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MODULES OVER THE STEENROD ALGEBRA

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§0. INTRODUCTION

IN THIS paper we will investigate certain aspects of the structure of the mod 2 Steenrod algebra, A, and of modules over it. Because of the central role taken by the cohomology groups of spaces considered as A-modules in much of recent algebraic topology (e.g. [1], [2], [7]), it is not unreasonable to suppose that the algebraic structure that we elucidate will be of benefit in the study of topological problems.

The main result of this paper, Theorem 3.1, gives a criterion for a module M to be free over the Steenrod algebra, or over some subHopf algebra B of A. To describe this criterion consider a ring R and a left module over R, M; for $e \in R$ we define the map $e: M \to M$ by $e(m) = em$. If the element e satisfies the condition that $e^2 = 0$ then $\operatorname{im} e \subset \ker e$, therefore we can define the homology group $H(M, e) = \ker e/\operatorname{im} e$. Then, for example, $H(R, e) = 0$ implies that $H(F, e) = 0$ for any free R-module F. We can now describe the criterion referred to above: for any subHopf algebra B of A there are elements e_i—with $e_i^2 = 0$ and $H(B, e_i) = 0$—such that a connected B-module M is free over B if and only if $H(M, e_i) = 0$ for all i. This criterion we attribute to C.T.C. Wall (unpublished); he studied the cases $B = A_1, A_2$ proving our Theorem 3.1 for $B = A_1$. Here A_n is the subalgebra of A generated by $Sq^1, \ldots, Sq^{2^n}$. It should also be noted that some of these homology groups have already been found useful in algebraic topology, for example in the work of Anderson, Brown and Peterson [2].

This paper is organized as follows. Section 1 is devoted to a brief description of the Steenrod algebra from the point of view taken by Milnor [6]. The section also quotes a characterization, due to the second author, of subHopf algebras of A. In Section 2 we develop the algebraic tools needed to prove the main theorem by considering the particular situation of exterior algebras. Section 3 is primarily devoted to the proof of the main theorem, Theorem 3.1, which was described above. The section also includes some of the more immediate corollaries, for example, if we are given a short exact sequence of A-modules and any two are free then so is the third. In Section 4 we begin a more detailed study of A-modules using the homology groups introduced in Section 3. The results are of two types, "global" and "local". We prove, for example, that free A-Modules are injective and use this to answer positively the question: is there a nice relation between A-modules M and N if there is an A-map between them that induces isomorphisms of the homologies?

A brief word about the genesis of this paper: all the results were originally proven by the second author [3]; however this paper incorporates a proof of the main theorem due to the first author that is substantially easier to follow.

§1. THE STRUCTURE OF THE Mod 2 STEENROD ALGEBRA

In this section we recall some basic properties of the Steenrod algebra primarily from the point of view of the Milnor basis. The mod 2 Steenrod algebra, A, is the algebra of stable operations in cohomology with Z_2-coefficients. We recall the description due to John Milnor of this algebra:

THEOREM 1.1. *The Steenrod algebra A is a locally finite Hopf algebra such that the dual Hopf algebra A^* satisfies*

(1) $A^* = Z_2[\xi_1, \xi_2, \ldots]$ $\deg \xi_i = 2^i - 1$, *and*

(2) $\psi\xi_i = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k \quad (\xi_0 = 1)$.

Therefore A has a Z_2-basis dual to the monomial basis of A^* which we denote $\{Sq(r_1, r_2, \ldots)\}$ with $Sq(r_1, r_2, \ldots)$ dual to $\xi_1^{r_1}\xi_2^{r_2} \ldots$. Further with respect to this basis the Hopf algebra structure of A is given by

(1) $\psi Sq(r_1, \ldots) = \sum_{s_i + t_i = r_i} Sq(s_1, \ldots) \otimes Sq(t_1, \ldots)$, and

(2) $Sq(r_1, \ldots) \cdot Sq(s_1, \ldots) = \sum_X \beta(X) Sq(t_1, \ldots)$, the summation being over all matrices

$$X = \begin{vmatrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & \cdots & \\ x_{20} & \vdots & & \\ \vdots & & & \end{vmatrix}$$ that satisfy $s_j = \sum x_{ij}$, $r_i = \sum 2^j x_{ij}$ and with $t_k = \sum_{i+j=k} x_{ij}$

and $\beta(X) = \prod_k \dfrac{t_k!}{x_{k0}! \cdots x_{0k}!} \in Z_2$.

Proof. See [6] for details.

Definition 1.1. We will be interested in certain particular elements of A to which we give special notation. Let $P_t(r) = Sq(0, \ldots, r)$ with the r in the tth position and $P_t^s = P_t(2^s)$. This notation agrees with May's [5].

Notation. We will say that $2^k \in r$ if 2^k appears in the dyadic expansion of r.

The following lemma is the reason that the multiplication described in Theorem 1.1 is not as bad as it looks.

LEMMA 1.2. *The coefficient $\beta(X) = 0$ if and only if for some $x_{i_1 j_1}, x_{i_2 j_2}$ with $i_1 + j_1 = i_2 + j_2$ and some k, $2^k \in x_{i_1 j_1}, x_{i_2 j_2}$.*

Proof. It suffices to show that the multinomial coefficient $(x_1, \ldots, x_n) = \dfrac{(x_1 + \cdots + x_n)!}{x_1! \ldots x_n!}$ is zero mod 2 if and only if for some i, j, k, $2^k \in x_i, x_j$. We will do this by induction on n.

If $n = 2$ we have the standard result that $\frac{(x_1 + x_2)!}{x_1!\,x_2!} \equiv \prod \binom{a_i}{b_i} \bmod 2$ where $x_1 + x_2 = \sum a_i 2^i$ and $x_1 = \sum b_i 2^i$ (see [9]). Since $\binom{a_i}{b_i} \equiv 0 \bmod 2$ if and only if $a_i = 0$ and $b_i = 1$, the coefficient is zero mod 2 if and only if $2^k \in x_1, x_2$ for some k. Assuming the result for $n - 1$ we note that $(x_1, \ldots, x_n) = (x_1 + \cdots + x_{n-1}, x_n) \cdot (x_1, \ldots, x_{n-1})$. Therefore $(x_1, \ldots, x_n) \equiv 0 \bmod 2$ if and only if $(x_1 + \cdots + x_{n-1}, x_n) \equiv 0 \bmod 2$ or $(x_1, \ldots, x_{n-1}) \equiv 0 \bmod 2$. If $(x_1, \ldots, x_{n-1}) \equiv 0 \bmod 2$, then for some $i, j < n$ and some k, $2^k \in x_i, x_j$ and if not then $(x_1 + \cdots + x_{n-1}, x_n) \equiv 0 \bmod 2$ implies for some $i < n$ and some k, $2^k \in x_i, x_n$.

By way of illustration we record the computations that arise later in this work.

LEMMA 1.3. (1) *If $r, s < 2^t$ then $[P_t(r), P_t(s)] = 0$.*

(2) *If $r < 2^t$ then $P_t(r)P_t(s) = \frac{(r+s)!}{r!\,s!} P_t(r + s)$, in particular $P_t(r)^2 = 0$.*

(3) $P_t^t \cdot P_t^t = P_t(2^t - 1)P_{2t}^0$.

(4) *If $1 \leq r < 2^t$ then $[P_t^t, P_t(r)] = P_t(r - 1)P_{2t}^0$.*

(5) $[P_t^t, P_{2t}^0] = 0$.

Proof. As a sample consider (3). Because of the restrictions on the rows of the matrices that we must consider, the only ones to look at are

$$\begin{vmatrix} * & 0 & \cdots & 2^t \\ 1 & & & \\ \vdots & & \iddots & \\ 2^t & & & \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} * & \cdots & 2^t - 1 \\ 0 & & \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{vmatrix}.$$

From Lemma 1.2 it is obvious that the multinomial coefficient associated with the first matrix is 0 and with the second matrix is 1. This gives (3).

In Section 3 we will be interested in the structure of the subHopf algebras of A. Therefore we quote the following proposition of [4] which gives a characterization of the subHopf algebras suitable for our purposes.

PROPOSITION 1.4. *Let $B \subset A$ be a subHopf algebra, let*

$$r_B(t) = \max\{s \mid 2^s \in b_t \text{ for } Sq(b_1, \ldots, b_t, \ldots) \in B\}$$

or $r_B(t) = \infty$ if there is no maximum or

$$r_B(t) = -1 \text{ if } b_t = 0 \text{ for all } Sq(b_1, \ldots, b_t, \ldots) \in B.$$

(1) *B has a Z_2-basis $\{Sq(r_1, \ldots) \mid r_t < 2^{r_B(t)+1}\}$.*

(2) *As an algebra B is generated by $\{P_t^s \mid s \leq r_B(t)\}$.*

Note. As the proposition makes clear, a subHopf algebra B is determined by its function $r_B(t)$. However not all such functions are so realizable—for example $r(1) = 1$, $r(2) = -1$ is not.

Examples. (1) Let A_n denote the subHopf algebra determined by the function

$$r(t) = \max\{n + 1 - t, -1\} \text{ then } \bigcup_n A_n = A.$$

(2) Let $A(0)$ denote the subHopf algebra of A determined by the function $r(t) = 0$ all t, then $A(0)$ is an exterior algebra on P_t^0 all t.

(3) Let B be a subHopf algebra of A, let $B(0)$ denote the subHopf algebra $A(0) \cap B$—it is determined by $r(t) = 0$ if $r_B(t) \geq 0$ and $r(t) = -1$ otherwise. Note that $B(0)$ is a normal subalgebra of B—a subalgebra C of B is normal if $B \cdot I(c)$ is a two-sided ideal of B.

Definition 1.2. Let B be a subHopf algebra of A, then htB is the maximum of $r_B(t)$ for all t—we say $htB = \infty$ if either $r_B(t) = \infty$ for some t or there is no maximum. So for example $htA_n = n$ and $htB(0) = 0$ (unless B is trivial).

The main result of this paper will be proved by induction on htB and in that induction the following proposition will play a key role.

PROPOSITION 1.5. *Let B be a subHopf algebra of A with $htB = n < \infty$. Then $B/\!/B(0)$ is isomorphic to a subHopf algebra of A, B', with $htB' = n - 1$. For instance $A_n/\!/A_n(0) \approx A_{n-1}$.*

Recall that for C normal in B, $B/\!/C$ is the algebra $B/BI(C) = Z_2 \otimes_C B$.

Proof. Let $\theta: A^* \to A^*$ be given by $\theta x = x^2$, then it is easy to check that θ is a map of Hopf algebras. Therefore there is a dual map $\theta^*: A \to A$ of Hopf algebras, θ^* halving degree and $\theta^* P_t^s = P_t^{s-1}$, $s > 0$, $\theta^* P_t^0 = 0$. Since $\ker \theta^* = A \cdot I(A(0))$, θ^* induces an isomorphism $\theta': A/\!/A(0) \to A$, so let $B' = \theta'(B/\!/B(0))$ (θ' being an isomorphism of Hopf algebras and $B/\!/B(0)$ a subHopf algebra of $A/\!/A(0)$ imply that B' is a subHopf algebra of A). Further since $\theta' P_t^s = P_t^{s-1}$ it follows from Proposition 1.4 that $r_{B'}(t) = r_B(t) - 1$ and so $htB' = htB - 1$ if $htB < \infty$.

§2. MODULES OVER EXTERIOR ALGEBRAS

For the remainder of this paper we will be working with categories of *connected* (left) modules over *connected* algebras—a graded algebra R over K is connected if $R_i = 0$ for $i < 0$ and $R_0 = K$ and a graded R-module M is connected if $M_i = 0$ for $i < r$ for some integer r.

Definition 2.1. Let M be such a module over the algebra R. For $e \in R$ we define the map $e: M \to M$ by $e(m) = em$. We say that *e is exact on M* if $M \xrightarrow{e} M \xrightarrow{e} M$ is exact. For $e \in R$ satisfying $e^2 = 0$ we also have $H(M; e) = \ker e|M / \operatorname{im} e|M$ and then e is exact on M if and only if $H(M; e) = 0$.

It is immediate that if e is exact on R and M is free over R then e is exact on M. This leads us to the following definition.

Definition 2.2. An algebra R is *pseudo-exterior* on $\{e_i\}$ with respect to a category of R-modules, $\mathscr{C}$, if there are elements $e_i \in R$ which are exact on R such that the following condition is satisfied:

an R-module $M \in \mathscr{C}$ is R-free if and only if $H(M, e_i) = 0$ for all i.

The name we have chosen is reasonable because we shall show that exterior algebras on generators of distinct degree are pseudo-exterior. However the subject of pseudo-exterior algebras only becomes truly non-trivial when we add that we will prove that every subHopf algebra of the mod 2 Steenrod algebra—including A itself—is pseudo-exterior.

THEOREM 2.1. *Let $R = E[x_1, \dots, x_n]$ the exterior algebra on generators $x_1, \dots, x_n$. We further assume that the dimensions of the generators are distinct. Then a connected R module M is R-free if and only if $H(M, x_i) = 0$ for all i; that is R is pseudo-exterior with respect to the category of connected R-modules.*

Note. (1) The condition on the degrees of the generators is essential as the following example shows. Let $R = E[x_1, x_2]$ where $\deg x_1 = \deg x_2$; we will exhibit an R-module M such that $H(M, x_1) = H(M, x_2) = H(M, x_3) = 0$ where $x_3 = x_1 + x_2$ (these are all the elements of R which are exact on R itself) but M is not free over R. One such M is given by generators a, b and relations $x_1 a + x_3 b$, $x_2 a + x_1 b$, $x_3 a + x_2 b$.

(2) Since the fact that the generators have distinct degrees is crucial, we will need the following in Section 3: $\deg P_t^s = \deg P_{t_1}^{s_1}$ if and only if $P_t^s = P_{t_1}^{s_1}$. That is $\deg P_t^s = 2^s(2^t - 1)$ and the t and s can be recovered from the dyadic expansion of this number.

In order to prove Theorem 2.1 we prove two propositions that will also be of use in §3. For an R-module M let $Q_R(M) = M/I(R)M = K \otimes_R M$ (then for $B \subset R$, $R/\!/B = Q_B(R)$ and $Q_B(M)$ is a $Q_B(R)$-module). Let $\{m_i\}$ be a set of elements in M whose images in $Q_R(M)$ form a K-base for $Q_R(M)$. Then the following facts are well known and easily proved.

(i) The elements m_i generate M as an R-module.

(ii) If M is free over R on any base, then the elements m_i form an R-base for M.

PROPOSITION 2.2. *Let B be normal in R and $C = R/\!/B$. If M is free over B and $Q_B(M)$ is free over C then M is free over R.*

Proof. Take elements b_i forming a K-base for B, and elements c_j in A whose images in C form a K-base for C. Also take elements m_k in M whose images in $Q_R(M) = Q_C(Q_B(M))$ form a K-base. Since $Q_B(M)$ is C-free, the elements $c_j m_k$ give a K-base there. Since M is B-free, the elements $b_i c_j m_k$ form a K-base in M. Either there are no elements m_k, in which case the result is trivial, or else the elements $b_i c_j$ are linearly independent in R; so we assume the latter. Since the elements $b_i c_j$ certainly span R they form a K-base there. So the elements m_k form an R-base for M.

Let $R = E[x, y]$ where $\deg x = d$, $\deg y = e$ with $d \neq e$. Let M be a connected R-module. Then both yM and M/yM are modules over $E[x]$.

LEMMA 2.3. *If x and y are exact on M then x is exact on $M/(yM)$.*

Proof. Since y is exact on M we have an isomorphism $M/(yM) \xrightarrow{y} yM$ commuting with x. Therefore $H_n(M/(yM); x) = H_{n+e}(yM; x)$. But also we have the following exact sequence of $E[x]$-modules: $0 \to yM \to M \to M/(yM) \to 0$. This yields an exact homology sequence and since $H(M; x) = 0$ we have an isomorphism $H_n(M/(yM); x) = H_{n+d}(yM; x)$. So $H_n(M/(yM); x) = H_{n+d-e}(M/(yM); x)$. But $M/(yM)$ is zero in degrees less than some degree, so $H_r(M/(yM); x) = 0$ for r small enough. Since $d - e \neq 0$ we can use this isomorphism to show that $H_r(M/(yM); x) = 0$ for all r.

Proof of Theorem 2.1. The proof is by induction on the number n of generators. The result is trivially true for $n = 1$; suppose it is true for $E[x_1, \ldots, x_{n-1}]$. Let M be a module over R such that $H(M, x_i) = 0$ for each x_i. We apply Proposition 2.2 taking $B = E[x_n]$, so that $C = E[x_1, x_2, \ldots, x_{n-1}]$. Then M is free over B by the trivial case $n = 1$. We have $Q_B(M) = M/(x_n M)$; the elements x_i are exact on $Q_B(M)$ for $1 \leq i \leq n - 1$ by Lemma 2.3. By induction $Q_B(M)$ is C-free and so by Proposition 2.2, M is free over R.

We end this section by proving one final result that will be needed in the next section. Let R be an exterior algebra $E[x_1, \ldots, x_n]$ in which the dimensions of the generators are all distinct. Let M be an R-module. Then the quotient module

$N = M/(x_1 M + \cdots + x_{n-1}M) = Q_B(M) \quad (B = E[x_1, \ldots, x_{n-1}])$ is a module over $E[x_n]$.

LEMMA 2.4. *If each x_i is exact on M then x_n is exact on N.*

Of course this follows immediately from Theorem 2.1; but the obvious direct proof is by induction over n. The result is true for $n = 2$ by Lemma 2.3, so suppose it true for $n - 1$. Let R and M be as above. Then $P = M/(x_1 M + \cdots + x_{n-2}M)$ is a module over $E[x_{n-1}, x_n]$. By the inductive hypothesis, x_{n-1} and x_n are exact on P. By Lemma 2.3 x_n is exact on $P/(x_{n-1}P) = N$.

§3. THE MAIN THEOREM AND COROLLARIES

The primary aim of this paper is to prove that a wide class of interesting algebras are pseudo-exterior. Recall that for an arbitrary subHopf algebra $B \subset A$ (as usual A the mod 2 Steenrod algebra), B is generated as an algebra by the P_t^s's in B.

THEOREM 3.1. *Let B be a subHopf algebra of A then B is pseudo-exterior* (*with respect to the category of connected B-modules*) *on the P_t^s's in B with $s < t$. In particular A is pseudo-exterior on $\{P_t^s \mid s < t\}$.*

The brunt of our work will be to prove the following theorem of which Theorem 3.1 is an easy corollary.

THEOREM 3.2. *Let B be a finite subHopf algebra of A, then B is pseudo-exterior on the P_t^s's in B with $s < t$.*

We observe that any particular application of Theorem 3.2 can be made independent of proposition 1.4. To do so, one simply has to check that the given subalgebra B, and the subalgebra B', B'' etc. which arise from it by using Proposition 1.5, are as described in Proposition 1.4. For example the reader who does not wish to check the proof of Proposition 1.4 can still be sure that Theorem 3.2 is true for A_n and Theorem 3.1 is true for A.

Proof of Theorem 3.1 *from Theorem* 3.2. There are two things that must be proven. First that for any B and $P_t^s \in B$ with $s < t$, P_t^s is exact on B. Second if a connected B-module satisfies $H(M, P_t^s) = 0$ for all $P_t^s \in B$ with $s < t$ then M is free over B.

To prove the first let C be the subalgebra of B generated by $P_t^{s'}$ with $s' \leq s$. Since $s < t$, C is an exterior on the $P_t^{s'}$'s with $s' \leq t$ (see for example Lemma 1.3) and since C

has a Z_2-basis given by all $P_t(r)$ with $r < 2^t$, it is in fact a subHopf algebra of B. Therefore by an oft-quoted result of Milnor and Moore [8], B is free over C. So P_t^s is exact on B if it is exact on C and C being an exterior algebra with P_t^s as one of the generators, this is obvious.

To prove the second statement assuming Theorem 3.2 we proceed as follows: let $B_n = B \cap A_n$, then the B_n's satisfy

(1) B_n is a finite subHopf algebra of B

(2) $B_n \subset B_{n+1}$ and $\cup B_n = B$.

Let M be a connected B-module satisfying $H(M, P_t^s) = 0$ for all $P_t^s \in B$ with $s < t$. In particular $H(M, P_t^s) = 0$ for all $P_t^s \in B_n$ for any n and so by Theorem 3.2 M is B_n-free for any n. So it suffices to show that if M is B_n-free for all n then M is B-free. Let $\{m_i\}$ be a set of elements in M whose images in $Q_B(M)$ form a Z_2-base for $Q_B(M)$. We will prove that M is B-free on $\{m_i\}$ so assume that there are elements $b_i \in B$ (with only a finite number non-zero) such that $\Sigma b_i m_i = 0$. For n large enough $b_i \in B_n$ for all i. Also since $M \to Q_{B_n}(M) \to Q_B(M)$, the images of the m_i's in $Q_{B_n}(M)$ are linearly independent so can be expanded to a base for $Q_{B_n}(M)$ over Z_2 coming from $\{m_j'\} \supset \{m_i\}$. Then since M is B_n-free we have that M has a B_n-base of $\{m_j'\}$. So in particular we must have that $\Sigma b_i m_i = 0$ implies $b_i = 0$ for all i.

Before proving Theorem 3.2 we state some of the more immediate corollaries.

COROLLARY 3.3. *Let B be a subHopf algebra of A. If we are given an exact sequence of B-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ such that any two of the modules are free over B then so is the third.*

Proof. For any $P_t^s \in B$ with $s < t$ we can regard M_i as a complex with differential P_t^s and maps of B-modules induce maps of complexes. Therefore the exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ induces long exact sequences in the P_t^s homologies. Then the freeness of two of the M_i's implies that their P_t^s homologies are both zero, so the same is true of the third which is therefore free by Theorem 3.1.

We recall the following definition from homological algebra.

Definition 3.1. An R-module M has *infinite homological degree* if there are no projective resolutions of M over R of finite length.

COROLLARY 3.4. *Let M be a connected B-module (B a subHopf algebra of A), then either M is B-free or M has infinite homological degree.*

Proof. If M_0 has a finite projective resolution over B then there is a sequence of exact sequences $0 \to M_i \to F_{i-1} \to M_{i-1} \to 0$ with F_{i-1} B-free and for i large enough M_i is B-free. Therefore iterated application of Corollary 3.3 gives us that M_0 is B-free.

Note. The results of Corollaries 3.3 and 3.4 as well as many of the results of the next section are true for an arbitrary pseudo-exterior algebra, as the proofs make evident.

We now commence with the proof of Theorem 3.2. The proof will be by induction on htB (see Definition 1.2), the inductive step based on Proposition 2.2. In order to apply the inductive hypothesis we need the following lemma, whose proof we postpone until the end of this section.

LEMMA 3.5. *Let B be a finite subHopf algebra of A and let M be a connected B-module. If P_t^s is exact on M for $P_t^s \in B$ with $s < t$ then P_t^s is exact on $M/\sum_{P_t^0 \in B} P_t^0 M = Q_{B(0)}(M)$ for $P_t^s \in B$ with $s \leq t$.*

Proof of Theorem 3.2. In the light of what has already been shown in proving Theorem 3.1 it suffices to show that if M is a connected B-module such that $H(M, P_t^s) = 0$ for all $P_t^s \in B$ with $s < t$, then M is B-free. As stated above the proof will be by induction on htB. If $htB = 0$ then B is an exterior algebra on a subset of the P_t^0's—this follows from Proposition 1.4—and so the theorem is true having been proven under the guise of Theorem 2.1. So assume the result for connected modules over B' with $htB' < n$ and let $htB = n$.

So consider M a module over B such that P_t^s is exact on M for $P_t^s \in B$, $s < t$. As in Section 1, let $B(0)$ be the exterior algebra generated by the P_t^0's in B and let $B' = B/\!/B(0)$. Then, as we have shown in Proposition 1.5, B' is a finite subHopf algebra of A with $htB' = n - 1$. Since the P_t^s's with $s \leq t$ in B project to the P_t^s's with $s < t$ in B' it follows from Lemma 3.5 that P_t^s is exact on $Q_{B(0)}(M)$ for all $P_t^s \in B'$ with $s < t$. Therefore by induction $Q_{B(0)}(M)$ is free over B'. In addition M is free over $B(0)$ as noted above. Therefore by Proposition 2.2 M is free over B, which completes the proof of the theorem modulo the proof of Lemma 3.5.

If we look at Lemma 3.5, it is clear that a special role is played by the operations P_t^s with $s = t$ because they have to be proven exact on the quotient although they are not given to be exact on M. We therefore study the elements $P_t^t \in B$. Since $P_t^t \in B$ and B is a subHopf algebra of A it follows that $P_t^0, P_t^1, \ldots, P_t^{t-1} \in B$ and since $[P_t^t, P_t^0] = P_{2t}^0$ we also have $P_{2t}^0 \in B$. As in Lemma 1.3 it is clear that $P_t^0, P_t^1, \ldots, P_t^{t-1}, P_{2t}$ generate a sub-exterior algebra of B. Let $D(t)$ denote the subalgebra of B generated by that exterior algebra and P_t^t. The structure of $D(t)$ is given by the following formulae of Lemma 1.3:

$$\begin{aligned}
&(P_t^t)^2 = P_t(2^t - 1)P_{2t}^0,\\
&[P_t^t, P_t(r)] = P_t(r-1)P_{2t}^0 \text{ for } 1 \leq r < 2^t,\\
&[P_t^t, P_{2t}^0] = 0.
\end{aligned}$$

It follows that $D(t)/\!/E[P_{2t}^0] = E[P_t^0, P_t^1, \ldots, P_t^{t-1}, P_t^t]$ and if M is a module over $D(t)$ then $Q_E(M)$ $(E = E[P_{2t}^0])$ is a module over $E[P_t^0, P_t^1, \ldots, P_t^{t-1}, P_t^t]$.

LEMMA 3.6. *Let M be a connected $D(t)$-module such that P_{2t}^0 is exact on M, then P_t^t is exact on $Q_E(M)$.*

Proof. The result to be proved may be expressed as follows. Suppose given an element x of M of degree n such that $P_t^t x = P_{2t}^0 y$ for some y in M. We have to show that we can write x in the form $x = P_t^t u + P_{2t}^0 v$ for some u, v in M.

The proof is by a double induction and the main induction is over n, so we assume as our main inductive hypothesis that the result is true for elements x' of degree less than n. Suppose given an element x of degree n such that $P_t^t x = P_{2t}^0 y$. Applying P_t^t we get

$P_t(2^t-1)P^0_{2t}x = P_t^tP^0_{2t}y$, that is $P^0_{2t}[P_t(2^t-1)x + P_t^ty] = 0$. Since P^0_{2t} is exact on M we have $P_t(2^t-1)x = P_t^t y + P^0_{2t}z$ for some z in M.

Now observe that

$$\begin{aligned} P_t(2^t-2)P^0_{2t}x &= P_t^tP_t(2^t-1)x + P_t(2^t-1)P_t^t x \\ &= P_t^tP_t^ty + P_t^tP^0_{2t}z + P_t(2^t-1)P^0_{2t}y \\ &= P_t^tP^0_{2t}z. \end{aligned}$$

Again by the exactness of P^0_{2t} we see that $P_t(2^t-2)x = P_t^tz + P^0_{2t}w$.

If $t=1$ this completes the proof since $P_t(0)=1$. Otherwise we continue by induction. More precisely we will prove by downwards induction over r that $P_t(r)x = P_t^tc + P^0_{2t}d$ for $0 \le r \le 2^t-1$. Then for $r=0$ this will express x in the required form. We have obtained this result for $r = 2^t-1$ and $r = 2^t-2$ above, so we assume as our subsidiary inductive hypothesis that $P_t(r)x$ has the desired form for some r in the range $1 \le r \le 2^t-2$. Let $s = 2^t-1-r$ $(s>0)$. We have $P_t(r)P_t(s) = P_t(s)P_t(r) = P_t(2^t-1)$.

So

$$\begin{aligned} P_t^ty + P^0_{2t}z &= P_t(2^t-1)x \\ &= P_t(s)P_t(r)x \\ &= P_t(s)P_t^tc + P_t(s)P^0_{2t}d \\ &= P_t^tP_t(s)c + P_t(s-1)P^0_{2t}c + P_t(s)P^0_{2t}(d). \end{aligned}$$

That is, $P_t^t(y + P_t(s)c) = P^0_{2t}(z + P_t(s-1)c + P_t(s)d)$. But the dimension of y is less than n, so the main inductive hypothesis shows that $y + P_t(s)c = P_t^te + P^0_{2t}f$. Now observe that

$$\begin{aligned} P_t(r-1)P^0_{2t}x &= P_t(r)P_t^tx + P_t^tP_t(r)x \\ &= P_t(r)P^0_{2t}y + P_t^tP_t^tc + P_t^tP^0_{2t}d \\ &= P_t(r)P^0_{2t}P_t(s)c + P_t(r)P^0_{2t}P_t^te + P_t^tP_t^tc + P_t^tP^0_{2t}d \\ &= P^0_{2t}P_t^tP_t(r)e + P^0_{2t}P_t^td. \end{aligned}$$

That is, $P^0_{2t}(P_t(r-1)x + P_t^th) = 0$ where $h = d + P_t(r)e$. Since P^0_{2t} is exact we have $P_t(r-1)x = P_t^th + P^0_{2t}k$. This completes the subsidiary induction over r, which completes the main induction and hence the lemma.

Proof of Lemma 3.5. Now let M be a module over B such that P_t^s is exact on M for $P_t^s \in B$ with $s<t$. Consider $P_t^s \in B$ with $s \le t$. We form first the quotient $M/P^0_{2t}M$. Then P_t^s is exact on $M/P^0_{2t}M$ by Lemma 3.6 if $s=t$ and by Lemma 2.4 if $s<t$. Also for $P_i^0 \in B$ with $i \neq 2t$, P_i^0 is exact on $M/P^0_{2t}M$ by Lemma 2.3.

We now consider the quotient $N = Q_{B_s}(M)$ where B_s is the subexterior algebra of B generated by all P_i^0 with $i>s$. Then N is a module over an exterior algebra C on generators $P_i^0 \in B$ with $i \le s$ and P_t^s; this being a subalgebra of $Q_{B_s}(B)$ as can be seen from the following relations:

$$\begin{aligned} &[P_i^0, P_t^s] = P^0_{i+t}P_t(2^s - 2^i) && \text{if } i \le s, \\ &P_t^sP_t^s = 0 && \text{if } s<t, \\ &P_t^tP_t^t = P^0_{2t}P_t(2^s-1). \end{aligned}$$

Further P_i^0 is exact on N for $1 \le i \le s$, as we see by applying Lemma 2.4 to M considered as a module over the exterior algebra generated by B_s and P_i^0. Again P_t^s is exact on N, as we see by applying Lemma 2.4 to $M/P_{2t}^0 M$ considered as a module over the exterior algebra generated by $P_i^0 \in B$ with $i > s$ and $i \neq 2t$ and P_t^s. These results allow us to apply Lemma 2.3 to N with respect to the exterior algebra C. We conclude that P_t^s is exact on $Q_{B(0)}(M)$.

§4. PROPERTIES OF THE HOMOLOGY GROUPS

This section is the beginning of a more detailed study of the homology groups introduced in Sections 2 and 3. The results are of two primary types.

On the one hand we would like to know something about the "global" nature of the homology groups. In particular we wish to answer the following question: let M and N be connected A-modules, and suppose that we are given an A-map $M \to N$ which induces isomorphisms of all the homology groups, is there a new relationship between M and N? Further, in the process of answering this question we prove that connected free A-modules are injective.

On the other hand we wish to consider properties that can perhaps be described as "local". In this work we consider the case in which B is a subHopf algebra of A. The localization results are of two types. If an A-module M is an extended B-module ($M = A \otimes_B N$ for some B-module N) then only the homology groups in B arise, i.e. $H(M, P_t^s) = 0$ if $P_t^s \notin B$. We can also localize with respect to degree and get that if B is finite and $H_i(M, P_t^s) = 0$ for $i < I$ and $P_t^s \in B$ then M is B-free through deg $I - \alpha$ (α independent of M or I).

THEOREM 4.1. *If F is a connected free A-module then F is injective in the category of arbitrary A-modules.**

Proof. (a) We first reduce the problem to one of showing that A itself is an injective A-module in the stated category. Suppose that A is injective then the arbitrary product of copies of A, $\Pi A x_\alpha$, is injective (in general the product of injectives is injective). Let $\Sigma A x_\alpha$ be a connected free A-module; then $\Pi A x_\alpha$ is connected. We have the short exact sequence $0 \to \Sigma A x_\alpha \to \Pi A x_\alpha \to M \to 0$ and since $\Pi A x_\alpha$ is injective it will suffice to show that this sequence splits (in general the direct summand of an injective is injective).

SUBLEMMA. *For all P_t^s, $H(\Pi A x_\alpha, P_t^s) = 0$ (therefore since $\Pi A x_\alpha$ is connected we have also that $\Pi A x_\alpha$ is A-free).*

Proof. Let $(a_\alpha) \in \ker P_t^s | \Pi A x_\alpha$ and let $\Pi_\alpha : \Pi A x_\alpha \to A x_\alpha$ be the projection. Then $P_t^s a_\alpha = P_t^s \Pi_\alpha(a_\alpha) = \Pi_\alpha P_t^s(a_\alpha) = 0$ and therefore $a_\alpha = P_t^s b_\alpha$. Thus $(a_\alpha) = P_t^s(b_\alpha)$.

From the sublemma and the sequence above we conclude that $H(M, P_t^s) = 0$ and since M is connected it is a free A-module. And so as we desired, we have the splitting of

$$0 \to \Sigma A x_\alpha \to \Pi A x_\alpha \to M \to 0.$$

(b) We must now show that A is injective in the category of arbitrary A-modules.

* We would like to thank J. C. Moore and F. Peterson for pointing out an error in the original proof of this result. Also the proof of part (b) is a modification due to F. Peterson of a proof of ours.

Consider $0 \to A \xrightarrow{\iota} M$ with M an arbitrary A-module. To show that ι splits consider all pairs (N, p) with $A \subset N \subset M$ and $p: N \to A$ satisfying $p\iota = 1$. Partially order these pairs by setting $(N, p) < (N', p')$ if $N \subset N'$ and $p = p' \mid N$. Every linearly ordered subset has an upper bound so we may extract a maximal element (N, p). Assume that $N \neq M$ and consider $m \in M - N$. We show how to extend p to $N + Am$.

As an A_n-module A is locally finite and free, and since A_n is a Poincare algebra we conclude that A is an injective A_n-module. So p extends to an A_n-map $p_n: M \to A$. Let $p_n(m) = a_n$, the sequence $\{a_n\}$ is a subset of A_k (where $k = \deg(m)$) which is a finitely generated vector space over Z_2. Therefore for some n, a_n occurs infinitely often and it is easy to check that $p': N + Am \to A$ defined by $p' \mid N = p$ and $p'(m) = a_n$ is an A-map (any relation in A occurs within some A_n).

We are now in a position to answer the question posed at the beginning of this section.

Definition 4.1. Two B-modules M and N are *stably equivalent* if there are free B-modules F_1 and F_2 such that $F_1 \oplus M$ and $F_2 \oplus N$ are isomorphic.

THEOREM 4.2. *Let $f: M \to N$ be a map of connected A-modules and suppose that for all P_t^s with $s < t$ $f_*: H(M, P_t^s) \to H(N, P_t^s)$ is an isomorphism. Then M and N are stably equivalent.*

Proof. First assume that f is epic, i.e. $0 \to K \to M \to N \to 0$. Then $H(K, P_t^s) = 0$ for all P_t^s and therefore by Theorem 3.1 K is A-free. By Theorem 4.1 we conclude that the sequence splits and $M \oplus K \approx N$.

Now consider the general case $0 \to K \to M \xrightarrow{f} N \to L \to 0$. Let $F \xrightarrow{g} N \to 0$ be exact with F a free A-module. Then $M \oplus F \xrightarrow{f+g} N$ is onto giving $0 \to K' \to M \oplus F \to N \to 0$. But since $H(F, P_t^s) = 0$ and $f_*: H(M, P_t^s) \to H(N, P_t^s)$ is an isomorphism, we get that

$$(f + g)_*: H(M \oplus F, P_t^s) \to H(N, P_t^s)$$

is an isomorphism, which completes the proof.

We come now to the "local" results described at the beginning of this section.

Definition 4.2. An A-module M is an *extended B-module* (B a subHopf algebra of A) if M is isomorphic to $A \otimes_B N$ for some B-module N.

THEOREM 4.3. *Let M be an extended B-module, then for $P_t^s \notin B$ (with $s < t$) $H(M, P_t^s) = 0$.*

The proof of this result can be found in [4].

The other type of localization that we wish to consider is that with respect to degree. In order to do this we make the following obvious definition.

Definition 4.3. Let M be an R-module (both R and M being graded) then M is *free through degree r* if there is a free R-module F and map $f: F \to M$ such that f is an isomorphism in degree $\leq r$.

THEOREM 4.4. *Let B be a finite subHopf algebra of A and let M be a connected B-module. If $H_i(M, P_t^s) = 0$ for $P_t^s \in B$, $s < t$, and for $i \leq r$ then M is B-free through degree $r - c$ where c is a constant depending on B.*

Since we are emphasizing the qualitative nature of the result, no attempt will be made to determine a best possible such c. Also we will only sketch the proof of this result since it is a straightforward modification of the proof of Theorem 3.2 (and the proofs of Proposition 2.2 and Lemmas 2.3, 2.4, 3.5, 3.6).

We first note that in the case of Lemmas 2.3, 2.4, 3.6 the desired modification can easily be made since each is essentially proved by induction on degree starting with $M_r = 0$ for r sufficiently small. In the case of Proposition 2.2 our proof involved an explicit construction of a basis of M over R. Since we are working with connected modules over connected algebras the same construction can be performed to give a basis through a range (in the sense of Definition 4.3).

With these results the modified version of Lemma 3.5 is easily provable. From this the proof of Theorem 4.4 follows.

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