THE SLICES OF QUATERNIONIC EILENBERG-MAC LANE SPECTRA

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ABSTRACT. We compute the slices and slice spectral sequence of integral suspensions of the equivariant Eilenberg-Mac Lane spectra $H\underline{\mathbb{Z}}$ for the group of equivariance Q_8 . Along the way, we compute the Mackey functors $\underline{\pi}_{k\rho}H\underline{\mathbb{Z}}$.

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1. INTRODUCTION

Let G be a finite group. The G-equivariant slice filtration was first defined in the context of G-equivariant stable homotopy theory by Dugger in [D]; it came to prominence as a result of its role in the proof of the Kervaire invariant conjecture by Hill, Hopkins, and Ravenel [HHR1]. The slice filtration is an analogue in the Gequivariant stable homotopy category of the classical Postnikov filtration of spectra. One can also define a G-equivariant Postnikov filtration; on passage to fixed points with respect to any subgroup $H \leq G$, this recovers the Postnikov filtration of the H-fixed point spectrum. However, there are many equivariant spectra which possess a periodicity with respect to suspension by a G-representation sphere, and this periodicity is not visible in the G-equivariant Postnikov filtration. The slice filtration was devised by Dugger in order to display this periodicity for the case of the C_2 -spectrum $K\mathbb{R}$.

Since the groundbreaking work [HHR1], a number of authors have calculated the slice filtration, as well as the associated slice spectral sequence, for G-spectra of interest. A few cases are understood for an arbitrary finite group G. If \underline{M} is a G-Mackey functor, then the equivariant Eilenberg-Mac Lane spectrum $H_G\underline{M}$ is always a 0-slice [HHR1] (in this article, we use the "regular" slice filtration, as introduced in [U]). The slice filtrations of $\Sigma^1 H_G\underline{M}$ and $\Sigma^{-1} H_G\underline{M}$ were described in [U]. The slices of certain suspensions of equivariant Eilenberg-Mac Lane spectra were determined for G an odd cyclic p-group in [HHR3], [Y2] and [A], for dihedral groups of order 2p, where p is odd, in [Z2], and for the Klein-four group in [GY] and [S1]. We extend this list by considering in this article the case of $G = Q_8$.

Some of the most far-reaching applications of the slice filtration and associated spectral sequence have come in the case of cyclic *p*-groups of equivariance. In addition to [HHR1], this also includes [HHR2], [MSZ], [S2], and [HSWX]. In particular, in [HSWX] the authors use slice technology to understand a C_4 -equivariant, height 4 Lubin-Tate theory at the prime 2. For each height *n*, there is a height *n* Lubin-Tate theory that comes equipped with an action of the height *n* (profinite) Morava stabilizer group. The homotopy fixed points with respect to this action gives a model for the K(n)-local sphere, a central object of study. More approachable are the homotopy fixed points with respect to finite subgroups. At height 4, the Morava stabilizer group contains a C_4 -subgroup (in fact a C_8), which gives the context for [HSWX]. On the other hand, at height 2m, where m is odd, the Morava stabilizer group contains a Q_8 -subgroup. Therefore it is possible that Q_8 -equivariant slice techniques will eventually shed light on the K(n)-local sphere when n = 2m and m is odd.

The focus of our article is the determination of the slices of $\Sigma^n H_{Q_8} \mathbb{Z}$. We list the slices in Section 6 and describe the associated spectral sequence in Section 8. We rely heavily on the computation of the slices of $\Sigma^n H_{K_4} \mathbb{Z}$ given by the second author in [S1]. The quotient map $Q_8 \longrightarrow K_4$ allows us to gain insight into the Q_8 -equivariant slices from the K_4 -case, as we now explain in greater generality.

Given a normal subgroup $N \trianglelefteq G$, there are several constructions that will produce a *G*-spectrum from a G/N-spectrum. First is the ordinary pullback, or inflation, functor. If $q: G \longrightarrow G/N$ is the quotient, then inflation is denoted $q^*: \mathbf{Sp}^{G/N} \longrightarrow \mathbf{Sp}^G$; it is left adjoint to the *N*-fixed point functor. This inflation functor plays an important role. For instance $q^*(S_{G/N}^0)$ is equivalent to S_G^0 . However, from our point of view, this construction has two deficiencies. First, the ordinary inflation does not interact well with the slice filtration. Secondly, the inflation of an $H_{G/N}\mathbb{Z}$ -module does not have a canonical $H_G\mathbb{Z}$ -module structure.

On the other hand, the "geometric inflation" functor ([H, Definition 4.1], [LMSM, Section II.9])

$$\phi_N^* \colon \mathbf{Sp}^{G/N} \longrightarrow \mathbf{Sp}^G,$$

which is right adjoint to the geometric fixed points functor, interacts well with slices. Namely, if N is a normal subgroup of order d and X is a G/N-spectrum, then

$$\phi_N^* P_k^k(X) \simeq P_{dk}^{dk} \left(\phi_N^* X \right),$$

by [U, Corollary 4-5] (see also [H, Section 4.2]). However, in general the geometric inflation of an $H_{G/N}\mathbb{Z}$ -module will not be an $H_G\mathbb{Z}$ -module.

The third variant is the $\underline{\mathbb{Z}}$ -module inflation functor ([Z1, Section 3.2])

$$\Psi_N^* \colon \operatorname{Mod}_{H_G/N\underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_G\underline{\mathbb{Z}}}.$$

By design, the $\underline{\mathbb{Z}}$ -module inflation of an $H_{G/N}\underline{\mathbb{Z}}$ -module has a canonical $H_G\underline{\mathbb{Z}}$ -module structure, though in general this functor does not interact well with the slice filtration.

In some cases, these constructions agree. For instance, if the underlying spectrum of the G/N-spectrum X is contractible, then $q^*X \simeq \phi_N^*X$. If X is furthermore an $H_{G/N}\mathbb{Z}$ -module, then the three inflation functors coincide on X (Proposition 3.18).

The above discussion applies to the slices of $\Sigma^n H_{G/N} \underline{\mathbb{Z}}$: all slices, except for the bottom slice, have trivial underlying spectrum. It follows that these inflate to give many of the slices of $\Sigma^n H_G \underline{\mathbb{Z}}$.

Our main result along these lines, Theorem 3.19, describes the higher slices of such an inflated $H_G \underline{\mathbb{Z}}$ -module. In the case of $G = Q_8$, $N = Z(Q_8)$, and $G/N = Q_8/Z \cong K_4$, it gives the following:

Theorem 1.1. Let $n \ge 0$. Then the nontrivial slices of $\Sigma^n H_{Q_8} \underline{\mathbb{Z}}$, above level 2n, are

$$P_{2k}^{2k}\left(\Sigma^{n}H_{Q_{8}}\underline{\mathbb{Z}}\right)\simeq\Psi_{Z}^{*}P_{k}^{k}\left(\Sigma^{n}H_{K_{4}}\underline{\mathbb{Z}}\right)\simeq\phi_{Z}^{*}P_{k}^{k}\left(\Sigma^{n}H_{K_{4}}\underline{\mathbb{Z}}\right)$$

for k > n. Furthermore,

$$P_n^{2k}\left(\Sigma^n H_{Q_8}\underline{\mathbb{Z}}\right) \simeq \Psi_Z^* P_n^k\left(\Sigma^n H_{K_4}\underline{\mathbb{Z}}\right)$$

As the slices of $\Sigma^n H_{K_4}\mathbb{Z}$ were determined by the second author in [S1], this immediately provides all of the slices of $\Sigma^n H_{Q_8}\mathbb{Z}$ above level 2n. The remaining slices of $\Sigma^n H_Q\mathbb{Z}$ are then given by analyzing the slice tower of $\Psi_N^*(P_n^n H_K\mathbb{Z})$. We perform this analysis in Section 6.1.

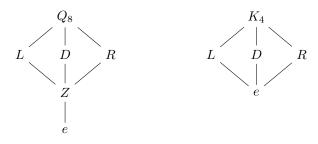
1.1. Notation. Throughout, whenever referencing the slice filtration, we will always mean the "regular" slice filtration of [U].

We will often write simply Q and K to denote the quaternion group Q_8 and Klein four group K_4 , respectively. We write Z for the central subgroup of Q of order two generated by z = -1. We write

$$L = \langle i \rangle, \qquad D = \langle k \rangle, \qquad \text{and} \qquad R = \langle j \rangle$$

for the normal, cyclic subgroups of Q of order 4. We also use the same names for the images of these subgroups in $Q/Z \cong K$. In other words, the subgroup lattices

of Q_8 and K_4 are



Our nomenclature for the order 4 subgroups of Q_8 amounts to a choice of isomorphism $Q/Z \cong K$.

The sign representation of C_2 will be denoted σ , and we will write \mathbb{Z}^{σ} for the corresponding C_2 -module.

1.2. **Organization.** The paper is organized as follows. In Section 2, we review the representations of C_4 , K_4 , and Q_8 , as well as Mackey functors over C_4 and K_4 . Then in Section 3, we introduce three inflation functors from a quotient group G/Nof some finite group G as well as several results that will aid in the calculation of the slices of $\Sigma^n H_{Q_8} H \underline{\mathbb{Z}}$. The relevant Q_8 -Mackey functors and the homology of $\Sigma^{k\rho_{Q_8}} H_{Q_8} \underline{\mathbb{Z}}$ are found in Section 4. The slices of $\Sigma^n H_{Q_8} \underline{\mathbb{Z}}$ must restrict to the appropriate slices of $\Sigma^n H_{C_4} \underline{\mathbb{Z}}$; thus, we review this information in Section 5. We provide some slice towers and describe all slices of $\Sigma^n H_{Q_8} \underline{\mathbb{Z}}$ in Section 6. We then compute the homotopy Mackey functors of the slices of $\Sigma^n H_{Q_8} \underline{\mathbb{Z}}$ in Section 7. Finally, we provide some examples of the slice spectral sequence for $\Sigma^n H_{C_4} \underline{\mathbb{Z}}$ and $\Sigma^n H_{Q_8} \underline{\mathbb{Z}}$ in Section 8.

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2. Background

2.1. Background for C_4 . The C_4 -sign representation σ_{C_4} is the inflation $p^*\sigma_{C_2}$ of the C_2 -sign representation along the surjection $C_4 \longrightarrow C_2$. We will simply write σ for σ_{C_4} . Then the regular representation for C_4 splits as

$$\rho_{C_4} = 1 \oplus \sigma \oplus \lambda,$$

where λ is the irreducible 2-dimensional rotation representation of C_4 . The $RO(C_4)$ graded homotopy Mackey functors of $H_{C_4}\mathbb{Z}$ are given in [HHR2]. More specifically, the homotopy Mackey functors of $\Sigma^{k\rho_{C_4}}H_{C_4}\mathbb{Z}$, $\Sigma^{k\lambda}H_{C_4}\mathbb{Z}$, and $\Sigma^{k\sigma}H_{C_4}\mathbb{Z}$ are given in Figures 3 and 6 of [HHR2]. Some C_4 -Mackey functors that will appear below are displayed in Table 1. All of these Mackey functors have trivial Weyl-group actions.

2.2. Background for K_4 . The Klein 4-group $K_4 = C_2 \times C_2$ has three sign representations, obtained as the inflation along the three surjections $K_4 \longrightarrow C_2$. We denote these three surjections by p_1 , m, and p_2 . Then the regular representation of K_4 splits as

$$\rho_{K_4} \cong 1 \oplus p_1^* \sigma \oplus m^* \sigma \oplus p_2^* \sigma.$$

Some K_4 -Mackey functors that will appear below are displayed in Table 2.

	*	T (a, t)	$\mathbf{D}(\mathbf{a}, \mathbf{a})$
$\Box = \underline{\mathbb{Z}}$	$\mathbb{X} = \underline{\mathbb{Z}}^*$	$\underline{\mathbb{Z}}(2,1)$	$\circ = \underline{B}(2,0)$
Z	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z}/4$
$1 \left(\begin{array}{c} 5 \\ 5 \end{array} \right)^2$	$2 \left(\begin{array}{c} 5 \\ 5 \end{array} \right)^{1}$	$2 \left(\int_{-\infty}^{\infty} 1 \right)$	$1 \left(\begin{array}{c} \\ \end{array} \right) 2$
\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z}/2$
$1 \left(\begin{array}{c} 5 \\ 5 \end{array} \right)^2$	$2 \left(\begin{array}{c} \\ \end{array} \right)^{1}$	$\begin{pmatrix} 1 \\ \end{pmatrix} \begin{pmatrix} 2 \\ \end{pmatrix}^2$	
Z	\mathbb{Z}	\mathbb{Z}	0
$\bullet = \underline{g}$	$\overline{\bullet} = \phi^* \underline{f}$	$ \bullet = \phi^* \underline{\mathbb{F}_2} $	$\phi^* \underline{\mathbb{F}_2}^*$
\mathbb{F}_2	0	\mathbb{F}_2	\mathbb{F}_2
		\downarrow 1	1
0	\mathbb{F}_2	\mathbb{F}_2	\mathbb{F}_2
0	0	0	0

TABLE 1. Some C_4 -Mackey functors

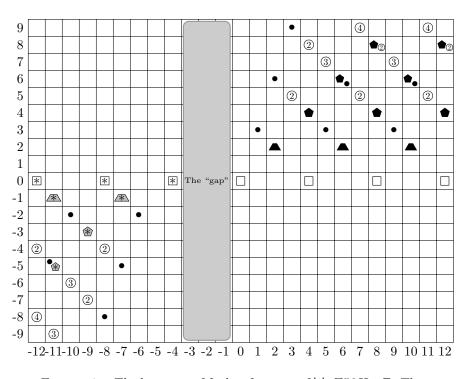
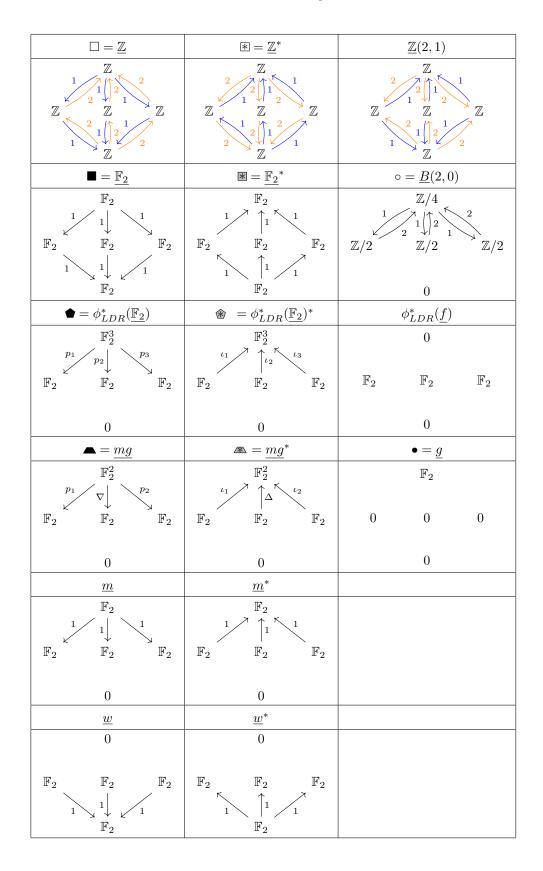


FIGURE 1. The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_{K_4} \underline{\mathbb{Z}}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_{K_4} \underline{\mathbb{Z}}$ appears in position (k, 4n - k).

The homotopy Mackey functors of $\Sigma^{n\rho}H_K\underline{\mathbb{Z}}$ were computed in [S1, Section 9]. They are displayed in Figure 1. The homotopy Mackey functors of $\Sigma^{n\rho}H_K\underline{\mathbb{F}}_2$ were computed in [GY, Section 7]. They are displayed in Figure 2.

TABLE 2. Some K_4 -Mackey functors



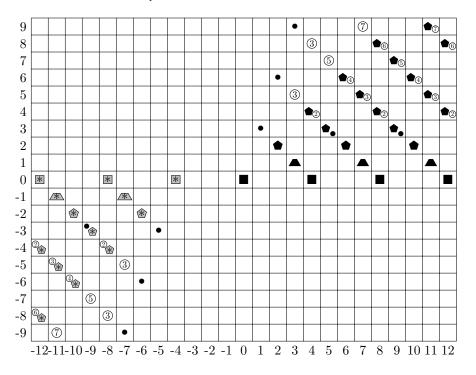


FIGURE 2. The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_{K_4} \underline{\mathbb{F}}_2$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_{K_4} \underline{\mathbb{F}}_2$ appears in position (k, 4n - k).

2.3. Background for Q_8 . The regular representation of Q splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

where \mathbb{H} is the 4-dimensional irreducible Q_8 -representation given by the action of the unit quaternions on the algebra of quaternions and ρ_K is the regular representation of K, inflated to Q along the quotient.

Denoting by C_4 any of the subgroups L, D, or R of Q_8 , we have that

$$\downarrow_{C_4}^{Q_8} \rho_K = 2 + 2\sigma \quad \text{and} \quad \downarrow_{C_4}^{Q_8} \mathbb{H} = 2\lambda.$$

3. INFLATION FUNCTORS

3.1. Inflation and the projection formula. Let $N \trianglelefteq G$ be a normal subgroup and $q: G \longrightarrow G/N$ the quotient map. Recall that there is an induced adjunction

$$\mathbf{Sp}^{G/N} \xleftarrow{q^*}{(-)^N} \mathbf{Sp}^G$$

where the pullback functor q^* , called inflation, is strong symmetric monoidal. We will also need a description of the *N*-fixed points of an Eilenberg-Mac Lane *G*-spectrum. First note that there is a functor

(3.1)
$$\operatorname{Mack}(G) \xrightarrow{q_*} \operatorname{Mack}(G/N)$$

given by

$$q_*(\underline{M})(H) = \underline{M}(H),$$

where $\overline{H} = H/N \leq G/N$ whenever $N \leq H$. The functor q_* is denoted $\beta^!$ in [TW, Lemma 5.4]. Then the homotopy Mackey functors of the *N*-fixed points of a *G*-spectrum *X* are given by

(3.2)
$$\underline{\pi}_n(X^N) \cong q_* \underline{\pi}_n(X).$$

In the case of an Eilenberg-Mac Lane spectrum this yields an equivalence

$$(H_G\underline{M})^N \simeq H_{G/N}(q_*\underline{M}).$$

The following result will be quite useful.

Proposition 3.3. [HK, Lemma 2.13] [BDS, Proposition 2.15] (Projection formula) Let $N \leq G$ be a normal subgroup and $q: G \longrightarrow G/N$ be the quotient map. Then for $X \in \mathbf{Sp}^{G/N}$ and $Y \in \mathbf{Sp}^G$, there is a natural equivalence of G/N-spectra

$$(q^*X \wedge Y)^N \simeq X \wedge Y^N.$$

We will frequently employ this in the case that $X = S^V$ for some G/N-representation V and $Y = H_G \underline{M}$ for some G-Mackey functor \underline{M} . Then the projection formula reads

(3.4)
$$(S^{q^*V} \wedge H_G\underline{M})^N \simeq S^V \wedge H_{G/N}(q_*\underline{M}).$$

See also [Z1, Corollary 5.8]

3.2. Geometric fixed points. For a normal subgroup $N \leq G$, we define the family of subgroups $\mathcal{F}[N]$ of G to consist of those subgroups that do not contain N. Recall that the N-geometric fixed points spectrum of a G-spectrum is defined as

$$\Phi^N(X) = \left(\widetilde{E\mathcal{F}[N]} \wedge X\right)^N$$

This notation is simultaneously used to denote the resulting G/N-spectrum as well as the underlying spectrum. The N-geometric fixed points has a right adjoint, given by the geometric inflation functor

$$\phi_N^*(Z) = \widetilde{E\mathcal{F}[N]} \wedge q^*Z.$$

To sum up, we have an adjunction

$$\mathbf{Sp}^G \xrightarrow{\Phi^N} \mathbf{Sp}^{G/N}.$$

3.3. Bottleneck subgroups. The subgroup $Z \trianglelefteq Q$ plays an important role in this article. The primary reason is that it satisfies the following property.

Definition 3.5. We say that $N \trianglelefteq G$ is a **bottleneck** subgroup if it is a nontrivial, proper subgroup such that, for any subgroup $H \le G$, either H contains N or N contains H.

We now demonstrate that bottleneck subgroups only occur in cyclic p-groups or quaternion groups. The following argument was sketched to us by Mike Geline.

Proposition 3.6. Let $N \leq G$ be a bottleneck subgroup of G. Then N is cyclic, and G is either a cyclic p-group or a generalized quaternion group.

Proof. We will refer to a subgroup $H \leq G$ which neither contains N nor is contained in N as "adjacent" to N. The assumption that N is a bottleneck subgroup means precisely that G has no subgroups that are adjacent to N. To see that N must be cyclic, note that if g is not in N, then $N \leq \langle g \rangle$, which implies that N is cyclic.

We next observe that G is necessarily a p-group. This is because if N is contained in some Sylow p-subgroup, then any Sylow q-subgroup, for a different prime q, would be adjacent. It follows that N contains all of the Sylow subgroups and therefore is all of G.

Next, we recall [B, Theorem 4.3] that for a p-group G, the group contains a unique subgroup of order p if and only if G is either cyclic or generalized quaternion. So we will argue that G contains a unique subgroup of order p. The first step is to note that G cannot contain a subgroup isomorphic to $C_p \times C_p$. This is because such a subgroup would necessarily contain N. This would imply that $N \cong C_p$, and then N would have a complement in $C_p \times C_p$, which would be a subgroup adjacent to N in G.

Finally, note that the center Z(G) contains a subgroup of order p. If G has another subgroup of order p, these two would generate a $C_p \times C_p$, contradicting the previous step.

Remark 3.7. It follows from Proposition 3.6 that if $N \trianglelefteq G$ is a bottleneck subgroup, then G/N is either a cyclic *p*-group or a dihedral 2-group.

If $N \leq G$ is a bottleneck subgroup, then geometric fixed points with respect to G can be computed in terms of geometric fixed points with respect to the quotient group G/N.

Proposition 3.8. Let $N \leq G$ be a bottleneck subgroup. Then $\Phi^G X \simeq \Phi^{G/N} X^N$ for any $X \in \mathbf{Sp}^G$.

Proof. If $N \trianglelefteq G$ is a bottleneck subgroup, then $q^* \widetilde{E\mathcal{P}_{G/N}} \simeq \widetilde{E\mathcal{P}_G}$. Thus $\Phi^G X = (\widetilde{E\mathcal{P}_G} \land X)^G \simeq ((q^* \widetilde{E\mathcal{P}_{G/N}} \land X)^N)^{G/N}.$

By the Projection Formula (Proposition 3.3), this is equivalent to

$$(\widetilde{E\mathcal{P}_{G/N}} \wedge X^N)^{G/N} = \Phi^{G/N} X^N.$$

Proposition 3.8 also follows from the more general [K, Proposition 9].

3.4. Inflation for $\underline{\mathbb{Z}}$ -modules. Given a surjection $q: G \longrightarrow G/N$, the inflation functor

$$\phi_N^* \colon \mathbf{Mack}(G/N) \longrightarrow \mathbf{Mack}(G)$$

does not send $\underline{\mathbb{Z}}$ -modules for G/N to $\underline{\mathbb{Z}}$ -modules for G. We now describe a modified inflation functor that exists at the level of $\underline{\mathbb{Z}}$ -modules. This functor previously appeared in [Z1, Section 3.2] and [BG, Section 3.10].

Definition 3.9. Let $\mathcal{B}\mathbb{Z}_G \subset \operatorname{Mod}_{\mathbb{Z}[G]}$ denote the full subcategory of permutation G-modules. Recall [Z1, Proposition 2.15] that \mathbb{Z}_G -modules correspond to additive functors $\mathcal{B}\mathbb{Z}_G^{op} \longrightarrow \operatorname{Ab}$. Then the \mathbb{Z} -module inflation functor

$$\Psi_N^* \colon \operatorname{Mod}_{\underline{\mathbb{Z}}_{G/N}} \longrightarrow \operatorname{Mod}_{\underline{\mathbb{Z}}_G}$$

is defined to be the left Kan extension along the inflation functor $\mathcal{B}\underline{\mathbb{Z}}_{G/N} \longrightarrow \mathcal{B}\underline{\mathbb{Z}}_{G}$.

The following is an immediate corollary of the definition as a left Kan extension.

Proposition 3.10. The functor Ψ_N^* is left adjoint to the functor $q_* \colon \operatorname{Mod}_{\mathbb{Z}_G} \longrightarrow \operatorname{Mod}_{\mathbb{Z}_G/N}$, defined as in (3.1).

Proposition 3.11 ([BG, (3.11)]). For $\underline{M} \in \operatorname{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, the $\underline{\mathbb{Z}}_{G}$ -module $\Psi_{N}^{*}(\underline{M})$ satisfies

- (1) $q_*(\Psi_N^*(\underline{M}))$ is \underline{M} and
- (2) $\downarrow_N^G (\Psi_N^*(\underline{M}))$ is the constant Mackey funtor at $\underline{M}(e)$.

Note that Proposition 3.11 completely describes $\Psi_N^*(\underline{M})$ if N is a bottleneck subgroup. The following result states that $\underline{\mathbb{Z}}$ -module inflation agrees with ordinary inflation on geometric Mackey functors.

Proposition 3.12. Let $\underline{M} \in \operatorname{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, and let $N \leq G$ be a bottleneck subgroup. If $\underline{M}(e) = 0$, then $\Psi_N^* \underline{M} \cong \phi_N^* \underline{M}$.

Proof. This follows immediately from Proposition 3.11.

Remark 3.13. Note that Proposition 3.12 is not true without the bottleneck hypothesis. For instance, in the case $N = C_3 \leq \Sigma_3$, then $\downarrow_{C_2}^{\Sigma_3} (\Psi_{C_3}^* \underline{M}) \cong \underline{M}$. In particular, it is not true that $\Psi_{C_3}^* \underline{M}$ is concentrated over $N = C_3$.

We now discuss the extension to equivariant spectra.

Proposition 3.14. The N-fixed points functor

$$(-)^N \colon \operatorname{Mod}_{H_G \mathbb{Z}} \longrightarrow \operatorname{Mod}_{H_G/N \mathbb{Z}}$$

for $H\underline{\mathbb{Z}}$ -modules has a left adjoint

 $\Psi_N^* \colon \operatorname{Mod}_{H_G/N\underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_G\underline{\mathbb{Z}}}.$

If $N \leq G$ is a bottleneck subgroup, then the spectrum-level functor Ψ_N^* extends the functor Ψ_N^* of Definition 3.9, in the sense that

(3.15)
$$\Psi_N^* H_{G/N} \underline{M} \simeq H_G(\Psi_N^* \underline{M})$$

for \underline{M} in $\operatorname{Mod}_{\underline{\mathbb{Z}}_{G/N}}$.

Proof. For an $H_{G/N}\mathbb{Z}$ -module X, the inflation q^*X is canonically a module over $q^*H_{G/N}\mathbb{Z}$. We then define the spectrum-level functor Ψ_N^* by the formula

$$\Psi_N^* X = H\underline{\mathbb{Z}} \wedge_{q^* H\underline{\mathbb{Z}}} (q^* X)$$

We leave it to the reader to verify that this is indeed left adjoint to the N-fixed points functor.

To see that (3.15) holds, we show first that this holds on the indecomposable projective $\underline{\mathbb{Z}}_{G/N}$ -modules. These are of the form $\uparrow_{K/N}^{G/N} \underline{\mathbb{Z}}$, and the diagram of commuting adjoint functors

$$\begin{array}{c} \operatorname{Mod}_{H_{G/N}\underline{\mathbb{Z}}} \xrightarrow{\Psi_N^*} \operatorname{Mod}_{H_G\underline{\mathbb{Z}}} \\ \uparrow_{K/N}^{G/N} & \uparrow_{K}^G \\ \downarrow \downarrow_{K/N}^{G/N} & \uparrow_{K}^G \\ \operatorname{Mod}_{H_{K/N}\underline{\mathbb{Z}}} \xrightarrow{\Psi_N^*} \operatorname{Mod}_{H_K\underline{\mathbb{Z}}} \end{array}$$

shows that

$$\Psi_N^* \left(H_{G/N} \uparrow_{K/N}^{G/N} \underline{\mathbb{Z}} \right) \simeq \uparrow_K^G \Psi_N^* (H_{K/N} \underline{\mathbb{Z}}) \simeq \uparrow_K^G H_K \underline{\mathbb{Z}} \simeq H_G \uparrow_K^G \underline{\mathbb{Z}} \simeq H_G \Psi_N^* \left(\uparrow_{K/N}^{G/N} \underline{\mathbb{Z}} \right).$$

Since the functor Ψ_N^* : $\operatorname{Mod}_{\mathbb{Z}_{G/N}} \longrightarrow \operatorname{Mod}_{\mathbb{Z}_G}$ is exact [Z1, Lemma 3.14], it follows that if $\operatorname{Mod}_{\mathbb{Z}_{G/N}}$ has finite global projective dimension, then (3.15) will hold for any $\mathbb{Z}_{G/N}$ -module \underline{M} . By [BSW, Theorem 1.7], this is the case precisely when G/N is as described in Remark 3.7.

Example 3.16. Let $X \in \mathbf{Sp}^{G/N}$ and $\underline{M} \in \mathbf{Mack}(G/N)$, with $\underline{M}(e) = 0$. Again assume that N is a bottleneck subgroup. Then Proposition 3.12 and Proposition 3.14 give that

$$\Psi_N^*(X \wedge H_{G/N}\underline{M}) \simeq q^*(X) \wedge \Psi_N^*(H_{G/N}\underline{M}) \simeq q^*(X) \wedge \phi_N^*H_{G/N}\underline{M}$$
$$\simeq \phi_N^*(X \wedge H_{G/N}\underline{M}).$$

We will employ this equivalence when X is a representation sphere.

Proposition 3.17. Let $N \trianglelefteq G$ be a bottleneck subgroup. Then for any G/N-representation V and $\underline{\mathbb{Z}}_{G/N}$ -module \underline{L} , we have

$$\underline{\pi}_n \left(\Psi_N^* \Sigma^V H_{G/N} \underline{L} \right) \cong \Psi_N^* \underline{\pi}_n \left(\Sigma^V H_{G/N} \underline{L} \right).$$

Proof. Let us write $X = \Psi_N^* \Sigma^V H_{G/N} \underline{L} \simeq \Sigma^{q^*V} H_G \Psi_N^* \underline{L}$. Since N is a bottleneck subgroup, it is enough to describe $\downarrow_N^G \underline{\pi}_n X$ and $q_* \underline{\pi}_n X$. Now

$$\downarrow_N^G \underline{\pi}_n X \cong \underline{\pi}_n \downarrow_N^G X = \underline{\pi}_n \Sigma^{\dim V} H_N \underline{L}(N/N).$$

This is a constant Mackey functor. On the other hand, by (3.2) and (3.4), we have

$$q_*\underline{\pi}_n X \cong \underline{\pi}_n(X^N) \cong \underline{\pi}_n(\Sigma^V H_{G/N}\underline{L}).$$

By Proposition 3.11, this agrees with $\Psi_N^* \underline{\pi}_n \left(\Sigma^V H_{G/N} \underline{L} \right)$.

More generally, we have an extension of Proposition 3.12 to $H\underline{\mathbb{Z}}$ -modules:

Proposition 3.18. Let $X \in \operatorname{Mod}_{H\underline{\mathbb{Z}}_{G/N}}$ and let $N \leq G$ be a bottleneck subgroup. If the underlying spectrum $\downarrow_e^{G/N} X$ is contractible, then $\Psi_N^*(X) \simeq \phi_N^* X$.

Proof. If the underlying spectrum of X is contractible, then $X \simeq \widetilde{E(G/N)} \wedge X$. The assumption that N is a bottleneck subgroup implies that $E(G/N) = q^*(E(G/N))$ is the universal space for the family of subgroups of N, so that $\widetilde{E(G/N)} \wedge \widetilde{E\mathcal{F}[N]} \simeq \widetilde{E(G/N)}$ and it follows that

$$q^*X \simeq \widetilde{E(G/N)} \wedge q^*X \simeq \widetilde{E(G/N)} \wedge \phi_N^*(X) \simeq \phi_N^*X.$$

Now

$$\Psi_N^*(X) = H_G \underline{\mathbb{Z}} \wedge_{q^* H_{G/N} \underline{\mathbb{Z}}} q^*(X)$$
$$\simeq H_G \underline{\mathbb{Z}} \wedge_{q^* H_{G/N} \underline{\mathbb{Z}}} (\widetilde{E(G/N)}) \wedge q^*(X)).$$

Since $\widetilde{E(G/N)}$ is smash idempotent, this can be rewritten as

$$\Psi_N^*(X) \simeq \widetilde{E(G/N)} \wedge H_G \underline{\mathbb{Z}} \wedge_{\widetilde{E(G/N)} \wedge q^* H_{G/N} \underline{\mathbb{Z}}} \widetilde{E(G/N)} \wedge q^*(X).$$

It remains only to show that

$$\widetilde{E(G/N)} \wedge H_G \underline{\mathbb{Z}} \simeq \widetilde{E(G/N)} \wedge q^* H_{G/N} \underline{\mathbb{Z}}$$

Both sides restrict trivially to an N-equivariant spectrum, so it suffices to show an equivalence on Φ^H , where H properly contains N. Without loss of generality, we may suppose H = G. Since $\Phi^G(\widetilde{E(G/N)}) \simeq S^0$, it suffices to show that

$$\Phi^G H_G \underline{\mathbb{Z}} \simeq \Phi^G q^* H_{G/N} \underline{\mathbb{Z}}.$$

According to Proposition 3.8, the left side is $\Phi^{G/N}H_{G/N}\mathbb{Z}$. Similarly, Proposition 3.8 and the Projection Formula (Proposition 3.3) show that the right side is

$$\Phi^{G}q^{*}H_{G/N}\underline{\mathbb{Z}} \simeq \Phi^{G/N} \left(H_{G/N}\underline{\mathbb{Z}} \wedge (S_{G}^{0})^{N}\right)$$
$$\simeq \Phi^{G/N}H_{G/N}\underline{\mathbb{Z}} \wedge \Phi^{G/N}(S_{G}^{0})^{N}$$
$$\simeq \Phi^{G/N}H_{G/N}\underline{\mathbb{Z}}.$$

Theorem 3.19. Let $n \ge 0$ and let $N \trianglelefteq G$ be a bottleneck subgroup of order p, a prime. Let $\underline{M} \in \operatorname{Mod}_{\mathbb{Z}_{G/N}}$ such that $P_n^n \Sigma^n H_{G/N} \underline{M}$ is of the form $\Sigma^V H_{G/N} \underline{L}$, for some G/N-representation V and $\underline{L} \in \operatorname{Mod}_{\mathbb{Z}_{G/N}}$. Then the nontrivial slices of the Eilenberg-Mac Lane G-spectrum $\Sigma^n H_G(\Psi_N^* \underline{M})$, above level pn, are

$$P_{pk}^{pk}\left(\Sigma^{n}H_{G}(\Psi_{N}^{*}\underline{M})\right)\simeq\Psi_{N}^{*}P_{k}^{k}\left(\Sigma^{n}H_{G/N}\underline{M}\right)\simeq\phi_{N}^{*}P_{k}^{k}\left(\Sigma^{n}H_{G/N}\underline{M}\right)$$

for k > n. Furthermore,

$$P_n^{pk}\left(\Sigma^n H_G(\Psi_N^*\underline{M})\right) \simeq \Psi_N^* P_n^k\left(\Sigma^n H_{G/N}\underline{M}\right).$$

Proof. Applying the functor Ψ_N^* to the slice tower for $\Sigma^n H_{G/N}\underline{M}$ produces a tower of fibrations whose layers are $\Psi_N^* P_k^k \left(\Sigma^n H_{G/N}\underline{M} \right)$ for $k \ge n$. We wish to show that this is a partial slice tower for $\Sigma^n H_G(\Psi_N^*\underline{M})$. For k > n, the k-slice $P_k^k \left(\Sigma^n H_{G/N}\underline{M} \right)$ has trivial underlying spectrum. It follows from Proposition 3.18 that

$$\Psi_N^* P_k^k \left(\Sigma^n H_{G/N} \underline{M} \right) \simeq \phi_N^* P_k^k \left(\Sigma^n H_{G/N} \underline{M} \right)$$

for k > n. As the geometric inflation of a k-slice, this is a pk-slice.

It remains to show that

$$\Psi_N^* P_n^n \left(\Sigma^n H_{G/N} \underline{M} \right) \simeq \Psi_N^* \Sigma^V H_{G/N} \underline{L} \simeq \Sigma^V H_G \Psi_N^* \underline{L}$$

has no slices above level pn. First, note that the restriction of $\Sigma^V H_G \Psi_N^* \underline{L}$ to N is the N-spectrum $\Sigma^n H_N \underline{L}(N)$, where $\underline{L}(N)$ is being considered as a constant N-Mackey functor at the value $\underline{L}(G/N)$. It follows that this N-spectrum has no slices above dimension $|N| \cdot n = pn$. Therefore, to show that $\Sigma^V H_G \Psi_N^* \underline{L}$ is less than pn, it suffices to show that

$$[G_+ \wedge_H S^{k\rho_H + r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G = 0$$

for any $N < H \leq G$ and integers $r \geq 0$ and k such that k|H| > pn. Without loss of generality we consider the case H = G.

Denote by U a complement of $\rho_{G/N}$ in ρ_G , so that

$$\rho_G \cong \rho_{G/N} \oplus U.$$

We then have a cofiber sequence

$$S(kU)_+ \wedge S^{k\rho_{G/N}} \longrightarrow S^{k\rho_{G/N}} \longrightarrow S^{k\rho_G}$$

and a resulting exact sequence

$$\begin{split} [\Sigma^1 S(kU)_+ \wedge S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G &\longrightarrow [S^{k\rho_G+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G \\ &\longrightarrow [S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G = 0. \end{split}$$

We must show that the left term vanishes. Note that the *G*-action on S(kU) is free, since *N* is order *p*. Then the desired vanishing follows from the fact that $\Sigma^1 S(kU)_+ \wedge S^{k\rho_G/N-V}$ is *G*-connected, since dim $k\rho_{G/N} > \dim V = n$.

4. Q_8 -Mackey functors and Bredon homology

We display a number of the Q_8 -Mackey functors that will be relevant in Table 3. In these Lewis diagrams, we are using the subgroup lattice of Q_8 as displayed in Section 1.1. We will also often abuse notation and write the name for a K_4 -Mackey functor, such as <u>m</u> or <u>mg</u>, to denote the resulting inflated Q_8 -Mackey functor. We will only write the symbol ϕ_Z^* when it is necessary to resolve an ambiguity, for instance between $\phi_Z^* \mathbb{F}_2$ and \mathbb{F}_2 .

In [HHR3, Section 2.1], the authors introduce "forms of $\underline{\mathbb{Z}}$ " Mackey functors $\underline{\mathbb{Z}}(i, j)$, where $i \geq j \geq 0$, in the case of $G = C_{p^n}$. From our point of view, Q_8 behaves very similarly to C_8 , and we similarly write $\underline{\mathbb{Z}}(i, j)$ for the Mackey functor that looks like $\underline{\mathbb{Z}}^*$ between the subgroups of order 2^i and 2^j and looks like $\underline{\mathbb{Z}}$ outside of this range. We will at times follow [HHR3] in denoting by $\underline{B}(i, j)$ the cokernel of $\underline{\mathbb{Z}}(i, j) \hookrightarrow \underline{\mathbb{Z}}$, although we will often instead use the descriptions given in Proposition 4.1.

These Mackey functors fit together in exact sequences as follows:

Proposition 4.1. There are exact sequences of Mackey functors

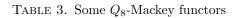
 $\begin{array}{ll} (1) & \underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g} \\ (2) & \underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{B}(2,0) \\ (3) & \underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \twoheadrightarrow \underline{m}^* \\ (4) & \underline{\mathbb{Z}}(2,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{m} \\ (5) & \underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{\mathbb{F}}_2 \\ (6) & \underline{\mathbb{Z}}^* \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{B}(3,0) \\ (7) & \underline{mg} \hookrightarrow \underline{mgw} \twoheadrightarrow \underline{w}. \end{array}$

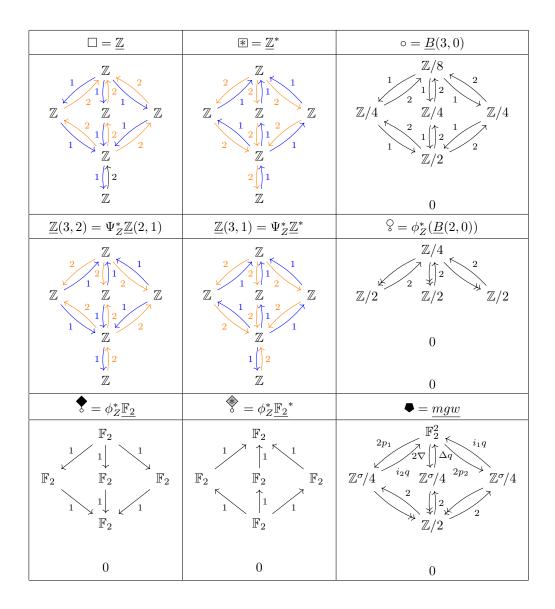
4.1. $RO(Q_8)$ -graded Mackey functor $\underline{\mathbb{Z}}$ -homology of a point. We will now compute the homology of $S^{k\rho_Q}$, with coefficients in $\underline{\mathbb{Z}}$, as a Mackey functor. The starting point is that the regular representation of Q splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

where \mathbb{H} is the 4-dimensional irreducible Q-representation given by the action of the unit quaternions on the algebra of quaternions and ρ_K is the regular representation of K, inflated to Q along the quotient. We begin by computing the homology of $S^{k\mathbb{H}}$. See also [L, Section 2] for an alternative viewpoint.

First, Proposition 3.3 and [S1, Proposition 9.1] combine to yield the following.





Proposition 4.2. For $k \ge 0$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho_K} H_Q \underline{\mathbb{Z}}$ are

$$\underline{\pi}_n \left(\Sigma^{k\rho_K} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 4k \\ \underline{mg} & n = 4k - 2 \\ \underline{g^{\frac{1}{2}(4k-n-1)}} & n \in [2k, 4k-3], n \text{ odd} \\ \underline{g^{\frac{1}{2}(4k-n-4)}} \oplus \phi^*_{LDR} \underline{\mathbb{F}}_2 & n \in [2k, 4k-3], n \text{ even} \\ \underline{g}^{n-k+1} & n \in [k, 2k-1]. \end{cases}$$

Next, we employ the cofiber sequence

(4.3) $S(\mathbb{H})_+ \longrightarrow S^0 \longrightarrow S^{\mathbb{H}}$

to obtain the homology of S^{ρ_Q} from that of S^{ρ_K} .

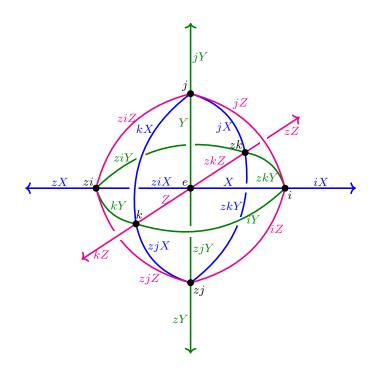


FIGURE 3. The 1-skeleton of $S(\mathbb{H})$.

Proposition 4.4. The nontrivial homotopy Mackey functors of $S(\mathbb{H}) \wedge H_Q\mathbb{Z}$ are

$$\underline{\pi}_n \left(S(\mathbb{H})_+ \wedge H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 3\\ \underline{mgw} & n = 1\\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

Proof. Since the action of Q on $S(\mathbb{H})$ is free, we can write down an equivariant cell structure using only free cells. Viewing $S(\mathbb{H})$ as the one-point compactification of \mathbb{R}^3 , there is a straight-forward cell structure in which the subgroups L, D, and R act freely on the x, y, and z-axes, respectively. We display the 1-skeleton in Figure 3, and the cell structure is described by the following complex of $\mathbb{Z}[Q]$ -modules:

$$\mathbb{Z}[Q]^2 \xrightarrow[\stackrel{e \quad j}{\longrightarrow} \mathbb{Z}[Q]^4 \xrightarrow[\stackrel{e \quad p}{\longrightarrow} \mathbb{Z}[Q]^4 \xrightarrow[\stackrel{e \quad e \quad e \quad k}{\xrightarrow[e \quad -j \quad -e \quad j]} \mathbb{Z}[Q]^3 \xrightarrow[(i-e \ j-e \ k-e)]{} \mathbb{Z}[Q].$$

This yields an associated complex of induced Mackey functors

$$\underline{\mathbb{Z}[Q]}^2 \longrightarrow \underline{\mathbb{Z}[Q]}^4 \longrightarrow \underline{\mathbb{Z}[Q]}^3 \longrightarrow \underline{\mathbb{Z}[Q]}$$

leading to the claimed homology Mackey functors.

Remark 4.5. A smaller chain complex for computing the homology of $S(\mathbb{H})$ is given by

$$\mathbb{Z}[Q] \xrightarrow{\binom{i-e}{e-k}} \mathbb{Z}[Q]^2 \xrightarrow{\binom{e+i}{-e-j} - e+i} \mathbb{Z}[Q]^2 \xrightarrow{(i-e-j-e)} \mathbb{Z}[Q].$$

We gave a less efficient chain complex in the proof of Proposition 4.4 for geometric reasons.

Using (4.3), this immediately yields the following.

Corollary 4.6. The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H}} H_Q \mathbb{Z}$ are

$$\underline{\pi}_n \left(\Sigma^{\mathbb{H}} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 4 \\ \underline{mgw} & n = 2 \\ \underline{B}(3, 0) & n = 0. \end{cases}$$

We will use this to compute the homology of $S^{\rho_Q},$ using the following periodicity result.

Proposition 4.7 ([W, Proposition 4.1]). For any orientable representation V of dimension d and free Q-space X, the orientation $u_V \in H_d(S^V; \underline{\mathbb{Z}})$ induces an equivalence

$$\Sigma^d X_+ \wedge H_Q \underline{\mathbb{Z}} \simeq \Sigma^V X_+ \wedge H_Q \underline{\mathbb{Z}}$$

We now compute the homology of S^{ρ_Q} .

Proposition 4.8. The nontrivial homotopy Mackey functors of $\Sigma^{\rho_Q} H_Q \mathbb{Z}$ are

$$\underline{\pi}_n \left(\Sigma^{\rho_Q} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 8\\ \underline{mgw} & n = 6\\ \underline{B}(3,0) & n = 4\\ \underline{mg} & n = 2\\ \underline{g} & n = 1. \end{cases}$$

Proof. The representation ρ_K is orientable. For example, using the basis $\{1, i, j, k\}$ for $\rho_K = \mathbb{R}[K]$, the matrix $\rho_K(i)$ is given by

$$\rho_K(i) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which has determinant equal to 1. By Proposition 4.7, we have

$$\underline{\pi}_n \left(S(\mathbb{H})_+ \wedge \Sigma^{\rho_K} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 7 \\ \underline{mgw} & n = 5 \\ \underline{\mathbb{Z}}^* & n = 4. \end{cases}$$

The result then follows from the cofiber sequence

$$S(\mathbb{H})_+ \wedge \Sigma^{\rho_K} H_Q \mathbb{Z} \longrightarrow \Sigma^{\rho_K} H_Q \mathbb{Z} \longrightarrow \Sigma^{\rho_Q} H_Q \mathbb{Z}.$$

Corollary 4.6 generalizes as follows.

Proposition 4.9. The nontrivial homotopy Mackey functors of $\Sigma^{k\mathbb{H}}H_Q\underline{\mathbb{Z}}$, for k > 0 are

$$\underline{\pi}_n \left(\Sigma^{k \mathbb{H}} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 4k \\ \underline{mgw} & 0 < n < 4k, n \equiv 2 \pmod{4} \\ \underline{B}(3, 0) & 0 \le n < 4k, n \equiv 0 \pmod{4}. \end{cases}$$

Proof. This follows by induction, using the cofiber sequence

$$S(\mathbb{H})_+ \wedge S^{(k-1)\mathbb{H}} \longrightarrow S^{(k-1)\mathbb{H}} \longrightarrow S^{k\mathbb{H}}$$

and Proposition 4.7. The latter applies since \mathbb{H} , and therefore also $(k-1)\mathbb{H}$, is orientable.

Combining this with the cofiber sequence

$$S(k\mathbb{H})_{+} \wedge \Sigma^{k\rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{k\rho_{K}} H_{Q} \underline{\mathbb{Z}} \longrightarrow \Sigma^{k\rho_{Q}} H_{Q} \underline{\mathbb{Z}}$$

and Proposition 4.7 gives the following result.

Proposition 4.10. The nontrivial homotopy Mackey functors of $\Sigma^{k\rho_Q} H_Q \underline{\mathbb{Z}}$, for k > 0, are

$$\underline{\pi}_n \left(\Sigma^{k\rho_Q} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}} & n = 8k \\ \underline{mgw} & 4k < n < 8k, n \equiv 2 \pmod{4} \\ \underline{B}(3,0) & 4k \le n < 8k, n \equiv 0 \pmod{4} \\ \phi_Z^* \underline{\pi}_n \left(\Sigma^{k\rho_K} H_K \underline{\mathbb{Z}} \right) & n < 4k, \end{cases}$$

where the latter Mackey functors are listed in Proposition 4.2.

The homotopy Mackey functors of $\Sigma^{k\rho_Q} H_Q \mathbb{Z}$ are displayed in Figure 4. When k is negative, the computation follows the same strategy. The initial input, which can again be computed using the chain complex given in Proposition 4.4, is that

(4.11)
$$\underline{\mathrm{H}}^{n}(S(\mathbb{H});\underline{\mathbb{Z}}) \cong \underline{\pi}_{-n}\left(F\left(S(\mathbb{H})_{+}, H_{Q}\underline{\mathbb{Z}}\right)\right) \cong \begin{cases} \underline{\mathbb{Z}}^{*} & n=3\\ \underline{mgw} & n=2\\ \underline{\mathbb{Z}} & n=0. \end{cases}$$

Using this and [S1, Proposition 9.2] leads to the following answer.

Proposition 4.12. The nontrivial homotopy Mackey functors of $\Sigma^{-k\rho_Q}H_Q\underline{\mathbb{Z}}$, for k > 0, are

$$\underline{\pi}_{-n} \left(\Sigma^{-k\rho_Q} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}}^* & n = 8k \\ \underline{mgw} & n \in [4k, 8k], n \equiv 3 \pmod{4} \\ \underline{B}(3, 0) & n \in [4k + 5, 8k], n \equiv 1 \pmod{4} \\ \phi_Z^* \underline{B}(2, 0) & n = 4k + 1 \\ \underline{mg^*} & n = 4k - 1 \\ \underline{g^{\frac{4k-n}{2}}} & n \in [2k + 4, 4k - 2], n \equiv 0 \pmod{2} \\ \underline{g^{\frac{4k-n-3}{2}}} \oplus \phi_{LDR}^* \underline{\mathbb{F}}_2^* & n \in [2k + 3, 4k - 2], n \equiv 1 \pmod{2} \\ \underline{g^{n-k-3}} & n \in [k + 4, 2k + 2]. \end{cases}$$

Remark 4.13. The "Gap Theorem" [HHR1, Proposition 3.20] predicts that the groups $\pi_n^Q \Sigma^{-k\rho} H \mathbb{Z}$ vanish for $k \geq 0$ and $n \in [-3, -1]$, as indicated in Figure 4. Actually, for $k \geq 2$ the argument there proves more. It tells us that for $k \geq 2$, the

cohomology groups $\mathrm{H}^{n}_{Q}(S^{k\rho};\underline{M})$ vanish for positive $n \leq k+1$. This is equivalent to saying that $\pi^{Q}_{-n}\Sigma^{-k\rho}H\underline{M}$ vanishes, with the same conditions on k and n.

4.2. Additional homology calculations. We will also need the following auxiliary calculations in Section 6.

Proposition 4.14. The nontrivial homotopy Mackey functors of $\Sigma^{\rho_{\kappa}-\mathbb{H}}H_Q\mathbb{Z}$ are

$$\underline{\pi}_n \left(\underline{\Sigma}^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}}_2 & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

Proof. The fiber sequence

$$\Sigma^{\rho_{K}-\mathbb{H}}H_{Q}\underline{\mathbb{Z}}\longrightarrow \Sigma^{\rho_{K}}H_{Q}\underline{\mathbb{Z}}\longrightarrow F(S(\mathbb{H})_{+},\Sigma^{\rho_{K}}H_{Q}\underline{\mathbb{Z}})\simeq \Sigma^{4}F(S(\mathbb{H})_{+},H_{Q}\underline{\mathbb{Z}})$$

yields an isomorphism $\underline{\pi}_0 \left(\Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}} \right) \cong \underline{\mathbb{Z}}^*$ and shows that the homotopy vanishes for *n* outside of [0,2]. Given that the restriction to any C_4 , which is the C_4 -spectrum $\Sigma^{2+2\sigma-2\lambda} H_{C_4} \underline{\mathbb{Z}}$, has a trivial $\underline{\pi}_2$ [Z1, Theorem 6.10], the long exact sequence further shows that $\underline{\pi}_2$ vanishes as well, and it implies that we have an extension

$$\underline{w} \hookrightarrow \underline{\pi}_1 \left(\Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}} \right) \twoheadrightarrow \underline{g}$$

It remains to show this is not the split extension. The fiber sequence

$$\uparrow_D^Q \Sigma^{1+2\sigma-2\lambda} H_{C_4} \underline{\mathbb{Z}} \longrightarrow \Sigma^{1+p_1^*\sigma+p_2^*\sigma-\mathbb{H}} H_Q \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_K-\mathbb{H}} H_Q \underline{\mathbb{Z}}$$

shows that $\underline{\pi}_1\left(\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}\right)$ injects into

$$\underline{\pi}_0 \left(\uparrow_D^Q \Sigma^{1+2\sigma-2\lambda} H_{C_4} \underline{\mathbb{Z}} \right) \cong \uparrow_D^Q \phi_{C_2}^* \underline{\mathbb{F}_2}$$

It follows that $\underline{\pi}_1\left(\Sigma^{\rho_K-\mathbb{H}}H_Q\underline{\mathbb{Z}}\right)\cong \phi_Z^*\underline{\mathbb{F}}_2$

Proposition 4.15. The nontrivial homotopy Mackey functors of $\Sigma^{\rho_{\kappa}} - \mathbb{H}H_Q\underline{\mathbb{Z}}(3,2)$ are

$$\underline{\pi}_n \left(\Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}}(3, 2) \right) \cong \begin{cases} \underline{w} & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

Proof. The short exact sequence

$$\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$$

gives rise to a cofiber sequence

$$\Sigma^{\rho_{K}-\mathbb{H}}H_{Q}\underline{\mathbb{Z}}(3,2)\longrightarrow\Sigma^{\rho_{K}-\mathbb{H}}H_{Q}\underline{\mathbb{Z}}\longrightarrow\Sigma^{\rho_{K}-\mathbb{H}}H_{Q}\underline{g}\simeq\Sigma^{1}H_{Q}\underline{g}.$$

Using a naturality square, the second map factors as

$$\Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}} \longrightarrow \Sigma^{\rho_K} H_Q \underline{\mathbb{Z}} \longrightarrow \Sigma^1 H_Q g,$$

where the first map is an epimorphism on $\underline{\pi}_1$ by the proof of Proposition 4.14 and the second is an isomorphism on $\underline{\pi}_1$. The conclusion follows.

Proposition 4.16. The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H}-\rho_K}H_Q\underline{\mathbb{Z}}(2,0)$ are

$$\underline{\pi}_n \left(\Sigma^{\mathbb{H}-\rho_K} H_Q \underline{\mathbb{Z}}(2,0) \right) \cong \begin{cases} \underline{\mathbb{Z}} & n=0\\ \underline{w}^* & n=-2. \end{cases}$$

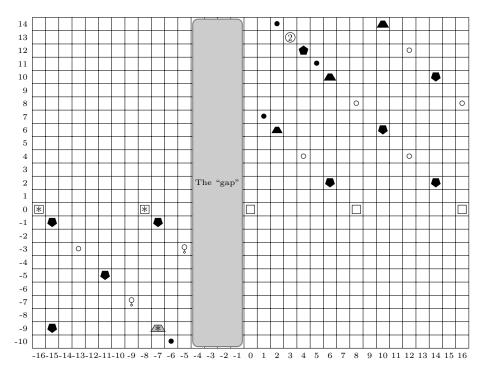


FIGURE 4. The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_Q \underline{\mathbb{Z}}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_Q \underline{\mathbb{Z}}$ appears in position (k, 8n - k).

Proof. This follows from Proposition 4.15 by duality. In more detail, Proposition 4.15 gives a fiber sequence

$$\Sigma^1 H_Q \underline{w} \longrightarrow \Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}}(3, 2) \longrightarrow H_Q \underline{\mathbb{Z}}^*$$

Applying Anderson duality (see [S1, Section 2.2]) gives a fiber sequence

$$I(\Sigma^1 H_Q \underline{w}) \longleftarrow I\left(\Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}}(3, 2)\right) \longleftarrow I(H_Q \underline{\mathbb{Z}}^*),$$

or in other words

$$\Sigma^{-1}I(H_Q\underline{w}) \longleftarrow \Sigma^{\mathbb{H}-\rho_K}H_Q\underline{\mathbb{Z}}(2,0) \longleftarrow H_Q\underline{\mathbb{Z}}.$$

But as the Mackey functor \underline{w} is torsion, the Anderson dual is the desuspension of the Brown-Comenetz dual. In other words, $I(H_Q\underline{w}) \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} H_Q\underline{w} \simeq \Sigma^{-1} H_Q\underline{w}^*$. \Box

5. Review of the C_4 -slices of $\Sigma^n H\mathbb{Z}$

In this section, we review the slices of $\Sigma^n H_{C_4} \mathbb{Z}$ from [Y1]. Note that the slices as listed in [Y1] are written using the classical slice filtration, whereas we use the regular slice filtration. The only difference is a suspension by one. The Mackey functors that appear here were introduced in Table 1.

According to [Y1, Section 4.2], the C_4 -spectrum $\Sigma^n H_{C_4} \mathbb{Z}$ is an *n*-slice for $0 \leq n \leq 4$. For $n \geq 5$, $\Sigma^n H_{C_4} \mathbb{Z}$ has a nontrivial slice tower. Yarnall's method for determining these slice towers is to splice together suspensions of the cofiber sequences

$$\Sigma^{-1}H_{C_4}g \longrightarrow \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \longrightarrow \Sigma^{2\sigma}H_{C_4}\underline{\mathbb{Z}},$$

$$\Sigma^{-1}H_{C_4}\phi_{C_2}^*\underline{\mathbb{F}_2}^*\longrightarrow \Sigma^2 H_{C_4}\underline{\mathbb{Z}}\longrightarrow \Sigma^{\lambda}H_{C_4}\underline{\mathbb{Z}}(2,1),$$

and

$$\Sigma^{-1}H_{C_4}\underline{B}(2,0)\longrightarrow \Sigma^2 H_{C_4}\underline{\mathbb{Z}}\longrightarrow \Sigma^{\lambda}H_{C_4}\underline{\mathbb{Z}}$$

in combination with the equivalences

$$\Sigma^2 H_{C_4} \underline{\mathbb{Z}} \simeq \Sigma^{2\sigma} H_{C_4} \underline{\mathbb{Z}}(2,1)$$

and

$$\Sigma^{-1} H_{C_4} \phi_{C_2}^* \underline{\mathbb{F}}_2^* \simeq \Sigma^{-\sigma} H_{C_4} \phi_{C_2}^* \underline{f} \simeq \Sigma^{1-2\sigma} H_{C_4} \phi_{C_2}^* \underline{\mathbb{F}}_2$$

We first review these slices for odd n.

Proposition 5.1. [Y1, Theorem 4.2.6] Let $n \ge 5$ be odd. The bottom slice of $\Sigma^n H_{C_4} \underline{\mathbb{Z}}$ is

$$P_n^n(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{\frac{n-5}{4}\rho+4+\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 1 \pmod{8} \\ \Sigma^{\frac{n-3}{4}\rho+3} H_{C_4}\underline{\mathbb{Z}} & n \equiv 3 \pmod{8} \\ \Sigma^{\frac{n-5}{4}\rho+3+2\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 5 \pmod{8} \\ \Sigma^{\frac{n-3}{4}\rho+2+\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 7 \pmod{8}. \end{cases}$$

Proposition 5.2. [Y1, Lemma 4.2.5] Let $n \ge 5$ be odd. The nontrivial 4k-slices of $\Sigma^n H_{C_4} \mathbb{Z}$ are

$$P_{4k}^{4k}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{k\rho} H_{C_4}\underline{\mathbb{B}}(2,0) & 4k \in [n+1,2(n-3)], \ k \ even \\ \Sigma^{k\rho} H_{C_4}\phi^*\underline{f} & 4k \in [n+1,2(n-3)], \ k \ odd \\ \Sigma^{k\rho} H_{C_4}\underline{g} & 4k \in [2(n-1),4(n-3)], \ k \ even \end{cases}$$

The 4k-slices can also be read off of [HHR2, Figure 3]. When n is odd, these are the only nontrivial slices of $\Sigma^n H_{C_4} \mathbb{Z}$.

We now recall the slices of $\Sigma^n H_{C_4} \mathbb{Z}$ for even n.

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Proposition 5.3. [Y1, Theorem 4.2.9] Let $n \ge 6$ be even. The bottom slice of $\Sigma^n H_{C_4} \mathbb{Z}$ is

$$P_n^n(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{\frac{n-4}{4}\rho+3+\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 0 \pmod{8} \\ \Sigma^{\frac{n-6}{4}\rho+3+3\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 2 \pmod{8} \\ \Sigma^{\frac{n-4}{4}\rho+4} H_{C_4}\underline{\mathbb{Z}} & n \equiv 4 \pmod{8} \\ \Sigma^{\frac{n-6}{4}\rho+4+2\sigma} H_{C_4}\underline{\mathbb{Z}} & n \equiv 6 \pmod{8}. \end{cases}$$

Proposition 5.4. [Y1, Lemma 4.2.7] Let $n \ge 6$ be even. The nontrivial 4k-slices of $\Sigma^n H_{C_4} \underline{\mathbb{Z}}$ are

$$P_{4k}^{4k}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^k H_{C_4}\underline{g}, \quad k \ odd$$

for 4k in the range [n + 2, 4n - 12].

Again, the 4k-slices can also be read off of [HHR2, Figure 3].

Proposition 5.5. [Y1, Theorem 4.2.9] Let $n \ge 6$ be even. The (4k + 2)-slices of $\Sigma^n H_{C_4} \underline{\mathbb{Z}} \ are$

$$P^{8k+2}_{8k+2}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^{1+2k\rho} H\phi^* \underline{\mathbb{F}}_2$$
$$P^{8k+6}_{8k+6}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^{3+2k\rho} H\phi^* \underline{\mathbb{F}}_2.$$

for 8k + 2 or 8k + 6 in the range [n + 2, 2n - 6]

We may also view these slices through the perspective of the \mathbb{Z} -module inflation functor. By Theorem 3.19,

$$\Psi_{C_2}^* : \operatorname{Mod}_{H_{C_2}\underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_{C_4}\underline{\mathbb{Z}}}$$

will provide all slices of $\Sigma^n H_{C_4}$ above level 2n. Let $r \equiv n \pmod{4}$ with $3 \leq r \leq 6$. It follows from [S1, Proposition 3.5] that the slices of $\Sigma^n H_{C_4} \mathbb{Z}$ in level at least 2n + 2r - 4 are

$$P_{4k}^{4k}\left(\Sigma^n H_{C_4}\underline{\mathbb{Z}}\right) \simeq \Psi_{C_2}^* \Sigma^k H_{C_2}\underline{g} \simeq \Sigma^k H_{C_4}\underline{g}$$

for $4k \in [2n + 2r - 4, 4(n - 3)]$. The rest of the slices then follow from determining the slices of

$$\Psi_{C_2}^* \Sigma^{\frac{n-r}{2}\rho_{C_2}+r} H_{C_2} \underline{\mathbb{Z}} \simeq \Sigma^{\frac{n+r}{2}+\frac{n-r}{2}\sigma} H_{C_4} \underline{\mathbb{Z}}$$

The slice tower for this C_4 -spectrum can be found by splicing together the cofiber sequences listed at the start of this section.

6. Q_8 -SLICES

The slices of $\Sigma^n H_K \underline{\mathbb{Z}}$ were determined by the second author in [S1, Section 8]. As stated in Theorem 3.19, it follows that the $\underline{\mathbb{Z}}$ -module inflation functor

$$\Psi_Z^* \colon \operatorname{Mod}_{H_K \underline{\mathbb{Z}}} \longrightarrow \operatorname{Mod}_{H_Q \underline{\mathbb{Z}}}$$

of Proposition 3.14 will produce all slices of $\Sigma^n H_Q \mathbb{Z}$ in degree larger than 2n, as the inflation of the slices of $\Sigma^n H_K \mathbb{Z}$ above degree n.

The remaining slices of $\Sigma^n H_Q \underline{\mathbb{Z}}$ will be given as the slices of $\Psi_Z^* (P_n^n(\Sigma^n H_K \underline{\mathbb{Z}}))$. By [S1, Proposition 8.5], these are of the form

$$\Psi_Z^*\left(\Sigma^{r+j\rho_K} H_K \underline{\mathbb{Z}}\right) \simeq \Sigma^{r+j\rho_K} H_Q \underline{\mathbb{Z}},$$

where $r \in \{3, 4, 5\}$, if $n \not\equiv 2 \pmod{4}$. In the case $n \equiv 2 \pmod{4}$, the same result states that this is

$$\Psi_Z^* \Big(\Sigma^{2+j\rho_K} H_K \underline{\mathbb{Z}}(1,0) \Big) \simeq \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1).$$

But the cofiber sequence (Proposition 4.1)

(6.1)
$$\Sigma^{1+j\rho_K} H_Q \underline{m} \longrightarrow \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1) \longrightarrow \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1)$$

reduces the computation of slices of $\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1)$ to the question of the slice tower for $\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}$, given that $\Sigma^{1+j\rho_K} H_Q \underline{m} \simeq \phi_Z^* (\Sigma^{1+j\rho_K} H_K \underline{m})$ is an 8j+4-slice [S1, Proposition 5.7]. We determine the slices of $\Sigma^{r+j\rho_K} H_Q \underline{\mathbb{Z}}$, for $r \in \{2, \ldots, 5\}$ in Section 6.1.

6.1. Slice towers for $\Sigma^{r+j\rho_K} H_Q \mathbb{Z}$. The K_4 -spectrum $\Sigma^{r+j\rho_K} H_K \mathbb{Z}$ is an *n*-slice for $r \in \{2, \ldots, 5\}$ [S1, Proposition 7.1]. However, the inflation of this to Q_8 is no longer a slice. We here determine the slice towers of these inflations. Throughout, we will implicitly use Proposition 6.6, which does not rely on the following material. 6.1.1. (r = 2). First, we observe that $\Sigma^{2+\rho_K} H_Q \mathbb{Z}$ is a 6-slice. To see this we first note that it restricts to a 6-slice at every proper subgroup by Proposition 5.3. It therefore remains only to show that it does not have any 8k-slices for $k \ge 1$. This is equivalent to showing that $\underline{\pi}_{-2} \left(\Sigma^{\rho_K - k\rho_Q} H_Q \mathbb{Z} \right)$ vanishes for $k \ge 1$. In the case k = 1, (4.11) shows that $\Sigma^{-\mathbb{H}} H_Q \mathbb{Z}$ is (-3)-truncated, in the sense that it has no homotopy Mackey functors above dimension -3. This remains true after further desuspending by copies of ρ_Q .

Next, the tower for $\Sigma^{2+2\rho_K} H_Q \mathbb{Z}$ is given by

$$\begin{split} P_{14}^{14} &= \Sigma^{-1+2\rho_Q} H_Q \underline{w}^* \longrightarrow \Sigma^{2+2\rho_K} H_Q \underline{\mathbb{Z}} \\ & \downarrow \\ P_{12}^{12} &= \Sigma^{1+\rho_Q} H_Q \underline{\mathbb{M}} \longrightarrow \Sigma^{2+\rho_Q} H_Q \underline{\mathbb{Z}}(2,0) \\ & \downarrow \\ P_{10}^{10} &= \Sigma^{2+\rho_Q} H_Q \underline{\mathbb{Z}}(1,0). \end{split}$$

This uses the computation (see Proposition 4.16)

$$\underline{\pi}_n \left(\Sigma^{\mathbb{H}-\rho_K} H_Q \underline{\mathbb{Z}}(2,0) \right) \cong \begin{cases} \underline{\mathbb{Z}} & n=0\\ \underline{w}^* & n=-2 \end{cases}$$

to produce the first cofiber sequence.

Finally, for $j \ge 3$, the tower may be obtained by recursively using

$$P_{8j-2}^{8j-2} = \Sigma^{-1+j\rho_Q} H_Q \underline{w}^* \longrightarrow \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}$$

$$\downarrow$$

$$P_{8j-4}^{8j-4} = \Sigma^{1+(j-1)\rho_Q} H_Q \underline{m} \longrightarrow \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q \underline{\mathbb{Z}}(2,0)$$

$$\downarrow$$

$$P_{8j-6}^{8j-6} = \Sigma^{1+(j-1)\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}}_2 \longrightarrow \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q \underline{\mathbb{Z}}(1,0)$$

$$\downarrow$$

$$\Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q \mathbb{Z}.$$

We have proved the following result.

Proposition 6.2. Let $j \ge 1$. The bottom slice of $\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}$ is

$$P_{2+4j}^{2+4j} \left(\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}} \right) \simeq \begin{cases} \Sigma^{1+\rho_K+\frac{j-1}{2}\rho_Q} H_Q \underline{\mathbb{Z}}^* & j \ odd \\ \Sigma^{2+\frac{j}{2}\rho_Q} H_Q \underline{\mathbb{Z}} & j \ even. \end{cases}$$

6.1.2. (r = 3). By (4.11), the cohomology of $S^{\mathbb{H}}$ is given by

$$\underline{\widetilde{H}}^{n}(S^{\mathbb{H}};\underline{\mathbb{Z}}) \cong \underline{\pi}_{-n} \left(\Sigma^{-\mathbb{H}} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \underline{\mathbb{Z}}^{*} & n = 4\\ \underline{mgw} & n = 3. \end{cases}$$

Suspending by $3+\rho_Q$ leads to the cofiber sequence

$$P_8^8 = \Sigma^{\rho_Q} H_Q \underline{mgw} \longrightarrow \Sigma^{3+\rho_K} H_Q \underline{\mathbb{Z}}$$

$$\downarrow$$

$$P_7^7 = \Sigma^{\rho_Q-1} H_Q \underline{\mathbb{Z}}^*$$

The tower for $\Sigma^{3+j\rho_K} H_Q \underline{\mathbb{Z}}$, where $j \geq 2$, is then given recursively by

$$\begin{split} P^{8j}_{8j} &= \Sigma^{j\rho_Q} H_Q \underline{mgw} \longrightarrow \Sigma^{3+j\rho_K} H_Q \underline{\mathbb{Z}} \\ & \downarrow \\ \Sigma^{(j-1)\rho_K + \rho_Q - 1} H_Q \underline{\mathbb{Z}}^* \\ & \parallel \\ P^{8j-4}_{8j-4} &= \Sigma^{2+(j-1)\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2} \longrightarrow \Sigma^{3+(j-2)\rho_K + \rho_Q} H_Q \underline{\mathbb{Z}}(1,0) \\ & \downarrow \\ \Sigma^{3+(j-2)\rho_K + \rho_Q} H_Q \underline{\mathbb{Z}}. \end{split}$$

The last cofiber sequence arises from Proposition 4.1. We have proved the following result.

Proposition 6.3. Let $j \ge 1$. The bottom slice of $\Sigma^{3+j\rho_K} H_Q \underline{\mathbb{Z}}$ is

$$P_{3+4j}^{3+4j}\left(\Sigma^{3+j\rho_{K}}H_{Q}\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{-1+\frac{j+1}{2}\rho_{Q}}H_{Q}\underline{\mathbb{Z}}^{*} & j \ odd\\ \Sigma^{3+\frac{j}{2}\rho_{Q}}H_{Q}\underline{\mathbb{Z}} & j \ even. \end{cases}$$

6.1.3. (r = 4). The tower for $\Sigma^{4+\rho_K} H_Q \underline{\mathbb{Z}}$ is given by

$$\begin{split} P_{12}^{12} &= \Sigma^{\rho_Q + 1} H_Q \underline{mg} \longrightarrow \Sigma^{4 + \rho_K} H_Q \underline{\mathbb{Z}} \simeq \Sigma^{2\rho_K} H_Q \underline{\mathbb{Z}}(3, 1) \\ & \downarrow \\ P_{10}^{10} &= \Sigma^{\rho_Q + 1} \underline{w} \longrightarrow \Sigma^{2\rho_K} H_Q \underline{\mathbb{Z}}(3, 2) \\ & \downarrow \\ P_8^8 &= \Sigma^{\rho_Q} H_Q \underline{\mathbb{Z}}^*. \end{split}$$

This uses the short exact sequence (Proposition 4.1)

$$\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \twoheadrightarrow \underline{m}^*,$$

the equivalence $\Sigma^{\rho_K} H_K \underline{m}^* \simeq \Sigma^2 H_K \underline{mg}$ ([GY, Proposition 4.8]), and the computation (see Proposition 4.15)

$$\underline{\pi}_n \left(\Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}}(3, 2) \right) \cong \begin{cases} \underline{w} & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

The tower for $\Sigma^{4+j\rho_K} H_Q \mathbb{Z}$, where $j \geq 2$, may then be obtained recursively from

Proposition 6.4. Let $j \geq 1$. The bottom slice of $\Sigma^{4+j\rho_K} H_Q \underline{\mathbb{Z}}$ is

$$P_{4+4j}^{4+4j}\left(\Sigma^{4+j\rho_{K}}H_{Q}\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{\frac{j+1}{2}\rho_{Q}}H_{Q}\underline{\mathbb{Z}}^{*} & j \ odd\\ \Sigma^{4+\frac{j}{2}\rho_{Q}}H_{Q}\underline{\mathbb{Z}} & j \ even \end{cases}$$

6.1.4. (r = 5). Here, we start with the slice tower for $\Sigma^5 H_Q \mathbb{Z}$, as this is not a slice. The short exact sequence

$$\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{B}(2,0)$$

gives rise to a cofiber sequence

$$P_8^8 = \Sigma^{\rho_Q} H_Q \phi_Z^* \underline{B}(2,0) \longrightarrow \Sigma^5 H_Q \underline{\mathbb{Z}} \simeq \Sigma^{1+\rho_K} H_Q \underline{\mathbb{Z}}(3,1) \longrightarrow \Sigma^{1+\rho_K} H_Q \underline{\mathbb{Z}}(3,1)$$

Now the argument showing that $\Sigma^{2+\rho_K} H_Q \underline{\mathbb{Z}}$ is a 6-slice, given above in Section 6.1.1, also applies to show that $\Sigma^{1+\rho_K} H_Q \underline{\mathbb{Z}}$ is a 5-slice. Thus, this cofiber sequence is the slice tower for $\Sigma^5 H_Q \underline{\mathbb{Z}}$. Next, the tower for $\Sigma^{5+\rho_K} H_Q \underline{\mathbb{Z}}$ is given by

where the bottom cofiber sequence arises from the computation (Proposition 4.14)

$$\underline{\pi}_n \left(\Sigma^{\rho_K - \mathbb{H}} H_Q \underline{\mathbb{Z}} \right) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}}_2 & n = 1 \\ \underline{\mathbb{Z}}^* & n = 0. \end{cases}$$

The tower for $\Sigma^{5+j\rho_K} H_Q \mathbb{Z}$, where $j \geq 2$, may then be obtained recursively from

Proposition 6.5. Let $j \ge 1$. The bottom slice of $\Sigma^{5+j\rho_K} H_Q \underline{\mathbb{Z}}$ is

$$P_{5+4j}^{5+4j}\left(\Sigma^{5+j\rho_{K}}H_{Q}\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{1+\frac{j+1}{2}\rho_{Q}}H_{Q}\underline{\mathbb{Z}}^{*} & j \text{ odd} \\ \Sigma^{1+\rho_{K}+\frac{j}{2}\rho_{Q}}H_{Q}\underline{\mathbb{Z}} & j \text{ even} \end{cases}$$

6.2. Slices of $\Sigma^n H_Q \mathbb{Z}$. In this section, we describe all slices of $\Sigma^n H_Q \mathbb{Z}$ for $n \ge 0$. **Proposition 6.6.** The Q_8 -spectrum $\Sigma^n H_Q \mathbb{Z}$ is an n-slice for $0 \le n \le 4$.

Proof. Since this is true after restricting to any C_4 (see Section 5), any higher slices would necessarily be geometric and therefore occurring in slice dimension at least 8. But we can show directly that $\Sigma^n H_Q \underline{\mathbb{Z}} < 8$ if $n \in [0, 4]$. This follows from the vanishing of $\pi_{\rho_Q} \Sigma^n H_Q \underline{\mathbb{Z}} \cong \pi_{-n} \Sigma^{-\rho_Q} H_Q \underline{\mathbb{Z}}$ as displayed in Figure 4.

It remains to determine the slices of $\Sigma^n H_Q \mathbb{Z}$ when $n \geq 5$. Note that Theorem 3.19 applies by [S1, Proposition 8.5]. We first describe the bottom slice.

Proposition 6.7 (The *n*-slice). For $n \ge 5$, write n = 8k + r, where $r \in [5, 12]$. Then the *n*-slice of $\Sigma^n H_Q \underline{\mathbb{Z}}$ is

$$P_n^n\left(\Sigma^n H_Q\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{1+\rho_K+k\rho_Q} H_Q\underline{\mathbb{Z}} & r=5\\ \Sigma^{2+\rho_K+k\rho_Q} H_Q\underline{\mathbb{Z}}(3,2) & r=6\\ \Sigma^{-1+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & r=7\\ \Sigma^{(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & r=8\\ \Sigma^{1+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & r=9\\ \Sigma^{2+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}(1,0) & r=10\\ \Sigma^{3+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}} & r=11\\ \Sigma^{4+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}} & r=12. \end{cases}$$

Proof. By Theorem 3.19, the *n*-slice of $\Sigma^n H_Q \mathbb{Z}$ is the *n*-slice of the \mathbb{Z} -module inflation of the *n*-slice of $\Sigma^n H_K \mathbb{Z}$. By [S1, Proposition 8.5], writing $n = 4j + r_4$ with $r_4 \in \{2, 3, 4, 5\}$, we have

$$\Psi_Z^* P_n^n \left(\Sigma^n H_{K_4} \underline{\mathbb{Z}} \right) \simeq \begin{cases} \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1) & n \equiv 2 \pmod{4} \\ \Sigma^{r_4+j\rho_K} H_Q \underline{\mathbb{Z}} & \text{else.} \end{cases}$$

If $n \not\equiv 2 \pmod{4}$, the slice tower was given in Section 6.1. For the case of $n \equiv 2$, since $\Sigma^{1+j\rho_K} H_Q \underline{m} \simeq \phi_Z^* (\Sigma^{1+j\rho_K} H_K \underline{m})$ is an 8j + 4-slice [S1, Proposition 5.7], the cofiber sequence (Proposition 4.1)

(6.8)
$$\Sigma^{1+j\rho_K} H_Q \underline{m} \longrightarrow \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1) \longrightarrow \Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1)$$

combines with the work of Section 6.1.1 to to show that

$$P_n^n\left(\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}(2,1)\right) \simeq P_n^n\left(\Sigma^{2+j\rho_K} H_Q \underline{\mathbb{Z}}\right).$$

The latter is given in Proposition 6.2.

Proposition 6.9 (The 8k-slices). For $n \ge 5$ and 8k > n, the 8k-slice of $\Sigma^n H_Q \underline{\mathbb{Z}}$ is

$$P_{8k}^{8k} \left(\Sigma^n H_Q \underline{Z} \right) \simeq \begin{cases} \Sigma^k H_Q \underline{g}^{n-k-3} & 8k \in [4n-8, 8n-32] \\ \Sigma^{k\rho_Q} H_Q \underline{g}^{\frac{4k-n}{2}} & 8k \in [2n+4, 4n-16] \\ and \quad n \equiv 0 \pmod{2} \\ \Sigma^{k\rho_Q} H_Q \underline{g}^{\frac{4k-n-3}{2}} \oplus \phi_{LDR}^* \underline{\mathbb{F}}_2^* & 8k \in [2n+4, 4n-12] \\ and \quad n \equiv 1 \pmod{2} \\ \Sigma^{k\rho_Q} H_Q \underline{mg}^* & 8k = 2n+2 \\ \Sigma^{k\rho_Q} H_Q \phi_Z^* \underline{B}(2,0) & 8k = 2n-2 \\ \Sigma^{k\rho_Q} H_Q \underline{\phi}_Z^* \underline{B}(2,0) & 8k \in [n+3, 2n-10] \\ and \quad n \equiv 1 \pmod{4} \\ \Sigma^{k\rho_Q} H_Q \underline{mgw} & 8k \in [n+1, 2n] \\ and \quad n \equiv 3 \pmod{4}. \end{cases}$$

Proof. This is a translation of Proposition 4.12. Alternatively, the slices above dimension 2n follow from Theorem 3.19 and [S1, Proposition 8.6]. The slices in dimensions 2n and lower follow from the towers computed in Section 6.1.

Proposition 6.10 (The 8k + 4-slices). For $n \ge 5$ and 8k + 4 > n, the 8k + 4-slices of $\Sigma^n H_Q \mathbb{Z}$ are

$$P_{8k+4}^{8k+4}\left(\Sigma^{n}H_{Q}\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{3+k\rho_{Q}}H_{Q}\phi_{LDR}^{*}\underline{\mathbb{F}}_{2} & 8k+4 \in [2n+4,4n-12], \quad n \text{ even} \\ \Sigma^{2+k\rho_{Q}}H_{Q}\phi_{Z}^{*}\underline{\mathbb{F}}_{2} & 8k+4 \in [n+1,2n-4], \quad n \text{ odd} \\ \Sigma^{1+k\rho_{Q}}H_{Q}\underline{m} & 8k+4 \in [n+2,2n], \quad n \equiv 2 \pmod{4} \\ \Sigma^{1+k\rho_{Q}}H_{Q}mg & 8k+4 \in [n+4,2n-4], \quad n \equiv 0 \pmod{4} \end{cases}$$

Proof. The first case follows from [S1, Proposition 8.7]. The remaining cases follow from (6.8) and Section 6.1.

Proposition 6.11 (The 4k + 2-slices). Let $n \ge 5$. If n is odd, then $\sum^n H_Q \underline{\mathbb{Z}}$ has no nontrivial 4k + 2-slices if 4k + 2 > n. If n is even and 8k + 2 > n, then the 8k + 2-slice of $\sum^n H_Q \underline{\mathbb{Z}}$ is nontrivial only if $8k + 2 \in [n + 1, 2n]$, in which case the slice is

$$P_{8k+2}^{8k+2}\left(\Sigma^{n}H_{Q}\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{1+k\rho_{Q}}H_{Q}\underline{w} & n \equiv 0 \pmod{4} \\ \Sigma^{1+k\rho_{Q}}H_{Q}\phi_{Z}^{*}\underline{\mathbb{F}}_{2} & n \equiv 2 \pmod{4} \end{cases}$$

Similarly, if n is even and 8k-2 > n, the 8k-2-slice is nontrivial only if $8k-2 \in [n+1,2n]$, in which case the slice is

$$P_{8k-2}^{8k-2}\left(\Sigma^{n}H_{Q}\underline{\mathbb{Z}}\right) \simeq \begin{cases} \Sigma^{-1+k\rho_{Q}}H_{Q}\phi_{Z}^{*}\underline{\mathbb{F}_{2}}^{*} & n \equiv 0 \pmod{4} \\ \Sigma^{-1+k\rho_{Q}}H_{Q}\underline{w}^{*} & n \equiv 2 \pmod{4}. \end{cases}$$

Proof. According to [S1], the K_4 -spectrum $\Sigma^n H_K \mathbb{Z}$ does not have any nontrivial slices in odd dimensions, except for the *n*-slice. By Theorem 3.19, this implies that $\Sigma^n H_Q \mathbb{Z}$ does not have any 4k + 2-slices above dimension 2n. The slices in dimensions below 2n are given by Section 6.1.

6.3. Slice towers for $\Sigma^n H_Q \mathbb{Z}$. By Proposition 6.6, $\Sigma^n H_Q \mathbb{Z}$ is an *n*-slice for $n \in \{0, \ldots, 4\}$. The slice tower for $\Sigma^5 H_Q \mathbb{Z}$ was given in Section 6.1.4. We now display a few more examples of slice towers.

Example 6.12. The slice tower for $\Sigma^6 H_Q \mathbb{Z}$ is

$$P_{16}^{16} = \Sigma^2 H_Q \underline{g} \longrightarrow \Sigma^6 H_Q \underline{\mathbb{Z}}$$

$$\downarrow$$

$$P_{12}^{12} = \Sigma^{1+\rho} H_Q \underline{m} \longrightarrow \Sigma^{2+\rho_K} H_Q \underline{\mathbb{Z}}(2,1)$$

$$\downarrow$$

$$P_6^6 = \Sigma^{2+\rho_K} H_Q \underline{\mathbb{Z}}(2,1)$$

This follows immediately from combining [S1, Example 8.2], (6.8), and Section 6.1.1.

Example 6.13. The slice tower for $\Sigma^7 H_Q \underline{\mathbb{Z}}$ is

$$P_{24}^{24} = \Sigma^3 H_Q \underline{g} \longrightarrow \Sigma^7 H_Q \underline{\mathbb{Z}}$$

$$\downarrow$$

$$P_{16}^{16} = \Sigma^{2+\rho_Q} H_Q \underline{m} \longrightarrow \Sigma^{3+\rho_K} H_Q \underline{\mathbb{Z}}(2,1)$$

$$\downarrow$$

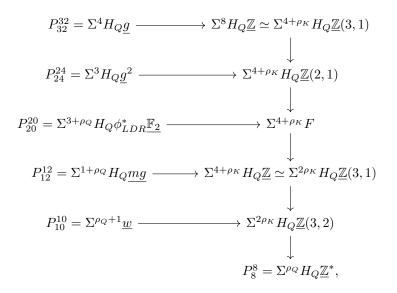
$$P_8^8 = \Sigma^{\rho_Q} H_Q \underline{mgw} \longrightarrow \Sigma^{3+\rho_K} H_Q \underline{\mathbb{Z}}$$

$$\downarrow$$

$$P_7^7 = \Sigma^{\rho_Q - 1} H_{Q_8} \underline{\mathbb{Z}}^*$$

This follows immediately from combining [S1, Example 8.3] and Section 6.1.2.

Example 6.14. The slices, but not the slice tower, for $\Sigma^8 H_K \mathbb{Z}$ were determined in [S1, Section 8]. Let us denote by F the fiber of the map $H_Q \mathbb{Z} \longrightarrow H_Q \phi_{LDR} \mathbb{F}_2$ induced by the map of Q_8 -Mackey functors $\mathbb{Z} \longrightarrow \phi_{LDR} \mathbb{F}_2$ that is surjective at \overline{L} , D, and R. Then the nontrivial homotopy Mackey functors of F are $\underline{\pi}_0(F) \simeq \mathbb{Z}(2,1)$ and $\underline{\pi}_{-1}(F) \cong \underline{g}^2$. The slice tower for $\Sigma^8 H_Q \underline{\mathbb{Z}}$ is



where the bottom of the tower comes from Section 6.1.3.

7. Homology calculations

In Section 6, we described the slices of $\Sigma^n H_Q \mathbb{Z}$. In Section 8 below, we will give the corresponding slice spectral sequences. The E_2 -pages of those spectral sequences are given by the homotopy Mackey functors of the slices. We describe those homotopy Mackey functors here.

7.1. The *n*-slice. We start with the *n*-slices in the order listed in Proposition 6.7. The homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \mathbb{Z}$ were calculated in Proposition 4.10. We use the same methods to determine the homotopy Mackey functors of $\Sigma^{\rho_{\kappa}+j\rho_Q} H_Q \mathbb{Z}$.

Proposition 7.1. For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{\rho_{\kappa}+j\rho_{Q}}H_{Q}\mathbb{Z}$ are

$$\underline{\pi}_i(\Sigma^{\rho_K+j\rho_Q}H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \begin{array}{ll} \underline{\mathbb{Z}} & i = 8j+4 \\ i \in [4j+4,8j+3], \\ \underline{mgw} & i \equiv 2 \pmod{4} \\ \underline{B}(3,0) & i \in [4j+4,8j+3], \\ \underline{B}(3,0) & i \equiv 0 \pmod{4} \\ \phi_Z^*\underline{\pi}_i(\Sigma^{(j+1)\rho_K}H_K\underline{\mathbb{Z}}) & i \in [j+1,4j+3]. \end{cases} \end{cases}$$

See Proposition 4.2 or Figure 1 for the homotopy Mackey functors of $\Sigma^{(j+1)\rho_K} H_K \underline{\mathbb{Z}}$.

We may now use Proposition 7.1 and the exact sequence $\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$ to get the homotopy Mackey functors of $\Sigma^{\rho_K+j\rho_Q}H_Q\underline{\mathbb{Z}}(3,2)$.

Proposition 7.2. For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{\rho_{K}+j\rho_{Q}}H_{Q}\underline{\mathbb{Z}}(3,2)$ are

$$\underline{\pi}_{i}(\Sigma^{\rho_{K}+j\rho_{Q}}H_{Q}\underline{\mathbb{Z}}(3,2)) \cong \begin{cases} \underline{\mathbb{Z}} & i = 8j+4 \\ i \in [4j+4,8j+3], \\ i \equiv 2 \pmod{4} \\ \underline{B}(3,0) & i \equiv 0 \pmod{4} \\ \phi_{Z}^{*}\underline{\pi}_{i}(\Sigma^{(j+1)\rho_{K}}H_{K}\underline{\mathbb{Z}}) & i \in [j+2,4j+3]. \end{cases}$$

The key point here is that the homotopy Mackey functors of $\Sigma^{\rho_{\kappa}+j\rho_{Q}}H_{Q}\underline{\mathbb{Z}}(3,2)$ are the same as that of $\Sigma^{\rho_{\kappa}+j\rho_{Q}}H_{Q}\underline{\mathbb{Z}}$, except that the \underline{g} in degree j+1 has been removed.

In Proposition 4.12 we list the homotopy Mackey functors of $\Sigma^{-j\rho_Q} H_Q \underline{\mathbb{Z}}$. And derson duality then provides us with the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{\mathbb{Z}}^*$.

Proposition 7.3. For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{\mathbb{Z}}^*$ are

$$\underline{\pi}_{i}(\Sigma^{j\rho_{Q}}H_{Q}\underline{\mathbb{Z}}^{*}) \cong \begin{cases} \underline{\mathbb{Z}} & i = 8j \\ i \in [4j+1,8j-1], \\ \underline{\mathbb{B}}(3,0) & i \equiv 2 \mod 4 \\ \underline{\mathbb{B}}(3,0) & i \equiv 0 \mod 4 \\ \phi_{Z}^{*}\underline{\mathbb{B}}(2,0) & i = 4j \\ \phi_{Z}^{*}\underline{\pi}_{i-4}(\Sigma^{(j-1)\rho_{K}}H_{K}\underline{\mathbb{Z}}) & i \in [j+3,4j-1]. \end{cases}$$

Finally, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{\mathbb{Z}}(1,0)$ follow from the exact sequence $\underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{\mathbb{F}}_2$.

Proposition 7.4. For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \mathbb{Z}(1,0)$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{\mathbb{Z}}(1,0)) \cong \begin{cases} \underline{\mathbb{Z}} & i = 8j \\ i \in [4j+1,8j-2] \\ \underline{mgw} & i \equiv 2 \pmod{4} \\ \underline{B}(3,0) & i \in [4j+1,8j-2] \\ \underline{B}(3,0) & i \equiv 0 \pmod{4} \\ \phi_Z^* \underline{B}(2,0) & i = 4j \\ \phi_Z^* \underline{\pi}_i(\Sigma^{j\rho_K} H_K \underline{\mathbb{Z}}) & i \in [j,4j-1]. \end{cases}$$

7.2. The 8k-slices. We now move on to the 8k-slices.

Proposition 7.5. For j = 1, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{B}(2,0)$ are

$$\underline{\pi}_i(\Sigma^{\rho_Q} H_Q \phi_Z^* \underline{B}(2,0)) \cong \begin{cases} \underline{mg} & i=2\\ \underline{g} & i=1. \end{cases}$$

For $j \geq 2$, they are

$$\underline{\pi}_{i}(\Sigma^{j\rho_{Q}}H_{Q}\phi_{Z}^{*}\underline{B}(2,0)) \cong \begin{cases} \phi_{LDR}^{*}\underline{\mathbb{F}}_{2} & i=2j\\ \underline{g}^{3} & i\in[j+2,2j-1]\\ \underline{g}^{2} & i=j+1\\ \underline{g} & i=j. \end{cases}$$

Proof. Because $\phi_Z^* \underline{B}(2,0)$ is a pullback,

$$\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{B}(2,0) \simeq \Sigma^{j\rho_K} H_Q \phi_Z^* \underline{B}(2,0).$$

The exact sequence of K-Mackey functors $\underline{m}^* \longrightarrow \underline{B}(2,0) \longrightarrow \underline{g}$ provides us with $\Sigma^{j\rho_K} H_K \underline{m}^* \longrightarrow \Sigma^{j\rho_K} H_K \underline{B}(2,0) \longrightarrow \Sigma^{j\rho_K} H_K \underline{g}$. The conclusion follows from [GY, Propositions 4.8 and 7.4] and the resulting long exact sequence in homotopy. \Box

We may again use this strategy of reducing the calculations from Q to K for determining the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$.

Proposition 7.6. For j = 1 the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$ are

$$\underline{\pi}_i(\Sigma^{\rho_K} H_K \underline{B}(3,0)) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i=4\\ \underline{mg} & i=2\\ \underline{g} & i=1. \end{cases}$$

For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$ are

$$\underline{\pi}_{i}(\Sigma^{j\rho_{Q}}H_{Q}\underline{B}(3,0)) \cong \begin{cases} \phi_{Z}^{*}\underline{\mathbb{F}}_{2} & i = 4j \\ \underline{mg} & i = 4j - 1 \\ \overline{\phi}_{LDR}^{*}\underline{\mathbb{F}}_{2} \oplus g^{4j-2-i} & i \in [2j+2,4j-2] \\ \underline{g}^{2(k-2)+1} & i = 2j + 1 \\ \overline{\phi}_{LDR}^{*}\underline{\mathbb{F}}_{2} \oplus \underline{g}^{2(j-3)+1} & i = 2j \\ \underline{g}^{2(i-j-1)} & i \in [j+3,2j-1] \\ \underline{g}^{i-j+1} & i \in [j,j+2]. \end{cases}$$

Proof. Because the underlying spectrum of $H_Q\underline{B}(3,0)$ is contractible,

$$\Sigma^{\rho_Q} H_Q \underline{B}(3,0) \simeq \Sigma^{\rho_K} H_Q \underline{B}(3,0).$$

Now, we may consider $\underline{B}(3,0)$ as a pullback $\phi_Z^* B := \underline{B}(3,0)$, thus the calculation is reduced to one of K-Mackey functors. The sequence of K-Mackey functors $\underline{\mathbb{Z}}^* \xrightarrow{2} \underline{\mathbb{Z}} \longrightarrow \underline{B}$ provides us with

$$\Sigma^{j\rho\kappa}H_K\underline{\mathbb{Z}}^*\longrightarrow \Sigma^{j\rho\kappa}H_K\underline{\mathbb{Z}}\longrightarrow \Sigma^{j\rho\kappa}H_K\underline{B}.$$

Except for i = 4j - 2, the result follows from the associated long exact sequence in homotopy. In degree 4j - 2 we have an extension

$$\underline{mg} \longrightarrow \underline{\pi}_{4j-2}(\Sigma^{j\rho_K} H\underline{B}) \longrightarrow \underline{g}$$

We need to show this is not the split extension. This follows from the exact sequence $\underline{B}(2,0) \longrightarrow \underline{B} \longrightarrow \underline{\mathbb{F}}_2$ of K-Mackey functors. \Box

Proposition 7.7. For j = 1 and j = 2, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{mgw}$ are

$$\underline{\pi}_{i}(\Sigma^{\rho_{Q}}H_{Q}\underline{mgw}) \cong \begin{cases} \phi_{Z}^{*}\underline{\mathbb{F}}_{2} & i=4\\ \phi_{Z}^{*}\underline{B}(2,0) & i=2 \end{cases}$$

and

$$\underline{\pi}_i(\Sigma^{2\rho_Q} H_Q \underline{mgw}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 8\\ \underline{mg} & i = 7\\ \phi_{LDR} \underline{\mathbb{F}_2} & i = 6\\ \underline{g} & i = 5\\ \underline{mg} & i = 4\\ \underline{g} & i = 3. \end{cases}$$

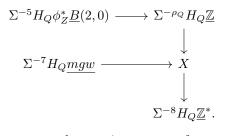
For $j \geq 3$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q mgw$ are

$$\underline{\pi}_{i}(\Sigma^{j\rho_{Q}}H_{Q}\underline{m}\underline{g}\underline{w}) \cong \begin{cases} \phi_{Z}^{*}\underline{\mathbb{F}_{2}} & i = 4j \\ \underline{m}\underline{g} & i = 4j - 1 \\ \phi_{LDR}\underline{\mathbb{F}_{2}} \oplus \underline{g}^{4j-i-2} & i \in [2j+2,4j-2] \\ \underline{g}^{2j-3} & i = 2j + 1 \\ \underline{g}^{2j-5} \oplus \phi_{LDR}\underline{\mathbb{F}_{2}} & i = 2j \\ \underline{g}^{2(i-j)-2} & i \in [j+2,2j-1] \\ \underline{g} & i = j+1 \end{cases}$$

Proof. We first deal with the case j = 1. The short exact sequence of Mackey functors

$$\underline{w}^* \hookrightarrow mgw \twoheadrightarrow mg^*$$

combines with Proposition 7.17 and Proposition 7.9 to show that the only nontrivial Mackey functors are $\phi_Z^* \underline{\mathbb{F}}_2$ in degree 4 and an extension of \underline{m} by \underline{g} in degree 2. It remains to see that this extension is $\phi_Z^* \underline{B}(2,0)$. According to Proposition 4.12, the Postnikov tower for $\Sigma^{-\rho_Q} H_Q \underline{\mathbb{Z}}$ is



Desuspending this diagram once by ρ_Q gives a tower for computing the homotopy Mackey functors of $\Sigma^{-2\rho_Q} H_Q \underline{\mathbb{Z}}$. The homotopy Mackey functors for $\Sigma^{-8-\rho_Q} H_Q \underline{\mathbb{Z}}^*$ and $\Sigma^{-5\rho_Q} H_Q \Psi^* \underline{B}(2,0)$ follow, using Anderson duality, from Proposition 4.10 and Proposition 7.5. Long exact sequences in homotopy then imply that

$$\underline{\pi}_{-9}(\Sigma^{-7-\rho_Q}H_Qmgw) \cong \phi_Z^*\underline{B}(2,0).$$

Dualizing gives that $\underline{\pi}_2(\Sigma^{\rho_Q}H_Q\underline{mgw})$ is $\phi_Z^*\underline{B}(2,0)$. We now have a fiber sequence

(7.8)
$$\Sigma^4 H_Q \phi_Z^* \underline{\mathbb{F}}_2 \longrightarrow \Sigma^{\rho_Q} H_Q \underline{mgw} \longrightarrow \Sigma^2 H_Q \phi_Z^* \underline{B}(2,0).$$

Suspending this sequence by ρ_Q immediately gives the homotopy Mackey functors of $\Sigma^{2\rho_Q} H_Q \underline{mgw}$. The same is true in the case j = 3, except that we have an extension

$$g \hookrightarrow \underline{\pi}_6 \Sigma^{3\rho_Q} H_Q m g w \twoheadrightarrow \phi_{LDR} \underline{\mathbb{F}}_2.$$

We claim that, more generally, any extension of \mathbb{Z} -modules

$$\underline{g}^m \hookrightarrow \underline{E} \twoheadrightarrow \phi_{LDR} \underline{\mathbb{F}}_2$$

is necessarily the split extension. To see this, first note that $\phi_{LDR}\underline{\mathbb{F}}_2$ is, by definition, the direct sum $\phi_L^*\underline{\mathbb{F}}_2 \oplus \phi_D^*\underline{\mathbb{F}}_2 \oplus \phi_R^*\underline{\mathbb{F}}_2$. It therefore suffices to show that the only $\underline{\mathbb{Z}}$ -module extension of $\phi_L^*\underline{\mathbb{F}}_2$ by \underline{g}^m is the split extension. Since any such extension will vanish at the subgroups D and R, the $\underline{\mathbb{Z}}$ -module structure forces the value at Q to be 2-torsion and therefore equal to $\underline{\mathbb{F}}_2^{m+1}$. Since there is a nontrivial

restriction to the subgroup L, the $\underline{\mathbb{Z}}$ -module structure forces the transfer from L to vanish. Thus the extension must be the split extension.

The suspension by $(j-1)\rho_Q$ of (7.8) gives the homotopy Mackey functors of $\Sigma^{j\rho_Q}H_Q\underline{mgw}$ in degrees 2j+1 and higher. Now we argue by induction that the Mackey functors for $\Sigma^{j\rho_Q}H_Q\underline{mgw}$ are as claimed, for $j \geq 3$. For instance, since the bottom Mackey functor is

$$\underline{\pi}_j(\Sigma^{(j-1)\rho_Q} H_Q \underline{mgw}) \cong \underline{g},$$

we see by decomposing $\Sigma^{(j-1)\rho_Q} H_Q mgw$ using the Postnikov tower that

$$\underline{\pi}_{j+1}(\Sigma^{j\rho_Q}H_Q\underline{mgw}) \cong \underline{g}.$$

The values of the Mackey functors $\underline{\pi}_i$, for $i \leq 2j - 2$, follow in a similar way. The values

$$\underline{\pi}_{2j-2}(\Sigma^{(j-1)\rho_Q}H_Q\underline{mgw}) \cong \underline{g}^{2j-7} \oplus \phi_{LDR}\underline{\mathbb{F}}_2,$$

and

$$\underline{\pi}_{2j-1}(\Sigma^{(j-1)\rho_Q}H_Q\underline{mgw}) \cong \underline{g}^{2j-5}$$

give that

$$\underline{\pi}_{2j-1}(\Sigma^{j\rho_Q}H_Q\underline{mgw}) \cong \underline{g}^{2j-4}$$

and that we have an extension of $\underline{\mathbb{Z}}\text{-modules}$

$$\underline{g}^{2j-5} \hookrightarrow \underline{\pi}_{2j}(\Sigma^{j\rho_Q} H_Q \underline{\mathbb{Z}}) \twoheadrightarrow \phi_{LDR} \underline{\mathbb{F}}_2.$$

By the argument given above, this must be the split extension.

The homotopy Mackey functors for the remaining 8k-slices follow from [S1, Propositions 9.5, 9.8].

Proposition 7.9 ([S1, Proposition 9.5], [GY, Proposition 4.8]). We have the equivalence $\Sigma^{\rho_Q} H_Q \underline{mg}^* \simeq \Sigma^2 H_Q \underline{m}$. For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q mg^*$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{mg}^*) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}_2} & i = 2j \\ \underline{g}^3 & i \in [j+2,2j-1] \\ \underline{g} & i = j+1. \end{cases}$$

Proposition 7.10 ([S1, Proposition 9.8]). We have equivalences

$$\Sigma^{j\rho_Q} H \phi_{LDR}^* \underline{\mathbb{F}}_2^* \simeq \begin{cases} \Sigma^2 H \phi_{LDR}^* \underline{f} & j = 1\\ \Sigma^4 H \phi_{LDR}^* \underline{\mathbb{F}}_2 & j = 2. \end{cases}$$

Then for $j \geq 3$, the nontrivial homotopy Mackey functors of $\Sigma^{j\rho_Q} H \phi^*_{LDR} \mathbb{F}_2^*$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi^*_{LDR} \underline{\mathbb{F}_2}^*) = \begin{cases} \phi^*_{LDR} \underline{\mathbb{F}_2} & i = 2j\\ \underline{g}^3 & i \in [j+2, 2j-1]. \end{cases}$$

7.3. The 8k + 4-slices. Similarly, the homotopy Mackey functors of the (8k + 4)-slices follow from [S1, Proposition 9.8] and [GY, Corollary 7.2, Propositions 7.3, 7.4].

Proposition 7.11 ([GY, Proposition 3.6]). For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi^*_{LDR} \underline{\mathbb{F}}_2$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi_{LDR}^* \underline{\mathbb{F}}_2) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}}_2 & i = 2j \\ \underline{g}^3 & i \in [j, 2j - 1]. \end{cases}$$

Proposition 7.12 ([GY, Corollary 7.2]). For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi_Z^* \mathbb{F}_2$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4j \\ \frac{mg}{\phi_{LDR}^* \underline{\mathbb{F}_2}} \oplus g^{4j-2-i} & i \in [2j, 4j-2] \\ \underline{g}^{2(i-j)+1} & i \in [j, 2j-1]. \end{cases}$$

Proposition 7.13 ([GY, Proposition 7.3]). For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{m}$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{m}) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}}_2 & i = 2j \\ \underline{g}^3 & i \in [j+1, 2j-1] \\ \underline{g} & i = j. \end{cases}$$

Proposition 7.14 ([GY, Proposition 7.4]). For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{mg}$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{mg}) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}}_2 & i = 2j\\ \underline{g}^3 & i \in [j+1, 2j-1].\\ \underline{g}^2 & i = j. \end{cases}$$

7.4. The 4k + 2-slices. The homotopy Mackey functors of the (4k + 2)-slice $\Sigma^{1+k\rho_Q}H_Q\phi_Z^*\mathbb{F}_2$ are given in Proposition 7.12. The homotopy Mackey functors of the remaining (4k + 2)-slices are as follows.

Proposition 7.15 ([GY, Proposition 4.8, Corollary 7.2]). We have the equivalence $\Sigma^{\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}}_2^* \simeq \Sigma^4 H_Q \phi_Z^* \underline{\mathbb{F}}_2$. For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}}_2^*$ are

$$\underline{\pi}_{i}(\Sigma^{j\rho_{Q}}H_{Q}\phi_{Z}^{*}\underline{\mathbb{F}_{2}}^{*}) \cong \begin{cases} \phi_{Z}^{*}\underline{\mathbb{F}_{2}} & i = 4j \\ \frac{mg}{\phi_{LDR}^{*}\underline{\mathbb{F}_{2}}} \oplus g^{4j-2-i} & i \in [2j+2,4j-2] \\ \underline{g}^{2(i-j)-5} & i \in [j+3,2j+1]. \end{cases}$$

Finally, we have the homotopy of $\Sigma^{j\rho_Q} H_Q \underline{w}$ and $\Sigma^{j\rho_Q} H_Q \underline{w}^*$.

Proposition 7.16. For $j \ge 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{w}$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & i = 4j \\ \underline{mg} & i = 4j - 1 \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus g^{4j-2-i} & i \in [2j, 4j-2] \\ g^{2(i-j)+1} & i \in [j+1, 2j-1]. \end{cases}$$

Proof. The underlying spectrum of $\Sigma^{j\rho_Q} H_Q \underline{w}$ is contractible; thus,

$$\Sigma^{j\rho_Q} H_Q \underline{w} \simeq \Sigma^{j\rho_K} H_Q \underline{w}.$$

Then, because \underline{w} is a pullback over Z, the calculation is essentially K-equivariant. Consider the short exact sequence of K-Mackey functors $\underline{w} \longrightarrow \underline{\mathbb{F}}_2 \longrightarrow \underline{g}$ and the corresponding cofiber sequence $\Sigma^{j\rho\kappa}H_K\underline{w} \longrightarrow \Sigma^{j\rho\kappa}H_K\underline{\mathbb{F}}_2 \longrightarrow \Sigma^{j\rho\kappa}H_K\underline{g}$. The statement follows immediately from the resulting long exact sequence in homotopy. **Proposition 7.17.** For j = 1, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{w}^*$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}^*) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}}_2 & i = 4\\ \underline{g} & i = 2. \end{cases}$$

For $j \geq 2$, they are

$$\underline{\pi}_{i}(\Sigma^{j\rho_{Q}}H_{Q}\underline{w}^{*}) \cong \begin{cases} \phi_{Z}^{*}\underline{\mathbb{F}_{2}} & i = 4j \\ \underline{mg} & i = 4j - 1 \\ \phi_{LDR}^{*}\underline{\mathbb{F}_{2}} \oplus g^{4j-2-i} & i \in [2j+2,4j-2] \\ \underline{g}^{2(i-j)-5} & i \in [j+3,2j+1] \\ \underline{g} & i = j+1. \end{cases}$$

Proof. The proof is the same as that in Proposition 7.16, except that we start with the exact sequence of K-Mackey functors $\underline{g} \longrightarrow \underline{\mathbb{F}_2}^* \longrightarrow \underline{w}^*$.

8. SLICE SPECTRAL SEQUENCES

Here we include the slice spectral sequences for $\Sigma^n H_Q \mathbb{Z}$ for several values of n between 5 and 15. In some cases, we use the restriction to the C_4 -subgroups to determine some of the slice differentials.

The grading is the same as that in [HHR1, Section 4.4.2]. The Mackey functor $\underline{E}_2^{t-n,t}$ is $\underline{\pi}_n P_t^t(X)$. We also follow the Adams convention, where $\underline{\pi}_n P_t^t(X)$ has coordinates (n, t - n) and the differential

$$d_r: \underline{E}_r^{s,t} \longrightarrow \underline{E}_r^{s+r,t+r-1}$$

points left one and up r.

The Q-Mackey functors that appear in these spectral sequences are listed in Table 4. We also display some companion C_4 -slice spectral sequences, and the C_4 -Mackey functors that appear are listed in Table 5.

TABLE 4.	Symbols	for	Q-Mackey	functors

$\Box = \underline{\mathbb{Z}}$	$ \mathbf{\diamondsuit} = \phi_Z^* \underline{\mathbb{F}_2} $	$\bullet = \phi_{LDR}^* \underline{\mathbb{F}_2}$
$\mathbf{\Phi} = \underline{mgw}$	$\circ = \underline{B}(3,0)$	${}^{\bigotimes}=\phi_Z^*\underline{B}(2,0)$
$\blacksquare = \underline{mg}$	$\textcircled{n} = \underline{g}^n$	

TABLE 5. Symbols for C_4 -Mackey functors

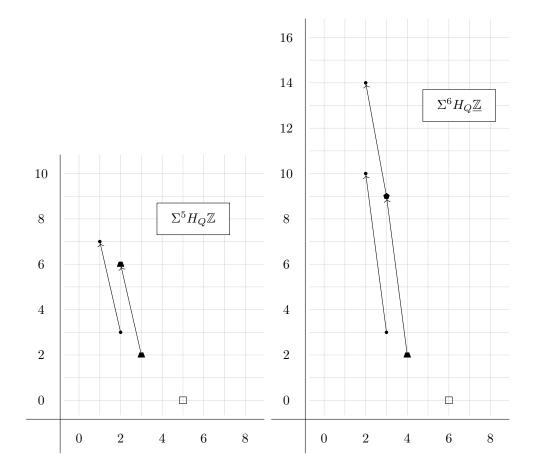
Example 8.1. In the spectral sequences for $\Sigma^5 H_Q \underline{\mathbb{Z}}$, $\Sigma^6 H_Q \underline{\mathbb{Z}}$, and $\Sigma^7 H_Q \underline{\mathbb{Z}}$, because we must be left with

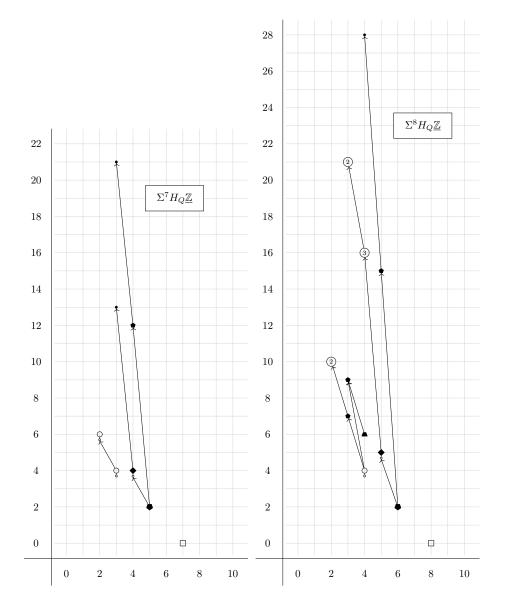
$$\underline{\pi}_n(P_n^n\Sigma^nH_Q\underline{\mathbb{Z}})\cong\underline{\mathbb{Z}},$$

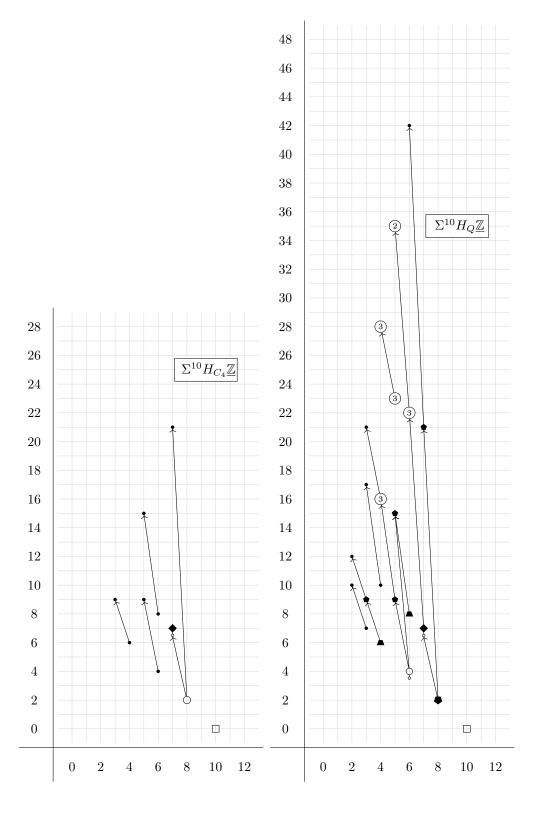
all differentials are forced.

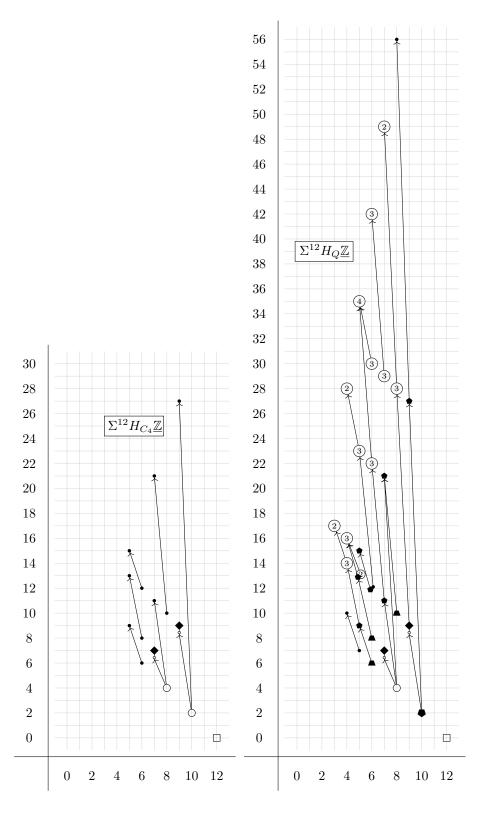
Example 8.2. For $\Sigma^8 H_Q \underline{\mathbb{Z}}$, the pattern of differentials emanating from the Mackey functor $\underline{\pi}_6(P_8^8 \Sigma^8 H_Q \underline{\mathbb{Z}})$ is forced; no other pattern of differentials wipes out all classes in this region. The shorter differentials clearing out the smaller region are then similarly forced.

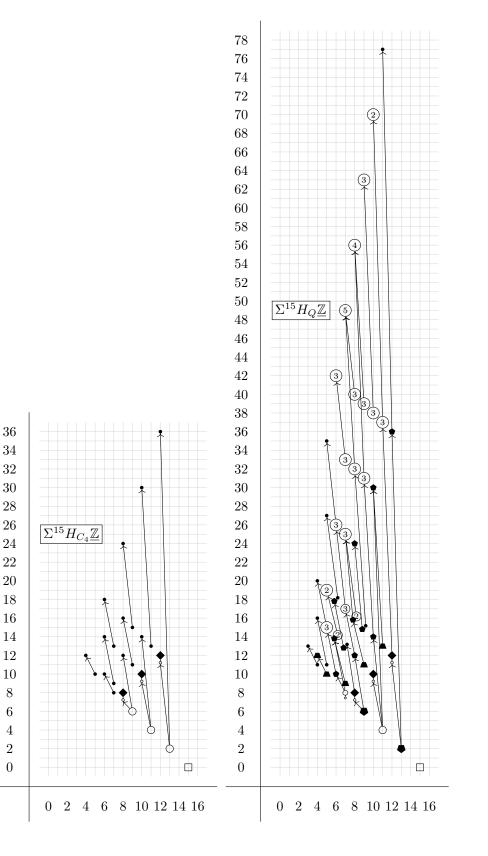
Example 8.3. In the cases of $\Sigma^n H_Q \mathbb{Z}$ for n = 10, 12, and 15, we also display the corresponding slice spectral sequence for $\Sigma^n H_{C_4} \mathbb{Z}$, where we use C_4 to indiscriminately refer to any of the subgroups $L, D, R \leq Q$. The slice differentials in the C_4 -case force many of the slice differentials for the Q-equivariant spectra.











 $\mathbf{6}$

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