THE GEOMETRY OF THE CYCLOTOMIC TRACE

DAVID AYALA, AARON MAZEL-GEE, AND NICK ROZENBLYUM

ABSTRACT. We provide a new construction of the topological cyclic homology $\mathsf{TC}(\mathcal{C})$ of any spectrally-enriched ∞-category $\mathcal{C}$, which affords a precise algebro-geometric interpretation of the cyclotomic trace map $\mathsf{K}(X) \to \mathsf{TC}(X)$ from algebraic K-theory to topological cyclic homology for any scheme X. This construction rests on a new identification of the cyclotomic structure on $\mathsf{THH}(\mathcal{C})$, which we find to be a consequence of (i) the geometry of 1-manifolds, and (ii) linearization (in the sense of Goodwillie calculus). Our construction of the cyclotomic trace likewise arises from the linearization of more primitive data.

CONTENTS

0. INTRODUCTION

0.1. **Overview.** The field of **_algebraic K-theory_** was initiated by Grothendieck in order to formulate his relative Riemann–Roch theorem [SGA71, BS58], and it has since found deep connections in such diverse areas as number theory, geometric topology, and motives [FG05]. Of course, profound implications necessitate an inherent complexity, and indeed algebraic K-theory is notoriously difficult to compute. The vast majority of known computations of algebraic K-theory result from its *infinitesimal* behavior: it admits a **_cyclotomic trace_** map

$$\mathsf{K} \longrightarrow \mathsf{TC}$$

to **_topological cyclic homology_** due to Bökstedt–Hsiang–Madsen [BHM93], which is "locally constant" by a celebrated theorem of Dundas–Goodwillie–McCarthy [Goo86, McC97, Dun97, DM94, DGM13]. Indeed, the resulting techniques – collectively known as "trace methods" – have produced a plethora of spectacular computations of algebraic K-theory [HM97b, HM97a, HM03, HM04, AGH09, AGHL14, KR97, Rog03, BM].

However, despite the central and enduring importance of the cyclotomic trace, its geometric nature – and indeed, that of TC itself – has remained mysterious. In this paper, we provide a new construction of TC which affords a precise conceptual description of the cyclotomic trace at the level of derived algebraic geometry. The broad idea can be summarized informally as follows.

Slogan 0.1. *The formation of $\mathsf{TC}(X)$ from $\mathsf{THH}(X) \simeq \mathscr{O}(\mathscr{L}X)$ amounts to cutting out precisely those functions on the free loopspace $\mathscr{L}X$*

- *which are $\mathbb{T}$-invariant, and*

- *whose value on any loop*

$$(S^1 \xrightarrow{\gamma} X) \in \mathscr{L}X$$

  determines its values on all its iterates

$$(S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X) \in \mathscr{L}X$$

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> *"up to universal indeterminacy", subject to all possible universal compatibility relations between iterates of these determinations.*

Moreover, the cyclotomic trace – the factorization

$$\mathsf{K}(X) \longrightarrow \mathsf{THH}(X)$$
$$\searrow \qquad \nearrow$$
$$\mathsf{TC}(X)$$

of the Dennis trace – amounts to the recognition that the trace-of-monodromy function of any vector bundle $E \downarrow X$ satisfies these conditions.

The concepts and assertions contained in Slogan 0.1 will be both explained and rigorized over the course of this introductory section.

The present paper is the centerpiece of a trilogy, whose overarching purpose is to formulate and prove a precise version of Slogan 0.1. However, in §0.6 we also speculate on some possible applications and future directions of inquiry. The two supporting papers are [AMGRb], in which we study cyclotomic spectra (of which THH is the primary example), and [AMGRa], in which we study enriched factorization homology (of which THH is the primary example). Our work, particularly our treatment of cyclotomic spectra, finds an important precedent in recent work of Nikolaus–Scholze [NS]; for a discussion of the relationship, see [AMGRb, Remark 0.11].

0.2. **The geometry of the Dennis trace.** By definition, the cyclotomic trace arises as an iterated factorization

$$\mathsf{K} \longrightarrow \mathsf{THH}$$
$$\mathsf{THC}^- := \mathsf{THH}^{h\mathbb{T}} \qquad (1)$$
$$\mathsf{TC} := \mathsf{THH}^{h\mathsf{Cyc}}$$

of the *Dennis trace* map to **topological Hochschild homology**. The latter carries a **cyclotomic structure**, i.e. it is naturally an object of a certain ∞-category $\mathsf{Cyc}(\mathbb{Sp})$ of *cyclotomic spectra*. This participates in an adjunction

$$
\begin{array}{ccc}
\mathbb{Sp} & \xrightleftharpoons[\;(-)^{h\mathsf{Cyc}}\;]{\;\mathsf{triv}\;} & \mathsf{Cyc}(\mathbb{Sp}) \\
\cup & & \cup \\
\mathsf{TC} & \longleftarrow & \mathsf{THH}
\end{array} \quad ,
$$

whose right adjoint – which we refer to informally and notationally as the (*homotopy*) *invariants* of the cyclotomic structure – takes THH to TC. Among other data which we will describe shortly, a cyclotomic structure consists of an action of the circle group $\mathbb{T}$, and we obtain **topological negative cyclic homology** by taking the (homotopy) fixedpoints of the underlying $\mathbb{T}$-action on THH.

In fact, it is well known how to geometrically interpret the entire upper triangle in diagram (1) [MSV97, TV09]. To explain this, suppose that X is a scheme, and let us write Perf_X for its stable ∞-category of vector bundles (i.e. perfect complexes).[1] Then, its algebraic K-theory

$$\mathsf{K}(X) := \mathsf{K}(\mathsf{Perf}_X)$$

---

[1] In fact, X can be a *spectral* scheme, or more generally any (geometric entity incarnated through its sheaves of some flavor (usually either perfect or coherent) as a) stable ∞-category. In particular, it is of course possible to take

has cocycles given by vector bundles over X, while we have an identification of its topological Hochschild homology

$$\mathsf{THH}(X) \simeq \mathscr{O}(\mathscr{L}X)$$

as the functions on the **free loopspace** of X (a derived mapping stack).[2] Under these identifications, the Dennis trace is given by the association

$$
\begin{array}{ccc}
\mathsf{K}(X) & \longrightarrow & \mathsf{THH}(X) \\
\cup & & \cup \\
\left(\begin{array}{c} \text{vector bundle} \\[4pt] E \downarrow X \end{array}\right) & \longmapsto & \left(\left(\begin{array}{c} \text{free loop} \\[4pt] S^1 \xrightarrow{\gamma} X \end{array}\right) \longmapsto \left(\begin{array}{c} \text{trace of monodromy of} \\[4pt] \gamma^* E \downarrow S^1 \end{array}\right)\right)
\end{array}^{.3} \qquad (2)
$$

From here, we observe further that the $\mathbb{T}$-action on THH corresponds to rotation of loops. Hence, the factorization of the Dennis trace through THC^- in diagram (1) – which we refer to as the **cyclic trace** – amounts to recognizing that for any vector bundle $E \downarrow X$, the function obtained by prescription (2) is invariant under precomposing the loop γ with a rotation of the circle.

0.3. **Cocycles for topological Hochschild homology.** Let us also take a moment to describe the Dennis trace more directly.

First of all, by definition, $\mathsf{THH}(X)$ is given by the (**enriched**) **factorization homology**

$$\mathsf{THH}(X) := \mathsf{THH}(\mathsf{Perf}_X) := \int_{S^1} \mathsf{Perf}_X$$

of the spectrally-enriched ∞-category Perf_X over the framed circle S^1. Thus, informally speaking, a cocycle for $\mathsf{THH}(X)$ is given by the data of

- a nonempty configuration of points $p_1, \ldots, p_n \in S^1$ on the framed circle (named cyclically),

- labelings of those points by vectors bundles $E_i \downarrow X$, and

- labelings of the framed intervals between them by maps $E_i \to E_{i+1}$.

The equivalence relation among cocycles comes from allowing points

- to disappear, in which case we compose the adjacent maps, and

- to anticollide, in which case we label the new interval by the identity map.

Somewhat more precisely, $\mathsf{THH}(X)$ is given by the colimit of a diagram indexed over such configurations and point-labelings, with value the smash product spectrum

$$\bigotimes_{i \in \mathbb{Z}/n} \underline{\mathsf{hom}}_{\mathsf{Perf}_X}(E_i, E_{i+1}) \ . \qquad (3)$$

X to be an ordinary scheme, but it is only *topological* (i.e. $\mathbb{S}$-linear) Hochschild homology which carries a cyclotomic structure. (See [AMGRb, Remark 3.9] for more on this point.)

[2]Actually, the latter identification only holds when X is sufficiently nice (e.g. a perfect stack [BZFN10]).

[3]After choosing a basepoint of the circle, a vector bundle thereover is determined by its monodromy, an automorphism of the fiber. Although different basepoints give different automorphisms, these will be conjugate to one another, and hence extracting their trace yields a well-defined invariant.

(Of course the structure maps of this diagram are just as described above, only they use *enriched* composition and unit maps of Perf_X.)

Note that among these two heuristic descriptions of $\mathsf{THH}(X)$, the former would suggest that we are simply taking a cartesian product of hom-spectra in Perf_X;[4] by contrast, the smash product (3) enforces *multilinearity*. This distinction will be crucial.

At the level of cocycles, the Dennis trace takes a vector bundle $E \downarrow X$ to the cocycle for $\mathsf{THH}(X)$ given by a single point on the circle labeled by E, with the framed interval labeled by its identity map. The identification $\mathsf{THH}(\mathsf{Perf}_X) \simeq \mathscr{O}(\mathscr{L}X)$ passes through *Morita invariance*, which yields an equivalence

$$\mathsf{THH}(\mathsf{Perf}_A) \simeq \mathsf{THH}(\mathfrak{B}A)$$

for any ring spectrum A, where $\mathfrak{B}A$ denotes its delooping to a one-object spectrally-enriched ∞-category (see Remarks 0.11 and 0.12).

0.4. **The geometry of the cyclotomic trace.** Whereas the cyclic trace arises from *automorphisms* of the circle, the cyclotomic trace arises from *endomorphisms* of the circle – that is, from the map

$$
\begin{array}{ccc}
\mathscr{L}X & \xrightarrow{\quad r^* \quad} & \mathscr{L}X \\
\cup\!\!\!| & & \cup\!\!\!| \\
\left(S^1 \xrightarrow{\gamma} X\right) & \longmapsto & \left(S^1 \xrightarrow{r} S^1 \xrightarrow{\gamma} X\right)
\end{array}
$$

given by precomposing with an r-fold self-covering map, for a natural number $r \in \mathbb{N}^\times$. Namely, given a vector bundle $E \downarrow X$, there is a sense in which the value of its trace-of-monodromy function on a loop γ determines its value on the loop $r^*\gamma$: not quite on-the-nose, but *up to universal indeterminacy*.

In fact, this phenomenon already manifests itself in the case of a square matrix $M \in \mathsf{M}_{d \times d}(R)$ over a ring R; if this matrix records the monodromy around γ, then its rth power records the monodromy around $r^*\gamma$. Let us therefore illustrate this phenomenon by considering the two elements

$$\mathsf{tr}\left(M^{\otimes r}\right), \mathsf{tr}(M)^{\otimes r} \in R^{\otimes r} \tag{4}$$

of the tensor power of the base ring. We first observe that the elements (4) are both cyclically invariant, the latter manifestly and the former by standard properties of traces of matrices; in other words, they actually lie in the fixedpoints

$$(R^{\otimes r})^{\mathsf{C}_r} \subset R^{\otimes r}$$

with respect to the action of the cyclic group C_r. Now, the elements (4) are not equal, but they differ by *norms*. The first nontrivial example occurs when $r = 2$. In this case, we have the **norm map**

$$
\begin{array}{ccc}
(R^{\otimes 2})_{\mathsf{C}_2} & \xrightarrow{\quad \mathsf{Nm} \quad} & (R^{\otimes 2})^{\mathsf{C}_2} \\
\cup\!\!\!| & & \cup\!\!\!| \\
[x \otimes y] & \longmapsto & \sum_{\sigma \in \mathsf{C}_2} \sigma(x \otimes y)
\end{array}
$$

---

[4]In fact, this is really more of a description of *unenriched* factorization homology, which in place of the smash product (3) of hom-spectra uses cartesian products of hom-spaces.

from the orbits to the fixedpoints, and we can compute directly that the difference

$$\mathsf{tr}\left(M^{\otimes 2}\right) - \mathsf{tr}(M)^{\otimes 2} = \sum_{i,j} m_{i,j} \otimes m_{j,i} - \sum_{k,l} m_{k,k} \otimes m_{l,l}$$

$$= \sum_{i \neq j} m_{i,j} \otimes m_{j,i} - \sum_{k \neq l} m_{k,k} \otimes m_{l,l}$$

$$= \sum_{i < j} \mathsf{Nm}([m_{i,j} \otimes m_{j,i}]) - \sum_{k < l} \mathsf{Nm}([m_{k,k} \otimes m_{l,l}])$$

does indeed land in its image. Thus, we obtain a canonical identification

$$\mathsf{tr}(M^{\otimes 2}) \equiv \mathsf{tr}(M)^{\otimes 2}$$

in the *cokernel* of the norm map. More generally, the elements (4) differ by norms not from just the trivial subgroup of C_r but from *all* proper subgroups; this assertion categorifies the combinatorial observation that for instance if

$$M = \begin{pmatrix} m_{1,1} & 0 \\ & \\ 0 & m_{2,2} \end{pmatrix}$$

is a 2×2 diagonal matrix, then the difference $\mathsf{tr}(M^{\otimes r}) - \mathsf{tr}(M)^{\otimes r}$ is governed by the binomial coefficients $\binom{r}{i}$ for $0 < i < r$, which are never coprime to r.[5]

Let us now return to considering $\mathsf{THH}(X)$, i.e. functions on the free loopspace of X. In this setting, we would like to compare two elements of

$$\mathsf{THH}(X)^{\tau \mathsf{C}_r} ,$$

the ("*generalized*", though we'll omit this for brevity) **Tate construction** for the natural C_r-action on $\mathsf{THH}(X)$ – a homotopical version of the "quotient the fixedpoints by norms from all proper subgroups" operation.[6,7] As we will soon see, this indeed receives two distinguished maps: one running

$$\mathsf{THH}(X)^{h\mathsf{C}_r} \longrightarrow \mathsf{THH}(X)^{\tau \mathsf{C}_r} \tag{5}$$

and corresponding to the construction "$M \mapsto \mathsf{tr}(M)^{\otimes r}$", and another running

$$\mathsf{THH}(X) \longrightarrow \mathsf{THH}(X)^{\tau \mathsf{C}_r} \tag{6}$$

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[5]For a more thorough and coordinate-independent treatment of these phenomena, we refer the reader to [NS].

[6]When $r = p$ is prime, then the only proper subgroup of C_p is the trivial subgroup. Correspondingly, in this case the generalized Tate construction $(-)^{\tau \mathsf{C}_p}$ reduces to the ordinary Tate construction, namely the cofiber

$$(-)_{h\mathsf{C}_p} \xrightarrow{\mathsf{Nm}} (-)^{h\mathsf{C}_p} \longrightarrow (-)^{t\mathsf{C}_p} .$$

[7]For a justification of this description of the generalized Tate construction, see [AMGRb, Remark 2.13].

and corresponding to the construction "$M \mapsto \mathsf{tr}\,(M^{\otimes r})$".[8] These are suitably equivariant, in such a way that they determine a pair of parallel maps

$$\mathsf{THH}(X)^{h\mathbb{T}} \rightrightarrows \left(\mathsf{THH}(X)^{\tau \mathsf{C}_r}\right)^{h\mathbb{T}} \quad ;$$

unwinding the definitions, it follows that the equalizer of this pair picks out

> $\mathbb{T}$-*equivariant functions on* $\mathscr{L}X$ *whose value on any loop* γ *determines its values on the iterate* $r^*\gamma$ *up to universal indeterminacy,*

thus embodying much of the algebro-geometric description of $\mathsf{TC}(X)$ asserted in Slogan 0.1.

But much more is true: these structure maps will come equipped with an *infinite hierarchy* of compatibilities, corresponding to all possible composable sequences of self-coverings of the circle as well as a choice at each stage of whether to enact the operation "$M \mapsto \mathsf{tr}(M)^{\otimes r}$" or the operation "$M \mapsto \mathsf{tr}\,(M^{\otimes r})$".[9] Collectively, the limit over all of these structure maps will impose

> *all possible universal compatibility relations between iterates of these determinations,*

which is the remaining assertion of Slogan 0.1 regarding the geometry of $\mathsf{TC}(X)$; these compatibilities will be spelled out more explicitly in Remark 0.5. All in all, we will see that in an admittedly revisionistic sense,

> $\mathsf{TC}(X)$ *is precisely built to encode all of the structure present on those functions on* $\mathscr{L}X$ *that arise as trace-of-monodromy functions of vector bundles over* X,

completing the explanation of Slogan 0.1.

0.5. **Main theorems.** We now describe the four central theorems of the trilogy to which this paper belongs, which collectively implement the geometric picture described in §0.4 above; the first and third are proved in [AMGRb], while the second and fourth are proved here.

First of all, we reidentify the ∞-category $\mathsf{Cyc}(\mathsf{Sp})$ of cyclotomic spectra in terms of the Tate construction.

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[8] It follows from an analysis at the level of cyclic bar constructions that these maps do indeed behave as claimed. The key observation is that the trace actually occurs in the equivalence $\mathsf{THH}(\mathfrak{B}(\mathsf{M}_{d \times d}(R))) \xrightarrow{\sim} \mathsf{THH}(\mathfrak{B}R)$ coming from Morita invariance (see e.g. [Lod98, Theorem 1.2.4]); then, we can see the map (5) as coming from the composite

$$\left(\mathsf{M}_{d \times d}(R)^{\otimes r(\bullet+1)}\right)^{h\mathsf{C}_r} \longrightarrow \left(R^{\otimes r(\bullet+1)}\right)^{h\mathsf{C}_r} \longrightarrow \left(R^{\otimes r(\bullet+1)}\right)^{\tau\mathsf{C}_r}$$

and the map (6) as coming from the composite

$$\mathsf{M}_{d \times d}(R)^{\otimes(\bullet+1)} \longrightarrow \left(\mathsf{M}_{d \times d}(R)^{\otimes r(\bullet+1)}\right)^{\tau\mathsf{C}_r} \longrightarrow \left(R^{\otimes r(\bullet+1)}\right)^{\tau\mathsf{C}_r} \quad .$$

(More generally we might consider arbitrary composable (not necessarily square) matrices, but this would only serve to make the notation more complicated, and is itself rendered unnecessary by Morita invariance; the prior reduction to free objects (instead of just their retracts) is likewise justified by Morita invariance.)

[9] In fact, there will be yet more compatibilities, accounting for the fact that the "quotienting by norms from all proper subgroups" operation doesn't strictly commute with multiplication of natural numbers (the orders of the various cyclic groups); see Remark 0.2.

Theorem A (A reidentification of cyclotomic spectra (proved in [AMGRb])). *The endofunctors*

$$\mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p}) \longrightarrow \mathsf{Fun}(B(\mathbb{T}/C_r), \mathcal{S}\mathsf{p}) \xrightarrow[\sim]{(\mathbb{T}/C_r)\simeq\mathbb{T}} \mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p})$$

$$\cup \qquad\qquad\qquad\qquad \cup \qquad\qquad\qquad\qquad \cup$$

$$(\mathbb{T} \curvearrowright E) \longmapsto \left(\mathbb{T}/C_r \curvearrowright E^{\tau C_r}\right) \longmapsto \left(\mathbb{T} \curvearrowright E^{\tau C_r}\right)$$

for $r \in \mathbb{N}^{\times}$ assemble into a left-lax right action

$$\mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p}) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times} \ .$$

Moreover, there is a canonical equivalence of ∞-categories

$$\mathsf{Cyc}(\mathcal{S}\mathsf{p}) \simeq \lim{}^{\mathsf{r.lax}} \left(\mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p}) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right)$$

with the right-lax limit of this left-lax right action. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 0.2. To say that the action is *left-lax* is to say that there are only comparison morphisms (rather than equivalences) governing its associativity. For instance, for each pair of elements $r, s \in \mathbb{N}^{\times}$ and any $E \in \mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p})$, there is a canonical map

$$E^{\tau C_{rs}} \xrightarrow{\chi_{r,s}} \left(E^{\tau C_r}\right)^{\tau C_s} \ .^{10} \tag{7}$$

Then, an object of the *right-lax* limit of this left-lax right action – that is, a cyclotomic spectrum – consists of the following data:

(0) an object

$$T \in \mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p}) \ ,$$

namely the **underlying $\mathbb{T}$-spectrum** of the cyclotomic spectrum;

(1) for every element $r \in \mathbb{N}^{\times}$, a **cyclotomic structure map**

$$T \xrightarrow{\sigma_r} T^{\tau C_r}$$

in $\mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p})$;

(2) for every pair of elements $r, s \in \mathbb{N}^{\times}$, the *data* of a commutative square

$$
\begin{array}{ccc}
T & \xrightarrow{\ \sigma_s\ } & T^{\tau C_s} \\
{\scriptstyle \sigma_{rs}}\downarrow & & \downarrow{\scriptstyle (\sigma_r)^{\tau C_s}} \\
T^{\tau C_{rs}} & \xrightarrow{\ \chi_{r,s}\ } & \left(T^{\tau C_r}\right)^{\tau C_s}
\end{array}
$$

in $\mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p})$, denoted $\sigma_{r,s}$;
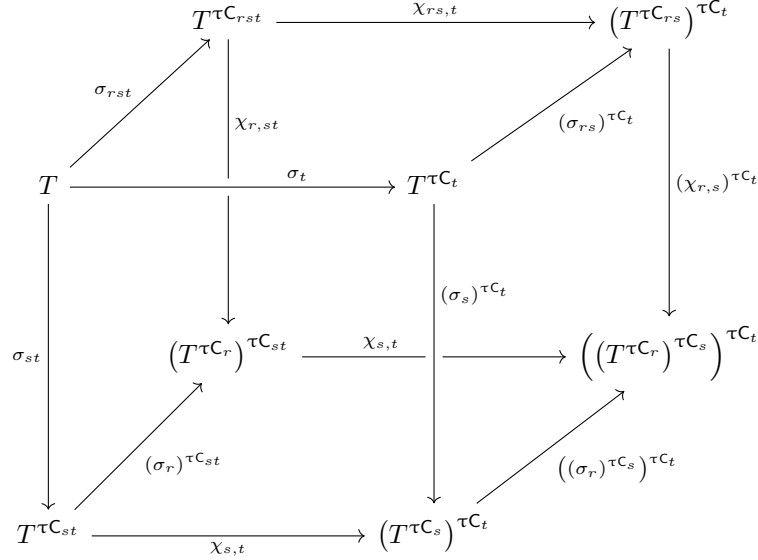
---

[10]More generally, for any word $W := (r_1, \ldots, r_n)$ of elements of $\mathbb{N}^{\times}$, writing $|W| := \prod_i r_i$ for the product of its elements, there is a canonical map

$$E^{\tau C_{|W|}} \xrightarrow{\chi_W} E^{\tau C_W} := E^{\tau C_{r_1} \cdots \tau C_{r_n}} \ .$$

These maps satisfy various compatibilities, a few of which will appear presently.

(3) for every triple of elements $r, s, t \in \mathbb{N}^\times$, the *data* of a commutative 3-cube

in $\mathsf{Fun}(\mathsf{B}\mathbb{T}, \mathcal{S}\mathsf{p})$, denoted $\sigma_{r,s,t}$,

- whose front, top, and left, and right faces are the commutative squares $\sigma_{s,t}$, $\sigma_{rs,t}$, $\sigma_{r,st}$, and $(\sigma_{r,s})^{\tau\mathsf{C}_t}$, respectively,

- whose bottom face is determined by the naturality of $\chi_{s,t}$, and

- whose back face is determined by the compatibility of the various χ maps (7);

$$\vdots$$

(n) in general, for each word $W := (r_1, \ldots, r_n)$ in $\mathbb{N}^\times$, the *data* of a commutative n-cube σ_W in $\mathsf{Fun}(\mathsf{B}\mathbb{T}, \mathcal{S}\mathsf{p})$, whose faces are given

- by previously specified data (inductively for all $n \geq 0$) and

- by the various coherences enjoyed by the "χ" comparison maps (7).

These data are subject to certain compatibility relations that apply whenever the unit element $1 \in \mathbb{N}^\times$ appears in a word.

Remark 0.3. The sorts of words in $\mathbb{N}^\times$ that are relevant to us are parametrized by the subdivision category $\mathsf{sd}(\mathsf{B}\mathbb{N}^\times)$: its objects are precisely the equivalence classes of words in $\mathbb{N}^\times$ under the relation that any instances of the element $1 \in \mathbb{N}^\times$ may be freely inserted or omitted.[11] In fact, this is not a coincidence: subdivision plays a key role in the definition of a *right*-lax limit of a *left*-lax action (i.e. it appears because these handednesses disagree), and it will reappear in our explicit formula for TC (see Theorem C and Remark 0.5).

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[11]Due to the equivalence relation, it doesn't matter mathematically whether we allow the empty word $\varnothing$ or not, since it's equivalent to the word (1). However, it will be convenient to use the evident "normal form" where we simply omit all 1's, which leads us to choose to allow the empty word.

In order to state our next theorem, we introduce/recall the following notation:

- $\mathcal{S}$ and $\mathcal{S}\mathsf{p}$ respectively denote the ∞-categories of spaces and of spectra;

- $\mathsf{Cat}(\mathcal{S}) \simeq \mathsf{Cat}$ and $\mathsf{Cat}(\mathcal{S}\mathsf{p})$ respectively denote the ∞-categories of spatially-enriched (i.e. unenriched) ∞-categories and of spectrally-enriched ∞-categories;

- $\mathsf{THH}_{\mathcal{S}}$ and $\mathsf{THH} := \mathsf{THH}_{\mathcal{S}\mathsf{p}}$ respectively denote spatially-enriched and spectrally-enriched factorization homology over the circle;

- $\mathsf{Cyc}^{\mathsf{h}}(\mathcal{S})$ and $\mathsf{Cyc}(\mathcal{S}\mathsf{p}) := \mathsf{Cyc}^{\tau}(\mathcal{S}\mathsf{p})$ respectively denote the ∞-categories of unstable cyclotomic spaces (recalled in §3 below) and of cyclotomic spectra.

Theorem B (The cyclotomic structure on THH (proved in Part I))**.**

(1) *There is a **diagonal package** for spaces, which induces a canonical lift*

$$\mathsf{Cat}(\mathcal{S}) \dashrightarrow \mathsf{Cyc}^{\mathsf{h}}(\mathcal{S})$$

*through the forgetful functor, yielding the **unstable cyclotomic structure** on* $\mathsf{THH}_{\mathcal{S}}$.

(2) *Via linearization (in the sense of Goodwillie calculus), the diagonal package for spaces gives rise to the **Tate package** for spectra, which induces a canonical lift*

$$\mathsf{Cat}(\mathcal{S}\mathsf{p}) \dashrightarrow \mathsf{Cyc}(\mathcal{S}\mathsf{p})$$

*through the forgetful functor, yielding the **cyclotomic structure** on* THH.

Remark 0.4. The term **package** is meant to emphasize that this object *packages* together the named maps, along with all of their requisite functorialities: the *diagonal package* packages the diagonal maps

$$V \longrightarrow \left(V^{\times r}\right)^{\mathsf{hC}_r}$$

for spaces $V \in \mathcal{S}$ (which are in fact equivalences), while the *Tate package* packages the **Tate diagonal** maps

$$E \longrightarrow \left(E^{\otimes r}\right)^{\tau \mathsf{C}_r}$$

for spectra $E \in \mathcal{S}\mathsf{p}$. The way in which diagonal maps for spaces induce the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{S}}$ is described in §3.8; the Tate diagonal maps for spectra induce the cyclotomic structure on THH in an essentially identical way.

Together, Theorems A and B go much of the way towards formalizing the assertions of Slogan 0.1 regarding TC. However, what still lacks is a concrete formula for the passage from THH to TC,

guaranteeing that it does indeed implement the asserted geometric description. Our next result fills this gap.

Theorem C (The formula for TC (proved in [AMGRb])). *There is a canonical factorization*

$$
\begin{array}{ccc}
\mathcal{S}\mathsf{p} & \xleftarrow{\quad (-)^{\mathsf{hCyc}} \quad} & \mathsf{Cyc}(\mathcal{S}\mathsf{p}) \\
& \searrow & \downarrow \\
\scriptstyle\lim & & \scriptstyle (-)^{\mathsf{h}\mathbb{T}} \\
& & \\
& \mathsf{Fun}(\mathsf{sd}(\mathsf{B}\mathbb{N}^{\times}), \mathcal{S}\mathsf{p}) &
\end{array}
\qquad , ^{12}
$$

where the first functor takes a cyclotomic spectrum $T \in \mathsf{Cyc}(\mathcal{S}\mathsf{p})$ to the diagram

$$
\begin{array}{ccc}
\mathsf{sd}(\mathsf{B}\mathbb{N}^{\times}) & \xrightarrow{\quad T^{\mathsf{h}\mathbb{T}} \quad} & \mathcal{S}\mathsf{p} \\
\cup & & \cup \\
(r_1, \ldots, r_k) =: W & \longmapsto & \left(T^{\tau\mathsf{C}_W}\right)^{\mathsf{h}\mathbb{T}} := \left(T^{\tau\mathsf{C}_{r_1} \cdots \tau\mathsf{C}_{r_k}}\right)^{\mathsf{h}\mathbb{T}}
\end{array}
\qquad (8)
$$

of spectra. □

Remark 0.5. The category $\mathsf{sd}(\mathsf{B}\mathbb{N}^{\times})$ and diagram (8) are described in detail in [AMGRb, Remark 3.35]. As described there, the morphisms in $\mathsf{sd}(\mathsf{B}\mathbb{N}^{\times})$ are generated by the operations of

- adding a letter to the beginning of a word,

- adding a letter to the end of a word, and

- factoring a letter in a word.

In particular, the empty word

$$(\ ) \in \mathsf{sd}(\mathsf{B}\mathbb{N}^{\times})$$

is an empty object (i.e. it receives no nonidentity maps), and moreover it maps to every other object in at least two distinct ways. Hence, the extraction of the limit

$$\lim(8) =: T^{\mathsf{hCyc}}$$

amounts to *cutting out equations* inside of the value

$$T^{\mathsf{h}\mathbb{T}} \in \mathcal{S}\mathsf{p}$$

of the diagram (8) at the empty word.

Let us now specialize to our particular case of interest, namely when we take our cyclotomic spectrum to be $T = \mathsf{THH}(X)$ for a scheme X. In this case, we have that

$$\mathsf{THH}(X)^{\mathsf{h}\mathbb{T}} \simeq (\mathcal{O}(\mathscr{L}X))^{\mathsf{h}\mathbb{T}}$$

consists of the $\mathbb{T}$-invariant functions on the free loopspace $\mathscr{L}X$. Then, the three operations described above can be interpreted *geometrically*: they correspond to operations on such functions of the form

- "$M \mapsto \mathsf{tr}\left(M^{\otimes r}\right)$",

---

[12]It would be slightly more correct to write $\mathsf{sd}\left((\mathsf{B}\mathbb{N}^{\times})^{\mathsf{op}}\right)$ instead of $\mathsf{sd}(\mathsf{B}\mathbb{N}^{\times})$, but these two categories are canonically equivalent (due to the commutativity of the monoid $\mathbb{N}^{\times}$) and so to simplify our notation we elide the distinction here.

- "$M \mapsto \mathsf{tr}(M)^{\otimes r}$", and

- "take a further quotient corresponding to the new universal indeterminacies that arise in factoring an rs-fold covering map as an r-fold covering map followed by an s-fold covering map".

Thus, Theorem C shows that the passage from $\mathsf{THH}(X)$ to $\mathsf{TC}(X)$ afforded by Theorem B does indeed implement the geometric description asserted in Slogan 0.1.

In order to state our final theorem, we introduce/recall the following notation.

- We write
$$\mathsf{TC}_{\mathcal{S}} : \mathsf{Cat} \xrightarrow{\mathsf{THH}_{\mathcal{S}}} \mathsf{Cyc}^{\mathsf{h}}(\mathcal{S}) \xrightarrow{(-)^{\mathsf{hW}}} \mathcal{S}$$
for the unstable topological cyclic homology functor (recalled in §5). This is the composite of spatially-enriched factorization homology over the circle with the "homotopy invariants of the unstable cyclotomic structure" functor; the notation $\mathbb{W}$ is explained in §3.

- We write
$$\mathsf{TC} : \mathsf{Cat}(\mathcal{S}\mathsf{p}) \xrightarrow{\mathsf{THH}} \mathsf{Cyc}(\mathcal{S}\mathsf{p}) \xrightarrow{(-)^{\mathsf{hCyc}}} \mathcal{S}\mathsf{p}$$
for the topological cyclic homology functor. This is the composite of spectrally-enriched factorization homology over the circle with the "homotopy invariants of the cyclotomic structure" functor.

- For any enriching ∞-category $\mathcal{V}$, we write
$$\mathsf{Cat}(\mathcal{V}) \xrightarrow{\iota} \mathcal{S}$$
for the "underlying ∞-groupoid" functor.

Theorem D (The cyclotomic trace (proved in Part II)).

(1) *The contravariant functoriality of spatially-enriched factorization homology with respect to the proper constructible bundle*
$$\mathbb{D}^0 \longleftarrow S^1$$
induces an **unstable Dennis trace**
$$\iota \simeq \int_{\mathbb{D}^0} \longrightarrow \int_{S^1} =: \mathsf{THH}_{\mathcal{S}}$$
in $\mathsf{Fun}(\mathsf{Cat}, \mathcal{S})$, *whose equivariance induces a factorization*

through the **unstable cyclotomic trace**.

(2) *Via linearization (in the sense of Goodwillie calculus) and the equivalence*
$$\Sigma^{\infty}_{+} \iota \simeq \int_{\mathbb{D}^0}$$

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in $\mathsf{Fun}(\mathsf{Cat}(\mathcal{S}p), \mathcal{S}p)$, *the unstable Dennis and cyclotomic trace maps give rise to the **Dennis pre-trace** and **cyclotomic pre-trace** maps*

$$
\begin{array}{ccc}
\Sigma_+^\infty \iota & \dashrightarrow & \mathsf{THH} \\
& \searrow & \uparrow \\
& & \mathsf{TC}
\end{array}
$$

in $\mathsf{Fun}(\mathsf{Cat}(\mathcal{S}p), \mathcal{S}p)$, *the latter of which when restricted to the subcategory*

$$\mathsf{StCat} \subset \mathsf{Cat}(\mathcal{S}p)$$

*of stable ∞-categories and exact functors between them induces the **cyclotomic trace** map*

$$
\begin{array}{ccc}
\Sigma_+^\infty \iota & \longrightarrow & \mathsf{THH} \\
\downarrow & \searrow & \uparrow \\
\mathsf{K} & \dashrightarrow & \mathsf{TC}
\end{array}
$$

in $\mathsf{Fun}(\mathsf{StCat}, \mathcal{S}p)$.

Remark 0.6. It is immediate that for a scheme X, the Dennis pre-trace

$$\Sigma_+^\infty \iota(\mathsf{Perf}_X) \longrightarrow \mathsf{THH}(\mathsf{Perf}_X) =: \mathsf{THH}(X)$$

behaves as described in §0.3: it takes a vector bundle $E \in \mathsf{Perf}_X$ to the cocycle for $\mathsf{THH}(X)$ represented by a single point on the circle labeled by E, with the framed interval labeled by its identity map. As under the identification

$$\mathsf{THH}(X) \simeq \mathscr{O}(\mathscr{L}X)$$

this cocycle corresponds to the trace-of-monodromy function of E, the cyclotomic (pre-)trace indeed amounts to the observation that such a function necessarily satisfies the conditions imposed by taking the homotopy invariants of the cyclotomic structure. Thus, Theorem D completes the rigorization of the assertions of Slogan 0.1.

0.6. **Miscellaneous remarks.**

Remark 0.7. There is an evident cardinality filtration

$$\mathsf{THH} \simeq \mathsf{colim}\left(\mathsf{THH}_{\leq 1} \longrightarrow \mathsf{THH}_{\leq 2} \longrightarrow \mathsf{THH}_{\leq 3} \longrightarrow \cdots\right)$$

of THH, obtained by restricting the number of points on the circle. This filtration is respected by the $\mathbb{T}$-action, and so it refines to a filtration

$$(\mathsf{THC}^-)_{\leq n} := (\mathsf{THH}_{\leq n})^{h\mathbb{T}}$$

of $\mathsf{THC}^- := \mathsf{THH}^{h\mathbb{T}}$.[13] In fact, our construction provides for a further refinement to a filtration of TC: for a word

$$W := (r_1, \ldots, r_k) \in \mathsf{sd}(B\mathbb{N}^\times) \ ,$$

---

[13]This will not generally be "exhaustive" (i.e. it won't generally recover THC^- in the colimit), since directed colimits don't generally commute with homotopy $\mathbb{T}$-fixedpoints.

if we write
$$|W| := \prod_i r_i \in \mathbb{N}^\times$$
for the product of its elements, then we set
$$\mathsf{TC}_{\leq n} := \lim_{\mathsf{sd}(\mathsf{B}\mathbb{N}^\times)} \left((\mathsf{THH}_{\leq n \cdot |W|})^{\tau \mathsf{C}_W} \right)^{\mathsf{h}\mathbb{T}} .[14]$$

Preliminary computations suggest that these filtrations should be closely related to the Goodwillie towers of the *relative* versions of the respective theories. This would give purchase on nonconnective and nonaffine generalizations of the Dundas–Goodwillie–McCarthy theorem, which at present only applies in the somewhat restrictive case of connective ring spectra.

Remark 0.8. Whereas most previous work on THH and TC relies on simplicial methods (e.g. the cyclic bar construction), our work uses the geometry of 1-manifolds. This brings conceptual advantages: it clarifies the connection with loopspaces, and it suggests the appropriate higher-dimensional generalization (see Remark 0.9). But it also brings an important technical advantage: the desired symmetries are built into the framework, rather than having to be put back in by hand (as e.g. an action of the simplicial circle). In effect, employing simplicial methods amounts to choosing a basepoint on the circle (see [AMGRa, Remark 1.28]), which destroys much of its symmetry.

Remark 0.9. Our work paves the way for *higher-dimensional* versions of TC, defined for any spectrally-enriched (∞, d)-category: this should simultaneously involve all compact framed d-manifolds and (framed) covering maps among them. One of our main technical results – the *proto Tate package* [AMGRb, Theorem 3.3], which mediates the passage from the diagonal package to the Tate package of Theorem B – is already sufficiently general to apply in this situation, since it handles arbitrary finite groups and their iterated extensions (appearing as the deck groups of covering maps). We expect that this should reduce to the notion of "covering homology" [BCD10] in the case of a connective commutative ring spectrum (viewed as a flagged (see Remark 0.11) spectrally-enriched (∞, d)-category with a single k-morphism for all $k < d$).

Remark 0.10. We phrase our work in terms of ∞-categories enriched in spaces and spectra, but in fact it applies word-for-word to the more general context of a (presentably) *cartesian* symmetric monoidal ∞-category $\mathcal{V}$ and its stabilization $\mathsf{Stab}(\mathcal{V})$. On the other hand, much of our development of enriched factorization homology is stated in terms of an *arbitrary* symmetric monoidal ∞-category $\mathcal{V}$; we only specialize to spaces and spectra when we start to use their more particular features.

Remark 0.11. We work not just with $\mathcal{V}$-enriched ∞-categories, but with *flagged* $\mathcal{V}$-enriched ∞-categories, i.e. "a $\mathcal{V}$-enriched ∞-category with a chosen space of objects".[15] In particular, a ring spectrum A determines a flagged spectrally-enriched ∞-category $\mathfrak{B}A$ whose "space of objects" consists of a single point.[16] This added generality is useful (and requires no extra work), since

---

[14]Note that this filtration of TC does not arise from a filtration of THH by cyclotomic spectra.

[15]An enriched ∞-category $\mathcal{C}$ has a canonically defined maximal subgroupoid, but one can also present it as having for its "space of objects" some other ∞-groupoid: to do so amounts to choosing a π_0-surjection from the latter to the former. Of course, taking this map to be an equivalence recovers the terminal such choice: this generalizes the relationship between Segal spaces and *complete* Segal spaces (see [AMGRa, Remark 2.10]).

[16]Indeed, these two types of data are manifestly equivalent; see §2.1.

it recovers the more classical definition of $\mathsf{THH}(A)$ via the cyclic bar construction (see [AMGRa, Remark 1.28]).

Remark 0.12. Enriched factorization homology over the circle satisfies Morita invariance. To illustrate this, we offer the following proof-by-picture. Let $\mathcal{C}$ and $\mathcal{D}$ be flagged $\mathcal{V}$-enriched ∞-categories, and suppose that

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{D} \xrightarrow{\mathcal{P}} \mathcal{V}$$

and

$$\mathcal{D}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{Q}} \mathcal{V}$$

are bimodules determining a Morita equivalence between $\mathcal{C}$ and $\mathcal{D}$. Then, we have the string of equivalences

in $\mathcal{V}$ among the indicated values of "$\mathcal{B}$-framed" factorization homology [AFT17], a generalization of factorization homology that allows distinct strata to be labeled by distinct algebraic data.[17]

Remark 0.13. An arbitrary functor

$$\mathcal{C} \longrightarrow \mathcal{D}$$

of flagged $\mathcal{V}$-enriched ∞-categories induces a $(\mathcal{C}, \mathcal{D})$-bimodule

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{D} \longrightarrow \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \xrightarrow{\underline{\mathsf{hom}}_{\mathcal{D}}(-,-)} \mathcal{V} .$$

This allows us to consider the ∞-category $\mathsf{fCat}(\mathcal{V})$ of flagged $\mathcal{V}$-enriched ∞-categories as a full subcategory of a certain Morita ∞-category.[18] Thus, bimodules can be considered as a generalization of ordinary functors, and in particular it is meaningful to assert that a morphism in $\mathsf{fCat}(\mathcal{V})$ is a Morita equivalence.

Let us note two examples of this phenomenon of particular interest.

(1) Let $A \in \mathsf{Alg}(\mathcal{S}\mathsf{p})$ be a ring spectrum, and write $\mathfrak{B}A$ for its corresponding one-object flagged $\mathcal{V}$-enriched ∞-category. Then there is a canonical spectrally-enriched functor

$$\mathfrak{B}A \longrightarrow \mathsf{Perf}_A ,$$

---

[17]It would require a nontrivial amount of rigorously merge the formalism of $\mathcal{B}$-framed factorization homology developed in [AFT17] with the formalism of enriched factorization homology developed in [AMGRa]. On the other hand, note that this assertion is proved in [BM12] (in the "set of objects" and spectrally-enriched setting).

[18]It is indeed merely a condition for a bimodule to be represented by an actual functor: this is the condition that its adjunct factors through the Yoneda embedding, a monomorphism.

which takes the unique object of $\mathfrak{B}A$ to the object $A \in \mathsf{Perf}_A$ and induces an equivalence $A \xrightarrow{\sim} \underline{\mathsf{hom}}_{\mathsf{Perf}_A}(A, A)$ on hom-objects. This is a Morita equivalence because Perf_A is the *stable idempotent-completion* of $\mathfrak{B}A$, and so the restriction

$$\underline{\mathsf{Fun}}(\mathfrak{B}A, \mathcal{Sp}) \xleftarrow{\sim} \underline{\mathsf{Fun}}(\mathsf{Perf}_A, \mathcal{Sp}) \simeq \mathsf{Fun}^{\mathsf{ex}}(\mathsf{Perf}_A, \mathcal{Sp})$$

is an equivalence (where $\underline{\mathsf{Fun}}$ denotes the ∞-category of spectrally-enriched functors and $\mathsf{Fun}^{\mathsf{ex}}$ denotes the ∞-category of exact functors).

(2) Let $\mathcal{C} \in \mathsf{fCat}(\mathcal{V})$ be any flagged $\mathcal{V}$-enriched ∞-category, and write $\overline{\mathcal{C}} \in \mathsf{Cat}(\mathcal{V})$ for its completion (see Footnote 15). Then, the canonical functor

$$\mathcal{C} \longrightarrow \overline{\mathcal{C}}$$

is a Morita equivalence: the restriction

$$\mathsf{Mod}_{\mathcal{C}} := \underline{\mathsf{Fun}}(\mathcal{C}, \mathcal{V}) \xleftarrow{\sim} \underline{\mathsf{Fun}}(\overline{\mathcal{C}}, \mathcal{V}) =: \mathsf{Mod}_{\overline{\mathcal{C}}}$$

is an equivalence since $\mathcal{V}$ itself is complete.[19]

In particular, combining example (2) with Remark 0.12 implies that our definition of THH (for spectrally-enriched ∞-categories) agrees with those defined in terms of a set of objects, as e.g. in [BM12].

Remark 0.14. The forgetful functor

$$\mathsf{Cyc}(\mathcal{Sp}) \xrightarrow{\mathsf{fgt}} \mathcal{Sp}$$

is conservative, as both functors in the composite

$$\mathsf{Cyc}(\mathcal{Sp}) \xrightarrow{\mathsf{fgt}} \mathsf{Fun}(\mathsf{B}\mathbb{T}, \mathcal{Sp}) \xrightarrow{\mathsf{fgt}} \mathcal{Sp}$$

are easily seen to be conservative. In particular, for any ring spectrum $A \in \mathsf{Alg}(\mathcal{Sp})$, the canonical functor

$$\mathfrak{B}A \longrightarrow \mathsf{Perf}_A$$

of Remark 0.13 induces an equivalence

$$\mathsf{THH}(\mathfrak{B}A) \xrightarrow{\sim} \mathsf{THH}(\mathsf{Perf}_A)$$

of cyclotomic spectra, and thereafter an equivalence

$$\mathsf{TC}(\mathfrak{B}A) \xrightarrow{\sim} \mathsf{TC}(\mathsf{Perf}_A) \ .$$

More generally, for any $\mathcal{C} \in \mathsf{fCat}(\mathcal{Sp})$, the Yoneda embedding $\mathcal{C} \to \mathsf{Perf}_{\mathcal{C}}$ (into the stable idempotent-completion) is a Morita equivalence. Moreover, Morita equivalences between stable idempotent-complete ∞-categories always arise from actual functors. Thus, *all* Morita equivalences in $\mathsf{fCat}(\mathcal{Sp})$ can be realized as zigzags of enriched functors; hence, THH takes them to equivalences of cyclotomic spectra, and TC takes them to equivalences of spectra.

---

[19]By passing from $\mathcal{V}$ to the ∞-category of presheaves on $\mathcal{V}$ equipped with Day convolution (and noting that the Yoneda embedding is a monoidal monomorphism), one can assume without loss of generality that $\mathcal{V}$ is presentably (and hence closed) monoidal.

0.7. **Outline.** This paper is entirely self-contained: over the course of our proofs, we recall everything that we need from the supporting papers [AMGRb, AMGRa]. It is organized as follows.

- In Part I, we prove Theorem B (the cyclotomic structure on THH). We begin with a review of stratified 1-manifolds in §1, and of enriched factorization homology in §2. We then study the unstable cyclotomic structure on spatially-enriched factorization homology over the circle in §3, in a way that anticipates its stable analog. Finally, in §4 we construct the cyclotomic structure on spectrally-enriched factorization homology over the circle (i.e. THH).

- In Part II, we deduce Theorem D (the cyclotomic trace). In §6, we review the unstable cyclotomic trace. Then, in §5 we use this to obtain the cyclotomic trace.

0.8. **Notation and conventions.**

(1) We work within the context of ∞-categories, taking [Lur09] and [Lur] as our standard references. We work model-independently (for instance, we make no reference to the simplices of a quasicategory), and we omit all technical uses of the word "essentially" (for instance, we shorten the term "essentially surjective" to "surjective"). We also make some light use of the theory of $(\infty, 2)$-categories, which is developed in the appendix of [GR17].

(2) We use the following decorations for our functors.

- An arrow $\longhookrightarrow$ denotes a monomorphism, i.e. the inclusion of a subcategory: a functor which is fully faithful on equivalences and induces inclusions of path components (i.e. monomorphisms) on all hom-spaces.

- An arrow $\xhookrightarrow{\text{f.f.}}$ denotes a fully faithful functor.

- An arrow $\longrightarrow\!\!\!\!\!\rightarrow$ denotes a surjection.

- An arrow $\downarrow$ denotes a functor considered as an object in the overcategory of its target (which will often be some sort of fibration).

More generally, we use the notation $\downarrow$ to denote a morphism in any ∞-category that we consider as defining an object in an overcategory.

(3) Given some datum in an ∞-category (such as an object or morphism), for clarity we may use the superscript $(-)^\circ$ to denote the corresponding datum in the opposite ∞-category.

(4) We write Cat for the ∞-category of ∞-categories, $\mathcal{S}$ for the ∞-category of spaces, and $\mathcal{Sp}$ for the ∞-category of spectra. These are related by the various adjoint functors

$$\mathsf{Cat} \underset{(-)^{\simeq}}{\overset{(-)^{\mathsf{gpd}}}{\underset{\perp}{\overset{\perp}{\rightleftarrows}}}} \mathcal{S} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\underset{\perp}{\rightleftarrows}}} \mathcal{Sp} \quad .$$

(5) For a base ∞-category $\mathcal{B}$, we define the commutative diagrams of monomorphisms among ∞-categories

$$
\begin{array}{ccc}
\mathsf{coCart}_{\mathcal{B}} & \hookrightarrow\!\!\!\rightarrow & \mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}} \\
{\scriptstyle \text{f.f.}}\downarrow & & \downarrow{\scriptstyle \text{f.f.}} \\
\mathsf{loc.coCart}_{\mathcal{B}} & \hookrightarrow\!\!\!\rightarrow & \mathsf{Cat}_{\mathsf{loc.cocart}/\mathcal{B}}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathsf{Cart}_{\mathcal{B}} & \hookrightarrow\!\!\!\rightarrow & \mathsf{Cat}_{\mathsf{cart}/\mathcal{B}} \\
{\scriptstyle \text{f.f.}}\downarrow & & \downarrow{\scriptstyle \text{f.f.}} \\
\mathsf{loc.Cart}_{\mathcal{B}} & \hookrightarrow\!\!\!\rightarrow & \mathsf{Cat}_{\mathsf{loc.cart}/\mathcal{B}}
\end{array}
$$

as follows:

- objects in the upper rows are co/cartesian fibrations over $\mathcal{B}$,

- objects in the lower rows are locally co/cartesian fibrations over $\mathcal{B}$,

- morphisms in the left columns are functors over $\mathcal{B}$ which preserve co/cartesian morphisms, and

- morphisms in the right columns are arbitrary functors over $\mathcal{B}$.

(6) We make use of the theory of *exponentiable fibrations* of [AF] (see also [AFR, §5]), an ∞-categorical analog of the "Conduché fibrations" of [Gir64, Con72]: these are the objects $(\mathcal{E} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$ satisfying the condition that there exists a right adjoint

$$
\mathsf{Cat}_{/\mathcal{B}} \xrightleftharpoons[\mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E},-)]{\overset{-\times_{\mathcal{B}}\mathcal{E}}{\underset{\perp}{\longrightarrow}}} \mathsf{Cat}_{/\mathcal{B}}
$$

to the pullback; by the adjoint functor theorem, these can be equivalently characterized as those objects for which the proposed left adjoint preserves colimits. We refer to this right adjoint as the ***relative functor ∞-category*** construction; it is analogous to the internal hom of presheaves. Thus, for any target object $(\mathcal{F} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$ and any test object $(\mathcal{K} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$, a lift

$$
\begin{array}{ccc}
& & \mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E},\mathcal{F}) \\
& \nearrow & \downarrow \\
\mathcal{K} & \longrightarrow & \mathcal{B}
\end{array}
$$

is equivalent data to a functor

$$
\begin{array}{ccc}
\mathcal{E}_{|\mathcal{K}} & \dashrightarrow & \mathcal{F}_{|\mathcal{K}} \\
& \searrow \quad \swarrow & \\
& \mathcal{B} &
\end{array}
$$

between pullbacks over $\mathcal{K}$. In particular, for

$$
\mathcal{F} := \mathcal{G} \times \mathcal{B} \longrightarrow \mathcal{B}
$$

a projection from a product we simply write

$$
\mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E},\underline{\mathcal{G}}) := \mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E},\mathcal{F})
$$

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(thinking of $\mathcal{F}$ as the "constant presheaf" at $\mathcal{G}$). This special case – which in fact will be the only case that we ever use – participates in the composite adjunction

$$\mathsf{Cat}_{/\mathcal{B}} \xrightarrow[\underset{\mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}\left(\mathcal{E},\underline{(-)}\right)}{\xleftarrow{\hspace{2cm}\perp\hspace{2cm}}}]{\overset{-\underset{\mathcal{B}}{\times}\mathcal{E}}{\xrightarrow{\hspace{2cm}}}} \mathsf{Cat} \quad .$$

We write

$$\mathsf{EFib}_{\mathcal{B}} \overset{\mathsf{f.f.}}{\lhook\joinrel\twoheadrightarrow} \mathsf{Cat}_{/\mathcal{B}}$$

for the full subcategory on the exponentiable fibrations, and we note once and for all that cocartesian fibrations and cartesian fibrations are exponentiable.

(7) In order not to overburden our language, at times we are slightly cavalier about exactly which limits and colimits we require to exist the ∞-category $\mathcal{V}$. This should not cause concern, however, as ultimately we will be specializing to the case that $\mathcal{V}$ is one of the ∞-categories $\mathcal{S}$ or Sp.

(8) Since we will refer to them regularly, we use the letters C and F to indicate references to [AMGRb] and [AMGRa], respectively: for example, we will refer to [AMGRb, Theorem 3.3] simply as Theorem C.3.3.

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Part I. The cyclotomic structure on THH

In this part of the paper, we prove Theorem B modulo certain technical details which are proved in the papers [AMGRb, AMGRa], giving precise and specific references for all such details as we proceed.

To begin, in §1 we describe the essential features of *stratified 1-manifolds*, which parametrize our entire story.

Although THH is given specifically by *spectrally*-enriched factorization homology over the circle, we will find its cyclotomic structure as a consequence of a more primitive structure on *spaces*-enriched factorization homology over the circle. Thus, in §2 we discuss $\mathcal{V}$-enriched factorization homology over the circle for an arbitrary symmetric monoidal ∞-category $(\mathcal{V}, \boxtimes)$, which we denote by

$$\mathsf{THH}_{\mathcal{V}} \ .$$

We write

$$\mathsf{THH}_{\mathcal{V}}^{\times}$$

when we specialize to the case that $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal, and we simply write

$$\mathsf{THH} := \mathsf{THH}_{\mathsf{Sp}}$$

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when we specialize to the particular case that $(\mathcal{V}, \boxtimes) = (\mathcal{S}\mathsf{p}, \otimes)$ is the ∞-category of spectra (equipped with its smash product symmetric monoidal structure).

In §3 and §4, we proceed to incorporate additional symmetries via those of 1-manifolds. There exists a canonical $\mathbb{T}$-action on $\mathsf{THH}_\mathcal{V}$ for any symmetric monoidal enriching ∞-category $\mathcal{V}$, and $\mathsf{THH}_\mathcal{V}^\times$ carries additional contravariant functoriality for covering maps of circles via diagonal maps in $\mathcal{V}$; we distill this state of affairs in §3, taking an expositional detour to indicate how the unstable cyclotomic structure arises in order to motivate the analogous construction in the stable case. Finally, in §4 we show how the diagonal maps for spaces induce Tate diagonal maps for spectra, which in turn endow THH with its cyclotomic structure.

1. STRATIFIED 1-MANIFOLDS

Factorization homology in dimension 1 is parametrized by a certain ∞-category

$$\mathbf{\mathcal{M}}$$

of *stratified 1-manifolds*. We describe its essential features in §1.1, and in §1.2 we show how it gives rise to various indexing categories of classical interest. The details omitted from this section are contained in [§F.1].

1.1. **Stratified 1-manifolds.** The objects of $\mathbf{\mathcal{M}}$ are simply finite disjoint unions of framed circles and finite directed connected graphs and framed circles. For any stratified 1-manifold $M \in \mathbf{\mathcal{M}}$, we write $M^{(0)}$ and $M^{(1)}$ for its 0- and 1-dimensional strata, which we'll often identify with their underlying spaces. The full subcategory

$$\mathbf{\mathcal{D}} \subset \mathbf{\mathcal{M}}$$

of *disk-stratified* 1-manifolds then consists of those objects whose strata are both disjoint unions of disks; equivalently, an object of $\mathbf{\mathcal{M}}$ is disk-stratified just when it doesn't contain any (unstratified) circles among its connected components.

While the morphisms in $\mathbf{\mathcal{M}}$ are somewhat more complicated to describe, it will suffice to highlight two distinguished classes thereof.

- The **refinement morphisms** correspond *covariantly* to homeomorphisms on underlying topological spaces, but where in passing from the source to the target we're allowed to forget some of the marked points (while the framings on the 1-dimensional strata must coincide).

- The **creation morphisms** correspond *contravariantly* to surjective proper constructible bundles.

We will generally denote an arbitrary object of $\mathbf{\mathcal{M}}$ by M; we will write S to specifically denote a (smooth) circle, and R to denote an object of $\mathbf{\mathcal{D}}$ (which will frequently be the source of a refinement morphism).

1.2. **Disk-refinements.** Factorization homology over a stratified 1-manifold $M \in \mathbf{\mathcal{M}}$ will be given by a colimit over the ∞-category $\mathbf{\mathcal{D}}(M)$ of *disk-refinements* of M, an object of which is simply a disk-stratified 1-manifold R equipped with a refinement morphism $R \to M$: writing

$$\mathbf{\mathcal{D}}/^{\mathsf{ref}}\mathbf{\mathcal{M}} \subset \mathsf{Ar}(\mathbf{\mathcal{M}})$$

for the full subcategory of the arrow ∞-category of $\mathcal{M}$ on the refinement morphisms with disk-stratified source, this is given by the fiber

$$
\begin{array}{ccc}
\mathcal{D}(M) & \longrightarrow & \mathcal{D}/^{\mathsf{ref}}\mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle t} \\
\{M\} & \longrightarrow & \mathcal{M}
\end{array} \quad .
$$

In the special case that we take $M = S^1$ to be the circle, we have an identification

$$\mathcal{D}(S^1) \simeq \boldsymbol{\Delta}_{\circlearrowright}^{\mathsf{op}}$$

with the **paracyclic indexing category**. This carries a left action of the circle group $\mathbb{T}$ coming from the functoriality of the construction $\mathcal{D}(-)$ for equivalences in $\mathcal{M}$, and its quotient thereby admits the further identification

$$(\boldsymbol{\Delta}_{\circlearrowright}^{\mathsf{op}})_{\mathsf{h}\mathbb{T}} \simeq \boldsymbol{\Lambda}^{\mathsf{op}}$$

with the **cyclic indexing category**. In fact, this – as well as a larger category of interest – can be recovered directly as follows. Let us respectively write

$$\mathsf{B}\mathbb{T} \lhook\joinrel\twoheadrightarrow \mathsf{B}\mathbb{W} \xrightarrow{\text{ f.f. }} \mathcal{M}$$

for the full subgroupoid and the full subcategory on the object $S^1 \in \mathcal{M}$; the monoid $\mathbb{W}$ can be identified as the semidirect product

$$\mathbb{W} \simeq \mathbb{T} \rtimes \mathbb{N}^{\times}$$

classified by the right action of the monoid $\mathbb{N}^{\times}$ on the group $\mathbb{T}$ by positive-degree endomorphisms.[20] Then, we have a diagram

$$
\begin{array}{ccccccc}
\boldsymbol{\Delta}_{\circlearrowright}^{\mathsf{op}} & \longrightarrow\joinrel\rightarrow & \boldsymbol{\Lambda}^{\mathsf{op}} & \longrightarrow\joinrel\rightarrow & \widetilde{\boldsymbol{\Lambda}}^{\mathsf{op}} & \xrightarrow{\text{ f.f. }} & \mathcal{D}/^{\mathsf{ref}}\mathcal{M} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle t} \\
\{S^1\} & \longrightarrow\joinrel\rightarrow & \mathsf{B}\mathbb{T} & \lhook\joinrel\rightarrow & \mathsf{B}\mathbb{W} & \underset{\text{f.f.}}{\lhook\joinrel\rightarrow} & \mathcal{M} \\
& & \downarrow & & \downarrow & & \\
& & \mathsf{pt} & \lhook\joinrel\rightarrow & \mathsf{B}\mathbb{N}^{\times} & &
\end{array}
\quad , \qquad (9)
$$

in which

- all squares are pullbacks,

- among the upper vertical maps, all but the rightmost are cocartesian fibrations,[21]

- both lower vertical maps are right fibrations, and

- $\widetilde{\boldsymbol{\Lambda}}^{\mathsf{op}}$ denotes the **epicyclic indexing category**.

---

[20]The notation $\mathbb{W}$ stems from the fact that this will keep track of Frobenius and Verschiebung operators, along the lines of [HM97b].

[21]The rightmost map $\mathcal{D}/^{\mathsf{ref}}\mathcal{M} \to \mathcal{M}$ is not a cocartesian fibration, but it is when restricted to a fairly large subcategory [Remark F.1.23].

In particular, the epicyclic category is given by the left-lax quotient

$$\widetilde{\mathbf{\Lambda}}^{\mathsf{op}} \simeq (\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowright})_{\mathsf{h}^{\mathsf{l.lax}}\mathbb{W}} \ ,$$

where the left action of the monoid $\mathbb{W}$ arises from the contravariant functoriality of the construction $\mathfrak{D}(-)$ for the positive-degree self-coverings of the circle: disk-refinements can be pulled back along covering maps.

Whereas the paracyclic category $\mathbf{\Delta}^{\mathsf{op}}_{\circlearrowright}$ might be thought of as being given by disk-refinements of *the* circle, the cyclic and epicyclic categories $\mathbf{\Lambda}^{\mathsf{op}}$ and $\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}$ might be thought of as being given by disk-refinements of *a* circle: the point being that in the morphisms in these latter two categories the underlying circle need not remain fixed, but can change either by an automorphism or an endomorphism in $\mathbf{M}$, respectively.

2. Factorization homology

In this section we define

$$\mathsf{THH}_{\mathcal{V}}(\mathcal{C}) \ ,$$

the **enriched factorization homology over the circle** of an arbitrary (flagged) $\mathcal{V}$-enriched ∞-category $\mathcal{C}$. This definition is given in §2.3. It relies on a *categorified* form of factorization homology, which we describe in §2.2. We begin in §2.1 by recalling the formalism of $\mathcal{V}$-enriched ∞-categories laid out in [GH15]. The details omitted from this section are contained in [§F.2 and §F.3].

2.1. Enriched ∞-categories. Let $\mathcal{V} := (\mathcal{V}, \boxtimes)$ be a monoidal ∞-category, and write

$$\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathfrak{B}\mathcal{V}} \mathsf{Cat}$$

for its *monoidal deloop* – that is, its bar construction. This is a *category object* in Cat, i.e. a simplicial object satisfying the Segal condition. Then, a $\mathcal{V}$-**enriched ∞-category** $\mathcal{C}$ can be specified as follows. First, we must specify its *underlying ∞-groupoid*, an object $\mathcal{C}^{\simeq} \in \mathcal{S}$. Let us write

$$\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{cd}(\mathcal{C}^{\simeq})} \mathcal{S}$$

for the "codiscrete category object" on $\mathcal{C}^{\simeq}$: the category object in $\mathcal{S} \subset \mathsf{Cat}$ whose space of objects is $\mathcal{C}^{\simeq}$ and whose hom-spaces are all contractible. Then, $\mathcal{C}$ is specified by its **enriched hom functor**, a right-lax functor

$$\mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\underline{\mathsf{hom}}_{\mathcal{C}}} \mathfrak{B}\mathcal{V} \tag{10}$$

of category objects in Cat. In simplicial degree n, this is given by

$$
\begin{array}{ccc}
(\mathcal{C}^{\simeq})^{\times(n+1)} & \xrightarrow{\hspace{3cm}} & \mathcal{V}^{\times n} \\
\cup & & \cup \\
(C_0, \ldots, C_n) & \longmapsto & (\underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_1), \ldots, \underline{\mathsf{hom}}_{\mathcal{C}}(C_{n-1}, C_n))
\end{array} \ ,
$$

22

and the right-laxness determines the categorical structure maps; for instance, restricting to the indicated morphisms in $\mathbf{\Delta}^{\mathsf{op}}$, this specifies the diagram

$$
\begin{array}{ccc}
[2]^\circ & (\mathcal{C}^\simeq)^{\times 3} \longrightarrow \mathcal{V}^{\times 2} \\
{\scriptstyle \delta_1}\downarrow & \downarrow \quad {\scriptstyle \swarrow} \quad \downarrow \\
[1]^\circ & (\mathcal{C}^\simeq)^{\times 2} \longrightarrow \mathcal{V} \\
{\scriptstyle \sigma_0}\uparrow & \uparrow \quad {\scriptstyle \nwarrow} \quad \uparrow \\
[0]^\circ & \mathcal{C}^\simeq \longrightarrow \mathsf{pt}
\end{array} \quad ,
$$

$$
\mathbf{\Delta}^{\mathsf{op}} \qquad \mathsf{cd}(\mathcal{C}^\simeq) \xrightarrow{\ \underline{\mathsf{hom}}_{\mathcal{C}}\ } \mathfrak{B}\mathcal{V}
$$

in which the upper square selects the composition maps

$$
\underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_1) \boxtimes \underline{\mathsf{hom}}_{\mathcal{C}}(C_1, C_2) \longrightarrow \underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_2) \tag{11}
$$

while the lower square selects the unit maps

$$
\mathbb{1}_\mathcal{V} \longrightarrow \underline{\mathsf{hom}}_{\mathcal{C}}(C_0, C_0) \ . \tag{12}
$$

In fact, the above data merely specify a **flagged** $\mathcal{V}$-enriched ∞-category; these assemble into an ∞-category

$$
\mathsf{fCat}(\mathcal{V}) \ .
$$

Among these, one can demand a *completeness* condition: there is an internally-defined "maximal subgroupoid" which receives a π_0-surjective map from the underlying ∞-groupoid $\mathcal{C}^\simeq$, and the condition is that this map be an equivalence.[22] These define a full subcategory

$$
\mathsf{Cat}(\mathcal{V}) \subset \mathsf{fCat}(\mathcal{V}) \ ,
$$

which admits a left adjoint when $\mathcal{V}$ is presentably monoidal. However, as mentioned in Remark 0.11, we will have no reason to restrict our attention to this subcategory, and indeed when we take $\mathcal{C}^\simeq = \mathsf{pt}$ we recover the definition of the monoidal deloop $\mathcal{C} = \mathfrak{B}A$ of an algebra object $A \in \mathsf{Alg}(\mathcal{V})$ (and thereafter the classical definition of $\mathsf{THH}(A)$).

2.2. **Categorified factorization homology.** Now, we can take factorization homology *of category objects* in Cat. In fact, we will only need this for disk-stratified 1-manifolds, and this is particularly easy to describe: for a disk-stratified 1-manifold $R \in \mathcal{D}$ and a category object $\mathcal{Y} \in \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat})$, the factorization homology

$$
\int_R \mathcal{Y} \in \mathsf{Cat}
$$

is given by a limit of copies of $\mathcal{Y}_{|[0]^\circ}$ and $\mathcal{Y}_{|[1]^\circ}$: one copy of $\mathcal{Y}_{|[0]^\circ}$ for each vertex, one copy of $\mathcal{Y}_{|[1]^\circ}$ for each edge, and with structure maps $s = \delta_1$ and $t = \delta_0$ determined by the incidence data of the

---

[22]This is analogous to the completeness condition for Segal spaces: the map from a Segal space to its completion is always a π_0-surjection in simplicial degree 0, and the Segal space is equivalent data to its completion (or equivalently its corresponding ∞-category) along with this π_0-surjection.

directed graph R.[23] This is suitably functorial for right-lax functors of category objects, so that a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ – that is, its enriched hom functor (10) – determines a morphism

$$
\begin{array}{ccc}
\int_{|\mathcal{D}} \mathsf{cd}(\mathcal{C}^{\simeq}) & \xrightarrow{\int_{|\mathcal{D}} \underline{\mathsf{hom}}_{\mathcal{C}}} & \int_{|\mathcal{D}} \mathfrak{B}\mathcal{V} \\
\downarrow & \swarrow & \\
\mathcal{D} & &
\end{array}
\tag{13}
$$

in $\mathsf{Cat}_{\mathsf{cocart}/\mathcal{D}}$.[24] In particular, pulling back along the composite

$$
\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}} \longrightarrow \mathcal{D}/^{\mathsf{ref}}\mathcal{M} \xrightarrow{s} \mathcal{D} \ ,
$$

we obtain a likewise morphism

$$
\begin{array}{ccc}
\int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \mathsf{cd}(\mathcal{C}^{\simeq}) & \xrightarrow{\int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \underline{\mathsf{hom}}_{\mathcal{C}}} & \int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \mathfrak{B}\mathcal{V} \\
\downarrow & \swarrow & \\
\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}} & &
\end{array}
\tag{14}
$$

in $\mathsf{Cat}_{\mathsf{cocart}/\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}}$. Over a disk-refinement $(R \to S^1) \in \boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}$, an object of its source is simply given by a labeling of its set of vertices $R^{(0)}$ by objects of the ∞-groupoid

$$
\mathcal{C}^{\simeq} =: \mathsf{cd}(\mathcal{C}^{\simeq})_{|[0]^{\circ}}
$$

(since the morphism-data in $\mathsf{cd}(\mathcal{C}^{\simeq})$ are canonically determined), while an object of its target is simply given by a labeling of its set of edges $R^{(1)}$ by objects of the ∞-category

$$
\mathcal{V} =: \mathfrak{B}\mathcal{V}_{|[1]^{\circ}}
$$

(since the object-data in $\mathfrak{B}\mathcal{V}$ are canonically determined). In other words, over this object of $\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}$ the map (14) restricts to a map

$$
(\mathcal{C}^{\simeq})^{\times R^{(0)}} \longrightarrow \mathcal{V}^{\times R^{(1)}} \ .
$$

Of course, this is given by nothing other than the enriched hom functor of $\mathcal{C}$.

2.3. **Enriched factorization homology.** When $\mathcal{V}$ is additionally *symmetric* monoidal, there exists a "tensor everything together" functor

$$
\int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \ .[25]
$$

Then, we define the ***enriched factorization homology*** of $\mathcal{C}$ over S^1 by the formula

$$
\mathsf{THH}_{\mathcal{V}}(\mathcal{C}) := \int_{S^1} \mathcal{C} := \mathsf{colim}\left(\int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \underline{\mathsf{hom}}_{\mathcal{C}}} \int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \right) \ .
\tag{15}
$$

---

[23]The composition in $\mathcal{Y}$ plays a role in the functoriality of this construction, and in particular in the definition of the factorization homology $\int_M \mathcal{Y}$ when $M \in \mathbf{M}$ is no longer disk-stratified.

[24]The morphism (13) does not lie in $\mathsf{coCart}_{\mathcal{D}}$, but it does preserve cocartesian morphisms over a fairly substantial class of morphisms in $\mathcal{D}$ [Observation F.2.27].

[25]In fact, in order to guarantee the existence of this functor, it suffices (definitionally) for $\mathcal{V}$ to be merely cyclically monoidal.

Note that this indeed precisely rigorizes the heuristic definition of §0.3. First of all, this colimit is indexed over the data of collections of points $p_1, \ldots, p_n \in S^1$ labeled by objects $C_1, \ldots, C_n \in \mathcal{C}$, with value the monoidal product

$$\boxtimes_{i \in \mathbb{Z}/n} \underline{\mathsf{hom}}_{\mathcal{C}}(C_i, C_{i+1}) \ .$$

Moreover, its structure maps are given either by disappearances of points (i.e. refinement morphisms among disk-refinements of S^1), which are taken to composition maps (as map (11)), or by anticollisions of points (i.e. refinement morphisms among disk-refinements of S^1), which are taken to unit maps (as map (12)).

This definition of enriched factorization homology has the key feature of being optimized for the isolation of the various specific properties of the enriching ∞-category $\mathcal{V}$: all natural operations thereon have only to do with the factorization homology of $\mathfrak{B}\mathcal{V}$, while the factorization homology of $\mathsf{cd}(\mathcal{C}^{\simeq})$ simply comes along for the ride.

3. The unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}$

In this section, we obtain the **unstable cyclotomic structure**

$$\mathsf{Cyc}^{\mathsf{h}} \curvearrowright \mathsf{THH}_{\mathcal{V}}^{\times}$$

in the case that $(\mathcal{V}, \boxtimes) = (\mathcal{V}, \times)$ is *cartesian* symmetric monoidal. In fact, this structure is nothing but an action of the monoid $\mathbb{W}$, which we will obtain in §3.2 after obtaining its underlying $\mathbb{T}$-action in §3.1. These are both special cases of more general constructions: §3.1 is based on material in [§F.3] while §3.2 is based on material in [§F.4], and all details omitted from these two subsections can be found there.

In order to motivate our construction of the cyclotomic structure on $\mathsf{THH} := \mathsf{THH}_{\mathsf{Sp}}$, we will rederive the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}$ as a consequence of the **diagonal package**, which packages the diagonal maps that exist in $\mathcal{V}$. We accomplish this as follows. First, in §3.5, we describe the diagonal package. Then, in §3.6, we reformulate our definition of unstable cyclotomic objects in similar terms. Finally, in §3.7, we describe the way in which the diagonal package induces the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}$.

The diagonal package will be a global section of a certain cartesian fibration over $\mathsf{B}\mathbb{N}^{\times}$. So before describing the diagonal package, we study this cartesian fibration: we provide a heuristic description thereof in §3.3, and we give a careful analysis (as well as a useful generalization) thereof in §3.4.

Given the underlying $\mathbb{T}$-action on $\mathsf{THH}_{\mathcal{V}}^{\times}$, the extension to a $\mathbb{W}$-action – that is, the data of its unstable cyclotomic structure – is manifestly equivalent data to a family of *unstable cyclotomic structure maps*

$$\mathsf{THH}_{\mathcal{V}}^{\times} \longrightarrow \left(\mathsf{THH}_{\mathcal{V}}^{\times}\right)^{\mathsf{hC}_r} \tag{16}$$

for all $r \in \mathbb{N}^{\times}$, which are compatible in the sense that they compose in the evident way.[26] For expository purposes, in §3.8 we show explicitly how the unstable cyclotomic structure maps (16) arise from diagonal maps in $\mathcal{V}$.

---

[26]Note the similarity with our description of a cyclotomic structure on a $\mathbb{T}$-spectrum in Remark 0.2.

3.1. **The $\mathbb{T}$-action on $\mathsf{THH}_{\mathcal{V}}$.** For any symmetric monoidal ∞-category $\mathcal{V}$, we can obtain a left $\mathbb{T}$-action on $\mathsf{THH}_{\mathcal{V}}$ by a direct extension of formula (15). Namely, rather than pulling the map (13) all the way back along the composite

$$\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}} \longrightarrow \boldsymbol{\Lambda}^{\mathsf{op}} \longrightarrow \widetilde{\boldsymbol{\Lambda}}^{\mathsf{op}} \longrightarrow \boldsymbol{\mathcal{D}}/^{\mathsf{ref}}\boldsymbol{\mathcal{M}} \overset{s}{\longrightarrow} \boldsymbol{\mathcal{D}}$$

(recall diagram (9)), we can instead pull it back only to the cyclic category $\boldsymbol{\Lambda}^{\mathsf{op}}$. We again have a "tensor everything together" functor

$$\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V} \ . \tag{17}$$

So, we obtain the desired action by taking the left Kan extension

$$; \tag{18}$$

as the two vertical functors are cocartesian fibrations then so is their composite, and hence this simply reduces to a fiberwise colimit.[27]

3.2. **The $\mathbb{W}$-action on $\mathsf{THH}_{\mathcal{V}}^{\times}$.** Now, one might hope to extend the construction of §3.1 yet further, from the cyclic category $\boldsymbol{\Lambda}^{\mathsf{op}}$ to the epicyclic category $\widetilde{\boldsymbol{\Lambda}}^{\mathsf{op}}$. However, one immediately runs into a problem: there is only a "tensor everything together" functor

$$\int_{|\widetilde{\boldsymbol{\Lambda}}^{\mathsf{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{\boxtimes} \mathcal{V}$$

when $\mathcal{V}$ is *cartesian* symmetric monoidal;[28] a simple case in which this issue arises is e.g. for the morphism

$$\tag{19}$$

---

[27]As the fiber of $\boldsymbol{\Lambda}^{\mathsf{op}}$ over each point in $\mathsf{B}\mathbb{T}$ is a copy of $\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}$, over each point in $\mathsf{B}\mathbb{T}$ we're taking a colimit over the fiber

$$\int_{|\boldsymbol{\Delta}_{\circlearrowleft}^{\mathsf{op}}} \mathsf{cd}(\mathcal{C}^{\simeq}) \ ;$$

thus, the underlying object is indeed $\mathsf{THH}_{\mathcal{V}}(\mathcal{C})$, as defined in formula (15).

[28]In fact, it suffices (definitionally) for $\mathcal{V}$ to be an augmented cyclically monoidal and cyclically comonoidal bialgebra object in Cat: this is precisely the data of a suitable functor generalizing the one labeled $\prod$ in diagram (20).

in $\int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}$, which must be sent to a natural morphism

$$V \longrightarrow V^{\boxtimes 3}$$

in $\mathcal{V}$. However, in this case, we can indeed obtain a $\mathbb{W}$-action on $\mathsf{THH}_{\mathcal{V}}^{\times}(\mathcal{C})$ by taking the left Kan extension

$$\tag{20}$$

again a fiberwise colimit.[29] This will turn out to be precisely the data of the *unstable cyclotomic structure* on $\mathsf{THH}_{\mathcal{V}}^{\times}(\mathcal{C})$.

3.3. **The local structure of the functor $\prod$.** To see more clearly what is going on, it is helpful to work "locally" over $\mathsf{B}\mathbb{N}^{\times}$. First of all, to specify the functor $\prod$ in diagram (20) is equivalently to specify a functor

over $\mathsf{B}\mathbb{N}^{\times}$. This, in turn, we can consider as a global section

$$\tag{21}$$

of the *relative functor ∞-category* (recall item (6) of §0.8). Now, the fiber of the downwards functor in diagram (21) over the unique object of $\mathsf{B}\mathbb{N}^{\times}$ is given by

$$\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) . \tag{22}$$

Moreover, as we will see in §3.4, a core computation reveals that this downwards functor is in fact a cartesian fibration: over a morphism $[1] \xrightarrow{r} \mathsf{B}\mathbb{N}^{\times}$, the cartesian monodromy is given heuristically

---

[29]An explicit description of the functoriality of this fiberwise colimit is given in §3.8.

by the functor

$$, \qquad (23)$$

which takes a functor F to the new functor that precomposes with pullback along an r-fold cover and postcomposes with homotopy fixedpoints for the resulting C_r-action. In other words, this cartesian fibration (the downwards functor in diagram (21)) is classified by a right $\mathbb{N}^\times$-action on the fiber (22), through which the element $r \in \mathbb{N}^\times$ acts as the endofunctor (23). To emphasize that this right action involves *homotopy* fixedpoints, we denote it by

$$\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^\times \ . \qquad (24)$$

3.4. **Cartesian monodromy via cocartesian monodromy in families.** We now describe the cartesian monodromy functor (23) more precisely. For this, let us first recall that an object of the fiber

$$
\begin{array}{ccc}
\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} & \lhook\joinrel\relbar\joinrel\twoheadrightarrow & \int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \\
\downarrow & & \downarrow \\
\mathsf{pt} & \lhook\joinrel\longrightarrow\joinrel\twoheadrightarrow & \mathsf{B}\mathbb{N}^\times
\end{array}
$$

is given by the data

$$\left(S \in \mathsf{B}\mathbb{W} \ , \ (R \to S) \in \mathbf{\mathcal{D}}(S) \ , \ R^{(1)} \to \mathcal{V}\right) \qquad (25)$$

of a circle, a disk-refinement thereof, and a labeling of the intervals of the latter by objects of $\mathcal{V}$. As both maps in the composite

$$
\begin{array}{c}
\int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \\
\downarrow \\
\widetilde{\mathbf{\Lambda}}^{\mathrm{op}} \\
\downarrow \\
\mathsf{B}\mathbb{W}
\end{array}
$$

are cocartesian fibrations, then so is their composite. Hence, for any map $S \to S'$ in $\mathsf{B}\mathbb{W}$ of degree $r \in \mathbb{N}^\times$, we obtain a cocartesian pushforward

$$\left(S' \in \mathsf{B}\mathbb{W} \ , \ (R' \to S') \in \mathbf{\mathcal{D}}(S) \ , \ R'^{(1)} \to R^{(1)} \to \mathcal{V}\right)$$

28

of the object (25): the disk-refinement R' is pulled back from R, and this induces an r-fold covering map $R'^{(1)} \to R^{(1)}$ of discrete spaces (i.e. we pull back the labelings of intervals by objects of $\mathcal{V}$ from R to R' as well).[30]

Now, the space of degree-r maps in $\mathbb{BW}$ with source S is a copy of BC_r. Hence, cocartesian pushforward *over all of these maps simultaneously* defines a C_r-object

$$\pi_r^*(25) : \mathsf{BC}_r \longrightarrow \int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} .$$

This construction assembles into a functor

$$\pi_r^* : \int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \longrightarrow \mathsf{Fun}\left(\mathsf{BC}_r, \int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}\right) , \tag{26}$$

and with this notation we can rewrite the cartesian monodromy functor (23) more formally as

$$
\begin{array}{ccc}
\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) & \longleftarrow & \mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) \\
\cup\!| & & \cup\!| \\
(F(\pi_r^*(-)))^{\mathsf{hC}_r} & \longleftarrow\!\!\dashv & F
\end{array} . \tag{27}
$$

This identification of cartesian monodromy in a relative functor ∞-category via "cocartesian monodromy in families" admits a useful generalization. Namely, let

$$(\mathcal{E} \downarrow \mathbb{BW}) \in \mathsf{coCart}_{\mathbb{BW}}$$

be an arbitrary cocartesian fibration over $\mathbb{BW}$, and write

$$
\begin{array}{ccc}
\mathcal{E}_0 & \lhook\joinrel\longrightarrow\!\!\!\!\rightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathbb{BT} & \lhook\joinrel\longrightarrow\!\!\!\!\rightarrow & \mathbb{BW}
\end{array}
$$

for the pullback (or equivalently, the fiber over the unique point in $\mathbb{BN}^\times$). Let us repurpose our notation π_r^* for the "all the cocartesian monodromy over degree-r maps in $\mathbb{BW}$ at once" functor (26) to still write

$$\pi_r^* : \mathcal{E}_0 \longrightarrow \mathsf{Fun}(\mathsf{BC}_r, \mathcal{E}_0)$$

in this more general setting. Then, by [Lemma C.3.22], the functor

$$
\begin{array}{c}
\mathsf{Fun}^{\mathsf{rel}}_{/\mathbb{BN}^\times}(\mathcal{E}, \underline{\mathcal{V}}) \\
\downarrow \\
\mathbb{BN}^\times
\end{array} \tag{28}
$$

is once again a cartesian fibration: its fiber is $\mathsf{Fun}(\mathcal{E}_0, \mathcal{V})$, and the cartesian monodromy functor over the morphism $[1] \xrightarrow{r} \mathbb{BN}^\times$ is given by the formula

$$
\begin{array}{ccc}
\mathsf{Fun}(\mathcal{E}_0, \mathcal{V}) & \longleftarrow & \mathsf{Fun}(\mathcal{E}_0, \mathcal{V}) \\
\cup\!| & & \cup\!| \\
(F(\pi_r^*(-)))^{\mathsf{hC}_r} & \longleftarrow\!\!\dashv & F
\end{array} .
$$

---

[30]In fact, we have already seen an example of such a cocartesian pushforward, namely the morphism (19).

We denote this right $\mathbb{N}^\times$-module by

$$\mathsf{Fun}(\mathcal{E}_0, \mathcal{V}) \curvearrowleft_{\mathsf{h}} \mathbb{N}^\times \ , \tag{29}$$

extending the notation (24). Evidently, this construction assembles into a functor

$$\mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{B}\mathbb{N}^\times}(-, \underline{\mathcal{V}}) : (\mathsf{coCart}_{\mathsf{B}\mathbb{W}})^{\mathsf{op}} \longrightarrow \mathsf{Cart}_{\mathsf{B}\mathbb{N}^\times} \ . \tag{30}$$

3.5. **The diagonal package.** Let us now return to the global section (21). By definition, this is an object

$$\left(\prod, \mathsf{h}\right) \in \lim{}^{\mathsf{r.lax}}\left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) \curvearrowleft_{\mathsf{h}} \mathbb{N}^\times\right) := \Gamma\left(\begin{array}{c} \mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{B}\mathbb{N}^\times}\left(\int_{|\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{V}}\right) \\ \downarrow \\ \mathsf{B}\mathbb{N}^\times \end{array}\right) \simeq \mathsf{Fun}\left(\int_{|\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) \tag{31}$$

of the right-lax limit of the right $\mathbb{N}^\times$-module (24): over the unique object of $\mathsf{B}\mathbb{N}^\times$ this selects the object

$$\prod \in \mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V}\right) \ ,$$

and over a morphism $[1] \xrightarrow{r} \mathsf{B}\mathbb{N}^\times$ this selects the diagonal map

$$\prod \longrightarrow \left(\prod \pi_r^*(-)\right)^{\mathsf{hC}_r} . {}^{[31]}$$

We include the "h" in the notation of the object (31) to emphasize that these structure maps take place within the ambient context of a right $\mathbb{N}^\times$-action involving homotopy fixedpoints. We refer to the object (31) as the **_diagonal package_**, since as we will see in §3.7 it encodes diagonal maps in $\mathcal{V}$ along with all of their requisite compatibilities necessary for defining the $\mathbb{W}$-action (20) on $\mathsf{THH}^\times_{\mathcal{V}}$.

3.6. **Unstable cyclotomic objects in $\mathcal{V}$.** Consider the functor

$$\begin{array}{c} \mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{B}\mathbb{N}^\times}(\mathsf{B}\mathbb{W}, \underline{\mathcal{V}}) \\ \downarrow \\ \mathsf{B}\mathbb{N}^\times \end{array} \ .$$

By the discussion of §3.4, this is a cartesian fibration, whose fiber over the unique object of $\mathsf{B}\mathbb{N}^\times$ is

$$\mathsf{Fun}(\mathsf{B}\mathbb{T}, \mathcal{V})$$

and whose cartesian monodromy over the morphism $[1] \xrightarrow{r} \mathsf{B}\mathbb{N}^\times$ is

$$\begin{array}{ccc} \mathsf{Fun}(\mathsf{B}\mathbb{T}, \mathcal{V}) & \longleftarrow & \mathsf{Fun}(\mathsf{B}\mathbb{T}, \mathcal{V}) \\ \cup & & \cup & ; \\ \left(\mathbb{T} \simeq (\mathbb{T}/\mathsf{C}_r) \curvearrowright V^{\mathsf{hC}_r}\right) & \longleftarrow\!\shortmid & (\mathbb{T} \curvearrowright V) \end{array}$$

---

[31]In fact, this map is an equivalence: for any objects $V_1, \ldots, V_k \in \mathcal{V}$, the map

$$\prod_i V_i \longrightarrow \left(\prod_i (V_i)^{\times r}\right)^{\mathsf{hC}_r}$$

is an equivalence. In other words, the global section (21) actually lands in the subcategory of _cartesian_ morphisms, and hence defines a point in the _strict_ limit: the ∞-category of cartesian sections of the corresponding cartesian fibration. However, this will not be relevant for us.

according to our notational convention (29), we write

$$\mathsf{Fun}(B\mathbb{T}, \mathcal{V}) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^{\times} \tag{32}$$

for the corresponding right $\mathbb{N}^{\times}$-module. Then, we define the ∞-category of **unstable cyclotomic objects** in $\mathcal{V}$ to be

$$\mathsf{Cyc}^{\mathsf{h}}(\mathcal{V}) := \lim{}^{\mathsf{r.lax}}\left(\mathsf{Fun}(B\mathbb{T}, \mathcal{V}) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^{\times}\right) := \Gamma\left(\begin{array}{c} \mathsf{Fun}^{\mathsf{rel}}_{/B\mathbb{N}^{\times}}(B\mathbb{W}, \underline{\mathcal{V}}) \\ \downarrow \\ B\mathbb{N}^{\times} \end{array}\right) \simeq \mathsf{Fun}(B\mathbb{W}, \mathcal{V}) \ ,$$

the right-lax limit of the right $\mathbb{N}^{\times}$-module (32).[32,33] In analogy with our notation for cyclotomic spectra, we denote an unstable cyclotomic structure on an object $V \in \mathcal{V}$ by

$$\mathsf{Cyc}^{\mathsf{h}} \curvearrowright V \ .$$

3.7. The unstable cyclotomic structure on $\mathsf{THH}^{\times}_{\mathcal{V}}$.
We now explain how the diagonal package (31) gives rise to the unstable cyclotomic structure on $\mathsf{THH}^{\times}_{\mathcal{V}}(\mathcal{C})$.

We begin by noting that the diagram

defines a morphism in $\mathsf{coCart}_{B\mathbb{W}}$ [Observation F.2.27]. Trivially, we obtain a span

$$\tag{33}$$

---

[32]One might alternatively refer to this as the ∞-category of **cyclohomic objects**, since an unstable cyclotomic structure involves *homotopy* fixedpoints whereas a *cyclotomic* structure on a spectrum will involve *Tate* fixedpoints.

[33]The simplest example of an unstable cyclotomic object is a *free loop object*, i.e. the cotensoring

$$S^1 \pitchfork V$$

of the circle into an arbitrary object $V \in \mathcal{V}$ (e.g. the free loopspace of a space); the unstable cyclotomic structure arises from the identifications $S^1 \simeq (S^1)_{\mathsf{hC}_r}$ (compatibly for all $r \in \mathbb{N}^{\times}$), so that its $\mathbb{T}$-equivariant structure maps

$$S^1 \pitchfork V \longrightarrow (S^1 \pitchfork V)^{\mathsf{hC}_r} \simeq (S^1)_{\mathsf{hC}_r} \pitchfork V$$

are actually *equivalences*. When $\mathcal{V} = \mathcal{S}$, this is nothing but $\mathsf{THH}^{\times}_{\mathcal{S}}(V)$, the spatially-enriched factorization homology of the ∞-groupoid $V \in \mathcal{S} \subset \mathsf{Cat}$ over the circle.

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in $\mathsf{coCart}_{\mathbb{BW}}$. Applying the functor (30) to this span gives a cospan

$$\mathsf{Fun}^{\mathsf{rel}}_{/\mathbb{BN}^\times}\left(\textstyle\int_{|\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}}\mathfrak{BV}, \underline{\mathcal{V}}\right) \longrightarrow \mathsf{Fun}^{\mathsf{rel}}_{/\mathbb{BN}^\times}\left(\textstyle\int_{|\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}}\mathsf{cd}(\mathcal{C}^\simeq), \underline{\mathcal{V}}\right) \longleftarrow \mathsf{Fun}^{\mathsf{rel}}_{/\mathbb{BN}^\times}(\mathbb{BW}, \underline{\mathcal{V}})$$

$$\mathbb{BN}^\times \tag{34}$$

in $\mathsf{Cart}_{\mathbb{BN}^\times}$.

Now, on fibers, the leftward functor of diagram (34) restricts to a functor admitting a left adjoint

$$\mathsf{Fun}\left(\textstyle\int_{|\mathbf{\Lambda}^{\mathsf{op}}}\mathsf{cd}(\mathcal{C}^\simeq), \mathcal{V}\right) \xleftrightarrow{\quad\perp\quad} \mathsf{Fun}(\mathbb{BW}, \mathcal{V}) \quad,$$

namely the left Kan extension appearing in diagram (18). By completely general principles [Lemma C.1.36], it follows that there exists a (necessarily unique) extension

$$\mathsf{Fun}\left(\textstyle\int_{|\mathbf{\Lambda}^{\mathsf{op}}}\mathsf{cd}(\mathcal{C}^\simeq), \mathcal{V}\right) \xleftrightarrow{\quad\perp\quad} \mathsf{Fun}(\mathbb{BW}, \mathcal{V})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \tag{35}$$

$$\mathsf{Fun}^{\mathsf{rel}}_{/\mathbb{BN}^\times}\left(\textstyle\int_{|\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}}\mathsf{cd}(\mathcal{C}^\simeq), \underline{\mathcal{V}}\right) \xleftrightarrow{\quad\perp\quad} \mathsf{Fun}^{\mathsf{rel}}_{/\mathbb{BN}^\times}(\mathbb{BW}, \underline{\mathcal{V}})$$

of this left adjoint, a morphism in $\mathsf{Cat}_{\mathsf{cart}/\mathbb{BN}^\times}$. In fact, the lower row of diagram (35) defines an adjunction in the $(\infty, 2)$-category $\mathsf{Cat}_{\mathsf{cart}/\mathbb{BN}^\times}$ (which justifies the notation), i.e. a functor

$$\mathsf{Adj} \longrightarrow \mathsf{Cat}_{\mathsf{cart}/\mathbb{BN}^\times} \tag{36}$$

of $(\infty, 2)$-categories from the walking adjunction (which is actually just a 2-category). Then, post-composing the functor (36) with the global sections functor

$$\mathsf{Cat}_{\mathsf{cart}/\mathbb{BN}^\times} \xrightarrow{\;\Gamma\;} \mathsf{Cat}$$

of $(\infty, 2)$-categories yields an adjunction

$$\mathsf{Fun}\left(\textstyle\int_{|\widetilde{\mathbf{\Lambda}}^{\mathsf{op}}}\mathsf{cd}(\mathcal{C}^\simeq), \mathcal{V}\right) \xleftrightarrow{\quad\perp\quad} \mathsf{Fun}(\mathbb{BW}, \mathcal{V})$$

in Cat, whose left adjoint is the left Kan extension appearing in diagram (20). Hence, we obtain the unstable cyclotomic structure on $\mathsf{THH}^\times_{\mathcal{V}}(\mathcal{C})$ as the image of the diagonal package, as illustrated in Figure 1.

In Figure 2, we summarize the constructions of this subsection in the language of modules of [§C.1]; our method for obtaining the cyclotomic structure on THH will be closely analogous.

3.8. **From fiberwise colimits to unstable cyclotomic structures.** Purely for expository purposes, we unwind here how the fiberwise colimit of diagram (20) gives rise to the unstable cyclotomic structure maps

$$\mathsf{THH}^\times_{\mathcal{V}}(\mathcal{C}) \longrightarrow \mathsf{THH}^\times_{\mathcal{V}}(\mathcal{C})^{\mathsf{hC}_r} \; ; \tag{37}$$

this is essentially just an exercise in the functoriality of fiberwise colimits along cocartesian fibrations. For this, let

$$S_s \xrightarrow{\;\varphi\;} S_t$$

FIGURE 1. Above: the diagram of ∞-categories housing the passage from the diagonal package to the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}(\mathcal{C})$. All solid horizontal morphisms are given by pullback; the upwards functors are given by restriction to underlying objects of right $\mathbb{N}^{\times}$-modules, and the diagram commutes when omitting either all left adjoints or all right adjoints. Below: the diagonal package gives rise to the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}(\mathcal{C})$, and the restriction of this passage recovers its underlying $\mathbb{T}$-action.

$$
\begin{array}{ccccc}
[1] \coprod_{1,\mathsf{pt},1} [1] & \xrightarrow{\left(\int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathcal{B}\mathcal{V} \leftarrow \int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathsf{cd}(\mathcal{C}^{\simeq}) \to \mathbb{B}\mathbb{W}\right)^{\circ}} & (\mathsf{LMod}_{\mathbb{W}})^{\mathrm{op}} & \xrightarrow{\left(\mathsf{Fun}((-)_0,\mathcal{V}) \curvearrowright_{\mathsf{h}}^{\mathbb{N}^{\times}}\right)} & \mathsf{RMod}_{\mathbb{N}^{\times}} \\
\Big\downarrow{\scriptstyle \mathsf{id}\coprod_{\mathsf{pt}}\mathsf{r.adjt}} & & & & \Big\downarrow \\
[1] \coprod_{1,\mathsf{pt},l} \mathsf{Adj} & \xdashrightarrow{\ \exists!\ [\text{Lemma C.1.36}]\ } & & & \mathsf{RMod}_{\mathbb{N}^{\times}}^{\mathsf{r.lax}} \\
\Big\uparrow{\scriptstyle \mathsf{id}\coprod_{\mathsf{pt}}\mathsf{l.adjt}} & & & & \\
[2] & & & & \Big\downarrow{\scriptstyle \mathsf{lim}^{\mathsf{r.lax}}} \\
\Big\uparrow{\scriptstyle \{0<2\}} & & & & \\
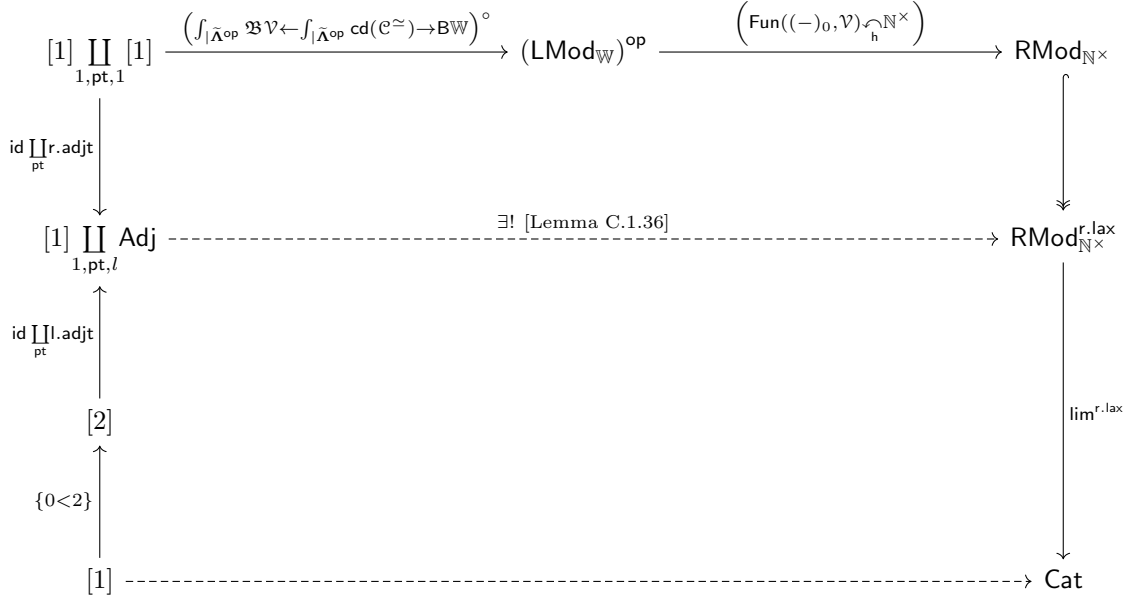[1] & \xdashrightarrow{\hspace{8cm}} & & & \mathsf{Cat}
\end{array}
$$

FIGURE 2. A diagram illustrating in the language of modules how to obtain the lower rightwards composite functor in Figure 1, which takes the diagonal package to the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}(\mathcal{C})$. Here, we write $l \in \mathsf{Adj}$ for the source of the walking left adjoint.

be a morphism in $\mathbb{B}\mathbb{W}$ corresponding contravariantly to an r-fold covering map of framed circles, and write

$$
\mathfrak{D}(S_s) \xrightarrow{\varphi_*} \mathfrak{D}(S_t)
$$

for the resulting functor on categories of disk-refinements (the *r-fold subdivision* functor on $\mathbf{\Delta}_{\circlearrowleft}^{\mathsf{op}}$). Then, the map (37) is given by the composite

$$
\mathsf{THH}_{\mathcal{V}}^{\times}(\mathcal{C}) := \int_{S_s} \mathcal{C}
$$

$$
:= \operatorname*{colim}_{\substack{(R_s \to S_s) \in \mathfrak{D}(S_s) \\ (C_i) \in (\mathcal{C}^{\simeq})^{\times R_s^{(0)}}}} \left(\prod_{i \in \mathbb{Z}/R_s^{(0)}} \underline{\mathsf{hom}}_{\mathcal{C}}(C_i, C_{i+1}) \right)
$$

$$
\longrightarrow \operatorname*{colim}_{\substack{(R_s \to S_s) \in \mathfrak{D}(S_s) \\ (C_i) \in (\mathcal{C}^{\simeq})^{\times R_s^{(0)}}}} \left(\left(\prod_{i \in \mathbb{Z}/R_s^{(0)}} \underline{\mathsf{hom}}_{\mathcal{C}}(C_i, C_{i+1})^{\times r} \right)^{\mathsf{hC}_r} \right) \tag{38}
$$

$$
\longrightarrow \left(\operatorname*{colim}_{\substack{(R_s \to S_s) \in \mathfrak{D}(S_s) \\ (C_i) \in (\mathcal{C}^{\simeq})^{\times R_s^{(0)}}}} \left(\prod_{i \in \mathbb{Z}/R_s^{(0)}} \underline{\mathsf{hom}}_{\mathcal{C}}(C_i, C_{i+1})^{\times r} \right) \right)^{\mathsf{hC}_r} \tag{39}
$$

$$
\longrightarrow \left(\operatorname*{colim}_{\substack{(R_t \to S_t) \in \mathfrak{D}(S_t) \\ (C_j) \in (\mathcal{C}^{\simeq})^{\times R_t^{(0)}}}} \left(\prod_{j \in \mathbb{Z}/R_t^{(0)}} \underline{\mathsf{hom}}_{\mathcal{C}}(C_j, C_{j+1}) \right) \right)^{\mathsf{hC}_r} \tag{40}
$$

$$=: \left(\int_{S_t} \mathcal{C} \right)^{\mathsf{hC}_r}$$

$$=: \left(\mathsf{THH}_{\mathcal{V}}^{\times}(\mathcal{C}) \right)^{\mathsf{hC}_r} \ ,$$

in which

- the map (38) arises from the diagonal maps in $\mathcal{V}$ (and is actually an equivalence),

- the map (39) arises from the universal property of colimits, and

- the map (40) arises from a map on colimits taking the term indexed by

$$\left(R_s \in \boldsymbol{\mathcal{D}}(S_s) \ , \ R_s^{(0)} \xrightarrow{(C_i)} \mathcal{C}^{\simeq} \right)$$

to the equivalent term indexed by

$$\left(\varphi_*(R_s) \in \boldsymbol{\mathcal{D}}(S_t) \ , \ (\varphi_*(R_s))^{(0)} \longrightarrow R_s^{(0)} \xrightarrow{(C_i)} \mathcal{C}^{\simeq} \right) \ .$$

4. The cyclotomic structure on THH

In this section, we specialize to the case that $(\mathcal{V}, \boxtimes) = (\mathcal{S}\mathsf{p}, \otimes)$ is the ∞-category of spectra equipped with its smash product monoidal structure, and we obtain the cyclotomic structure

$$\mathsf{Cyc} \curvearrowright \mathsf{THH} \ .$$

We achieve this in §4.4, in a nearly identical way to how we obtained the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}$ in §3.7. In particular, whereas the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}$ arose from the *diagonal package*, the cyclotomic structure on THH will arise from the ***Tate package***; we explain this analogy further in §4.1. We will obtain the Tate package from the diagonal package via linearization (in the sense of Goodwillie calculus); we achieve this in §4.3, after describing certain auxiliary constructions that collectively accommodate this maneuver in §4.2.

4.1. **Locating the Tate package.** As described in §3.8, the unstable cyclotomic structure on $\mathsf{THH}_{\mathcal{V}}^{\times}$ consists of a system of maps

$$\mathsf{THH}_{\mathcal{V}}^{\times} \longrightarrow (\mathsf{THH}_{\mathcal{V}}^{\times})^{\mathsf{hC}_r}$$

which are ultimately induced by the natural diagonal maps

$$\prod_i V_i \longrightarrow \prod_i (V_i)^{\times r}$$
$$\xrightarrow{\sim} \left(\prod_i (V_i)^{\times r} \right)^{\mathsf{hC}_r}$$

that exist for any objects $V_1, \ldots, V_k \in \mathcal{V}$. In §3.5, we organized these maps into the *diagonal package*, an object

$$\left(\prod, \mathsf{h} \right) \in \lim \left(\mathsf{Fun} \left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{V}, \mathcal{V} \right) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^{\times} \right)$$

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(originally (31)) in the limit of a certain right $\mathbb{N}^\times$-action by homotopy fixedpoints in which the element $r \in \mathbb{N}^\times$ acts as the endofunctor

$$
\begin{array}{ccc}
\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^\mathrm{op}} \mathcal{BV}, \mathcal{V}\right) & \longleftarrow & \mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^\mathrm{op}} \mathcal{BV}, \mathcal{V}\right) \\
\cup & & \cup \\
(F(\pi_r^*(-)))^{\mathsf{hC}_r} & \longleftarrow & F
\end{array}
$$

(originally (27)).

By contrast, the cyclotomic structure on THH will be induced in a formally identical way by the natural *Tate diagonal* maps

that exist for any spectra $E_1, \ldots, E_k \in \mathcal{Sp}$. But in fact, it turns out that the formula

$$
\begin{array}{ccc}
\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^\mathrm{op}} \mathcal{BSp}, \mathcal{Sp}\right) & \longleftarrow & \mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^\mathrm{op}} \mathcal{BSp}, \mathcal{Sp}\right) \\
\cup & & \cup \\
(F(\pi_r^*(-)))^{\tau\mathsf{C}_r} & \longleftarrow & F
\end{array}
$$

only defines a *left-lax* right $\mathbb{N}^\times$-action. Thus we find that the main ingredient in obtaining the cyclotomic structure on THH will be the **Tate package**, an object

$$
\left(\bigotimes, \tau\right) \in \lim{}^{\mathrm{r.lax}} \left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^\mathrm{op}} \mathcal{BSp}, \mathcal{Sp}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times\right)
$$

of the right-lax limit of a certain left-lax right $\mathbb{N}^\times$-action by Tate fixedpoints.

4.2. **Auxiliary constructions.** In order to obtain the Tate package, we make the following three constructions; the first two follow from [Theorem C.3.3],[34] while the third is simply a definition. As in §3.4, we write $(\mathcal{E} \downarrow B\mathbb{W}) \in \mathsf{coCart}_{B\mathbb{W}}$ for an arbitrary object, and we write $\mathcal{E}_0 = \mathcal{E}_{|B\mathbb{T}}$.

---

[34]Using the language of [AMGRb], these two deductions are supported by the functor

$$
\mathsf{coCart}_{B\mathbb{W}} \longrightarrow \mathsf{Fun}\left((B\mathbb{N}^\times)^\mathrm{op}, G\mathsf{Span}\right)
$$

which takes $(\mathcal{E} \downarrow B\mathbb{W}) \in \mathsf{coCart}_{B\mathbb{W}}$ to the functor $(B\mathbb{N}^\times)^\mathrm{op} \to G\mathsf{Span}$ defined as follows:

- it takes the unique object of $(B\mathbb{N}^\times)^\mathrm{op}$ to the object $\mathcal{E}_0 \in G\mathsf{Span}$, and

- it takes the morphism $[1] \xrightarrow{r^\circ} (B\mathbb{N}^\times)^\mathrm{op}$ to the span

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(1) We construct a left-lax right $\mathbb{N}^\times$-action on

$$\mathsf{Fun}\,(\mathcal{E}_0, \mathcal{S}\mathsf{p})\ ,$$

under which the element $r \in \mathbb{N}^\times$ acts by the functor

$$
\begin{array}{ccc}
\mathsf{Fun}\,(\mathcal{E}_0, \mathcal{S}\mathsf{p}) & \longleftarrow & \mathsf{Fun}\,(\mathcal{E}_0, \mathcal{S}\mathsf{p}) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
(F(\pi_r^*(-)))^{\tau \mathsf{C}_r} & \longleftarrow & F
\end{array}\ .
$$

We denote this by

$$\mathsf{Fun}\,(\mathcal{E}_0, \mathcal{S}\mathsf{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times\ . \tag{41}$$

This defines a functor

$$(\mathsf{coCart}_{\mathbb{BW}})^{\mathsf{op}} \xrightarrow{\left(\mathsf{Fun}((-)_0, \mathcal{S}\mathsf{p})\overset{\text{l.lax}}{\underset{\tau}{\curvearrowleft}}\mathbb{N}^\times\right)} \mathsf{RMod}_{\mathsf{l.lax}.\mathbb{N}^\times}\ . \tag{42}$$

(2) We construct a right-lax $\mathbb{N}^\times$-equivariant functor

$$
\left(
\begin{array}{c}
\mathsf{Fun}^{\mathsf{rel}}_{/\mathbb{BN}^\times}(\mathcal{E}, \mathcal{S}\mathsf{p}) \\
\downarrow \\
\mathbb{BN}^\times
\end{array}
\right)
=: \left(\mathsf{Fun}\,(\mathcal{E}_0, \mathcal{S}\mathsf{p}) \underset{\mathsf{h}}{\curvearrowleft} \mathbb{N}^\times\right) \xrightarrow{\text{r.lax}} \left(\mathsf{Fun}\,(\mathcal{E}_0, \mathcal{S}\mathsf{p}) \overset{\text{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times\right) \tag{43}
$$

of left-lax right $\mathbb{N}^\times$-modules. This is the identity functor

$$\mathsf{Fun}(\mathcal{E}_0, \mathcal{S}\mathsf{p}) \xrightarrow{\text{id}} \mathsf{Fun}(\mathcal{E}_0, \mathcal{S}\mathsf{p})$$

on their common underlying ∞-category $\mathsf{Fun}(\mathcal{E}_0, \mathcal{S}\mathsf{p})$, and for an object

$$F \in \mathsf{Fun}(\mathcal{E}_0, \mathcal{S}\mathsf{p})$$

(considered as living in the source), the right-laxness determines structure maps

$$\mathsf{id}\left(F(\pi_r^*(-))^{\mathsf{hC}_r}\right) \simeq F(\pi_r^*(-))^{\mathsf{hC}_r} \longrightarrow F(\pi_r^*(-))^{\tau \mathsf{C}_r} \simeq \mathsf{id}(F)(\pi_r^*(-))^{\tau \mathsf{C}_r}$$

in $\mathcal{S}\mathsf{p}$ (where we use "id" in our notation only to clarify where these various constructions are taking place).

---

from the full subcategory $\Gamma_{[1]}^{\mathsf{cocart}/\mathbb{BW}}(\mathcal{E}) \subset \Gamma_{[1]}(\mathcal{E})$ on those sections

$$
\begin{array}{ccc}
 & & \mathcal{E} \\
 & \nearrow & \downarrow \\
 & & \mathbb{BW} \\
 & & \downarrow \\
[1] & \xrightarrow{\ r\ } & \mathbb{BN}^\times
\end{array}
$$

which select morphisms in $\mathcal{E}$ that are cocartesian over their images in $\mathbb{BW}$.

(Note that this prescription respects composition since $\mathcal{E} \downarrow \mathbb{BW}$ is a cocartesian fibration, i.e. the cocartesian morphisms are stable under composition.)

(3) For any symmetric monoidal ∞-category $\mathcal{V}$ admitting pushouts, we define a full subcategory

$$\mathsf{Lin}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{Sp}\right) \subset \mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{Sp}\right) \tag{44}$$

of **fiberwise multilinear** functors, as follows. Recall first that a functor is *linear* if it takes pushouts to pullbacks, and a multifunctor is *multilinear* if it is linear separately in each variable. Then, we define the full subcategory (44) to consist of those functors

$$\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{F} \mathcal{Sp}$$

that become multilinear when restricted to every fiber

$$
\begin{array}{ccc}
\mathcal{V}^{\times R^{(1)}} & \longrightarrow & \int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V} \xrightarrow{F} \mathcal{Sp} \\
\downarrow & & \downarrow \\
\{(R \to S)\} & \longrightarrow & \mathbf{\Lambda}^{\mathrm{op}}
\end{array}
\quad .
$$

The subcategory (44) is stable under the left-lax right $\mathbb{N}^{\times}$-action (41): this is immediate from the fundamental calculus-theoretic feature of the Tate construction [Lemma C.2.19], that the endofunctor

$$E \longmapsto \left(E^{\otimes r}\right)^{\tau\mathsf{C}_r}$$

of $\mathcal{Sp}$ is linear. The construction of the subobject

$$\left(\mathsf{Lin}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{Sp}\right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^{\times}\right) \subset \left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{V}, \mathcal{Sp}\right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowright}} \mathbb{N}^{\times}\right)$$

is evidently contravariantly functorial in $\mathcal{V}$ for linear symmetric monoidal functors.

4.3. **The Tate package.** In order to construct the Tate package, we will first construct the *finite* Tate package, i.e. its restriction to finite spectra, and then we will extend this to the Tate package itself. This two-step process is necessary because the endofunctor

$$E \longmapsto \left(E^{\otimes r}\right)^{\tau\mathsf{C}_r}$$

on $\mathcal{Sp}$ is linear but does not preserve filtered colimits [Remark C.2.20].

We construct the finite Tate package from the finite diagonal package (which is simply the restriction of the diagonal package to finite spaces) as in Figure 3. A number of comments are in order.

(1) Perhaps unexpectedly, it is possible to take the left- or right-lax limit of any left- or right-lax left or right $\mathbb{N}^{\times}$-module (with all three instances of "or" interpreted independently); on the other hand, a right-lax $\mathbb{N}^{\times}$-equivariant morphism only induces a map on right-lax limits (and dually), while a strict morphism induces maps on all limits [§C.1]. Thus, it is indeed meaningful to apply the functor $\mathsf{lim}^{\mathsf{r.lax}}$ to the diagram in Figure 3.

(2) The first morphism

$$\left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}}, \mathcal{S}\right) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times}\right) \xrightarrow[\mathsf{r.lax}]{\Sigma^{\infty}_{+}\circ-} \left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}}, \mathcal{Sp}\right) \underset{\mathsf{h}}{\curvearrowright} \mathbb{N}^{\times}\right)$$

$$\left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\mathcal{S}^{\mathrm{fin}},\mathcal{S}\right)\underset{\mathsf{h}}{\curvearrowright}\mathbb{N}^{\times}\right)\xrightarrow[\mathrm{r.lax}]{\Sigma^{\infty}_{+}\circ-}\left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\mathcal{S}^{\mathrm{fin}},\mathcal{S}\mathsf{p}\right)\underset{\mathsf{h}}{\curvearrowright}\mathbb{N}^{\times}\right)\xrightarrow[\mathrm{r.lax}]{(43)}\left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\mathcal{S}^{\mathrm{fin}},\mathcal{S}\mathsf{p}\right)\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowright}}\mathbb{N}^{\times}\right)\xleftarrow{-\circ\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\Sigma^{\infty}_{+}\right)}\left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\mathcal{S}\mathsf{p}^{\mathrm{fin}},\mathcal{S}\mathsf{p}\right)\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowright}}\mathbb{N}^{\times}\right)$$

$$\left(\mathsf{Lin}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\mathcal{S}^{\mathrm{fin}},\mathcal{S}\mathsf{p}\right)\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowright}}\mathbb{N}^{\times}\right)\xleftarrow{\sim}\left(\mathsf{Lin}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\mathcal{S}\mathsf{p}^{\mathrm{fin}},\mathcal{S}\mathsf{p}\right)\overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowright}}\mathbb{N}^{\times}\right)$$

$$(\textstyle\prod,\mathsf{h})^{\mathrm{fin}}\longmapsto\left(\bigotimes\circ\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\Sigma^{\infty}_{+},\mathsf{h}\right)^{\mathrm{fin}}$$

$$\left(\bigotimes\circ\int_{|\mathbf{\Lambda}^{\mathrm{op}}}\mathcal{B}\Sigma^{\infty}_{+},\tau\right)^{\mathrm{fin}}\longmapsto\left(\bigotimes,\tau\right)^{\mathrm{fin}}$$

FIGURE 3. Above: a commutative diagram of right-lax equivariant functors among left-lax right $\mathbb{N}^{\times}$-modules. Below: upon passage to right-lax limits, the finite diagonal package $(\prod,\mathsf{h})^{\mathrm{fin}}$ is sent to the finite Tate package $(\bigotimes,\tau)^{\mathrm{fin}}$.

in Figure 3 is induced by the covariant functoriality of the relative functor ∞-category in the second variable. It is only right-lax $\mathbb{N}^\times$-equivariant because the suspension spectrum functor Σ_+^∞ doesn't commute with the homotopy fixedpoints functors $(-)^{\mathsf{h}C_r}$: given an object

$$F \in \mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}}, \mathcal{S}\right)$$

of the source, the right-laxness of this morphism determines structure maps

$$\Sigma_+^\infty\left((F(\pi_r^*(-)))^{\mathsf{h}C_r}\right) \longrightarrow \left(\Sigma_+^\infty\left(F(\pi_r^*(-))\right)\right)^{\mathsf{h}C_r} \ ,$$

which are induced by the universal property of limits.

(3) As the suspension spectrum functor is symmetric monoidal, the diagram

$$
\begin{array}{ccc}
\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}} & \xrightarrow{\ \ \prod\ \ } & \mathcal{S} \\
{\scriptstyle \int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\Sigma_+^\infty}\Big\downarrow & & \Big\downarrow{\scriptstyle \Sigma_+^\infty} \\
\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}} & \xrightarrow[\ \ \otimes\ \ ]{} & \mathcal{S}\mathsf{p}
\end{array}
\tag{45}
$$

canonically commutes. This explains the first association

$$\left(\prod, \mathsf{h}\right)^{\mathsf{fin}} \longmapsto \left(\Sigma_+^\infty \circ \prod, \mathsf{h}\right)^{\mathsf{fin}} \simeq \left(\bigotimes \circ \int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\Sigma_+^\infty, \mathsf{h}\right)^{\mathsf{fin}}$$

of Figure 3.

(4) It is immediate that both (equivalent) composites in commutative square (45) lie in the subcategory

$$\mathsf{Lin}\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}}, \mathcal{S}\mathsf{p}\right) \subset \mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}}, \mathcal{S}\mathsf{p}\right) \ ,$$

as the upper functor $\prod$ is fiberwise multi-cocontinuous while the right functor Σ_+^∞ is linear. Thus, the second association indeed factors as indicated.

(5) The restriction

$$\mathsf{Lin}\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}}, \mathcal{S}\mathsf{p}\right) \xleftarrow{\ \ -\circ\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\Sigma_+^\infty\right)\ \ } \mathsf{Lin}\left(\int_{|\boldsymbol{\Lambda}^{\mathrm{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}}, \mathcal{S}\mathsf{p}\right) \tag{46}$$

is an equivalence due to a multivariate elaboration of the universal property of $\mathcal{S}\mathsf{p}^{\mathsf{fin}}$ as the *stabilization* of $\mathcal{S}^{\mathsf{fin}}$, namely that the suspension spectrum functor

$$\mathcal{S}^{\mathsf{fin}} \xrightarrow{\ \Sigma_+^\infty\ } \mathcal{S}\mathsf{p}^{\mathsf{fin}}$$

is the initial linear functor from $\mathcal{S}^{\mathsf{fin}}$ into a *stable* ∞-category (considered in StCat, i.e. with respect to exact functors between stable ∞-categories).[35] Moreover, the forgetful functor

$$\mathsf{RMod}_{\mathsf{l.lax}.\mathbb{N}^\times} \longrightarrow \mathsf{Cat}$$

from left-lax right $\mathbb{N}^\times$-modules (and *strict* $\mathbb{N}^\times$-equivariant functors) to their underlying ∞-categories is conservative.

Now, to obtain the Tate package from the finite Tate package, we observe that the functor (42) takes the morphism

$$\begin{array}{ccc}
\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}} & \xhookrightarrow{\quad \text{f.f.} \quad} & \int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p} \\
& \searrow \qquad \swarrow & \\
& \boldsymbol{\Lambda}^{\mathsf{op}} & \\
& \downarrow & \\
& \mathrm{B}\mathbb{W} &
\end{array} \tag{47}$$

in $\mathsf{coCart}_{\mathrm{B}\mathbb{W}}$ to a morphism

$$\left(\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}}, \mathcal{S}\mathsf{p} \right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times \right) \longleftarrow \left(\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}, \mathcal{S}\mathsf{p} \right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times \right)$$

in $\mathsf{RMod}_{\mathsf{l.lax}.\mathbb{N}^\times}$ which is a right adjoint on underlying ∞-categories, and so extends to an adjunction

$$\left(\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}}, \mathcal{S}\mathsf{p} \right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times \right) \xrightleftharpoons[\quad\perp\quad]{} \left(\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}, \mathcal{S}\mathsf{p} \right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times \right) \tag{48}$$

in $\mathsf{RMod}_{\mathsf{l.lax}.\mathbb{N}^\times}^{\mathsf{r.lax}}$ [Lemma C.1.36]. Taking right-lax limits of this left adjoint, we obtain a functor

$$\begin{array}{ccc}
\lim^{\mathsf{r.lax}}\left(\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}}, \mathcal{S}\mathsf{p} \right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times \right) & \longrightarrow & \lim^{\mathsf{r.lax}}\left(\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}, \mathcal{S}\mathsf{p} \right) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^\times \right) \\
\cup & & \cup \\
(\bigotimes, \tau)^{\mathsf{fin}} & \longmapsto & (\bigotimes, \tau)
\end{array}$$

taking the finite Tate package to the Tate package.

To see that this object indeed overlies the "tensor everything together" functor (17), we first observe that on underlying ∞-categories, the adjunction (48) in $\mathsf{RMod}_{\mathsf{l.lax}.\mathbb{N}^\times}^{\mathsf{r.lax}}$ recovers an adjunction

$$\mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}}, \mathcal{S}\mathsf{p} \right) \xrightleftharpoons[\quad\perp\quad]{} \mathsf{Fun}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}, \mathcal{S}\mathsf{p} \right)$$

---

[35]To see this multivariate elaboration, we work fiberwise over $\boldsymbol{\Lambda}^{\mathsf{op}}$. Namely, writing $\mathcal{V}$ simultaneously for $\mathcal{S}^{\mathsf{fin}}$ and $\mathcal{S}\mathsf{p}^{\mathsf{fin}}$, let us consider the full subcategory

$$\mathsf{Lin}_{/\boldsymbol{\Lambda}^{\mathsf{op}}}^{\mathsf{rel}}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{S}\mathsf{p}} \right) \subset \mathsf{Fun}_{/\boldsymbol{\Lambda}^{\mathsf{op}}}^{\mathsf{rel}}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{V}, \underline{\mathcal{S}\mathsf{p}} \right)$$

on the multilinear functors. The restriction (46) arises as the global sections of a morphism

$$\begin{array}{ccc}
\mathsf{Lin}_{/\boldsymbol{\Lambda}^{\mathsf{op}}}^{\mathsf{rel}}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}^{\mathsf{fin}}, \underline{\mathcal{S}\mathsf{p}} \right) & \longleftarrow & \mathsf{Lin}_{/\boldsymbol{\Lambda}^{\mathsf{op}}}^{\mathsf{rel}}\left(\int_{|\boldsymbol{\Lambda}^{\mathsf{op}}} \mathfrak{B}\mathcal{S}\mathsf{p}^{\mathsf{fin}}, \underline{\mathcal{S}\mathsf{p}} \right) \\
& \searrow \qquad \swarrow & \\
& \boldsymbol{\Lambda}^{\mathsf{op}} &
\end{array}$$

in $\mathsf{Cat}_{/\boldsymbol{\Lambda}^{\mathsf{op}}}$, which it therefore suffices to show is an equivalence. But this morphism evidently lies in the subcategory $\mathsf{Cart}_{\boldsymbol{\Lambda}^{\mathsf{op}}} \subset \mathsf{Cat}_{/\boldsymbol{\Lambda}^{\mathsf{op}}}$, and so it suffices to show that it is an equivalence on fibers. But this is clear.

in Cat. This left adjoint is given by left Kan extension along the functor (47) (i.e. along the image in Cat of that morphism in $\mathsf{coCart}_{B\mathbb{W}}$); considering that functor as a morphism in $\mathsf{coCart}_{\boldsymbol{\Lambda}^{\mathsf{op}}}$, we see that this left Kan extension may be computed fiberwise over $\boldsymbol{\Lambda}^{\mathsf{op}}$. We then conclude by noting that for any $r \geq 1$, the commutative triangle

$$
\begin{array}{ccc}
\left(\mathcal{S}\mathsf{p}^{\mathsf{fin}}\right)^{\times r} & \xrightarrow{\ \otimes\ } & \mathcal{S}\mathsf{p} \\
{\scriptstyle \mathsf{f.f.}}\downarrow & \nearrow {\scriptstyle \otimes} & \\
\mathcal{S}\mathsf{p}^{\times r} & &
\end{array}
$$

is a left Kan extension diagram: for any $E \in \mathcal{S}\mathsf{p}$ the ∞-category

$$
\left(\mathcal{S}\mathsf{p}^{\mathsf{fin}}\right)_{/E} := \mathcal{S}\mathsf{p}^{\mathsf{fin}} \underset{\mathcal{S}\mathsf{p}}{\times} \mathcal{S}\mathsf{p}_{/E}
$$

is sifted (in fact filtered), so for any $(E_i) \in \mathcal{S}\mathsf{p}^{\times r}$ the ∞-category

$$
\left(\left(\mathcal{S}\mathsf{p}^{\mathsf{fin}}\right)^{\times r}\right)_{/(E_i)} := \left(\mathcal{S}\mathsf{p}^{\mathsf{fin}}\right)^{\times r} \underset{\mathcal{S}\mathsf{p}^{\times r}}{\times} \left(\mathcal{S}\mathsf{p}^{\times r}\right)_{/(E_i)} \simeq \prod_i \left(\mathcal{S}\mathsf{p}^{\mathsf{fin}} \underset{\mathcal{S}\mathsf{p}}{\times} \mathcal{S}\mathsf{p}_{/E_i}\right)
$$

is sifted as well, and the tensor product functor

$$
\mathcal{S}\mathsf{p}^{\times r} \xrightarrow{\ \otimes\ } \mathcal{S}\mathsf{p}
$$

commutes with sifted colimits.

4.4. The cyclotomic structure on THH.
By Theorem A, we can identify the ∞-category of cyclotomic spectra as the right-lax limit

$$
\mathsf{Cyc}(\mathcal{S}\mathsf{p}) \simeq \lim{}^{\mathsf{r.lax}} \left(\mathsf{Fun}(B\mathbb{T}, \mathcal{S}\mathsf{p}) \overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \ .
$$

Then, for any spectrally-enriched ∞-category $\mathcal{C}$, the Tate package is taken to $\mathsf{THH}(\mathcal{C})$ equipped with its cyclotomic structure as illustrated in Figure 4. In analogy with Figure 2, we also summarize the construction in Figure 5.

Unlike in Figure 1, the commutativity of the upper diagram in Figure 4 (after omitting either both left adjoints or both right adjoints) – which guarantees that the underlying spectrum of the cyclotomic spectrum that is constructed is indeed $\mathsf{THH}(\mathcal{C})$ – is not trivial. However, it can be seen as follows. From [Definition C.1.26 and Observation C.1.27], we have a (nearly definitionally) commutative diagram

$$
\begin{array}{ccc}
\mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.}\mathbb{N}^{\times}} & \xrightarrow{\quad\quad\quad \lim^{\mathsf{r.lax}} \quad\quad\quad} & \mathsf{Cat} \\
{\scriptstyle \mathsf{f.f.}}\searrow & & \nearrow {\scriptstyle \mathsf{ev}_{\mathsf{sd}((B\mathbb{N}^{\times})^{\mathsf{op}})}} \\
& \mathsf{Fun}\left(\left(\mathsf{Cat}_{/(B\mathbb{N}^{\times})^{\mathsf{op}}}\right)^{\mathsf{op}}, \mathsf{Cat}\right) &
\end{array}
\quad .
$$

Note that evaluating the image of a left-lax right $\mathbb{N}^{\times}$-module on a morphism

$$
\begin{array}{ccc}
\mathsf{pt} & \xrightarrow{\quad\quad\quad\quad} & \mathsf{sd}((B\mathbb{N}^{\times})^{\mathsf{op}}) \\
& \searrow \quad\quad \swarrow & \\
& (B\mathbb{N}^{\times})^{\mathsf{op}} &
\end{array}
\tag{49}
$$

$$\mathrm{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathcal{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}\right) \longrightarrow \mathrm{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathrm{cd}(\mathcal{C}^{\simeq}), \mathcal{S}\mathrm{p}\right) \xleftarrow{\quad\perp\quad} \mathrm{Fun}(B\mathbb{T}, \mathcal{S}\mathrm{p})$$

$$\lim^{\mathrm{r.lax}}\left(\mathrm{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathcal{B}\mathcal{S}\mathrm{p}, \mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \longrightarrow \lim^{\mathrm{r.lax}}\left(\mathrm{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathrm{cd}(\mathcal{C}^{\simeq}), \mathcal{S}\mathrm{p}\right) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \xleftarrow{\quad\perp\quad} \lim^{\mathrm{r.lax}}\left(\mathrm{Fun}(B\mathbb{T}, \mathcal{S}\mathrm{p}) \overset{\mathrm{l.lax}}{\underset{\tau}{\curvearrowleft}} \mathbb{N}^{\times}\right) \simeq \mathrm{Cyc}(\mathcal{S}\mathrm{p})$$

$$\otimes \xmapsto{\qquad\qquad (18) \qquad\qquad} (\mathbb{T} \curvearrowright \mathrm{THH}(\mathcal{C}))$$

$$(\otimes, \tau) \longmapsto (\mathrm{Cyc} \curvearrowright \mathrm{THH}(\mathcal{C}))$$

FIGURE 4. Above: the diagram of ∞-categories housing the passage from the Tate package to the cyclotomic structure on $\mathrm{THH}(\mathcal{C})$. All solid horizontal morphisms are given by pullback; the upwards functors are given by restriction to underlying objects of left-lax right $\mathbb{N}^{\times}$-modules, and the diagram commutes with omitting either both left adjoints or both right adjoints. Below: the Tate package gives rise to the cyclotomic structure on $\mathrm{THH}(\mathcal{C})$, and the restriction of this passage recovers its underlying $\mathbb{T}$-action.

$$
\begin{array}{ccccc}
[1] \coprod_{1,\mathsf{pt},1} [1] & \xrightarrow{\left(\int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathcal{BS}\mathsf{p} \leftarrow \int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathsf{cd}(\mathcal{C}^{\simeq}) \to B\mathbb{W}\right)^{\circ}} & (\mathsf{LMod}_{\mathbb{W}})^{\mathrm{op}} & \xrightarrow{\left(\mathsf{Fun}((-)_0,\mathcal{S}\mathsf{p})^{\mathsf{l.lax}}_{\tau}\overset{\curvearrowright}{}\mathbb{N}^{\times}\right)} & \mathsf{RMod}_{\mathsf{l.lax}.\mathbb{N}^{\times}} \\
\Big\downarrow{\scriptstyle \mathsf{id}\coprod_{\mathsf{pt}}\mathsf{r.adjt}} & & & & \Big\downarrow \\
[1] \coprod_{1,\mathsf{pt},l} \mathsf{Adj} & \dashrightarrow{\quad \exists! \ [\text{Lemma C.1.36}]\quad} & & & \mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathbb{N}^{\times}} \\
\Big\uparrow{\scriptstyle \mathsf{id}\coprod_{\mathsf{pt}}\mathsf{l.adjt}} & & & & \\
[2] & & & & \Big\downarrow{\scriptstyle \lim^{\mathsf{r.lax}}} \\
\Big\uparrow{\scriptstyle \{0<2\}} & & & & \\
[1] & \dashrightarrow{\qquad\qquad\qquad} & & & \mathsf{Cat}
\end{array}
$$

FIGURE 5. A diagram illustrating in the language of modules how to obtain the lower rightwards composite functor in Figure 4, which takes the Tate package to the cyclotomic structure on $\mathsf{THH}(\mathcal{C})$. Here, we write $l \in \mathsf{Adj}$ for the source of the walking left adjoint.

in $\mathsf{Cat}_{/(B\mathbb{N}^{\times})^{\mathrm{op}}}$ determines the canonical map from its right-lax limit to its underlying object – the upwards functors in Figure 4. Then, the commutativity of the left square in Figure 4 is obtained by precomposing the adjoint of the composite

$$
[1] \xrightarrow{\left(\int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathcal{BS}\mathsf{p} \leftarrow \int_{|\widetilde{\mathbf{\Lambda}}^{\mathrm{op}}} \mathsf{cd}(\mathcal{C}^{\simeq})\right)^{\circ}} (\mathsf{LMod}_{\mathbb{W}})^{\mathrm{op}} \xrightarrow{\left(\mathsf{Fun}((-)_0,\mathcal{S}\mathsf{p})^{\mathsf{l.lax}}_{\tau}\overset{\curvearrowright}{}\mathbb{N}^{\times}\right)} \mathsf{RMod}_{\mathsf{l.lax}.\mathbb{N}^{\times}} \longhookrightarrow \mathsf{Fun}\left(\left(\mathsf{Cat}_{/(B\mathbb{N}^{\times})^{\mathrm{op}}}\right)^{\mathrm{op}},\mathsf{Cat}\right)
$$

to obtain the composite

$$
[1] \times [1] \xrightarrow{\mathsf{id}\times(49)^{\circ}} [1] \times \left(\mathsf{Cat}_{/(B\mathbb{N}^{\times})^{\mathrm{op}}}\right)^{\mathrm{op}} \longrightarrow \mathsf{Cat} \ .
$$

Similarly, the commutativity of the right square in Figure 4 is obtained by precomposing the adjoint of the composite

$$
\mathsf{Adj} \xrightarrow{\left(\mathsf{Fun}\left(\int_{|\mathbf{\Lambda}^{\mathrm{op}}} \mathsf{cd}(\mathcal{C}^{\simeq}),\mathcal{S}\mathsf{p}\right)\overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowright}}\mathbb{N}^{\times}\right) \xleftarrow{\perp} \left(\mathsf{Fun}(B\mathbb{T},\mathcal{S}\mathsf{p})\overset{\mathsf{l.lax}}{\underset{\tau}{\curvearrowright}}\mathbb{N}^{\times}\right)} \mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathbb{N}^{\times}} \xrightarrow{\mathsf{f.f.}} \mathsf{Fun}\left(\left(\mathsf{Cat}_{/(B\mathbb{N}^{\times})^{\mathrm{op}}}\right)^{\mathrm{op}},\mathsf{Cat}\right)
$$

to obtain the composite

$$
\mathsf{Adj} \times [1] \xrightarrow{\mathsf{id}\times(49)^{\circ}} \mathsf{Adj} \times \left(\mathsf{Cat}_{/(B\mathbb{N}^{\times})^{\mathrm{op}}}\right)^{\mathrm{op}} \longrightarrow \mathsf{Cat} \ . \qquad\qquad \square
$$

PART II. THE CYCLOTOMIC TRACE

In this part of the paper, we prove Theorem D modulo certain key maneuvers which are contained in the papers [AMGRb, AMGRa]. To begin, in §5 we recall the unstable cyclotomic trace for flagged spatially-enriched ∞-categories, as constructed in [§F.5]. Then, in §6, through a key

maneuver which ultimately relies on the fundamental calculus-theoretic feature of the Tate construction [Lemma C.2.19], we use this to obtain a cyclotomic pre-trace map for spectrally-enriched ∞-categories and thereafter the cyclotomic trace map for stable ∞-categories.

5. The unstable cyclotomic trace

Factorization homology enriched in a cartesian symmetric monoidal ∞-category $(\mathcal{V}, \times)$ assembles into a bifunctor

$$\int_{(-)} (-) : \mathbf{M} \times \mathsf{fCat}(\mathcal{V}) \longrightarrow \mathcal{V}$$

[Definition F.4.12 and Proposition F.4.13]. Note the fully faithful embedding

$$\mathrm{B}\mathbb{W}^{\triangleleft} \xrightarrow{\mathsf{f.f.}} \mathbf{M}$$

extending the defining inclusion $\mathrm{B}\mathbb{W} \hookrightarrow \mathbf{M}$ over the initial object $\mathbb{D}^0 \in \mathbf{M}$. This determines the **unstable cyclotomic trace** map

$$\int_{\mathbb{D}^0} \simeq \iota \longrightarrow \mathsf{TC}_{\mathcal{S}}^{\times} := \left(\mathsf{THH}_{\mathcal{V}}^{\times}\right)^{\mathrm{h}\mathbb{W}} , \tag{50}$$

which assembles into a natural transformation

$$\mathsf{fCat}(\mathcal{S}) \overset{\iota}{\underset{\mathsf{TC}_{\mathcal{S}}^{\times}}{\Longrightarrow}} \mathcal{S} \ .$$

6. The cyclotomic trace

Given any flagged spectrally-enriched ∞-category $\mathcal{C} \in \mathsf{fCat}(\mathcal{S}\mathrm{p})$, consider its underlying ∞-groupoid

$$\mathcal{C}^{\simeq} \in \mathcal{S} \subset \mathsf{Cat} \simeq \mathsf{Cat}(\mathcal{S}) \subset \mathsf{fCat}(\mathcal{S})$$

as a flagged spatially-enriched ∞-category, with enriched hom functor

$$\mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\underline{\mathsf{hom}}_{\mathcal{C}^{\simeq}}} \mathfrak{B}\mathcal{S} \ .$$

The postcomposition

$$\underline{\mathsf{hom}}_{(\mathfrak{B}\Sigma_+^{\infty})(\mathcal{C}^{\simeq})} : \mathsf{cd}(\mathcal{C}^{\simeq}) \xrightarrow{\underline{\mathsf{hom}}_{\mathcal{C}^{\simeq}}} \mathfrak{B}\mathcal{S} \xrightarrow{\mathfrak{B}\Sigma_+^{\infty}} \mathfrak{B}\mathcal{S}\mathrm{p}$$

then determines the enriched hom functor of a flagged spectrally-enriched ∞-category

$$(\mathfrak{B}\Sigma_+^{\infty})(\mathcal{C}^{\simeq}) \in \mathsf{fCat}(\mathcal{S}\mathrm{p}) \ ,$$

which is the initial flagged spectrally-enriched ∞-category whose underlying ∞-groupoid is $\mathcal{C}^{\simeq}$.[36] Hence, we obtain a canonical functor

$$(\mathfrak{B}\Sigma_+^{\infty})(\mathcal{C}^{\simeq}) \longrightarrow \mathcal{C} \tag{51}$$

in $\mathsf{fCat}(\mathcal{S}\mathrm{p})$. This determines the *cyclotomic pre-trace*, namely the composite morphism

$$\Sigma_+^{\infty}(\mathcal{C}^{\simeq}) \xrightarrow{\Sigma_+^{\infty}(50)} \Sigma_+^{\infty}(\mathsf{TC}_{\mathcal{S}}^{\times}(\mathcal{C}^{\simeq}))$$

---

[36]To see this, note first that $\mathcal{C}^{\simeq}$ is evidently the initial flagged spatially-enriched ∞-category with underlying ∞-groupoid given by $\mathcal{C}^{\simeq}$ (e.g. using the equivalence between flagged spatially-enriched ∞-categories and Segal spaces [GH15, Theorem 4.4.7]). Then, observe that in the adjunction $\Sigma_+^{\infty} \dashv \Omega^{\infty}$ the left adjoint is symmetric monoidal while the right adjoint is left-lax symmetric monoidal, so that taking right-lax functors of category objects out of $\mathsf{cd}(\mathcal{C}^{\simeq})$ gives another adjunction, whose left adjoint (which preserves initial objects) takes $\mathcal{C}^{\simeq}$ to $(\mathfrak{B}\Sigma_+^{\infty})(\mathcal{C}^{\simeq})$.

$$=: \Sigma_+^\infty \left(\mathsf{THH}_\mathcal{S}^\times (\mathcal{C}^\simeq)^{\mathsf{h}\mathbb{W}} \right)$$

$$\longrightarrow \left(\Sigma_+^\infty \mathsf{THH}_\mathcal{S}^\times (\mathcal{C}^\simeq) \right)^{\mathsf{h}\mathbb{W}} \tag{52}$$

$$\longrightarrow \left(\Sigma_+^\infty \mathsf{THH}_\mathcal{S}^\times (\mathcal{C}^\simeq) \right)^{\mathsf{hCyc}} \tag{53}$$

$$\simeq \left(\mathsf{THH}_{\mathsf{Sp}}((\mathfrak{B}\Sigma_+^\infty)(\mathcal{C}^\simeq)) \right)^{\mathsf{hCyc}} \tag{54}$$

$$\xrightarrow{(\mathsf{THH}_{\mathsf{Sp}}(51))^{\mathsf{hCyc}}} (\mathsf{THH}_{\mathsf{Sp}}(\mathcal{C}))^{\mathsf{hCyc}}$$

$$=: \mathsf{TC}(\mathcal{C})$$

in Sp, as we now explain.[37]

- The morphism (52) arises from the universal property of limits.

- To obtain the morphism (53), recall first that $\mathsf{THH}_\mathcal{S}^\times(\mathcal{C}^\simeq)$ is an unstable cyclotomic space, so that $\Sigma_+^\infty \mathsf{THH}_\mathcal{S}^\times(\mathcal{C}^\simeq)$ is an unstable cyclotomic *spectrum*: the composite

$$\mathsf{B}\mathbb{W} \xrightarrow{\mathsf{THH}_\mathcal{S}^\times(\mathcal{C}^\simeq)} \mathcal{S} \xrightarrow{\Sigma_+^\infty} \mathsf{Sp} \ .$$

By [Corollary C.3.23], this may also be identified as a ***cyclotomic spectrum with Frobenius lifts*** [Definition C.3.16]: a cyclotomic spectrum $T \in \mathsf{Cyc}(\mathsf{Sp})$ equipped with suitably equivariant lifts

compatibly for all $r \in \mathbb{N}^\times$. Then, the morphism (53) is guaranteed by [Observation C.3.34].

- The equivalence (54) results from the fact that the suspension spectrum functor

$$\mathcal{S} \xrightarrow{\Sigma_+^\infty} \mathsf{Sp}$$

is symmetric monoidal and commutes with colimits.

This construction is natural in the variable $\mathcal{C} \in \mathsf{fCat}(\mathcal{C})$, so that it determines the natural transformation in the diagram

By the universal property of algebraic K-theory for stable ∞-categories [BGT13], this induces a unique factorization

$$
\begin{array}{c}
\mathsf{StCat} \xrightarrow{\ \ \mathsf{K}\ \ } \mathsf{Sp}
\end{array}
\quad,
$$

with $\Sigma_+^\infty \iota$ (top), $\Downarrow$, K, $\Downarrow$, TC (bottom).

namely the cyclotomic trace. $\qquad\square$

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