\mathbb{Z}$ presents the morphism q_0^{a-1} in $\mathsf{Mod}_\mathbb{Z}$. From this, we easily identify the morphisms (A.15) and (A.16) of graded abelian groups. Moreover, it is clear that the isomorphism

$$\pi_*(\mathbb{Z}^{\mathsf{hC}_{p^a}}) \cong \mathbb{Z}[c_a']/(p^a c_a')$$

of graded abelian groups is in fact one of graded-commutative rings. It follows that the homomorphism (A.15) of graded $\pi_*(\mathbb{Z}^{\mathsf{hC}_{p^a}})$-modules must coincide with the homomorphism (A.16) of graded $\mathbb{Z}[c_a']/(p^a c_a')$-modules. Now, it suffices to observe that the commutative ring structure on the graded abelian group $(\mathbb{Z}/p^a)[c_a^\pm]$ is the only one that lifts the homomorphism (A.16) of graded $\mathbb{Z}[c_a']/(p^a c_a')$-modules to one of graded-commutative rings. $\square$

Notation A.36. Given a stably symmetric monoidal ∞-category $\mathcal{C}$, a commutative algebra object $A \in \mathsf{CAlg}(\mathcal{C})$, an A-module $M \in \mathsf{Mod}_A(\mathcal{C})$, and a morphism $\mathbb{1}_\mathcal{C} \xrightarrow{f} \Sigma^k A$ in $\mathcal{C}$ for any $k \in \mathbb{Z}$, we simply write $M \xrightarrow{f} \Sigma^k M$ for the morphism given by multiplication by f, i.e. the composite

$$M \simeq \mathbb{1}_\mathcal{C} \otimes M \xrightarrow{f \otimes \mathsf{id}_M} \Sigma^k A \otimes M \longrightarrow \Sigma^k M \ .$$

In particular, we apply this to the commutative algebra objects $\mathbb{Z}^{\mathsf{hC}_p}, \mathbb{Z}^{\mathsf{tC}_p} \in \mathsf{CAlg}(\mathsf{Mod}_\mathbb{Z}^{\mathsf{hC}_{p^r}})$ (guaranteed by Observation 4.13(2)) and the morphisms

$$c_{r+1}' \in \underline{\mathsf{hom}}_{\mathsf{Mod}_\mathbb{Z}^{\mathsf{hC}_{p^r}}}\left(\mathbb{Z}, \Sigma^2 \mathbb{Z}^{\mathsf{hC}_p}\right) \simeq \Sigma^2 \mathbb{Z}^{\mathsf{hC}_{p^{r+1}}} \qquad \text{and} \qquad c_{r+1} \in \underline{\mathsf{hom}}_{\mathsf{Mod}_\mathbb{Z}^{\mathsf{hC}_{p^r}}}\left(\mathbb{Z}, \Sigma^2 \mathbb{Z}^{\mathsf{tC}_p}\right) \simeq \Sigma^2 (\mathbb{Z}^{\mathsf{tC}_p})^{\mathsf{hC}_{p^r}}$$

of Notation A.34.

Notation A.37. For any $i, r \geq 0$ we write

$$\mathsf{T}_r^i \xrightarrow{\mathsf{c}_r^i} \Sigma^2 \mathsf{T}_r^i$$

for the evident isomorphism in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ that is the identity in all degrees.

---

[37]It is not hard to see that more generally these same elements are respectively represented by the cycles

$$N \in \mathbb{Z}[\mathsf{C}_{p^r}] =: (\mathbb{Z}_r^a)_{-2} \qquad \text{and} \qquad N \in \mathbb{Z}[\mathsf{C}_{p^r}] =: (\mathsf{T}_r^{a-1})_{-2}$$

(again using Lemma A.24).

Observation A.38. For any $i \geq 0$ and $r \geq 1$ we have an evident identification

$$\left(\mathrm{T}_r^i \xrightarrow{\mathrm{c}_r^i} \Sigma^2 \mathrm{T}_r^i \right)^{\mathsf{C}_p} \cong \left(\mathrm{T}_{r-1}^{i+1} \xrightarrow{\mathrm{c}_{r-1}^{i+1}} \Sigma^2 \mathrm{T}_{r-1}^{i+1} \right)$$

in $\mathsf{Ar}(\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r-1}}]})$.

Lemma A.39. *The morphism*

$$\mathrm{T}_r^0 \xrightarrow{\mathrm{c}_r^0} \Sigma^2 \mathrm{T}_r^0$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ *is a presentation of the morphism*

$$\mathbb{Z}^{\mathsf{tC}_p} \xrightarrow{c_{r+1}} \Sigma^2 \mathbb{Z}^{\mathsf{tC}_p}$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$ *(using Notation A.36).*

Proof. For simplicity, we write

$$c' := c'_{r+1} , \qquad c := c_{r+1} , \qquad \text{and} \qquad \mathsf{c} := \mathsf{c}_r^0 .$$

It follows from Lemma A.35 (and Observation 4.13(2)) that we have a commutative square

$$
\begin{array}{ccc}
\mathbb{Z}^{\mathsf{hC}_p} & \xrightarrow{\;\mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z})\;} & \mathbb{Z}^{\mathsf{tC}_p} \\
{\scriptstyle c'} \downarrow & & \downarrow {\scriptstyle c' \simeq c} \\
\Sigma^2 \mathbb{Z}^{\mathsf{hC}_p} & \xrightarrow[\;\mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z})\;]{} & \Sigma^2 \mathbb{Z}^{\mathsf{tC}_p}
\end{array}
\tag{A.17}
$$

in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^r}}$. By definition, the left vertical morphism in diagram (A.17) is given by

$$\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+1}}}} \left(\mathbb{Z}[\mathsf{C}_{p^r}], \mathbb{Z} \xrightarrow{c'} \Sigma^2 \mathbb{Z} \right) \simeq \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+1}}}} \left(\mathbb{Z}[\mathsf{C}_{p^r}] \otimes \left(\mathbb{Z} \xrightarrow{c'} \Sigma^2 \mathbb{Z} \right)^{\vee}, \mathbb{Z} \right)$$

$$\simeq \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+1}}}} \left(\mathbb{Z}[\mathsf{C}_{p^r}] \otimes \left(\mathbb{Z} \xleftarrow{c'} \Sigma^{-2} \mathbb{Z} \right), \mathbb{Z} \right) . \tag{A.18}$$

We give a chain-level presentation of the morphism (A.18). For this, consider the commutative triangle

$$
\begin{array}{ccc}
 & & {}_0\mathsf{S}_{r+1} \\
 & \nearrow & \downarrow {\scriptstyle \simeq} \\
\Sigma^{-2} {}_0\mathsf{S}_{r+1} & \longrightarrow & \underset{\simeq}{\mathbb{Z}}
\end{array}
\tag{A.19}
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r+1}}]}$, in which both morphisms to $\underset{\simeq}{\mathbb{Z}}$ are characterized by the fact that they act as $\mathbb{Z}[\mathsf{C}_{p^{r+1}}] \xrightarrow{\sigma \mapsto 1} \mathbb{Z}$ in degree 0 and the diagonal morphism is characterized by the fact that it acts as the identity in all nonnegative degrees (and the vertical morphism is evidently a quasi-isomorphism). Through the evident isomorphism

$$\mathsf{Z}_0^{r+1} \cong \underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r+1}}]}} ({}_0\mathsf{S}_{r+1}, \underset{\simeq}{\mathbb{Z}})$$

in $\mathsf{Ch}_{\mathbb{Z}}$, the horizontal morphism in diagram (A.19) in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r+1}}]}$ represents the morphism $\Sigma^{-2} \mathbb{Z} \xrightarrow{c'} \mathbb{Z}$ in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+1}}}$. Therefore, the diagonal morphism in diagram (A.19) in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r+1}}]}$ represents the same morphism in $\mathsf{Mod}_{\mathbb{Z}}^{\mathsf{hC}_{p^{r+1}}}$. Hence, the morphism (A.18) in $\mathsf{Mod}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ is represented by the morphism

$$\underline{\mathsf{hom}}_{\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^{r+1}}]}} \left(\underset{\sim}{\mathbb{Z}[\mathsf{C}_{p^r}]} \otimes \left({}_0\mathsf{S}_{r+1} \longleftarrow \Sigma^{-2} {}_0\mathsf{S}_{r+1} \right), \underset{\simeq}{\mathbb{Z}} \right) \tag{A.20}$$

in $\mathsf{Ch}_{\mathbb{Z}}[\mathsf{C}_{p^r}]$ (using that ${}_0\mathsf{S}_r$ is levelwise free and concentrated in nonnegative degrees).

We have just shown that the morphism (A.20) in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ is a presentation of the left vertical morphism in diagram (A.17) in $\mathsf{Mod}_{\mathbb{Z}}^{h\mathsf{C}_{p^r}}$. Unwinding the definitions, we find that the former is the morphism

$$\mathsf{z}_r^1 \xrightarrow{\ c'\ } \Sigma^2 \mathsf{z}_r^1$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$ characterized by the fact that it acts as the identity in all nonpositive degrees. To proceed, we define the mapping telescopes

$$\mathsf{tel}(\mathsf{c}') := \mathsf{colim}\left(\mathsf{z}_r^1 \xrightarrow{\ c'\ } \Sigma^2 \mathsf{z}_r^1 \xrightarrow{\ c'\ } \Sigma^4 \mathsf{z}_r^1 \xrightarrow{\ c'\ } \cdots \right) \in \mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$$

and

$$\mathsf{tel}(c') := \mathsf{colim}\left(\mathbb{Z}^{h\mathsf{C}_p} \xrightarrow{\ c'\ } \Sigma^2 \mathbb{Z}^{h\mathsf{C}_p} \xrightarrow{\ c'\ } \Sigma^4 \mathbb{Z}^{h\mathsf{C}_p} \xrightarrow{\ c'\ } \cdots \right) \in \mathsf{Mod}_{\mathbb{Z}}^{h\mathsf{C}_{p^r}} .$$

Now, by Lemma A.35, the morphism $\mathbb{Z}^{t\mathsf{C}_p} \xrightarrow{c} \Sigma^2 \mathbb{Z}^{t\mathsf{C}_p}$ in $\mathsf{Mod}_{\mathbb{Z}}^{h\mathsf{C}_{p^r}}$ is an equivalence. Therefore the commutative square (A.17) in $\mathsf{Mod}_{\mathbb{Z}}^{h\mathsf{C}_{p^r}}$ extends to a commutative diagram

$$
\begin{array}{ccccc}
 & & \mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z}) & & \\
 & & \overbrace{\hspace{6cm}} & & \\
\mathbb{Z}^{h\mathsf{C}_p} & \dashrightarrow & \mathsf{tel}(c') & \xdashrightarrow{\ \sim\ } & \mathbb{Z}^{t\mathsf{C}_p} \\
{\scriptstyle c'}\downarrow & & {\scriptstyle c'}\downarrow & & \downarrow{\scriptstyle c'\simeq c} \\
\Sigma^2\mathbb{Z}^{h\mathsf{C}_p} & \dashrightarrow & \Sigma^2\mathsf{tel}(c') & \xdashrightarrow{\ \sim\ } & \Sigma^2\mathbb{Z}^{t\mathsf{C}_p} \\
 & & \underbrace{\hspace{6cm}} & & \\
 & & \mathsf{Q}_{\mathsf{C}_p}(\mathbb{Z}) & &
\end{array}
\quad , \tag{A.21}
$$

in which the equivalences also follow from Lemma A.35. On the other hand, using Lemma A.24 (and the fact that the functor $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]} \xrightarrow{\Pi_\infty} \mathsf{Mod}_{\mathbb{Z}}^{h\mathsf{C}_{p^r}}$ commutes with filtered colimits) we see that the diagram (A.21) in $\mathsf{Mod}_{\mathbb{Z}}^{h\mathsf{C}_{p^r}}$ is presented by the diagram

$$
\begin{array}{ccccc}
 & & \mathsf{q}_r^0 & & \\
 & & \overbrace{\hspace{6cm}} & & \\
\mathsf{z}_r^1 & \longrightarrow & \mathsf{tel}(\mathsf{c}') & \xrightarrow{\ \cong\ } & \mathsf{T}_r^0 \\
{\scriptstyle c'}\downarrow & & {\scriptstyle c'}\downarrow & & \downarrow{\scriptstyle c} \\
\Sigma^2\mathsf{z}_r^1 & \longrightarrow & \Sigma^2\mathsf{tel}(\mathsf{c}') & \xrightarrow{\ \cong\ } & \Sigma^2\mathsf{T}_r^0 \\
 & & \underbrace{\hspace{6cm}} & & \\
 & & \mathsf{q}_r^0 & &
\end{array}
$$

in $\mathsf{Ch}_{\mathbb{Z}[\mathsf{C}_{p^r}]}$, in which the square on the right commutes by inspection. In particular, the claim follows. $\qquad\square$

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