# THE SLICES OF QUATERNIONIC EILENBERG-MAC LANE SPECTRA 

BERTRAND J. GUILLOU AND CARISSA SLONE


#### Abstract

We compute the slices and slice spectral sequence of integral suspensions of the equivariant Eilenberg-Mac Lane spectra $H \underline{\mathbb{Z}}$ for the group of equivariance $Q_{8}$. Along the way, we compute the Mackey functors $\underline{\pi}_{k \rho} H \underline{\mathbb{Z}}$.


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## 1. Introduction

Let $G$ be a finite group. The $G$-equivariant slice filtration was first defined in the context of $G$-equivariant stable homotopy theory by Dugger in [D]; it came to prominence as a result of its role in the proof of the Kervaire invariant conjecture by Hill, Hopkins, and Ravenel [HHR1]. The slice filtration is an analogue in the $G$ equivariant stable homotopy category of the classical Postnikov filtration of spectra. One can also define a $G$-equivariant Postnikov filtration; on passage to fixed points with respect to any subgroup $H \leq G$, this recovers the Postnikov filtration of the $H$-fixed point spectrum. However, there are many equivariant spectra which possess a periodicity with respect to suspension by a $G$-representation sphere, and this periodicity is not visible in the $G$-equivariant Postnikov filtration. The slice filtration was devised by Dugger in order to display this periodicity for the case of the $C_{2}$-spectrum $K \mathbb{R}$.

Since the groundbreaking work [HHR1], a number of authors have calculated the slice filtration, as well as the associated slice spectral sequence, for $G$-spectra of interest. A few cases are understood for an arbitrary finite group $G$. If $\underline{M}$ is a $G$-Mackey functor, then the equivariant Eilenberg-Mac Lane spectrum $H_{G} \bar{M}$ is always a 0-slice [HHR1] (in this article, we use the "regular" slice filtration, as introduced in [U]). The slice filtrations of $\Sigma^{1} H_{G} \underline{M}$ and $\Sigma^{-1} H_{G} \underline{M}$ were described in [U]. The slices of certain suspensions of equivariant Eilenberg-Mac Lane spectra were determined for $G$ an odd cyclic $p$-group in [HHR3], [Y2] and [A], for dihedral groups of order $2 p$, where $p$ is odd, in [Z2], and for the Klein-four group in [GY] and [S1]. We extend this list by considering in this article the case of $G=Q_{8}$.

Some of the most far-reaching applications of the slice filtration and associated spectral sequence have come in the case of cyclic $p$-groups of equivariance. In addition to [HHR1], this also includes [HHR2], [MSZ], [S2], and [HSWX]. In particular, in [HSWX] the authors use slice technology to understand a $C_{4}$-equivariant, height 4 Lubin-Tate theory at the prime 2 . For each height $n$, there is a height $n$ LubinTate theory that comes equipped with an action of the height $n$ (profinite) Morava stabilizer group. The homotopy fixed points with respect to this action gives a model for the $K(n)$-local sphere, a central object of study. More approachable are the homotopy fixed points with respect to finite subgroups. At height 4, the Morava stabilizer group contains a $C_{4}$-subgroup (in fact a $C_{8}$ ), which gives the context for [HSWX]. On the other hand, at height $2 m$, where $m$ is odd, the Morava stabilizer group contains a $Q_{8}$-subgroup. Therefore it is possible that $Q_{8}$-equivariant slice techniques will eventually shed light on the $K(n)$-local sphere when $n=2 m$ and $m$ is odd.

The focus of our article is the determination of the slices of $\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}$. We list the slices in Section 6 and describe the associated spectral sequence in Section 8. We rely heavily on the computation of the slices of $\Sigma^{n} H_{K_{4}} \underline{\mathbb{Z}}$ given by the second author in [S1]. The quotient map $Q_{8} \longrightarrow K_{4}$ allows us to gain insight into the $Q_{8}$-equivariant slices from the $K_{4}$-case, as we now explain in greater generality.

Given a normal subgroup $N \unlhd G$, there are several constructions that will produce a $G$-spectrum from a $G / N$-spectrum. First is the ordinary pullback, or inflation, functor. If $q: G \longrightarrow G / N$ is the quotient, then inflation is denoted $q^{*}: \mathbf{S p}^{G / N} \longrightarrow$ $\mathbf{S p}{ }^{G}$; it is left adjoint to the $N$-fixed point functor. This inflation functor plays an important role. For instance $q^{*}\left(S_{G / N}^{0}\right)$ is equivalent to $S_{G}^{0}$. However, from our point
of view, this construction has two deficiencies. First, the ordinary inflation does not interact well with the slice filtration. Secondly, the inflation of an $H_{G / N \underline{\mathbb{Z}}}$-module does not have a canonical $H_{G} \underline{\mathbb{Z}}$-module structure.

On the other hand, the "geometric inflation" functor ([H, Definition 4.1], [LMSM, Section II.9])

$$
\phi_{N}^{*}: \mathbf{S p}^{G / N} \longrightarrow \mathbf{S p}^{G},
$$

which is right adjoint to the geometric fixed points functor, interacts well with slices. Namely, if $N$ is a normal subgroup of order $d$ and $X$ is a $G / N$-spectrum, then

$$
\phi_{N}^{*} P_{k}^{k}(X) \simeq P_{d k}^{d k}\left(\phi_{N}^{*} X\right)
$$

by [U, Corollary 4-5] (see also [H, Section 4.2]). However, in general the geometric inflation of an $H_{G / N} \underline{\mathbb{Z}}$-module will not be an $H_{G} \mathbb{Z}$-module.

The third variant is the $\underline{\mathbb{Z}}$-module inflation functor ([Z1, Section 3.2])

$$
\Psi_{N}^{*}: \operatorname{Mod}_{H_{G / N \underline{\mathbb{Z}}}} \longrightarrow \operatorname{Mod}_{H_{G} \underline{\mathbb{Z}}} .
$$

By design, the $\underline{\mathbb{Z}}$-module inflation of an $H_{G / N} \underline{\mathbb{Z}}$-module has a canonical $H_{G} \underline{\mathbb{Z}}$ module structure, though in general this functor does not interact well with the slice filtration.

In some cases, these constructions agree. For instance, if the underlying spectrum of the $G / N$-spectrum $X$ is contractible, then $q^{*} X \simeq \phi_{N}^{*} X$. If $X$ is furthermore an $H_{G / N \underline{\mathbb{Z}}}$-module, then the three inflation functors coincide on $X$ (Proposition 3.18).

The above discussion applies to the slices of $\Sigma^{n} H_{G / N} \underline{\mathbb{Z}}$ : all slices, except for the bottom slice, have trivial underlying spectrum. It follows that these inflate to give many of the slices of $\Sigma^{n} H_{G} \underline{\mathbb{Z}}$.

Our main result along these lines, Theorem 3.19, describes the higher slices of such an inflated $H_{G} \underline{\mathbb{Z}}$-module. In the case of $G=Q_{8}, N=Z\left(Q_{8}\right)$, and $G / N=$ $Q_{8} / Z \cong K_{4}$, it gives the following:

Theorem 1.1. Let $n \geq 0$. Then the nontrivial slices of $\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}$, above level $2 n$, are

$$
P_{2 k}^{2 k}\left(\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}\right) \simeq \Psi_{Z}^{*} P_{k}^{k}\left(\Sigma^{n} H_{K_{4}} \underline{\mathbb{Z}}\right) \simeq \phi_{Z}^{*} P_{k}^{k}\left(\Sigma^{n} H_{K_{4}} \underline{\mathbb{Z}}\right)
$$

for $k>n$. Furthermore,

$$
P_{n}^{2 k}\left(\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}\right) \simeq \Psi_{Z}^{*} P_{n}^{k}\left(\Sigma^{n} H_{K_{4}} \underline{\mathbb{Z}}\right)
$$

As the slices of $\Sigma^{n} H_{K_{4}} \underline{\mathbb{Z}}$ were determined by the second author in [S1], this immediately provides all of the slices of $\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}$ above level $2 n$. The remaining slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ are then given by analyzing the slice tower of $\Psi_{N}^{*}\left(P_{n}^{n} H_{K} \underline{\mathbb{Z}}\right)$. We perform this analysis in Section 6.1.
1.1. Notation. Throughout, whenever referencing the slice filtration, we will always mean the "regular" slice filtration of [U].

We will often write simply $Q$ and $K$ to denote the quaternion group $Q_{8}$ and Klein four group $K_{4}$, respectively. We write $Z$ for the central subgroup of $Q$ of order two generated by $z=-1$. We write

$$
L=\langle i\rangle, \quad D=\langle k\rangle, \quad \text { and } \quad R=\langle j\rangle
$$

for the normal, cyclic subgroups of $Q$ of order 4 . We also use the same names for the images of these subgroups in $Q / Z \cong K$. In other words, the subgroup lattices
of $Q_{8}$ and $K_{4}$ are


Our nomenclature for the order 4 subgroups of $Q_{8}$ amounts to a choice of isomorphism $Q / Z \cong K$.

The sign representation of $C_{2}$ will be denoted $\sigma$, and we will write $\mathbb{Z}^{\sigma}$ for the corresponding $C_{2}$-module.
1.2. Organization. The paper is organized as follows. In Section 2, we review the representations of $C_{4}, K_{4}$, and $Q_{8}$, as well as Mackey functors over $C_{4}$ and $K_{4}$. Then in Section 3, we introduce three inflation functors from a quotient group $G / N$ of some finite group $G$ as well as several results that will aid in the calculation of the slices of $\Sigma^{n} H_{Q_{8}} H \underline{Z}$. The relevant $Q_{8}$-Mackey functors and the homology of $\Sigma^{k \rho_{Q_{8}}} H_{Q_{8}} \underline{\mathbb{Z}}$ are found in Section 4. The slices of $\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}$ must restrict to the appropriate slices of $\Sigma^{n} H_{C_{4}} \mathbb{Z}$; thus, we review this information in Section 5 . We provide some slice towers and describe all slices of $\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}$ in Section 6. We then compute the homotopy Mackey functors of the slices of $\Sigma^{n} H_{Q_{8}} \underline{\mathbb{Z}}$ in Section 7. Finally, we provide some examples of the slice spectral sequence for $\Sigma^{n} H_{C_{4}} \underline{Z}$ and $\Sigma^{n} H_{Q_{8}} \mathbb{Z}$ in Section 8.
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## 2. Background

2.1. Background for $C_{4}$. The $C_{4}$-sign representation $\sigma_{C_{4}}$ is the inflation $p^{*} \sigma_{C_{2}}$ of the $C_{2}$-sign representation along the surjection $C_{4} \longrightarrow C_{2}$. We will simply write $\sigma$ for $\sigma_{C_{4}}$. Then the regular representation for $C_{4}$ splits as

$$
\rho_{C_{4}}=1 \oplus \sigma \oplus \lambda,
$$

where $\lambda$ is the irreducible 2-dimensional rotation representation of $C_{4}$. The $R O\left(C_{4}\right)$ graded homotopy Mackey functors of $H_{C_{4}} \underline{\mathbb{Z}}$ are given in [HHR2]. More specifically, the homotopy Mackey functors of $\Sigma^{k \rho_{C_{4}}} H_{C_{4}} \underline{\mathbb{Z}}, \Sigma^{k \lambda} H_{C_{4}} \underline{\mathbb{Z}}$, and $\Sigma^{k \sigma} H_{C_{4}} \underline{\mathbb{Z}}$ are given in Figures 3 and 6 of [HHR2]. Some $C_{4}$-Mackey functors that will appear below are displayed in Table 1. All of these Mackey functors have trivial Weyl-group actions.
2.2. Background for $K_{4}$. The Klein 4-group $K_{4}=C_{2} \times C_{2}$ has three sign representations, obtained as the inflation along the three surjections $K_{4} \longrightarrow C_{2}$. We denote these three surjections by $p_{1}, m$, and $p_{2}$. Then the regular representation of $K_{4}$ splits as

$$
\rho_{K_{4}} \cong 1 \oplus p_{1}^{*} \sigma \oplus m^{*} \sigma \oplus p_{2}^{*} \sigma .
$$

Some $K_{4}$-Mackey functors that will appear below are displayed in Table 2.

Table 1. Some $C_{4}$-Mackey functors

| $\square=\underline{\mathbb{Z}}$ | 困 $=\underline{\mathbb{Z}}^{*}$ | $\underline{\mathbb{Z}}(2,1)$ | $\bigcirc=\underline{B}(2,0)$ |
| :---: | :---: | :---: | :---: |
| $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z} / 4$ |
| ${ }^{1}()^{2}$ | ${ }^{2}()^{1}$ | $2()^{1}$ | $1()_{2}$ |
| $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z} / 2$ |
| $\underbrace{1}_{\mathbb{Z}} \int_{2}^{2}$ | ${ }^{2}\left(\int_{\mathbb{Z}}^{1}\right.$ | ${ }^{1}\left(\int_{\pi}^{2}\right.$ | 0 |
| - ${ }^{\text {g }}$ | - $=\phi^{*} f$ | $=\phi^{*} \mathbb{F}_{2}$ | $\phi^{*} \mathbb{F}_{2}{ }^{*}$ |
| $\mathbb{F}_{2}$ | 0 | $\mathbb{F}_{2}$ | $\mathbb{F}_{2}$ |
|  |  | $\downarrow_{1}^{2}$ | $1 \uparrow$ |
| 0 | $\mathbb{F}_{2}$ | $\mathbb{F}_{2}$ | $\mathbb{F}_{2}$ |
| 0 | 0 | 0 | 0 |



Figure 1. The homotopy Mackey functors of $\bigvee_{n} \Sigma^{n \rho} H_{K_{4}} \underline{Z}$. The Mackey functor $\underline{\pi}_{k} \Sigma^{n \rho} H_{K_{4}} \underline{\mathbb{Z}}$ appears in position $(k, 4 n-k)$.

The homotopy Mackey functors of $\Sigma^{n \rho} H_{K} \underline{\mathbb{Z}}$ were computed in [S1, Section 9]. They are displayed in Figure 1. The homotopy Mackey functors of $\Sigma^{n \rho} H_{K} \underline{\mathbb{F}_{2}}$ were computed in [GY, Section 7]. They are displayed in Figure 2.

Table 2．Some $K_{4}$－Mackey functors

| $\square=\underline{\mathbb{Z}}$ | 図 $=\underline{\mathbb{Z}}^{*}$ | $\underline{Z}(2,1)$ |
| :---: | :---: | :---: |
|  |  |  |
| ■ $=\mathbb{F}_{2}$ | 囷 $=\underline{F}_{2}{ }^{*}$ | －$=\underline{B}(2,0)$ |
|  |  |  |
| －$=\phi_{L D R}^{*}\left(\underline{\mathbb{F}_{2}}\right)$ | 柬 $=\phi_{L D R}^{*}\left(\underline{\mathbb{F}_{2}}\right)^{*}$ | $\phi_{L D R}^{*}(\underline{\underline{f}})$ |
|  |  | $\begin{array}{ccc} \hline & 0 & \\ & & \\ \mathbb{F}_{2} & \mathbb{F}_{2} & \mathbb{F}_{2} \end{array}$ |
| 0 | 0 | 0 |
| $\boldsymbol{\triangle}=\underline{m g}$ | ＊$=\underline{m g}{ }^{*}$ | －$=\underline{g}$ |
|  |  |  $\mathbb{F}_{2}$  <br> 0 0 0 |
| 0 | 0 | 0 |
| $\underline{m}$ | $\underline{\underline{m}}$ |  |
|  |  |  |
| 0 | 0 |  |
| $\underline{w}$ | $\underline{w}^{*}$ |  |
| 0 | 0 |  |
|  |  |  |



Figure 2. The homotopy Mackey functors of $\bigvee_{n} \Sigma^{n \rho} H_{K_{4}} \underline{\mathbb{F}_{2}}$.
The Mackey functor $\underline{\pi}_{k} \Sigma^{n \rho} H_{K_{4}} \mathbb{F}_{2}$ appears in position $(k, 4 n-\bar{k})$.
2.3. Background for $Q_{8}$. The regular representation of $Q$ splits as

$$
\rho_{Q} \cong \mathbb{H} \oplus \rho_{K},
$$

where $\mathbb{H}$ is the 4 -dimensional irreducible $Q_{8}$-representation given by the action of the unit quaternions on the algebra of quaternions and $\rho_{K}$ is the regular representation of $K$, inflated to $Q$ along the quotient.

Denoting by $C_{4}$ any of the subgroups $L, D$, or $R$ of $Q_{8}$, we have that

$$
\downarrow_{C_{4}}^{Q_{8}} \rho_{K}=2+2 \sigma \quad \text { and } \quad \downarrow_{C_{4}}^{Q_{8}} \mathbb{H}=2 \lambda .
$$

## 3. Inflation functors

3.1. Inflation and the projection formula. Let $N \unlhd G$ be a normal subgroup and $q: G \longrightarrow G / N$ the quotient map. Recall that there is an induced adjunction

$$
\mathbf{S} \mathbf{p}^{G / N} \underset{(-)^{N}}{\stackrel{q^{*}}{\leftrightarrows}} \mathbf{S p}^{G}
$$

where the pullback functor $q^{*}$, called inflation, is strong symmetric monoidal. We will also need a description of the $N$-fixed points of an Eilenberg-Mac Lane $G$ spectrum. First note that there is a functor

$$
\begin{equation*}
\operatorname{Mack}(G) \xrightarrow{q_{*}} \operatorname{Mack}(G / N) \tag{3.1}
\end{equation*}
$$

given by

$$
q_{*}(\underline{M})(\bar{H})=\underline{M}(H)
$$

where $\bar{H}=H / N \leq G / N$ whenever $N \leq H$. The functor $q_{*}$ is denoted $\beta^{!}$in [TW, Lemma 5.4]. Then the homotopy Mackey functors of the $N$-fixed points of a $G$-spectrum $X$ are given by

$$
\begin{equation*}
\underline{\pi}_{n}\left(X^{N}\right) \cong q_{*} \underline{\pi}_{n}(X) \tag{3.2}
\end{equation*}
$$

In the case of an Eilenberg-Mac Lane spectrum this yields an equivalence

$$
\left(H_{G} \underline{M}\right)^{N} \simeq H_{G / N}\left(q_{*} \underline{M}\right)
$$

The following result will be quite useful.
Proposition 3.3. [HK, Lemma 2.13] [BDS, Proposition 2.15] (Projection formula) Let $N \unlhd G$ be a normal subgroup and $q: G \longrightarrow G / N$ be the quotient map. Then for $X \in \mathbf{S} \mathbf{p}^{G / N}$ and $Y \in \mathbf{S p}^{G}$, there is a natural equivalence of $G / N$-spectra

$$
\left(q^{*} X \wedge Y\right)^{N} \simeq X \wedge Y^{N}
$$

We will frequently employ this in the case that $X=S^{V}$ for some $G / N$-representation $V$ and $Y=H_{G} \underline{M}$ for some $G$-Mackey functor $\underline{M}$. Then the projection formula reads

$$
\begin{equation*}
\left(S^{q^{*} V} \wedge H_{G} \underline{M}\right)^{N} \simeq S^{V} \wedge H_{G / N}\left(q_{*} \underline{M}\right) \tag{3.4}
\end{equation*}
$$

See also [Z1, Corollary 5.8]
3.2. Geometric fixed points. For a normal subgroup $N \unlhd G$, we define the family of subgroups $\mathcal{F}[N]$ of $G$ to consist of those subgroups that do not contain $N$. Recall that the $N$-geometric fixed points spectrum of a $G$-spectrum is defined as

$$
\Phi^{N}(X)=(\widetilde{E \mathcal{F}[N]} \wedge X)^{N}
$$

This notation is simultaneously used to denote the resulting $G / N$-spectrum as well as the underlying spectrum. The $N$-geometric fixed points has a right adjoint, given by the geometric inflation functor

$$
\phi_{N}^{*}(Z)=\widetilde{E \mathcal{F}[N]} \wedge q^{*} Z
$$

To sum up, we have an adjunction

$$
\mathbf{S p}^{G} \underset{\phi_{N}^{*}}{\stackrel{\Phi^{N}}{\rightleftarrows}} \mathbf{S p}^{G / N}
$$

3.3. Bottleneck subgroups. The subgroup $Z \unlhd Q$ plays an important role in this article. The primary reason is that it satisfies the following property.

Definition 3.5. We say that $N \unlhd G$ is a bottleneck subgroup if it is a nontrivial, proper subgroup such that, for any subgroup $H \leq G$, either $H$ contains $N$ or $N$ contains $H$.

We now demonstrate that bottleneck subgroups only occur in cyclic p-groups or quaternion groups. The following argument was sketched to us by Mike Geline.

Proposition 3.6. Let $N \unlhd G$ be a bottleneck subgroup of $G$. Then $N$ is cyclic, and $G$ is either a cyclic p-group or a generalized quaternion group.

Proof. We will refer to a subgroup $H \leq G$ which neither contains $N$ nor is contained in $N$ as "adjacent" to $N$. The assumption that $N$ is a bottleneck subgroup means precisely that $G$ has no subgroups that are adjacent to $N$. To see that $N$ must be cyclic, note that if $g$ is not in $N$, then $N \leq\langle g\rangle$, which implies that $N$ is cyclic.

We next observe that $G$ is necessarily a $p$-group. This is because if $N$ is contained in some Sylow $p$-subgroup, then any Sylow $q$-subgroup, for a different prime $q$, would be adjacent. It follows that $N$ contains all of the Sylow subgroups and therefore is all of $G$.

Next, we recall [B, Theorem 4.3] that for a $p$-group $G$, the group contains a unique subgroup of order $p$ if and only if $G$ is either cyclic or generalized quaternion. So we will argue that $G$ contains a unique subgroup of order $p$. The first step is to note that $G$ cannot contain a subgroup isomorphic to $C_{p} \times C_{p}$. This is because such a subgroup would necessarily contain $N$. This would imply that $N \cong C_{p}$, and then $N$ would have a complement in $C_{p} \times C_{p}$, which would be a subgroup adjacent to $N$ in $G$.

Finally, note that the center $Z(G)$ contains a subgroup of order $p$. If $G$ has another subgroup of order $p$, these two would generate a $C_{p} \times C_{p}$, contradicting the previous step.

Remark 3.7. It follows from Proposition 3.6 that if $N \unlhd G$ is a bottleneck subgroup, then $G / N$ is either a cyclic $p$-group or a dihedral 2 -group.

If $N \unlhd G$ is a bottleneck subgroup, then geometric fixed points with respect to $G$ can be computed in terms of geometric fixed points with respect to the quotient group $G / N$.
Proposition 3.8. Let $N \unlhd G$ be a bottleneck subgroup. Then $\Phi^{G} X \simeq \Phi^{G / N} X^{N}$ for any $X \in \mathbf{S p}{ }^{G}$.
Proof. If $N \unlhd G$ is a bottleneck subgroup, then $q^{*} \widetilde{E \mathcal{P}_{G / N}} \simeq \widetilde{E \mathcal{P}_{G}}$. Thus

$$
\Phi^{G} X=\left(\widetilde{E \mathcal{P}_{G}} \wedge X\right)^{G} \simeq\left(\left(q^{*} \widetilde{E \mathcal{P}_{G / N}} \wedge X\right)^{N}\right)^{G / N} .
$$

By the Projection Formula (Proposition 3.3), this is equivalent to

$$
\left(\widetilde{E \mathcal{P}_{G / N}} \wedge X^{N}\right)^{G / N}=\Phi^{G / N} X^{N}
$$

Proposition 3.8 also follows from the more general [K, Proposition 9].
3.4. Inflation for $\underline{Z}$-modules. Given a surjection $q: G \longrightarrow G / N$, the inflation functor

$$
\phi_{N}^{*}: \operatorname{Mack}(G / N) \longrightarrow \operatorname{Mack}(G)
$$

does not send $\underline{\mathbb{Z}}$-modules for $G / N$ to $\underline{\mathbb{Z}}$-modules for $G$. We now describe a modified inflation functor that exists at the level of $\underline{\mathbb{Z}}$-modules. This functor previously appeared in [Z1, Section 3.2] and [BG, Section 3.10].
Definition 3.9. Let $\mathcal{B} \underline{\mathbb{Z}}_{G} \subset \operatorname{Mod}_{\mathbb{Z}[G]}$ denote the full subcategory of permutation $G$-modules. Recall [Z1, Proposition 2.15] that $\underline{\mathbb{Z}}_{G}$-modules correspond to additive functors $\mathcal{B} \underline{\mathbb{Z}}_{G}^{o p} \longrightarrow \mathrm{Ab}$. Then the $\underline{\mathbb{Z}}$-module inflation functor

$$
\Psi_{N}^{*}: \operatorname{Mod}_{\underline{\underline{Z}}_{G / N}} \longrightarrow \operatorname{Mod}_{\underline{\underline{Z}}_{G}}
$$

is defined to be the left Kan extension along the inflation functor $\mathcal{B} \underline{\mathbb{Z}}_{G / N} \longrightarrow \mathcal{B} \underline{Z}_{G}$.
The following is an immediate corollary of the definition as a left Kan extension.

Proposition 3.10. The functor $\Psi_{N}^{*}$ is left adjoint to the functor $q_{*}: \operatorname{Mod}_{\underline{Z}_{G}} \longrightarrow$ $\operatorname{Mod}_{\underline{Z}_{G / N}}$, defined as in (3.1).

Proposition 3.11 ([BG, (3.11)]). For $\underline{M} \in \operatorname{Mod}_{\underline{\underline{Z}}_{G / N}}$, the $\underline{\mathbb{Z}}_{G}$-module $\Psi_{N}^{*}(\underline{M})$ satisfies
(1) $q_{*}\left(\Psi_{N}^{*}(\underline{M})\right)$ is $\underline{M}$ and
(2) $\downarrow_{N}^{G}\left(\Psi_{N}^{*}(\underline{M})\right)$ is the constant Mackey funtor at $\underline{M}(e)$.

Note that Proposition 3.11 completely describes $\Psi_{N}^{*}(\underline{M})$ if $N$ is a bottleneck subgroup. The following result states that $\underline{\mathbb{Z}}$-module inflation agrees with ordinary inflation on geometric Mackey functors.

Proposition 3.12. Let $\underline{M} \in \operatorname{Mod}_{\underline{\underline{Z}}_{G / N}}$, and let $N \unlhd G$ be a bottleneck subgroup. If $\underline{M}(e)=0$, then $\Psi_{N}^{*} \underline{M} \cong \phi_{N}^{*} \underline{M}$.
Proof. This follows immediately from Proposition 3.11.
Remark 3.13. Note that Proposition 3.12 is not true without the bottleneck hypothesis. For instance, in the case $N=C_{3} \unlhd \Sigma_{3}$, then $\downarrow_{C_{2}}^{\Sigma_{3}}\left(\Psi_{C_{3}}^{*} \underline{M}\right) \cong \underline{M}$. In particular, it is not true that $\Psi_{C_{3}}^{*} \underline{M}$ is concentrated over $N=C_{3}$.

We now discuss the extension to equivariant spectra.
Proposition 3.14. The $N$-fixed points functor

$$
(-)^{N}: \operatorname{Mod}_{H_{G} \underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_{G / N \underline{\mathbb{Z}}}}
$$

for HZ్Z-modules has a left adjoint

$$
\Psi_{N}^{*}: \operatorname{Mod}_{H_{G / N} \underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_{G} \underline{\mathbb{Z}}}
$$

If $N \unlhd G$ is a bottleneck subgroup, then the spectrum-level functor $\Psi_{N}^{*}$ extends the functor $\Psi_{N}^{*}$ of Definition 3.9, in the sense that

$$
\begin{equation*}
\Psi_{N}^{*} H_{G / N} \underline{M} \simeq H_{G}\left(\Psi_{N}^{*} \underline{M}\right) \tag{3.15}
\end{equation*}
$$

for $\underline{M}$ in $\operatorname{Mod}_{\underline{\mathbb{Z}}_{G / N}}$.
Proof. For an $H_{G / N \mathbb{Z}} \underline{\text {-module }} X$, the inflation $q^{*} X$ is canonically a module over $q^{*} H_{G / N} \underline{\mathbb{Z}}$. We then define the spectrum-level functor $\Psi_{N}^{*}$ by the formula

$$
\Psi_{N}^{*} X=H \underline{\mathbb{Z}} \wedge_{q^{*} H \underline{\mathbb{Z}}}\left(q^{*} X\right) .
$$

We leave it to the reader to verify that this is indeed left adjoint to the $N$-fixed points functor.

To see that (3.15) holds, we show first that this holds on the indecomposable projective $\underline{\mathbb{Z}}_{G / N}$-modules. These are of the form $\uparrow_{K / N}^{G / N} \underline{\mathbb{Z}}$, and the diagram of commuting adjoint functors

$$
\begin{aligned}
& \operatorname{Mod}_{H_{G / N} \mathbb{Z}} \underset{(-)^{N}}{\stackrel{\Psi_{N}^{*}}{\leftrightarrows}} \operatorname{Mod}_{H_{G} \mathbb{Z}} \\
& \left.\uparrow_{K / N}^{G / N} \uparrow\right|_{\downarrow_{K / N}^{G / N}} \uparrow_{K}^{G} \uparrow \mid \downarrow_{K}^{G} \\
& \operatorname{Mod}_{H_{K / N} \underline{\mathbb{Z}}}^{\stackrel{\Psi_{N}^{*}}{\underset{(-)^{N}}{\leftrightarrows}} \operatorname{Mod}_{H_{K} \underline{\mathbb{Z}}}}
\end{aligned}
$$

shows that
$\Psi_{N}^{*}\left(H_{G / N} \uparrow_{K / N}^{G / N} \underline{\mathbb{Z}}\right) \simeq \uparrow_{K}^{G} \Psi_{N}^{*}\left(H_{K / N} \underline{\mathbb{Z}}\right) \simeq \uparrow_{K}^{G} H_{K} \underline{\mathbb{Z}} \simeq H_{G} \uparrow_{K}^{G} \underline{\mathbb{Z}} \simeq H_{G} \Psi_{N}^{*}\left(\uparrow_{K / N}^{G / N} \underline{\mathbb{Z}}\right)$.

Since the functor $\Psi_{N}^{*}: \operatorname{Mod}_{\underline{Z}_{G / N}} \longrightarrow \operatorname{Mod}_{\underline{Z}_{G}}$ is exact [Z1, Lemma 3.14], it follows that if $\operatorname{Mod}_{\underline{\underline{Z}}_{G / N}}$ has finite global projective dimension, then (3.15) will hold for any $\underline{\mathbb{Z}}_{G / N}$-module $\underline{M}$. By [BSW, Theorem 1.7], this is the case precisely when $G / N$ is as described in Remark 3.7.

Example 3.16. Let $X \in \mathbf{S p}^{G / N}$ and $\underline{M} \in \operatorname{Mack}(G / N)$, with $\underline{M}(e)=0$. Again assume that $N$ is a bottleneck subgroup. Then Proposition 3.12 and Proposition 3.14 give that

$$
\begin{aligned}
\Psi_{N}^{*}\left(X \wedge H_{G / N \underline{M}}\right) & \simeq q^{*}(X) \wedge \Psi_{N}^{*}\left(H_{G / N} \underline{M}\right) \simeq q^{*}(X) \wedge \phi_{N}^{*} H_{G / N} \underline{M} \\
& \simeq \phi_{N}^{*}\left(X \wedge H_{G / N \underline{M}}\right)
\end{aligned}
$$

We will employ this equivalence when $X$ is a representation sphere.
Proposition 3.17. Let $N \unlhd G$ be a bottleneck subgroup. Then for any $G / N-$ representation $V$ and $\underline{\mathbb{Z}}_{G / N}$-module $\underline{L}$, we have

$$
\underline{\pi}_{n}\left(\Psi_{N}^{*} \Sigma^{V} H_{G / N} \underline{L}\right) \cong \Psi_{N}^{*} \underline{\pi}_{n}\left(\Sigma^{V} H_{G / N} \underline{L}\right) .
$$

Proof. Let us write $X=\Psi_{N}^{*} \Sigma^{V} H_{G / N} \underline{L} \simeq \Sigma^{q^{*} V} H_{G} \Psi_{N}^{*} \underline{L}$. Since $N$ is a bottleneck subgroup, it is enough to describe $\downarrow_{N}^{G} \underline{\pi}_{n} X$ and $q_{*} \underline{\pi}_{n} X$. Now

$$
\downarrow_{N}^{G} \underline{\pi}_{n} X \cong \underline{\pi}_{n} \downarrow_{N}^{G} X=\underline{\pi}_{n} \Sigma^{\operatorname{dim} V} H_{N} \underline{L}(N / N)
$$

This is a constant Mackey functor. On the other hand, by (3.2) and (3.4), we have

$$
q_{*} \underline{\pi}_{n} X \cong \underline{\pi}_{n}\left(X^{N}\right) \cong \underline{\pi}_{n}\left(\Sigma^{V} H_{G / N} \underline{L}\right) .
$$

By Proposition 3.11, this agrees with $\Psi_{N}^{*} \underline{\pi}_{n}\left(\Sigma^{V} H_{G / N} \underline{L}\right)$.
More generally, we have an extension of Proposition 3.12 to $H \underline{\mathbb{Z}}$-modules:
Proposition 3.18. Let $X \in \operatorname{Mod}_{H \underline{Z}_{G / N}}$ and let $N \unlhd G$ be a bottleneck subgroup. If the underlying spectrum $\downarrow_{e}^{G / N} X$ is contractible, then $\Psi_{N}^{*}(X) \simeq \phi_{N}^{*} X$.
Proof. If the underlying spectrum of $X$ is contractible, then $X \simeq \widetilde{E(G / N)} \wedge X$. The assumption that $N$ is a bottleneck subgroup implies that $E(G / N)=q^{*}(E(G / N))$ is the universal space for the family of subgroups of $N$, so that $\widetilde{E(G / N)} \wedge \widetilde{E \mathcal{F}[N]} \simeq$ $\widetilde{E(G / N)}$ and it follows that

$$
q^{*} X \simeq \widetilde{E(G / N)} \wedge q^{*} X \simeq \widetilde{E(G / N)} \wedge \phi_{N}^{*}(X) \simeq \phi_{N}^{*} X
$$

Now

$$
\begin{aligned}
\Psi_{N}^{*}(X) & =H_{G} \underline{\mathbb{Z}} \wedge_{q^{*} H_{G / N} \underline{\mathbb{Z}}} q^{*}(X) \\
& \left.\simeq H_{G} \underline{\mathbb{Z}} \wedge_{q^{*} H_{G / N} \underline{\mathbb{Z}}}(\widetilde{E(G / N}) \wedge q^{*}(X)\right) .
\end{aligned}
$$

Since $\widetilde{E(G / N)}$ is smash idempotent, this can be rewritten as

$$
\Psi_{N}^{*}(X) \simeq \widetilde{E(G / N)} \wedge H_{G} \underline{\mathbb{Z}} \wedge_{E(G / N) \wedge q^{*} H_{G / N \mathbb{Z}}} \widetilde{E(G / N)} \wedge q^{*}(X)
$$

It remains only to show that

$$
\widetilde{E(G / N)} \wedge H_{G} \underline{\mathbb{Z}} \simeq \widetilde{E(G / N)} \wedge q^{*} H_{G / N \underline{\mathbb{Z}}}
$$

Both sides restrict trivially to an $N$-equivariant spectrum, so it suffices to show an equivalence on $\Phi^{H}$, where $H$ properly contains $N$. Without loss of generality, we may suppose $H=G$. Since $\Phi^{G}(\widetilde{(G / N)}) \simeq S^{0}$, it suffices to show that

$$
\Phi^{G} H_{G} \underline{\mathbb{Z}} \simeq \Phi^{G} q^{*} H_{G / N} \underline{\mathbb{Z}}
$$

According to Proposition 3.8, the left side is $\Phi^{G / N} H_{G / N} \underline{\mathbb{Z}}$. Similarly, Proposition 3.8 and the Projection Formula (Proposition 3.3) show that the right side is

$$
\begin{aligned}
\Phi^{G} q^{*} H_{G / N} \underline{\mathbb{Z}} & \simeq \Phi^{G / N}\left(H_{G / N} \underline{\mathbb{Z}} \wedge\left(S_{G}^{0}\right)^{N}\right) \\
& \simeq \Phi^{G / N} H_{G / N \underline{\mathbb{Z}} \wedge \Phi^{G / N}\left(S_{G}^{0}\right)^{N}} \\
& \simeq \Phi^{G / N} H_{G / N} \underline{\mathbb{Z}}
\end{aligned}
$$

Theorem 3.19. Let $n \geq 0$ and let $N \unlhd G$ be a bottleneck subgroup of order $p$, a prime. Let $\underline{M} \in \operatorname{Mod}_{\underline{\underline{Z}}_{G / N}}$ such that $P_{n}^{n} \Sigma^{n} H_{G / N} \underline{M}$ is of the form $\Sigma^{V} H_{G / N} \underline{L}$, for some $G / N$-representation $V$ and $\underline{L} \in \operatorname{Mod}_{\underline{Z}_{G / N}}$. Then the nontrivial slices of the Eilenberg-Mac Lane $G$-spectrum $\Sigma^{n} H_{G}\left(\Psi_{N}^{*} \underline{M}\right)$, above level pn, are

$$
P_{p k}^{p k}\left(\Sigma^{n} H_{G}\left(\Psi_{N}^{*} \underline{M}\right)\right) \simeq \Psi_{N}^{*} P_{k}^{k}\left(\Sigma^{n} H_{G / N} \underline{M}\right) \simeq \phi_{N}^{*} P_{k}^{k}\left(\Sigma^{n} H_{G / N} \underline{M}\right)
$$

for $k>n$. Furthermore,

$$
P_{n}^{p k}\left(\Sigma^{n} H_{G}\left(\Psi_{N}^{*} \underline{M}\right)\right) \simeq \Psi_{N}^{*} P_{n}^{k}\left(\Sigma^{n} H_{G / N} \underline{M}\right)
$$

Proof. Applying the functor $\Psi_{N}^{*}$ to the slice tower for $\Sigma^{n} H_{G / N} \underline{M}$ produces a tower of fibrations whose layers are $\Psi_{N}^{*} P_{k}^{k}\left(\Sigma^{n} H_{G / N} \underline{M}\right)$ for $k \geq n$. We wish to show that this is a partial slice tower for $\Sigma^{n} H_{G}\left(\Psi_{N}^{*} \underline{M}\right)$. For $k>n$, the $k$-slice $P_{k}^{k}\left(\Sigma^{n} H_{G / N} \underline{M}\right)$ has trivial underlying spectrum. It follows from Proposition 3.18 that

$$
\Psi_{N}^{*} P_{k}^{k}\left(\Sigma^{n} H_{G / N} \underline{M}\right) \simeq \phi_{N}^{*} P_{k}^{k}\left(\Sigma^{n} H_{G / N} \underline{M}\right)
$$

for $k>n$. As the geometric inflation of a $k$-slice, this is a $p k$-slice.
It remains to show that

$$
\Psi_{N}^{*} P_{n}^{n}\left(\Sigma^{n} H_{G / N} \underline{M}\right) \simeq \Psi_{N}^{*} \Sigma^{V} H_{G / N} \underline{L} \simeq \Sigma^{V} H_{G} \Psi_{N}^{*} \underline{L}
$$

has no slices above level $p n$. First, note that the restriction of $\Sigma^{V} H_{G} \Psi_{N}^{*} \underline{L}$ to $N$ is the $N$-spectrum $\Sigma^{n} H_{N} L(N)$, where $L(N)$ is being considered as a constant $N$ Mackey functor at the value $\underline{\underline{L}}(G / N)$. It follows that this $N$-spectrum has no slices above dimension $|N| \cdot n=p n$. Therefore, to show that $\Sigma^{V} H_{G} \Psi_{N}^{*} \underline{L}$ is less than $p n$, it suffices to show that

$$
\left[G_{+} \wedge_{H} S^{k \rho_{H}+r}, \Sigma^{V} H_{G} \Psi_{N}^{*} \underline{L}\right]^{G}=0
$$

for any $N<H \leq G$ and integers $r \geq 0$ and $k$ such that $k|H|>p n$. Without loss of generality we consider the case $H=G$.

Denote by $U$ a complement of $\rho_{G / N}$ in $\rho_{G}$, so that

$$
\rho_{G} \cong \rho_{G / N} \oplus U
$$

We then have a cofiber sequence

$$
S(k U)_{+} \wedge S^{k \rho_{G / N}} \longrightarrow S^{k \rho_{G / N}} \longrightarrow S^{k \rho_{G}}
$$

and a resulting exact sequence

$$
\begin{aligned}
{\left[\Sigma^{1} S(k U)_{+} \wedge S^{k \rho_{G / N}+r}, \Sigma^{V} H_{G} \Psi_{N}^{*} \underline{L}\right]^{G} } & \longrightarrow\left[S^{k \rho_{G}+r}, \Sigma^{V} H_{G} \Psi_{N}^{*} \underline{L}\right]^{G} \\
& \longrightarrow\left[S^{k \rho_{G / N}+r}, \Sigma^{V} H_{G} \Psi_{N}^{*} \underline{L}\right]^{G}=0
\end{aligned}
$$

We must show that the left term vanishes. Note that the $G$-action on $S(k U)$ is free, since $N$ is order $p$. Then the desired vanishing follows from the fact that $\Sigma^{1} S(k U)_{+} \wedge S^{k \rho_{G / N}-V}$ is $G$-connected, since $\operatorname{dim} k \rho_{G / N}>\operatorname{dim} V=n$.

## 4. $Q_{8}$-Mackey functors and Bredon homology

We display a number of the $Q_{8}$-Mackey functors that will be relevant in Table 3 . In these Lewis diagrams, we are using the subgroup lattice of $Q_{8}$ as displayed in Section 1.1. We will also often abuse notation and write the name for a $K_{4}$-Mackey functor, such as $\underline{m}$ or $m g$, to denote the resulting inflated $Q_{8}$-Mackey functor. We will only write the symbol $\phi_{Z}^{*}$ when it is necessary to resolve an ambiguity, for instance between $\phi_{Z}^{*} \underline{\mathbb{F}_{2}}$ and $\underline{\mathbb{F}_{2}}$.

In [HHR3, Section 2.1], the authors introduce "forms of $\underline{\mathbb{Z}}$ " Mackey functors $\underline{\mathbb{Z}}(i, j)$, where $i \geq j \geq 0$, in the case of $G=C_{p^{n}}$. From our point of view, $Q_{8}$ behaves very similarly to $C_{8}$, and we similarly write $\underline{\mathbb{Z}}(i, j)$ for the Mackey functor that looks like $\underline{\mathbb{Z}}^{*}$ between the subgroups of order $2^{i}$ and $2^{j}$ and looks like $\underline{\mathbb{Z}}$ outside of this range. We will at times follow [HHR3] in denoting by $\underline{B}(i, j)$ the cokernel of $\underline{Z}(i, j) \hookrightarrow \underline{\mathbb{Z}}$, although we will often instead use the descriptions given in Proposition 4.1.

These Mackey functors fit together in exact sequences as follows:
Proposition 4.1. There are exact sequences of Mackey functors
(1) $\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \underline{g}$
(2) $\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \bar{\phi}_{Z}^{*} \underline{B}(2,0)$
(3) $\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \rightarrow \underline{m}^{*}$
(4) $\underline{\mathbb{Z}}(2,1) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \underline{m}$
(5) $\underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \phi_{Z}^{*} \underline{\mathbb{F}_{2}}$
(6) $\underline{\mathbb{Z}}^{*} \hookrightarrow \underline{\mathbb{Z}} \rightarrow \underline{B}(3,0)$
(7) $\underline{m g} \hookrightarrow \underline{m g w} \rightarrow \underline{w}$.
4.1. $R O\left(Q_{8}\right)$-graded Mackey functor $\mathbb{Z}$-homology of a point. We will now compute the homology of $S^{k \rho_{Q}}$, with coefficients in $\underline{Z}$, as a Mackey functor. The starting point is that the regular representation of $Q$ splits as

$$
\rho_{Q} \cong \mathbb{H} \oplus \rho_{K},
$$

where $\mathbb{H}$ is the 4 -dimensional irreducible $Q$-representation given by the action of the unit quaternions on the algebra of quaternions and $\rho_{K}$ is the regular representation of $K$, inflated to $Q$ along the quotient. We begin by computing the homology of $S^{k \mathbb{H}}$. See also [L, Section 2] for an alternative viewpoint.

First, Proposition 3.3 and [S1, Proposition 9.1] combine to yield the following.

Table 3. Some $Q_{8}$-Mackey functors

| $\square=\underline{\mathbb{Z}}$ | 図 $=\underline{\mathbb{Z}}^{*}$ | - $=\underline{B}(3,0)$ |
| :---: | :---: | :---: |
|  |  | $0$ |
| $\underline{\mathbb{Z}}(3,2)=\Psi_{Z}^{*} \underline{\mathbb{Z}}(2,1)$ | $\underline{\mathbb{Z}}(3,1)=\Psi_{Z}^{*} \underline{\mathbb{Z}}^{*}$ | $\mathcal{Q}=\phi_{Z}^{*}(\underline{B}(2,0))$ |
|  |  | 0 <br> 0 |
| $\hat{\delta}=\phi_{Z}^{*} \underline{\underline{\mathbb{F}_{2}}}$ | $\stackrel{*}{*}=\phi_{Z}^{*}{\underline{\underline{F_{2}}}}^{*}$ | - $=\underline{m g w}$ |
| 0 | 0 |  |

Proposition 4.2. For $k \geq 0$, the nontrivial homotopy Mackey functors of $\Sigma^{k \rho_{K}} H_{Q} \underline{\mathbb{Z}}$ are

$$
\underline{\pi}_{n}\left(\Sigma^{k \rho_{K}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}} & n=4 k \\ \frac{m g}{g^{\frac{1}{2}}(4 k-n-1)} & n=4 k-2 \\ \underline{g}^{\frac{1}{2}(4 k-n-4)} \oplus \phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & n \in[2 k, 4 k-3], n \text { odd } \\ \underline{g}^{n-k+1} & n \in[k, 4 k-3], n \text { even } \\ \left.\underline{g}^{2}\right] .\end{cases}
$$

Next, we employ the cofiber sequence

$$
\begin{equation*}
S(\mathbb{H})_{+} \longrightarrow S^{0} \longrightarrow S^{\mathbb{H}} \tag{4.3}
\end{equation*}
$$

to obtain the homology of $S^{\rho_{Q}}$ from that of $S^{\rho_{K}}$.


Figure 3. The 1 -skeleton of $S(\mathbb{H})$.

Proposition 4.4. The nontrivial homotopy Mackey functors of $S(\mathbb{H}) \wedge H_{Q} \underline{Z}$ are

$$
\underline{\pi}_{n}\left(S(\mathbb{H})_{+} \wedge H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}} & n=3 \\ \frac{m g w}{\mathbb{Z}^{*}} & n=1 \\ \underline{\underline{x}}^{2}=0\end{cases}
$$

Proof. Since the action of $Q$ on $S(\mathbb{H})$ is free, we can write down an equivariant cell structure using only free cells. Viewing $S(\mathbb{H})$ as the one-point compactification of $\mathbb{R}^{3}$, there is a straight-forward cell structure in which the subgroups $L, D$, and $R$ act freely on the $x, y$, and $z$-axes, respectively. We display the 1 -skeleton in Figure 3, and the cell structure is described by the following complex of $\mathbb{Z}[Q]$-modules:

$$
\mathbb{Z}[Q]^{2} \xrightarrow{\left(\begin{array}{cc}
e & j \\
-e & -i \\
e & k \\
-e & -e
\end{array}\right)} \mathbb{Z}[Q]^{4} \xrightarrow{\left(\begin{array}{cccc}
k & e & e & k \\
-e & -e & i & i \\
e & -j & -e & j
\end{array}\right)} \mathbb{Z}[Q]^{3} \xrightarrow{(i-e ~ j-e k-e)} \mathbb{Z}[Q] .
$$

This yields an associated complex of induced Mackey functors

$$
\underline{\mathbb{Z}[Q]^{2}} \longrightarrow \underline{\mathbb{Z}[Q]^{4}} \longrightarrow \underline{\mathbb{Z}[Q]^{3}} \longrightarrow \underline{\mathbb{Z}[Q]}
$$

leading to the claimed homology Mackey functors.
Remark 4.5. A smaller chain complex for computing the homology of $S(\mathbb{H})$ is given by

$$
\mathbb{Z}[Q] \xrightarrow{\binom{i-e}{e-k}} \mathbb{Z}[Q]^{2} \xrightarrow{\left(\begin{array}{cc}
e+i & e+k \\
-e-j & -e+i
\end{array}\right)} \mathbb{Z}[Q]^{2} \xrightarrow{(i-e j-e)} \mathbb{Z}[Q] .
$$

We gave a less efficient chain complex in the proof of Proposition 4.4 for geometric reasons.

Using (4.3), this immediately yields the following.
Corollary 4.6. The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H}} H_{Q} \underline{\mathbb{Z}}$ are

$$
\underline{\pi}_{n}\left(\Sigma^{\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}} & n=4 \\ \frac{m g w}{B(3,0)} & n=2 \\ \underline{B}=0\end{cases}
$$

We will use this to compute the homology of $S^{\rho_{Q}}$, using the following periodicity result.

Proposition 4.7 ([W, Proposition 4.1]). For any orientable representation $V$ of dimension d and free $Q$-space $X$, the orientation $u_{V} \in \mathrm{H}_{d}\left(S^{V} ; \underline{\mathbb{Z}}\right)$ induces an equivalence

$$
\Sigma^{d} X_{+} \wedge H_{Q} \underline{\mathbb{Z}} \simeq \Sigma^{V} X_{+} \wedge H_{Q} \underline{\mathbb{Z}}
$$

We now compute the homology of $S^{\rho_{Q}}$.
Proposition 4.8. The nontrivial homotopy Mackey functors of $\Sigma^{\rho_{Q}} H_{Q} \underline{\mathbb{Z}}$ are

$$
\underline{\pi}_{n}\left(\Sigma^{\rho_{Q}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}} & n=8 \\ \frac{m g w}{B}(3,0) & n=6 \\ \underline{m g} & n=4 \\ \frac{m}{g} & n=1\end{cases}
$$

Proof. The representation $\rho_{K}$ is orientable. For example, using the basis $\{1, i, j, k\}$ for $\rho_{K}=\mathbb{R}[K]$, the matrix $\rho_{K}(i)$ is given by

$$
\rho_{K}(i)=\left(\begin{array}{cccc}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

which has determinant equal to 1 . By Proposition 4.7, we have

$$
\underline{\pi}_{n}\left(S(\mathbb{H})_{+} \wedge \Sigma^{\rho_{K}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}} & n=7 \\ \frac{m g w}{} & n=5 \\ \underline{\mathbb{Z}^{*}} & n=4\end{cases}
$$

The result then follows from the cofiber sequence

$$
S(\mathbb{H})_{+} \wedge \Sigma^{\rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_{Q}} H_{Q} \underline{\mathbb{Z}}
$$

Corollary 4.6 generalizes as follows.

Proposition 4.9. The nontrivial homotopy Mackey functors of $\Sigma^{k \mathbb{H}} H_{Q} \underline{\mathbb{Z}}$, for $k>0$ are

$$
\underline{\pi}_{n}\left(\Sigma^{k \mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \cong\left\{\begin{array}{lll}
\underline{\mathbb{Z}} & n=4 k \\
\frac{m g w}{B}(3,0) & 0<n<4 k, n \equiv 2 & (\bmod 4) \\
\underline{B} \leq n<4 k, n \equiv 0 & (\bmod 4)
\end{array}\right.
$$

Proof. This follows by induction, using the cofiber sequence

$$
S(\mathbb{H})_{+} \wedge S^{(k-1) \mathbb{H}} \longrightarrow S^{(k-1) \mathbb{H}} \longrightarrow S^{k \mathbb{H}}
$$

and Proposition 4.7. The latter applies since $\mathbb{H}$, and therefore also $(k-1) \mathbb{H}$, is orientable.

Combining this with the cofiber sequence

$$
S(k \mathbb{H})_{+} \wedge \Sigma^{k \rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{k \rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}
$$

and Proposition 4.7 gives the following result.
Proposition 4.10. The nontrivial homotopy Mackey functors of $\Sigma^{k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$, for $k>0$, are

$$
\underline{\pi}_{n}\left(\Sigma^{k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}\right) \cong\left\{\begin{array}{lll}
\underline{\mathbb{Z}} & n=8 k \\
\frac{m g w}{B}(3,0) & 4 k<n<8 k, n \equiv 2 \quad(\bmod 4) \\
\underline{\phi_{Z}^{*} \underline{\pi}_{n}\left(\Sigma^{k \rho_{K}} H_{K} \underline{\mathbb{Z}}\right)} & 4 k \leq n<8 k, n \equiv 0 \quad(\bmod 4) \\
\hline
\end{array}\right.
$$

where the latter Mackey functors are listed in Proposition 4.2.
The homotopy Mackey functors of $\Sigma^{k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$ are displayed in Figure 4. When $k$ is negative, the computation follows the same strategy. The initial input, which can again be computed using the chain complex given in Proposition 4.4, is that

$$
\underline{\mathrm{H}}^{n}(S(\mathbb{H}) ; \underline{\mathbb{Z}}) \cong \underline{\pi}_{-n}\left(F\left(S(\mathbb{H})_{+}, H_{Q} \underline{\mathbb{Z}}\right)\right) \cong \begin{cases}\underline{\mathbb{Z}}^{*} & n=3  \tag{4.11}\\ \frac{m g w}{} & n=2 \\ \underline{\underline{Z}} & n=0\end{cases}
$$

Using this and [S1, Proposition 9.2] leads to the following answer.
Proposition 4.12. The nontrivial homotopy Mackey functors of $\Sigma^{-k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$, for $k>0$, are

$$
\underline{\pi}_{-n}\left(\Sigma^{-k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}}^{*} & n=8 k \\ \frac{m g w}{B(3,0)} & n \in[4 k, 8 k], n \equiv 3 \quad(\bmod 4) \\ \phi_{Z}^{*} \underline{B}(2,0) & n \in[4 k+5,8 k], n \equiv 1 \quad(\bmod 4) \\ \frac{m g^{*}}{g^{\frac{4 k-n}{2}}} & n=4 k+1 \\ \underline{g}^{\frac{4 k-n-3}{2}} \oplus \phi_{L D R}^{*} \underline{\mathbb{F}}_{2}^{*} & n \in 4 k-1 \\ \underline{g}^{n-k-3} & n \in[2 k+4,4 k-2], n \equiv 0 \quad(\bmod 2) \\ \left.\underline{m}^{*}+3,4 k-2\right], n \equiv 1 \quad(\bmod 2) \\ & n \in[k+4,2 k+2] .\end{cases}
$$

Remark 4.13. The "Gap Theorem" [HHR1, Proposition 3.20] predicts that the groups $\pi_{n}^{Q} \Sigma^{-k \rho} H \underline{\mathbb{Z}}$ vanish for $k \geq 0$ and $n \in[-3,-1]$, as indicated in Figure 4. Actually, for $k \geq 2$ the argument there proves more. It tells us that for $k \geq 2$, the
cohomology groups $\mathrm{H}_{Q}^{n}\left(S^{k \rho} ; \underline{M}\right)$ vanish for positive $n \leq k+1$. This is equivalent to saying that $\pi_{-n}^{Q} \Sigma^{-k \rho} H \underline{M}$ vanishes, with the same conditions on $k$ and $n$.
4.2. Additional homology calculations. We will also need the following auxiliary calculations in Section 6.

Proposition 4.14. The nontrivial homotopy Mackey functors of $\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}$ are

$$
\underline{\pi}_{n}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}}_{2} & n=1 \\ \underline{\mathbb{Z}}^{*} & n=0\end{cases}
$$

Proof. The fiber sequence

$$
\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow F\left(S(\mathbb{H})_{+}, \Sigma^{\rho_{K}} H_{Q} \underline{\mathbb{Z}}\right) \simeq \Sigma^{4} F\left(S(\mathbb{H})_{+}, H_{Q} \underline{\mathbb{Z}}\right)
$$

yields an isomorphism $\underline{\pi}_{0}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \cong \underline{\mathbb{Z}}^{*}$ and shows that the homotopy vanishes for $n$ outside of $[0,2]$. Given that the restriction to any $C_{4}$, which is the $C_{4}$-spectrum $\Sigma^{2+2 \sigma-2 \lambda} H_{C_{4}} \underline{Z}$, has a trivial $\underline{\pi}_{2}$ [Z1, Theorem 6.10], the long exact sequence further shows that $\underline{\pi}_{2}$ vanishes as well, and it implies that we have an extension

$$
\underline{w} \hookrightarrow \underline{\pi}_{1}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \rightarrow \underline{g} .
$$

It remains to show this is not the split extension. The fiber sequence

$$
\uparrow_{D}^{Q} \Sigma^{1+2 \sigma-2 \lambda} H_{C_{4}} \underline{\mathbb{Z}} \longrightarrow \Sigma^{1+p_{1}^{*} \sigma+p_{2}^{*} \sigma-\mathbb{H}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}
$$

shows that $\underline{\pi}_{1}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right)$ injects into

$$
\underline{\pi}_{0}\left(\uparrow_{D}^{Q} \Sigma^{1+2 \sigma-2 \lambda} H_{C_{4}} \underline{\mathbb{Z}}\right) \cong \uparrow_{D}^{Q} \phi_{C_{2}}^{*} \underline{\mathbb{F}_{2}} .
$$

It follows that $\underline{\pi}_{1}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \cong \phi_{Z}^{*} \underline{\mathbb{F}_{2}}$
Proposition 4.15. The nontrivial homotopy Mackey functors of $\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}(3,2)$ are

$$
\underline{\pi}_{n}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}(3,2)\right) \cong \begin{cases}\underline{w} & n=1 \\ \underline{\mathbb{Z}}^{*} & n=0\end{cases}
$$

Proof. The short exact sequence

$$
\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \underline{g}
$$

gives rise to a cofiber sequence

$$
\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}(3,2) \longrightarrow \Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{g} \simeq \Sigma^{1} H_{Q} \underline{g}
$$

Using a naturality square, the second map factors as

$$
\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{1} H_{Q} \underline{g}
$$

where the first map is an epimorphism on $\underline{\pi}_{1}$ by the proof of Proposition 4.14 and the second is an isomorphism on $\underline{\pi}_{1}$. The conclusion follows.
Proposition 4.16. The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H}-\rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,0)$ are

$$
\underline{\pi}_{n}\left(\Sigma^{\mathbb{H}-\rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,0)\right) \cong \begin{cases}\underline{\mathbb{Z}} & n=0 \\ \underline{w}^{*} & n=-2\end{cases}
$$



Figure 4. The homotopy Mackey functors of $\bigvee_{n} \Sigma^{n \rho} H_{Q} \underline{Z}$. The Mackey functor $\underline{\pi}_{k} \Sigma^{n \rho} H_{Q} \underline{\mathbb{Z}}$ appears in position $(k, 8 n-k)$.

Proof. This follows from Proposition 4.15 by duality. In more detail, Proposition 4.15 gives a fiber sequence

$$
\Sigma^{1} H_{Q} \underline{w} \longrightarrow \Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}(3,2) \longrightarrow H_{Q} \underline{\mathbb{Z}}^{*}
$$

Applying Anderson duality (see [S1, Section 2.2]) gives a fiber sequence

$$
I\left(\Sigma^{1} H_{Q} \underline{w}\right) \longleftarrow I\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}(3,2)\right) \longleftarrow I\left(H_{Q} \underline{\mathbb{Z}}^{*}\right)
$$

or in other words

$$
\Sigma^{-1} I\left(H_{Q} \underline{w}\right) \longleftarrow \Sigma^{\mathbb{H}-\rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,0) \longleftarrow H_{Q} \underline{\mathbb{Z}}
$$

But as the Mackey functor $\underline{w}$ is torsion, the Anderson dual is the desuspension of the Brown-Comenetz dual. In other words, $I\left(H_{Q} \underline{w}\right) \simeq \Sigma^{-1} I_{\mathbb{Q} / \mathbb{Z}} H_{Q} \underline{w} \simeq \Sigma^{-1} H_{Q} \underline{w}^{*}$.

## 5. Review of the $C_{4}$-SLICES of $\Sigma^{n} H \mathbb{Z}$

In this section, we review the slices of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ from [Y1]. Note that the slices as listed in [Y1] are written using the classical slice filtration, whereas we use the regular slice filtration. The only difference is a suspension by one. The Mackey functors that appear here were introduced in Table 1.

According to [Y1, Section 4.2], the $C_{4}$-spectrum $\Sigma^{n} H_{C_{4}} \mathbb{Z}$ is an $n$-slice for $0 \leq$ $n \leq 4$. For $n \geq 5, \Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ has a nontrivial slice tower. Yarnall's method for determining these slice towers is to splice together suspensions of the cofiber sequences

$$
\Sigma^{-1} H_{C_{4}} \underline{g} \longrightarrow \Sigma^{2} H_{C_{4}} \underline{\mathbb{Z}} \longrightarrow \Sigma^{2 \sigma} H_{C_{4}} \underline{\mathbb{Z}}
$$

$$
\Sigma^{-1} H_{C_{4}} \phi_{C_{2}}^{*} \underline{\mathbb{F}}^{*} \longrightarrow \Sigma^{2} H_{C_{4}} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\lambda} H_{C_{4}} \underline{\mathbb{Z}}(2,1)
$$

and

$$
\Sigma^{-1} H_{C_{4}} \underline{B}(2,0) \longrightarrow \Sigma^{2} H_{C_{4}} \underline{\mathbb{Z}} \longrightarrow \Sigma^{\lambda} H_{C_{4}} \underline{\mathbb{Z}}
$$

in combination with the equivalences

$$
\Sigma^{2} H_{C_{4}} \underline{\mathbb{Z}} \simeq \Sigma^{2 \sigma} H_{C_{4}} \underline{\mathbb{Z}}(2,1)
$$

and

$$
\Sigma^{-1} H_{C_{4}} \phi_{C_{2}}^{*}{\underline{\mathbb{F}_{2}}}^{*} \simeq \Sigma^{-\sigma} H_{C_{4}} \phi_{C_{2}}^{*} \underline{f} \simeq \Sigma^{1-2 \sigma} H_{C_{4}} \phi_{C_{2}}^{*} \underline{\mathbb{F}_{2}}
$$

We first review these slices for odd $n$.
Proposition 5.1. [Y1, Theorem 4.2.6] Let $n \geq 5$ be odd. The bottom slice of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ is

$$
P_{n}^{n}\left(\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}\right) \simeq\left\{\begin{array}{lll}
\Sigma^{\frac{n-5}{4} \rho+4+\sigma} H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 1 & (\bmod 8) \\
\Sigma^{\frac{n-3}{4} \rho+3} H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 3 & (\bmod 8) \\
\Sigma^{\frac{n-5}{4} \rho+3+2 \sigma} H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 5 & (\bmod 8) \\
\Sigma^{\frac{n-3}{4} \rho+2+\sigma} H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 7 & (\bmod 8)
\end{array}\right.
$$

Proposition 5.2. [Y1, Lemma 4.2.5] Let $n \geq 5$ be odd. The nontrivial $4 k$-slices of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ are

$$
P_{4 k}^{4 k}\left(\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{k \rho} H_{C_{4}} \underline{B}(2,0) & 4 k \in[n+1,2(n-3)], k \text { even } \\ \Sigma^{k \rho} H_{C_{4}} \phi^{*} \underline{f} & 4 k \in[n+1,2(n-3)], k \text { odd } \\ \Sigma^{k \rho} H_{C_{4}} \underline{g} & 4 k \in[2(n-1), 4(n-3)], k \text { even }\end{cases}
$$

The $4 k$-slices can also be read off of [HHR2, Figure 3]. When $n$ is odd, these are the only nontrivial slices of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$.

We now recall the slices of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ for even $n$.
Proposition 5.3. [Y1, Theorem 4.2.9] Let $n \geq 6$ be even. The bottom slice of $\sum^{n} H_{C_{4}} \underline{\mathbb{Z}}$ is

$$
P_{n}^{n}\left(\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}\right) \simeq\left\{\begin{array}{lll}
\Sigma^{\frac{n-4}{4} \rho+3+\sigma} H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 0 & (\bmod 8) \\
\Sigma^{\frac{n-6}{4}} \rho+3+3 \sigma & H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 2 \\
(\bmod 8) \\
\sum^{\frac{n-4}{4} \rho+4} H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 4 & (\bmod 8) \\
\Sigma^{\frac{n-6}{4} \rho+4+2 \sigma} H_{C_{4}} \underline{\mathbb{Z}} & n \equiv 6 & (\bmod 8) .
\end{array}\right.
$$

Proposition 5.4. [Y1, Lemma 4.2.7] Let $n \geq 6$ be even. The nontrivial $4 k$-slices of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ are

$$
P_{4 k}^{4 k}\left(\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}\right) \simeq \Sigma^{k} H_{C_{4}} \underline{g}, \quad k \text { odd }
$$

for $4 k$ in the range $[n+2,4 n-12]$.
Again, the $4 k$-slices can also be read off of [HHR2, Figure 3].
Proposition 5.5. [Y1, Theorem 4.2.9] Let $n \geq 6$ be even. The $(4 k+2)$-slices of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ are

$$
\begin{aligned}
& P_{8 k+2}^{8 k+2}\left(\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}\right) \simeq \Sigma^{1+2 k \rho} H \phi^{*} \underline{\mathbb{F}_{2}} \\
& P_{8 k+6}^{8 k+6}\left(\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}\right) \simeq \Sigma^{3+2 k \rho} H \phi^{*} \underline{\mathbb{F}_{2}} .
\end{aligned}
$$

for $8 k+2$ or $8 k+6$ in the range $[n+2,2 n-6]$

We may also view these slices through the perspective of the $\underline{\mathbb{Z}}$-module inflation functor. By Theorem 3.19,

$$
\Psi_{C_{2}}^{*}: \operatorname{Mod}_{H_{C_{2}} \underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_{C_{4}} \underline{\mathbb{Z}}}
$$

will provide all slices of $\Sigma^{n} H_{C_{4}}$ above level $2 n$. Let $r \equiv n(\bmod 4)$ with $3 \leq r \leq 6$. It follows from [S1, Proposition 3.5] that the slices of $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$ in level at least $2 n+2 r-4$ are

$$
P_{4 k}^{4 k}\left(\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}\right) \simeq \Psi_{C_{2}}^{*} \Sigma^{k} H_{C_{2}} \underline{g} \simeq \Sigma^{k} H_{C_{4}} \underline{g}
$$

for $4 k \in[2 n+2 r-4,4(n-3)]$. The rest of the slices then follow from determining the slices of

$$
\Psi_{C_{2}}^{*} \Sigma^{\frac{n-r}{2} \rho_{C_{2}}+r} H_{C_{2}} \underline{\mathbb{Z}} \simeq \Sigma^{\frac{n+r}{2}+\frac{n-r}{2} \sigma} H_{C_{4}} \underline{\mathbb{Z}}
$$

The slice tower for this $C_{4}$-spectrum can be found by splicing together the cofiber sequences listed at the start of this section.

## 6. $Q_{8}$-SLICES

The slices of $\Sigma^{n} H_{K} \underline{\mathbb{Z}}$ were determined by the second author in [S1, Section 8]. As stated in Theorem 3.19, it follows that the $\underline{\mathbb{Z}}$-module inflation functor

$$
\Psi_{Z}^{*}: \operatorname{Mod}_{H_{K} \underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_{Q} \underline{\mathbb{Z}}}
$$

of Proposition 3.14 will produce all slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ in degree larger than $2 n$, as the inflation of the slices of $\Sigma^{n} H_{K} \underline{\mathbb{Z}}$ above degree $n$.

The remaining slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ will be given as the slices of $\Psi_{Z}^{*}\left(P_{n}^{n}\left(\Sigma^{n} H_{K} \underline{\mathbb{Z}}\right)\right)$. By [S1, Proposition 8.5], these are of the form

$$
\Psi_{Z}^{*}\left(\Sigma^{r+j \rho_{K}} H_{K} \underline{\mathbb{Z}}\right) \simeq \Sigma^{r+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}
$$

where $r \in\{3,4,5\}$, if $n \not \equiv 2(\bmod 4)$. In the case $n \equiv 2(\bmod 4)$, the same result states that this is

$$
\Psi_{Z}^{*}\left(\Sigma^{2+j \rho_{K}} H_{K} \underline{\mathbb{Z}}(1,0)\right) \simeq \Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,1)
$$

But the cofiber sequence (Proposition 4.1)

$$
\begin{equation*}
\Sigma^{1+j \rho_{K}} H_{Q} \underline{m} \longrightarrow \Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,1) \longrightarrow \Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}} \tag{6.1}
\end{equation*}
$$

reduces the computation of slices of $\Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,1)$ to the question of the slice tower for $\Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}$, given that $\Sigma^{1+j \rho_{K}} H_{Q} \underline{m} \simeq \phi_{Z}^{*}\left(\Sigma^{1+j \rho_{K}} H_{K} \underline{m}\right)$ is an $8 j+4$-slice [S1, Proposition 5.7]. We determine the slices of $\Sigma^{r+j \rho_{K}} H_{Q} \underline{Z}$, for $r \in\{2, \ldots, 5\}$ in Section 6.1.
6.1. Slice towers for $\Sigma^{r+j \rho_{K}} H_{Q} \underline{Z}$. The $K_{4}$-spectrum $\Sigma^{r+j \rho_{K}} H_{K} \underline{\mathbb{Z}}$ is an $n$-slice for $r \in\{2, \ldots, 5\}\left[\mathrm{S} 1\right.$, Proposition 7.1]. However, the inflation of this to $Q_{8}$ is no longer a slice. We here determine the slice towers of these inflations. Throughout, we will implicitly use Proposition 6.6 , which does not rely on the following material.
6.1.1. $(r=2)$. First, we observe that $\Sigma^{2+\rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is a 6 -slice. To see this we first note that it restricts to a 6 -slice at every proper subgroup by Proposition 5.3. It therefore remains only to show that it does not have any $8 k$-slices for $k \geq 1$. This is equivalent to showing that $\underline{\pi}-2\left(\Sigma^{\rho_{K}-k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}\right)$ vanishes for $k \geq 1$. In the case $k=1$, (4.11) shows that $\Sigma^{-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}$ is (-3)-truncated, in the sense that it has no homotopy Mackey functors above dimension -3 . This remains true after further desuspending by copies of $\rho_{Q}$.

Next, the tower for $\Sigma^{2+2 \rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is given by


This uses the computation (see Proposition 4.16)

$$
\underline{\pi}_{n}\left(\Sigma^{\mathbb{H}-\rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,0)\right) \cong \begin{cases}\underline{\mathbb{Z}} & n=0 \\ \underline{w}^{*} & n=-2\end{cases}
$$

to produce the first cofiber sequence.
Finally, for $j \geq 3$, the tower may be obtained by recursively using


We have proved the following result.
Proposition 6.2. Let $j \geq 1$. The bottom slice of $\Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is

$$
P_{2+4 j}^{2+4 j}\left(\Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{1+\rho_{K}+\frac{j-1}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*} & j \text { odd } \\ \Sigma^{2+\frac{j}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}} & j \text { even } .\end{cases}
$$

6.1.2. $(r=3)$. By (4.11), the cohomology of $S^{H 1}$ is given by

$$
\underline{\widetilde{H}}^{n}\left(S^{\mathbb{H}} ; \underline{\mathbb{Z}}\right) \cong \underline{\pi}_{-n}\left(\Sigma^{-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}}^{*} & n=4 \\ \underline{m g w} & n=3 .\end{cases}
$$

Suspending by $3+\rho_{Q}$ leads to the cofiber sequence


The tower for $\Sigma^{3+j \rho_{K}} H_{Q} \underline{Z}$, where $j \geq 2$, is then given recursively by


The last cofiber sequence arises from Proposition 4.1. We have proved the following result.

Proposition 6.3. Let $j \geq 1$. The bottom slice of $\Sigma^{3+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is

$$
P_{3+4 j}^{3+4 j}\left(\Sigma^{3+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{-1+\frac{j+1}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*} & j \text { odd } \\ \Sigma^{3+\frac{j}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}} & j \text { even } .\end{cases}
$$

6.1.3. $(r=4)$. The tower for $\Sigma^{4+\rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is given by


This uses the short exact sequence (Proposition 4.1)

$$
\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \rightarrow \underline{m}^{*}
$$

the equivalence $\Sigma^{\rho_{K}} H_{K} \underline{m}^{*} \simeq \Sigma^{2} H_{K} \underline{m g}$ ([GY, Proposition 4.8]), and the computation (see Proposition 4.15)

$$
\underline{\pi}_{n}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}(3,2)\right) \cong \begin{cases}\underline{w} & n=1 \\ \underline{\mathbb{Z}}^{*} & n=0\end{cases}
$$

The tower for $\Sigma^{4+j \rho_{K}} H_{Q} \underline{Z}$, where $j \geq 2$, may then be obtained recursively from


Proposition 6.4. Let $j \geq 1$. The bottom slice of $\Sigma^{4+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is

$$
P_{4+4 j}^{4+4 j}\left(\Sigma^{4+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{\frac{j+1}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*} & j \text { odd } \\ \Sigma^{4+\frac{j}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}} & j \text { even } .\end{cases}
$$

6.1.4. $(r=5)$. Here, we start with the slice tower for $\Sigma^{5} H_{Q} \underline{\mathbb{Z}}$, as this is not a slice. The short exact sequence

$$
\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \phi_{Z}^{*} \underline{B}(2,0)
$$

gives rise to a cofiber sequence

$$
P_{8}^{8}=\Sigma^{\rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{B}(2,0) \longrightarrow \Sigma^{5} H_{Q} \underline{\mathbb{Z}} \simeq \Sigma^{1+\rho_{K}} H_{Q} \underline{\mathbb{Z}}(3,1) \longrightarrow \Sigma^{1+\rho_{K}} H_{Q} \underline{\mathbb{Z}}
$$

Now the argument showing that $\Sigma^{2+\rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is a 6 -slice, given above in Section 6.1.1, also applies to show that $\Sigma^{1+\rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is a 5 -slice. Thus, this cofiber sequence is the slice tower for $\Sigma^{5} H_{Q} \underline{\mathbb{Z}}$.

Next, the tower for $\Sigma^{5+\rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is given by

where the bottom cofiber sequence arises from the computation (Proposition 4.14)

$$
\underline{\pi}_{n}\left(\Sigma^{\rho_{K}-\mathbb{H}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}_{2}} & n=1 \\ \underline{\mathbb{Z}}^{*} & n=0\end{cases}
$$

The tower for $\Sigma^{5+j \rho_{K}} H_{Q} \underline{Z}$, where $j \geq 2$, may then be obtained recursively from


Proposition 6.5. Let $j \geq 1$. The bottom slice of $\Sigma^{5+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}$ is

$$
P_{5+4 j}^{5+4 j}\left(\Sigma^{5+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{1+\frac{j+1}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*} & j \text { odd } \\ \Sigma^{1+\rho_{K}+\frac{j}{2} \rho_{Q}} H_{Q} \underline{\mathbb{Z}} & j \text { even }\end{cases}
$$

6.2. Slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$. In this section, we describe all slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ for $n \geq 0$.

Proposition 6.6. The $Q_{8}$-spectrum $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ is an $n$-slice for $0 \leq n \leq 4$.
Proof. Since this is true after restricting to any $C_{4}$ (see Section 5), any higher slices would necessarily be geometric and therefore occurring in slice dimension at least 8. But we can show directly that $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}<8$ if $n \in[0,4]$. This follows from the vanishing of $\pi_{\rho_{Q}} \Sigma^{n} H_{Q} \underline{\mathbb{Z}} \cong \pi_{-n} \Sigma^{-\rho_{Q}} H_{Q} \underline{\mathbb{Z}}$ as displayed in Figure 4.

It remains to determine the slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ when $n \geq 5$. Note that Theorem 3.19 applies by [S1, Proposition 8.5]. We first describe the bottom slice.

Proposition 6.7 (The $n$-slice). For $n \geq 5$, write $n=8 k+r$, where $r \in[5,12]$. Then the $n$-slice of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ is

$$
P_{n}^{n}\left(\Sigma^{n} H_{Q} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{1+\rho_{K}+k \rho_{Q}} H_{Q} \underline{\mathbb{Z}} & r=5 \\ \Sigma^{2+\rho_{K}+k \rho_{Q}} H_{Q} \underline{\mathbb{Z}}(3,2) & r=6 \\ \Sigma^{-1+(k+1) \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*} & r=7 \\ \Sigma^{(k+1) \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*} & r=8 \\ \Sigma^{1+(k+1) \rho_{Q}} H_{Q} \underline{\mathbb{Z}^{*}} & r=9 \\ \Sigma^{2+(k+1) \rho_{Q}} H_{Q} \underline{\mathbb{Z}}(1,0) & r=10 \\ \Sigma^{3+(k+1) \rho_{Q}} H_{Q} \underline{\mathbb{Z}} & r=11 \\ \Sigma^{4+(k+1) \rho_{Q}} H_{Q} \underline{\mathbb{Z}} & r=12\end{cases}
$$

Proof. By Theorem 3.19, the $n$-slice of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ is the $n$-slice of the $\underline{\mathbb{Z}}$-module inflation of the $n$-slice of $\Sigma^{n} H_{K} \underline{\mathbb{Z}}$. By [S1, Proposition 8.5], writing $n=4 j+r_{4}$ with $r_{4} \in\{2,3,4,5\}$, we have

$$
\Psi_{Z}^{*} P_{n}^{n}\left(\Sigma^{n} H_{K_{4}} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,1) & n \equiv 2 \\ \Sigma^{r_{4}+j \rho_{K}} H_{Q} \underline{\mathbb{Z}} & \text { else }\end{cases}
$$

If $n \not \equiv 2(\bmod 4)$, the slice tower was given in Section 6.1. For the case of $n \equiv 2$, since $\Sigma^{1+j \rho_{K}} H_{Q} \underline{m} \simeq \phi_{Z}^{*}\left(\Sigma^{1+j \rho_{K}} H_{K} \underline{m}\right)$ is an $8 j+4$-slice [S1, Proposition 5.7], the cofiber sequence (Proposition 4.1)

$$
\begin{equation*}
\Sigma^{1+j \rho_{K}} H_{Q} \underline{m} \longrightarrow \Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,1) \longrightarrow \Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}, \tag{6.8}
\end{equation*}
$$

combines with the work of Section 6.1.1 to to show that

$$
P_{n}^{n}\left(\Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}(2,1)\right) \simeq P_{n}^{n}\left(\Sigma^{2+j \rho_{K}} H_{Q} \underline{\mathbb{Z}}\right)
$$

The latter is given in Proposition 6.2.
Proposition 6.9 (The $8 k$-slices). For $n \geq 5$ and $8 k>n$, the $8 k$-slice of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ is

$$
P_{8 k}^{8 k}\left(\Sigma^{n} H_{Q} \underline{\mathbb{Z}}\right) \simeq\left\{\begin{array}{ll}
\Sigma^{k} H_{Q} \underline{g}^{n-k-3} & 8 k \in[4 n-8,8 n-32] \\
\Sigma^{k \rho_{Q}} H_{Q} \underline{g}^{\frac{4 k-n}{2}} & 8 k \in[2 n+4,4 n-16] \\
\Sigma^{k \rho_{Q}} H_{Q} \underline{g}^{\underline{4 k-n-3}} 2
\end{array} \phi_{L D R}^{*} \underline{\mathbb{F}}_{2}^{*} \quad \begin{array}{ll}
\text { and } \quad n \equiv 0 \quad(\bmod 2) \\
& \text { and } \quad n \equiv 12 n+4,4 n-12] \\
\Sigma^{k \rho_{Q}} H_{Q} \underline{m g^{*}} & 8 k=2 n+2 \\
\Sigma^{k \rho_{Q}} H_{Q} \underline{\phi}_{Z}^{*} \underline{B}(2,0) & 8 k=2 n-2 \\
\Sigma^{k \rho_{Q}} H_{Q} \underline{B}(3,0) & 8 k \in[n+3,2 n-10] \\
& \text { and } n \equiv 1 \quad(\bmod 4) \\
\Sigma^{k \rho_{Q}} H_{Q} \underline{m g w} & 8 k \in[n+1,2 n] \\
& \text { and } n \equiv 3 \quad(\bmod 4) .
\end{array}\right.
$$

Proof. This is a translation of Proposition 4.12. Alternatively, the slices above dimension $2 n$ follow from Theorem 3.19 and [S1, Proposition 8.6]. The slices in dimensions $2 n$ and lower follow from the towers computed in Section 6.1.

Proposition 6.10 (The $8 k+4$-slices). For $n \geq 5$ and $8 k+4>n$, the $8 k+4$-slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ are

$$
P_{8 k+4}^{8 k+4}\left(\Sigma^{n} H_{Q} \underline{\mathbb{Z}}\right) \simeq \begin{cases}\Sigma^{3+k \rho_{Q}} H_{Q} \phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & 8 k+4 \in[2 n+4,4 n-12], \quad n \text { even } \\ \Sigma^{2+k \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}_{2}} & 8 k+4 \in[n+1,2 n-4], \quad n \text { odd } \\ \Sigma^{1+k \rho_{Q}} H_{Q} \underline{m} & 8 k+4 \in[n+2,2 n], \quad n \equiv 2 \quad(\bmod 4) \\ \Sigma^{1+k \rho_{Q}} H_{Q} \underline{m g} & 8 k+4 \in[n+4,2 n-4], \quad n \equiv 0 \quad(\bmod 4)\end{cases}
$$

Proof. The first case follows from [S1, Proposition 8.7]. The remaining cases follow from (6.8) and Section 6.1.

Proposition 6.11 (The $4 k+2$-slices). Let $n \geq 5$. If $n$ is odd, then $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ has no nontrivial $4 k+2$-slices if $4 k+2>n$. If $n$ is even and $8 k+2>n$, then the $8 k+2$-slice of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ is nontrivial only if $8 k+2 \in[n+1,2 n]$, in which case the slice is

$$
P_{8 k+2}^{8 k+2}\left(\Sigma^{n} H_{Q} \underline{\mathbb{Z}}\right) \simeq\left\{\begin{array}{lll}
\Sigma^{1+k \rho_{Q}} H_{Q} \underline{w} & n \equiv 0 & (\bmod 4) \\
\Sigma^{1+k \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}_{2}} & n \equiv 2 & (\bmod 4)
\end{array}\right.
$$

Similarly, if $n$ is even and $8 k-2>n$, the $8 k-2$-slice is nontrivial only if $8 k-2 \in$ $[n+1,2 n]$, in which case the slice is

$$
P_{8 k-2}^{8 k-2}\left(\Sigma^{n} H_{Q} \underline{\mathbb{Z}}\right) \simeq\left\{\begin{array}{lll}
\Sigma^{-1+k \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}}_{2}^{*} & n \equiv 0 & (\bmod 4) \\
\Sigma^{-1+k \rho_{Q}} H_{Q} \underline{w}^{*} & n \equiv 2 & (\bmod 4)
\end{array}\right.
$$

Proof. According to [S1], the $K_{4}$-spectrum $\Sigma^{n} H_{K} \underline{\mathbb{Z}}$ does not have any nontrivial slices in odd dimensions, except for the $n$-slice. By Theorem 3.19, this implies that $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ does not have any $4 k+2$-slices above dimension $2 n$. The slices in dimensions below $2 n$ are given by Section 6.1.
6.3. Slice towers for $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$. By Proposition 6.6, $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ is an $n$-slice for $n \in$ $\{0, \ldots, 4\}$. The slice tower for $\Sigma^{5} H_{Q} \underline{\mathbb{Z}}$ was given in Section 6.1.4. We now display a few more examples of slice towers.

Example 6.12. The slice tower for $\Sigma^{6} H_{Q} \mathbb{Z}$ is


This follows immediately from combining [S1, Example 8.2], (6.8), and Section 6.1.1.
Example 6.13. The slice tower for $\Sigma^{7} H_{Q} \underline{\mathbb{Z}}$ is


This follows immediately from combining [S1, Example 8.3] and Section 6.1.2.
Example 6.14. The slices, but not the slice tower, for $\Sigma^{8} H_{K} \underline{\mathbb{Z}}$ were determined in [S1, Section 8]. Let us denote by $F$ the fiber of the map $H_{Q} \underline{\mathbb{Z}} \longrightarrow H_{Q} \phi_{L D R} \underline{\mathbb{F}_{2}}$ induced by the map of $Q_{8}$-Mackey functors $\underline{Z} \longrightarrow \phi_{L D R} \underline{\mathbb{F}_{2}}$ that is surjective at $\bar{L}$, $D$, and $R$. Then the nontrivial homotopy Mackey functors of $F$ are $\underline{\pi}_{0}(F) \simeq \underline{\mathbb{Z}}(2,1)$
and $\underline{\pi}_{-1}(F) \cong \underline{g}^{2}$. The slice tower for $\Sigma^{8} H_{Q} \underline{\mathbb{Z}}$ is

where the bottom of the tower comes from Section 6.1.3.

## 7. Homology calculations

In Section 6, we described the slices of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$. In Section 8 below, we will give the corresponding slice spectral sequences. The $E_{2}$-pages of those spectral sequences are given by the homotopy Mackey functors of the slices. We describe those homotopy Mackey functors here.
7.1. The $n$-slice. We start with the $n$-slices in the order listed in Proposition 6.7. The homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$ were calculated in Proposition 4.10. We use the same methods to determine the homotopy Mackey functors of $\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$.

Proposition 7.1. For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}\right) \cong \begin{cases}\underline{\mathbb{Z}} & i=8 j+4 \\ \underline{m g w} & i \in[4 j+4,8 j+3] \\ \underline{B}(3,0) & i \equiv 2(\bmod 4) \\ \phi_{Z}^{*} \underline{\pi}_{i}\left(\Sigma^{(j+1) \rho_{K}} H_{K} \underline{\mathbb{Z}}\right) & i \in[4 j+4,8 j+3], \\ & i \equiv 0(\bmod 4) \\ \hline 1,4 j+3] .\end{cases}
$$

See Proposition 4.2 or Figure 1 for the homotopy Mackey functors of $\Sigma^{(j+1) \rho_{K}} H_{K} \underline{\mathbb{Z}}$.
We may now use Proposition 7.1 and the exact sequence $\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \underline{g}$ to get the homotopy Mackey functors of $\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}(3,2)$.

Proposition 7.2. For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}(3,2)$ are

$$
\underline{\pi}_{i}\left(\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}(3,2)\right) \cong \begin{cases}\underline{\mathbb{Z}} & i=8 j+4 \\ \underline{m g w} & i \in[4 j+4,8 j+3] \\ \underline{B}(3,0) & i \in[4 j+4,8 j+3] \\ \underline{\phi_{Z}^{*}} \underline{\pi}_{i}\left(\Sigma^{(j+1) \rho_{K}} H_{K} \underline{\mathbb{Z}}\right) & i \in[j=2,4 j+3]\end{cases}
$$

The key point here is that the homotopy Mackey functors of $\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{Z}(3,2)$ are the same as that of $\Sigma^{\rho_{K}+j \rho_{Q}} H_{Q} \underline{Z}$, except that the $g$ in degree $j+1$ has been removed.

In Proposition 4.12 we list the homotopy Mackey functors of $\Sigma^{-j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$. Anderson duality then provides us with the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*}$.

Proposition 7.3. For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*}\right) \cong \begin{cases}\underline{\mathbb{Z}} & i=8 j \\ \underline{m g w} & i \in[4 j+1,8 j-1], \\ \underline{B}(3,0) & i \in[4 j=2 \bmod 4 \\ \underline{y} \quad i \equiv 8 j-1] \\ \phi_{Z}^{*} \underline{B}(2,0) & i=4 j \\ \phi_{Z}^{*} \underline{\pi}_{i-4}\left(\Sigma^{(j-1) \rho_{K}} H_{K} \underline{\mathbb{Z}}\right) & i \in[j+3,4 j-1] .\end{cases}
$$

Finally, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \mathbb{Z}(1,0)$ follow from the exact sequence $\underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \rightarrow \phi_{Z}^{*} \underline{\mathbb{F}_{2}}$.
Proposition 7.4. For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}(1,0)$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}(1,0)\right) \cong \begin{cases}\underline{Z} & i=8 j \\ \underline{m g w} & i \in[4 j+1,8 j-2], \\ \underline{B}(3,0) & i \in[4 j+1,8 j-2], \\ \underline{y}(\bmod 4) \\ \phi_{Z}^{*} \underline{B}(2,0) & i=4 j \\ \phi_{Z}^{*} \underline{\pi}_{i}\left(\Sigma^{j \rho_{K}} H_{K} \underline{\mathbb{Z}}\right) & i \in[j, 4 j-1] .\end{cases}
$$

7.2. The $8 k$-slices. We now move on to the $8 k$-slices.

Proposition 7.5. For $j=1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{B}(2,0)$ are

$$
\underline{\pi}_{i}\left(\Sigma^{\rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{B}(2,0)\right) \cong \begin{cases}\frac{m g}{g} & i=2 \\ \underline{g} & i=1 .\end{cases}
$$

For $j \geq 2$, they are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{B}(2,0)\right) \cong \begin{cases}\phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & i=2 j \\ \underline{g}^{3} & i \in[j+2,2 j-1] \\ \underline{g} & i=j+1 \\ \underline{g} & i=j\end{cases}
$$

Proof. Because $\phi_{Z}^{*} \underline{B}(2,0)$ is a pullback,

$$
\Sigma^{j \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{B}(2,0) \simeq \Sigma^{j \rho_{K}} H_{Q} \phi_{Z}^{*} \underline{B}(2,0)
$$

The exact sequence of $K$-Mackey functors $\underline{m}^{*} \longrightarrow \underline{B}(2,0) \longrightarrow \underline{g}$ provides us with $\Sigma^{j \rho_{K}} H_{K} \underline{m}^{*} \longrightarrow \Sigma^{j \rho_{K}} H_{K} \underline{B}(2,0) \longrightarrow \Sigma^{j \rho_{K}} H_{K} \underline{g}$. The conclusion follows from [GY, Propositions 4.8 and 7.4] and the resulting long exact sequence in homotopy.

We may again use this strategy of reducing the calculations from $Q$ to $K$ for determining the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{B}(3,0)$.

Proposition 7.6. For $j=1$ the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{B}(3,0)$ are

$$
\underline{\pi}_{i}\left(\Sigma^{\rho_{K}} H_{K} \underline{B}(3,0)\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}}_{2} & i=4 \\ \frac{m g}{g} & i=2 \\ \underline{g} & i=1\end{cases}
$$

For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{B}(3,0)$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{B}(3,0)\right) \cong \begin{cases}\phi_{Z}^{*} \mathbb{F}_{2} & i=4 j \\ \frac{m g}{\phi_{L D R}^{*}} \mathbb{F}_{2} \oplus g^{4 j-2-i} & i=4 j-1 \\ g^{2(k-2)+1} & i=[2 j+2,4 j-2] \\ \phi_{L D R}^{*} \mathbb{F}_{2} \oplus \underline{g}^{2(j-3)+1} & i=2 j \\ \underline{g}^{2(i-j-1)} & i \in[j+3,2 j-1] \\ \underline{g}^{i-j+1} & i \in[j, j+2]\end{cases}
$$

Proof. Because the underlying spectrum of $H_{Q} \underline{B}(3,0)$ is contractible,

$$
\Sigma^{\rho_{Q}} H_{Q} \underline{B}(3,0) \simeq \Sigma^{\rho_{K}} H_{Q} \underline{B}(3,0)
$$

Now, we may consider $\underline{B}(3,0)$ as a pullback $\phi_{Z}^{*} B:=\underline{B}(3,0)$, thus the calculation is reduced to one of $K$-Mackey functors. The sequence of $K$-Mackey functors $\underline{\mathbb{Z}}^{*} \xrightarrow{2} \underline{\mathbb{Z}} \longrightarrow \underline{B}$ provides us with

$$
\Sigma^{j \rho_{K}} H_{K} \underline{\mathbb{Z}}^{*} \longrightarrow \Sigma^{j \rho_{K}} H_{K} \underline{\mathbb{Z}} \longrightarrow \Sigma^{j \rho_{K}} H_{K} \underline{B}
$$

Except for $i=4 j-2$, the result follows from the associated long exact sequence in homotopy. In degree $4 j-2$ we have an extension

$$
\underline{m g} \longrightarrow \underline{\pi}_{4 j-2}\left(\Sigma^{j \rho_{K}} H \underline{B}\right) \longrightarrow \underline{g} .
$$

We need to show this is not the split extension. This follows from the exact sequence $\underline{B}(2,0) \longrightarrow \underline{B} \longrightarrow \underline{F}_{2}$ of $K$-Mackey functors.

Proposition 7.7. For $j=1$ and $j=2$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{m g w}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{\rho_{Q}} H_{Q} \underline{m g w}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}_{2}} & i=4 \\ \phi_{Z}^{*} \underline{B}(2,0) & i=2\end{cases}
$$

and

$$
\underline{\pi}_{i}\left(\Sigma^{2 \rho_{Q}} H_{Q} \underline{m g w}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{F}_{2} & i=8 \\ \underline{m g} & i=7 \\ \phi_{L D R} \underline{\mathbb{F}_{2}} & i=6 \\ \underline{g} & i=5 \\ \underline{m g} & i=4 \\ \underline{g} & i=3\end{cases}
$$

For $j \geq 3$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{m g w}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{m g w}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}}_{2} & i=4 j \\ \underline{m g} & i=4 j-1 \\ \phi_{L D R} \underline{\mathbb{F}_{2}} \oplus \underline{g}^{4 j-i-2} & i \in[2 j+2,4 j-2] \\ \underline{g}^{2 j-3} & i=2 j+1 \\ \underline{g}^{2 j-5} \oplus \phi_{L D R} \underline{\mathbb{F}_{2}} & i=2 j \\ \underline{g}^{2(i-j)-2} & i \in[j+2,2 j-1] \\ \underline{g} & i=j+1\end{cases}
$$

Proof. We first deal with the case $j=1$. The short exact sequence of Mackey functors

$$
\underline{w}^{*} \hookrightarrow \underline{m g w} \longrightarrow \underline{m g}^{*}
$$

combines with Proposition 7.17 and Proposition 7.9 to show that the only nontrivial Mackey functors are $\phi_{Z}^{*} \underline{\mathbb{F}_{2}}$ in degree 4 and an extension of $\underline{m}$ by $\underline{g}$ in degree 2. It remains to see that this extension is $\phi_{Z}^{*} \underline{B}(2,0)$. According to Proposition 4.12, the Postnikov tower for $\Sigma^{-\rho_{Q}} H_{Q} \underline{\mathbb{Z}}$ is


Desuspending this diagram once by $\rho_{Q}$ gives a tower for computing the homotopy Mackey functors of $\Sigma^{-2 \rho_{Q}} H_{Q} \underline{\mathbb{Z}}$. The homotopy Mackey functors for $\Sigma^{-8-\rho_{Q}} H_{Q} \underline{\mathbb{Z}}^{*}$ and $\Sigma^{-5 \rho_{Q}} H_{Q} \Psi^{*} \underline{B}(2,0)$ follow, using Anderson duality, from Proposition 4.10 and Proposition 7.5. Long exact sequences in homotopy then imply that

$$
\underline{\pi}_{-9}\left(\Sigma^{-7-\rho_{Q}} H_{Q} \underline{m g w}\right) \cong \phi_{Z}^{*} \underline{B}(2,0)
$$

Dualizing gives that $\underline{\pi}_{2}\left(\Sigma^{\rho_{Q}} H_{Q} \underline{m g w}\right)$ is $\phi_{Z}^{*} \underline{B}(2,0)$.
We now have a fiber sequence

$$
\begin{equation*}
\Sigma^{4} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}_{2}} \longrightarrow \Sigma^{\rho_{Q}} H_{Q} \underline{m g w} \longrightarrow \Sigma^{2} H_{Q} \phi_{Z}^{*} \underline{B}(2,0) \tag{7.8}
\end{equation*}
$$

Suspending this sequence by $\rho_{Q}$ immediately gives the homotopy Mackey functors of $\Sigma^{2 \rho_{Q}} H_{Q}$ mgw. The same is true in the case $j=3$, except that we have an extension

$$
\underline{g} \hookrightarrow \underline{\pi}_{6} \Sigma^{3 \rho_{Q}} H_{Q} \underline{m g w} \rightarrow \phi_{L D R} \underline{\mathbb{F}_{2}} .
$$

We claim that, more generally, any extension of $\underline{\mathbb{Z}}$-modules

$$
\underline{g}^{m} \hookrightarrow \underline{E} \rightarrow \phi_{L D R} \underline{\mathbb{F}_{2}}
$$

is necessarily the split extension. To see this, first note that $\phi_{L D R} \underline{\mathbb{F}_{2}}$ is, by definition, the direct sum $\phi_{L}^{*} \underline{\mathbb{F}_{2}} \oplus \phi_{D}^{*} \underline{\mathbb{F}_{2}} \oplus \phi_{R}^{*} \underline{\mathbb{F}_{2}}$. It therefore suffices to show that the only $\underline{\mathbb{Z}}$-module extension of $\phi_{L}^{*} \underline{\mathbb{F}_{2}}$ by $\underline{g}^{m}$ is the split extension. Since any such extension will vanish at the subgroups $D$ and $R$, the $\mathbb{Z}$-module structure forces the value at $Q$ to be 2 -torsion and therefore equal to $\underline{\mathbb{F}}_{2}{ }^{m+1}$. Since there is a nontrivial
restriction to the subgroup $L$, the $\underline{\mathbb{Z}}$-module structure forces the transfer from $L$ to vanish. Thus the extension must be the split extension.

The suspension by $(j-1) \rho_{Q}$ of (7.8) gives the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{m g w}$ in degrees $2 j+1$ and higher. Now we argue by induction that the Mackey functors for $\Sigma^{j \rho_{Q}} H_{Q} \underline{m g w}$ are as claimed, for $j \geq 3$. For instance, since the bottom Mackey functor is

$$
\underline{\pi}_{j}\left(\Sigma^{(j-1) \rho_{Q}} H_{Q} \underline{m g w}\right) \cong \underline{g},
$$

we see by decomposing $\Sigma^{(j-1) \rho_{Q}} H_{Q} \underline{m g w}$ using the Postnikov tower that

$$
\underline{\pi}_{j+1}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{m g w}\right) \cong \underline{g} .
$$

The values of the Mackey functors $\underline{\pi}_{i}$, for $i \leq 2 j-2$, follow in a similar way. The values

$$
\underline{\pi}_{2 j-2}\left(\Sigma^{(j-1) \rho_{Q}} H_{Q} \underline{m g w}\right) \cong \underline{g}^{2 j-7} \oplus \phi_{L D R} \underline{\mathbb{F}_{2}}
$$

and

$$
\underline{\pi}_{2 j-1}\left(\Sigma^{(j-1) \rho_{Q}} H_{Q} \underline{m g w}\right) \cong \underline{g}^{2 j-5}
$$

give that

$$
\underline{\pi}_{2 j-1}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{m g w}\right) \cong \underline{g}^{2 j-4}
$$

and that we have an extension of $\underline{\mathbb{Z}}$-modules

$$
\underline{g}^{2 j-5} \hookrightarrow \underline{\pi}_{2 j}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{\mathbb{Z}}\right) \rightarrow \phi_{L D R} \underline{\mathbb{F}_{2}} .
$$

By the argument given above, this must be the split extension.
The homotopy Mackey functors for the remaining $8 k$-slices follow from [S1, Propositions 9.5, 9.8].

Proposition 7.9 ([S1, Proposition 9.5], [GY, Proposition 4.8]). We have the equivalence $\Sigma^{\rho_{Q}} H_{Q} \underline{m g^{*}} \simeq \Sigma^{2} H_{Q} \underline{m}$. For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{m g^{*}}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{m g}^{*}\right) \cong \begin{cases}\phi_{L D R}^{*} \underline{F}_{2} & i=2 j \\ \underline{g}^{3} & i \in[j+2,2 j-1] \\ \underline{g} & i=j+1\end{cases}
$$

Proposition 7.10 ([S1, Proposition 9.8]). We have equivalences

$$
\Sigma^{j \rho_{Q}} H \phi_{L D R}^{*}{\underline{\mathbb{F}_{2}}}^{*} \simeq \begin{cases}\Sigma^{2} H \phi_{L D R}^{*} \underline{f} & j=1 \\ \Sigma^{4} H \phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & j=2\end{cases}
$$

Then for $j \geq 3$, the nontrivial homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H \phi_{L D R}^{*} \underline{\mathbb{F}}_{2}^{*}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \phi_{L D R}^{*} \underline{\mathbb{F}}_{2}^{*}\right)= \begin{cases}\phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & i=2 j \\ \underline{g}^{3} & i \in[j+2,2 j-1]\end{cases}
$$

7.3. The $8 k+4$-slices. Similarly, the homotopy Mackey functors of the $(8 k+4)$ slices follow from [S1, Proposition 9.8] and [GY, Corollary 7.2, Propositions 7.3, 7.4].

Proposition 7.11 ([GY, Proposition 3.6]). For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \phi_{L D R}^{*} \underline{\mathbb{F}_{2}}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \phi_{L D R}^{*} \underline{\mathbb{F}_{2}}\right) \cong \begin{cases}\phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & i=2 j \\ \underline{g}^{3} & i \in[j, 2 j-1]\end{cases}
$$

Proposition 7.12 ([GY, Corollary 7.2]). For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}_{2}}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}_{2}}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}_{2}} & i=4 j \\ \frac{m g}{\phi_{L D R}^{*}} \mathbb{F}_{2} \oplus g^{4 j-2-i} & i=4 j-1 \\ \underline{g}^{2(i-j)+1} & i \in[2 j, 4 j-2] \\ \underline{x}^{2} \in[j, 2 j-1]\end{cases}
$$

Proposition 7.13 ([GY, Proposition 7.3]). For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{m}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{m}\right) \cong \begin{cases}\phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & i=2 j \\ \underline{g}^{3} & i \in[j+1,2 j-1] \\ \underline{g} & i=j\end{cases}
$$

Proposition 7.14 ([GY, Proposition 7.4]). For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{m g}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{m g}\right) \cong \begin{cases}\phi_{L D R}^{*} \underline{\mathbb{F}_{2}} & i=2 j \\ \underline{g}^{3} & i \in[j+1,2 j-1] \\ \underline{g}^{2} & i=j\end{cases}
$$

7.4. The $4 k+2$-slices. The homotopy Mackey functors of the $(4 k+2)$-slice $\Sigma^{1+k \rho_{Q}} H_{Q} \phi_{Z}^{*} \mathbb{F}_{2}$ are given in Proposition 7.12. The homotopy Mackey functors of the remaining $(4 k+2)$-slices are as follows.

Proposition 7.15 ([GY, Proposition 4.8, Corollary 7.2]). We have the equivalence $\Sigma^{\rho_{Q}} H_{Q} \phi_{Z}^{*}{\underline{\mathbb{F}_{2}}}^{*} \simeq \Sigma^{4} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}_{2}}$. For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \phi_{Z}^{*}{\underline{\mathbb{F}_{2}}}^{*}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \phi_{Z}^{*} \underline{\mathbb{F}}_{2}^{*}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}_{2}} & i=4 j \\ \frac{m g}{\phi_{L D R}^{*}} \mathbb{F}_{2} \oplus g^{4 j-2-i} & i=4 j-1 \\ \underline{g}^{2(i-j)-5} & i \in[2 j+2,4 j-2] \\ \underline{S}^{*}[j+3,2 j+1]\end{cases}
$$

Finally, we have the homotopy of $\Sigma^{j \rho_{Q}} H_{Q} \underline{w}$ and $\Sigma^{j \rho_{Q}} H_{Q} \underline{w}^{*}$.
Proposition 7.16. For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{w}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q \underline{w}}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}_{2}} & i=4 j \\ \frac{m g}{\phi_{L D R}^{*}} \mathbb{F}_{2} \oplus g^{4 j-2-i} & i=4 j-1 \\ \underline{g}^{2(i-j)+1} & i \in[2 j, 4 j-2] \\ \left.\underline{x}^{*}+1,2 j-1\right]\end{cases}
$$

Proof. The underlying spectrum of $\Sigma^{j \rho_{Q}} H_{Q} \underline{w}$ is contractible; thus,

$$
\Sigma^{j \rho_{Q}} H_{Q} \underline{w} \simeq \Sigma^{j \rho_{K}} H_{Q} \underline{w}
$$

Then, because $\underline{w}$ is a pullback over $Z$, the calculation is essentially $K$-equivariant. Consider the short exact sequence of $K$-Mackey functors $\underline{w} \longrightarrow \underline{\mathbb{F}_{2}} \longrightarrow \underline{g}$ and the corresponding cofiber sequence $\Sigma^{j \rho_{K}} H_{K} \underline{w} \longrightarrow \Sigma^{j \rho_{K}} H_{K} \underline{\mathbb{F}_{2}} \longrightarrow \Sigma^{j \rho_{K}} H_{K} \underline{g}$. The statement follows immediately from the resulting long exact sequence in homotopy.

Proposition 7.17. For $j=1$, the homotopy Mackey functors of $\Sigma^{j \rho_{Q}} H_{Q} \underline{w}^{*}$ are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{w}^{*}\right) \cong \begin{cases}\phi_{Z}^{*} \mathbb{F}_{2} & i=4 \\ \underline{g} & i=2\end{cases}
$$

For $j \geq 2$, they are

$$
\underline{\pi}_{i}\left(\Sigma^{j \rho_{Q}} H_{Q} \underline{w}^{*}\right) \cong \begin{cases}\phi_{Z}^{*} \underline{\mathbb{F}_{2}} & i=4 j \\ \frac{m g}{\phi_{L D R}^{*}} & i=4 j-1 \\ \underline{F_{2}} \oplus g^{4 j-2-i} & i \in[2 j+2,4 j-2] \\ \underline{g}(i-j)-5 & i \in[j+3,2 j+1] \\ \underline{g} & i=j+1\end{cases}
$$

Proof. The proof is the same as that in Proposition 7.16, except that we start with the exact sequence of $K$-Mackey functors $\underline{g} \longrightarrow \underline{\mathbb{F}}_{2}{ }^{*} \longrightarrow \underline{w}^{*}$.

## 8. Slice spectral sequences

Here we include the slice spectral sequences for $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ for several values of $n$ between 5 and 15. In some cases, we use the restriction to the $C_{4}$-subgroups to determine some of the slice differentials.

The grading is the same as that in [HHR1, Section 4.4.2]. The Mackey functor $\underline{E}_{2}^{t-n, t}$ is $\underline{\pi}_{n} P_{t}^{t}(X)$. We also follow the Adams convention, where $\underline{\pi}_{n} P_{t}^{t}(X)$ has coordinates $(n, t-n)$ and the differential

$$
d_{r}: \underline{E}_{r}^{s, t} \longrightarrow \underline{E}_{r}^{s+r, t+r-1}
$$

points left one and up $r$.
The $Q$-Mackey functors that appear in these spectral sequences are listed in Table 4. We also display some companion $C_{4}$-slice spectral sequences, and the $C_{4}$-Mackey functors that appear are listed in Table 5.

TABLE 4. Symbols for $Q$-Mackey functors

$$
\begin{array}{|c|c|c|}
\hline \square=\underline{\mathbb{Z}} & \dot{\delta}=\phi_{Z}^{*} \underline{\mathbb{F}_{2}} & \wedge=\phi_{L D R}^{*} \underline{\mathbb{F}_{2}} \\
\hline=\underline{m g w} & \circ=\underline{B}(3,0) & \mathcal{Q}=\phi_{Z}^{*} \underline{B}(2,0) \\
\hline \boldsymbol{\square}=\underline{m g} & \left(n=\underline{g}^{n}\right. & \\
\hline
\end{array}
$$

Table 5. Symbols for $C_{4}$-Mackey functors

| $\square=\underline{\mathbb{Z}}$ | $\bullet=\phi_{C_{2}}^{*} \underline{\mathbb{F}_{2}}$ | $\circ=\underline{B}(2,0)$ | $\bullet=\underline{g}$ |
| :--- | :--- | :--- | :--- |

Example 8.1. In the spectral sequences for $\Sigma^{5} H_{Q} \underline{\mathbb{Z}}, \Sigma^{6} H_{Q} \underline{\mathbb{Z}}$, and $\Sigma^{7} H_{Q} \underline{\mathbb{Z}}$, because we must be left with

$$
\underline{\pi}_{n}\left(P_{n}^{n} \Sigma^{n} H_{Q} \underline{\mathbb{Z}}\right) \cong \underline{\mathbb{Z}}
$$

all differentials are forced.

Example 8.2. For $\Sigma^{8} H_{Q} \underline{\mathbb{Z}}$, the pattern of differentials emanating from the Mackey functor $\underline{\pi}_{6}\left(P_{8}^{8} \Sigma^{8} H_{Q} \underline{\mathbb{Z}}\right)$ is forced; no other pattern of differentials wipes out all classes in this region. The shorter differentials clearing out the smaller region are then similarly forced.

Example 8.3. In the cases of $\Sigma^{n} H_{Q} \underline{\mathbb{Z}}$ for $n=10,12$, and 15 , we also display the corresponding slice spectral sequence for $\Sigma^{n} H_{C_{4}} \underline{\mathbb{Z}}$, where we use $C_{4}$ to indiscriminately refer to any of the subgroups $L, D, R \leq Q$. The slice differentials in the $C_{4}$-case force many of the slice differentials for the $Q$-equivariant spectra.







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Department of Mathematics, The University of Kentucky, Lexington, KY 40506-0027
Email address: bertguillou@uky.edu
Department of Mathematics, The University of Kentucky, Lexington, KY 40506-0027
Email address: c.slone@uky.edu

