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The Klein four slices of $\Sigma^n H\underline{\mathbb{F}}_2$

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Abstract
We describe the slices of positive integral suspensions of the equivariant Eilenberg–MacLane spectrum $H\underline{\mathbb{F}}_2$ for the constant Mackey functor over the Klein four-group $C_2 \times C_2$.

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Contents

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1 Introduction

The slice filtration is a filtration of equivariant spectra developed by Hill, Hopkins, and Ravenel, as a generalization of Dugger's filtration [1], in their solution to the Kervaire invariant-one problem [4]. It is an equivariant analogue of the Postnikov tower and was modeled on the motivic filtration of Voevodsky [10].

Since its inception, there have been a few reformulations and new understandings of the structure of the slice filtration. Some properties and useful results in this setting are summarized in Sect. 2.5. In this paper, we use the *regular* slice filtration (cf. [6,9]) on equivariant spectra and note that this filtration differs from the original filtration from [4] by a shift by one.

Let G be a finite group and let Sp^G be the category of genuine G-spectra.

Definition 1.1 Let $\tau^G_{\geq n} \subseteq \mathrm{Sp}^G$ be the localizing subcategory generated by G-spectra of the form $\Sigma^\infty_G G/H_+ \wedge S^{k\rho_H}$, where $H \subset G$, ρ_H is the regular representation of H and $k \cdot |H| \geq n$. We write $X \geq n$ to mean that $X \in \tau^G_{\geq n}$.

We use $P^{n-1}(-)$ to denote the localization functor associated to $\tau^G_{\geq n}$. There are natural transformations $P^n(-) \longrightarrow P^{n-1}(-)$ that give the slice tower of X

$$\cdots \longrightarrow P^{n+1}X \longrightarrow P^n X \longrightarrow P^{n-1}X \longrightarrow \ldots,$$

and the fiber at each level

$$P^n_n X \longrightarrow P^n X \longrightarrow P^{n-1} X$$

is known as the *n-slice* of X.

While in the nonequivariant setting the relationship between the Postnikov tower of a spectrum and the spectrum's homotopy groups is clear, there is a much more complicated story for homotopy groups and the slice tower when working equivariantly. Furthermore, such homotopy groups enjoy a richer structure. For a G-spectrum X, the homotopy groups $\pi_n(X^H)$, as H varies over the subgroups of G, define a G-Mackey functor. An underline will denote a Mackey functor, and we will display such functors $\underline{M}$ according to their Lewis diagrams. The general form of such diagrams for $G = C_2$ and $G = C_2 \times C_2$ are displayed below, where we write $\underline{M}(L)$ for what would be typically written as $\underline{M}((C_2 \times C_2)/L)$.

$$\begin{tikzcd} \underline{M}(C_2) \arrow[d, bend right] \\ \underline{M}(e) \arrow[u, bend right] \end{tikzcd} \qquad \begin{tikzcd} & \underline{M}(C_2 \times C_2) \arrow[dl, bend right] \arrow[d, bend right] \arrow[dr, bend left] & \\ \underline{M}(L) \arrow[ur, bend right] \arrow[dr, bend right] & \underline{M}(D) \arrow[u, bend right] \arrow[d, bend right] & \underline{M}(R) \arrow[ul, bend left] \arrow[dl, bend left] \\ & \underline{M}(e) \arrow[ul, bend right] \arrow[u, bend right] \arrow[ur, bend right] & \end{tikzcd}$$

Here L, D, and R are the left, diagonal, and right cyclic subgroups of $C_2 \times C_2$ of order two. We have not drawn in the Weyl group actions on the intermediate groups or the G-action on $\underline{M}(e)$. The maps pointing down are called restriction, and the maps pointing up are called transfers.

Associated to every G-Mackey functor $\underline{M}$, there is an Eilenberg–MacLane G-spectrum $H_G\underline{M}$, which we will usually write simply as $H\underline{M}$. While $H\underline{M}$ is always a 0-slice, and thus has a trivial slice tower, suspensions of Eilenberg–MacLane spectra produce interesting slices and corresponding towers. For instance, when $G = C_{p^n}$ the slices of $\Sigma^n H\underline{\mathbb{Z}}$ and $\Sigma^{n\lambda} H\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the constant Mackey functor at $\mathbb{Z}$ and λ is an irreducible C_{p^n}-representation, were presented in [12] and [5], respectively.

We will primarily work with the constant functor $\underline{\mathbb{F}_2}$ for the Klein four-group $C_2 \times C_2$ (which we will often denote by $\mathcal{K}$). This Mackey functor takes on the value $\mathbb{F}_2$ at each subgroup. The restriction maps are all the identity, and the transfers are all zero. In this paper we present the slices of $\Sigma^n H\underline{\mathbb{F}_2}$ for the Klein four-group $\mathcal{K} = C_2 \times C_2$ and $n \geq 0$. A summary of our main results is as follows:

Main result: *For* $\mathcal{K} = C_2 \times C_2$ *and* $n \geq 0$, *all nontrivial (regular) slices of* $\Sigma^n H_{\mathcal{K}}\underline{\mathbb{F}_2}$ *are given by:*

$$P_i^i(\Sigma^n H_{\mathcal{K}}\underline{\mathbb{F}_2}) = \Sigma^V H_{\mathcal{K}}\underline{M},$$

where $\dim V = i$ *and* i *is either equal to* n *or congruent to either* $0 \pmod 4$ *in the case* i *is in the range* $[n, 4n-12]$ *or to* $2 \pmod 4$ *if* i *is in the range* $[n, 2n-4]$. *The precise representations* V *and Mackey functors* $\underline{M}$ *are completely described in Proposition 5.5, Theorem 5.6, and Proposition 5.12.*

The paper is organized as follows. We begin with some background material in Sect. 2. In Sect. 3, we review results from [4] for the case of $H_{C_2}\underline{\mathbb{F}_2}$. We present the relevant $\mathcal{K}$-Mackey functors in Sect. 4. Our main results, which describe all of the slices of $\Sigma^n H_{\mathcal{K}}\underline{\mathbb{F}_2}$ are given in Sect. 5. In Sect. 6, we present the first few slice towers (up to $n = 8$). The homotopy Mackey functors of the slices are computed in Sect. 7. Finally, in Sect. 8, we display a few examples of the slice spectral sequence for $\Sigma^n H_{\mathcal{K}}\underline{\mathbb{F}_2}$. For convenience, we also list the important $\mathcal{K}$-Mackey functors in the "Appendix: Mackey functors".

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2 Background

2.1 $(C_2 \times C_2)$-Representations

Recall that the real representation ring for the group $\mathcal{K} = C_2 \times C_2$ is

$$RO(\mathcal{K}) \cong \mathbb{Z}\{1, \alpha_{1,0}, \alpha_{1,1}, \alpha_{0,1}\},$$

where 1 is the trivial one-dimensional representation and the other representations are defined by

$$\begin{aligned} \mathbb{Z}/2 \times \mathbb{Z}/2 \xrightarrow{\alpha_{i,j}} \mathbb{Z}/2 &\overset{\sigma}{\hookrightarrow} \mathrm{Gl}_1(\mathbb{R}) \\ (k, n) &\mapsto ik + jn. \end{aligned}$$

Thus $\alpha_{1,0}$ is the projection onto the left factor. To avoid cluttering notation, we prefer to write $\alpha = \alpha_{1,0}$, $\beta = \alpha_{0,1}$, $\gamma = \alpha_{1,1}$. We denote by ρ or $\rho_{\mathcal{K}}$ the regular representation, and we have

$$\rho = 1 + \alpha + \beta + \gamma$$

in $RO(\mathcal{K})$. The left, diagonal, and right cyclic subgroups of $\mathcal{K}$ will be denoted by L, D, and R, respectively. We have

$$L = \ker \beta, \qquad D = \ker \gamma, \qquad R = \ker \alpha.$$

It will often be important to consider restriction to the cyclic subgroups. Given that $RO(C_2) \cong \mathbb{Z}\{1, \sigma\}$, the restrictions of representations are given by

$$RO(\mathcal{K}) \xrightarrow{\iota^*} RO(C_2),$$
$$\iota_L^* = \left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{smallmatrix}\right), \qquad \iota_D^* = \left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right), \qquad \iota_R^* = \left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right)$$

In particular, we have $\iota^*(\rho_{\mathcal{K}}) = 2\rho_{C_2}$ in $RO(C_2)$.

Since the subgroups $H = L, D, R \trianglelefteq \mathcal{K}$ are all normal, we get an induced action of $C_2 \cong \mathcal{K}/H$ on the H-fixed points of any $\mathcal{K}$-representation. These fixed point homomorphisms are given by

$$RO(\mathcal{K}) \xrightarrow{(-)^H} RO(C_2),$$
$$(-)^L = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right), \qquad (-)^D = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right), \qquad (-)^R = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right)$$

In particular, for any of these index two subgroups $H \leq \mathcal{K}$, we have $(\rho_{\mathcal{K}})^H = \rho_{C_2}$.

2.2 Mackey functors

For G-spectra W and X, the collection of abelian groups $[G/H_+ \wedge W, X]^G$, as H varies, defines a G-Mackey functor. In the case $W = S^V$ for a (virtual) G-representation V, this is the Mackey functor $\underline{\pi}_V(X)$. We give examples of G-Mackey functors for $G = C_2$ in Sect. 3.1 and for $G = C_2 \times C_2$ in Sect. 4.

Notation 2.1 We will typically denote Mackey functor restriction maps by

$$r\downarrow_H^K \colon \underline{M}(K) \longrightarrow \underline{M}(H)$$

and transfers by

$$\tau\uparrow_H^K \colon \underline{M}(H) \longrightarrow \underline{M}(K).$$

Mackey functors are required to satisfy the so-called double coset formula. Since our group $G = C_2 \times C_2$ is abelian, this means that for any two (distinct) H_1 and H_2 of the nontrivial cyclic subgroups, the restriction maps commute with the transfer maps, in the sense that

$$r\downarrow_{H_1}^G \circ \tau\uparrow_{H_2}^G = \tau\uparrow_e^{H_1} \circ r\downarrow_e^{H_2} \tag{2.2}$$

Definition 2.3 Given a surjection $\phi_N : G \longrightarrow G/N$ of groups with kernel $N \trianglelefteq G$, there is a pullback for Mackey functors

$$\phi_N^* : \mathrm{Mack}(G/N) \longrightarrow \mathrm{Mack}(G)$$

defined by

$$\phi_N^*(\underline{M})(H) = \begin{cases} \underline{M}(H/N) & N \leq H \\ 0 & N \not\leq H. \end{cases}$$

In the Mackey functor literature, this pullback is known as inflation along the quotient $G \longrightarrow G/N$.

Example 2.4 Let

$$\begin{tikzcd} \underline{M}(C_2) \arrow[d, bend right] \\ \underline{M}(e) \arrow[u, bend right] \end{tikzcd} ,$$

be a C_2-Mackey functor, where we assume trivial Weyl group action for simplicity. Then, under the quotient map $\mathcal{K} \longrightarrow \mathcal{K}/R \cong C_2$, this pulls back to the $\mathcal{K}$-Mackey functor

$$\begin{tikzcd} & \underline{M}(R) \arrow[dr, bend right] & \\ 0 & 0 & \underline{M}(e) \arrow[ul, bend right] \\ & 0. & \end{tikzcd}$$

Notation 2.5 We will often encounter Mackey functors which are direct sums of inflations along the projections to different subgroups, and it will be convenient to use the notation

$$\phi^*_{LD}\underline{M} := \phi^*_L\underline{M} \oplus \phi^*_D\underline{M}, \qquad \phi^*_{LDR}\underline{M} := \phi^*_L\underline{M} \oplus \phi^*_D\underline{M} \oplus \phi^*_R\underline{M}.$$

There is a related construction in the world of spectra. Given a surjection $\phi : G \longrightarrow G/N$, there is a geometric pullback functor $\phi^*_N : \mathrm{Sp}^{G/N} \longrightarrow \mathrm{Sp}^G$ ([8, Theorem II.9.5], [3, Proposition 4.3]). For our purposes, the important property is its behavior on suspensions of Eilenberg–MacLane spectra. This is given by

$$\phi^*_N(S^{V^N} \wedge H_{G/N}\underline{M}) \simeq S^V \wedge H_G\phi^*_N\underline{M} \tag{2.6}$$

for $V \in RO(G)$ and $\underline{M} \in \mathrm{Mack}(G/N)$ ([3, Proposition 4.2, Corollary 4.6]).

2.3 Relationship between twisted (de)suspensions, transfers, and restrictions

Consider the $\mathcal{K}$-cofiber sequence $\mathcal{K}/R_+ \longrightarrow S^0 \longrightarrow S^\alpha$. For any $\mathcal{K}$-spectrum X, this induces a cofiber sequence

$$(\mathcal{K}/R_+ \wedge X)^{\mathcal{K}} \simeq X^R \xrightarrow{\tau\uparrow_R^{\mathcal{K}}} X^{\mathcal{K}} \longrightarrow (\Sigma^\alpha X)^{\mathcal{K}}, \tag{2.7}$$

where the map τ is the transfer. We similarly get that $(\Sigma^{-\alpha}X)^{\mathcal{K}}$ is the fiber of the restriction:

$$(\Sigma^{-\alpha}X)^{\mathcal{K}} \longrightarrow X^{\mathcal{K}} \xrightarrow{r\downarrow_R^{\mathcal{K}}} X^R. \tag{2.8}$$

We have similar fiber sequences relating the L transfer and restriction to (de)suspension by β, and the D transfer and restriction to (de)suspension by γ.

2.4 Anderson duality

In this section, G can be any finite group. By Brown Representability in the category of $H\underline{\mathbb{F}_2}$-modules, the functor

$$X \mapsto \mathrm{Hom}_{\mathbb{F}_2}(\pi_*^G X, \mathbb{F}_2)$$

on the category of $H\underline{\mathbb{F}_2}$-modules is represented by some $H\underline{\mathbb{F}_2}$-module, which we write $\mathbb{F}_2^{H\underline{\mathbb{F}_2}}$. As in [2, Lemma 3.1], plugging in the $H\underline{\mathbb{F}_2}$-modules $G/H_+ \wedge H\underline{\mathbb{F}_2}$ shows that in fact $\mathbb{F}_2^{H\underline{\mathbb{F}_2}} \simeq H\underline{\mathbb{F}_2}^*$. The following more general result, whose proof was explained to us by John Greenlees, will be quite useful.

Proposition 2.9 *Let $\underline{M}$ be an $\underline{\mathbb{F}_2}$-module. Then*

$$\underline{\pi}_V(H\underline{M}^*) \cong \left(\underline{\pi}_{-V} H\underline{M}\right)^*.$$

Proof By Brown Representability in the category of $H\underline{\mathbb{F}_2}$-modules, the functor

$$X \mapsto \mathrm{Hom}_{\mathbb{F}_2}(\pi_*^G(X \wedge_{H\underline{\mathbb{F}_2}} H\underline{M}), \mathbb{F}_2)$$

on the category of $H\underline{\mathbb{F}_2}$-modules is represented by some $H\underline{\mathbb{F}_2}$-module, which we write $\mathbb{F}_2^{H\underline{M}}$. By plugging in $X = G/H_+ \wedge H\underline{\mathbb{F}_2}$, we see that $\mathbb{F}_2^{H\underline{M}} \simeq H\underline{M}^*$. In other words,

$$[X, H\underline{M}^*]^{H\underline{\mathbb{F}_2}-\mathrm{mod}} \cong \mathrm{Hom}_{\mathbb{F}_2}(\pi_*^G(X \wedge_{H\underline{\mathbb{F}_2}} H\underline{M}), \mathbb{F}_2).$$

Plugging in $X = S^V \wedge H\underline{\mathbb{F}_2}$ gives the result.

2.5 The slice filtration

We have already defined $X \geq n$ for a G-spectrum X, and we have a notion of "less than" as well.

Definition 2.10 We say that $X < n$ if

$$[S^{k\rho_H + r}, X]^H = 0$$

for all $r \geq 0$ and all subgroups $H \leq G$ such that $k|H| \geq n$.

In other words, $X < n$ if and only if the restriction $X \downarrow_H^G$ is less than n for all proper subgroups $H < G$ and

$$[S^{k\rho_G + r}, X]^G = 0$$

for all $r \geq 0$ and $k \geq \frac{n}{|G|}$. More generally, restriction to subgroups is compatible with the slice dimension, in the following sense.

Proposition 2.11 ([3, Cor. 2.6]) *Suppose that $X \in \mathrm{Sp}^G$ satisfies $k \leq X \leq n$ and $H \leq G$. Then $k \leq X \downarrow_H^G \leq n$ as an H-spectrum.*

The following characterization of the subcategory $\tau_{\geq n}^G$ in terms of connectivity of fixed points is useful.

Theorem 2.12 ([6, Corollary 2.9, Theorem 2.10]) *Let $n \geq 0$. Then $X \geq n$ if and only if*

$$\pi_k(X^H) = 0 \quad \text{for } k < \frac{n}{|H|}.$$

An immediate corollary is

Corollary 2.13 *If $n \geq 0$ and X is n-connective, in the sense that $\pi_k(X^H) = 0$ for all subgroups and all $k < n$, then $X \geq n$.*

For a few values of n, the category of n-slices is well-understood.

Proposition 2.14 (1) [4, Proposition 4.50] *X is a 0-slice if and only if $X \simeq H\underline{M}$ for $\underline{M}$ an arbitrary Mackey functor.*
(2) [4, Proposition 4.50] *X is a 1-slice if and only if $X \simeq \Sigma^1 H\underline{M}$ for $\underline{M}$ a Mackey functor with injective restrictions.*
(3) [9, Theorem 6-4] *X is a (-1)-slice if and only if $X \simeq \Sigma^{-1} H\underline{M}$ for $\underline{M}$ a Mackey functor with surjective transfers.*

Though these characterizations are not enough to determine all slices in every case as the slice tower does not commute with taking ordinary suspensions, it does commute with suspensions by the regular representation of G.

Proposition 2.15 ([4, Corollary 4.25]) *For any $k \in Z$,*

$$P^{k+|G|}_{k+|G|}(\Sigma^\rho X) \simeq \Sigma^\rho P^k_k(X).$$

Additionally, we understand the relationship between the slice filtration and taking pull-backs.

Proposition 2.16 ([9, Corollary 4–5]) *Let $N \trianglelefteq G$ be normal of index k and let X be a G/N-spectrum. Then*

$$\phi^*_N(P^n_n X) \simeq P^{kn}_{kn}(\phi^*_N X).$$

In particular, the pullback of an n-slice is a kn-slice.

Proposition 2.17 *Let $d \in \mathbb{Z}$ and let*

$$X \xrightarrow{f} Y \longrightarrow Z$$

be a fiber sequence of G-spectra such that $P^d_d(Z) \simeq * \simeq P^d_d(\Sigma^{-1}Z)$. Then f induces an equivalence on n-slices.

Proof This follows from [11, Proposition 2.32].

2.6 Review of Holler–Kriz

In [7], the authors compute the homotopy of $(\Sigma^V H\underline{\mathbb{F}_2})^G$ for any elementary abelian group G. Their answer is given as the Poincaré series of the graded $\mathbb{F}_2$-vector space.

Theorem 2.18 ([7, Section 6]) *Let $\ell, n \geq 0$ and $i, j \geq 1$. The Poincaré series for $\pi_*((\Sigma^V H\underline{\mathbb{F}_2})^{\mathcal{K}})$ is*

(1) $V = 0$: $\quad 1$
(2) $V = n\alpha$: $\quad 1 + x + \cdots + x^n$
(3) $V = -j\alpha$: $\quad x^{-j} + \cdots + x^{-3} + x^{-2}$
(4) $V = n\alpha + \ell\beta$: $\quad (1 + \cdots + x^n) \cdot (1 + \cdots + x^\ell)$
(5) $V = n\alpha - j\beta$: $\quad (1 + \cdots + x^n) \cdot (x^{-j} + \cdots + x^{-2})$
(6) $V = -i\alpha - j\beta$: $\quad (x^{-i} + \cdots + x^{-2}) \cdot (x^{-j} + \cdots + x^{-2})$

If either i or j is equal to 1, then the above series should be interpreted as zero. The answer is more complicated when all three nontrivial irreducible representations are involved, so we state those cases separately.

Theorem 2.19 ([7, Section 6]) *Let $\ell, m, n \geq 1$. The Poincaré series for $\pi_*((\Sigma^{\ell\alpha+m\beta+n\gamma} H\underline{\mathbb{F}_2})^{\mathcal{K}})$ is*

$$(1 + \cdots + x^\ell)(1 + \cdots + x^m) + x(1 + \cdots + x^{\ell+m})(1 + \cdots + x^{n-1})$$

Expanded out, this polynomial can be described as follows, assuming $\ell \leq m \leq n$: The constant coefficient is 1. Then the coefficients increase by 2 until x^ℓ. Thereafter, they increase by 1 until x^m. They then stay constant until x^n, and finally decrease (by 1) to 1, which is the coefficient of $x^{\ell+m+n}$.

Theorem 2.20 ([7, Section 6]) *Let $\ell, m \geq 1$. If $k \geq 2$, then the Poincaré series for $\pi_*((\Sigma^{\ell\alpha+m\beta-k\gamma} H\underline{\mathbb{F}_2})^{\mathcal{K}})$ is*

$$\left(\frac{1}{x^k} + \cdots + \frac{1}{x}\right)(1 + x + \cdots + x^{k-2}) + x^k(1 + \cdots + x^{\ell-k})(1 + \cdots + x^{m-k})$$

In the case $k = 1$, the series is

$$x(1 + \cdots + x^{\ell-1})(1 + \cdots + x^{m-1}).$$

Theorem 2.21 ([7, Section 6]) *Let $j, k, \ell \geq 1$. Then the Poincaré series for $\pi_*((\Sigma^{\ell\alpha-j\beta-k\gamma} H\underline{\mathbb{F}_2})^{\mathcal{K}})$ is*

(1)
$$\frac{1}{x^{j+k-\ell}}(1 + \cdots + x^{j-\ell-2})(1 + \cdots + x^{k-\ell-2}) + \frac{1}{x^{\ell+1}}(1 + \cdots + x^\ell)(1 + \cdots + x^{\ell-1})$$
if $j, k \geq \ell + 1$ or

(2)
$$\frac{1}{x^j}(1 + \cdots + x^{j-2})(1 + \cdots + x^{\ell-k}) + \frac{1}{x^k}(1 + \cdots + x^{\ell-1})(1 + \cdots + x^{k-1})$$
if $\ell \geq k$.

Swapping the role of j and k gives the case $\ell \geq j$ in Theorem 2.21.

Theorem 2.22 ([7, Section 6]) *Let $i, j, k \geq 1$. Then the Poincaré series for $\pi_*((\Sigma^{-i\alpha-j\beta-k\gamma} H\underline{\mathbb{F}_2})^{\mathcal{K}})$ is*

$$\frac{1}{x^{i+j+k}}[(1 + x + \cdots + x^{j+k-2})(1 + \cdots + x^{i-2}) + x^{i-1}(1 + \cdots + x^{k-1})(1 + \cdots + x^{j-1})]$$

Corollary 2.23 *Let $k \geq 1$. The Poincaré series for $\pi_*((\Sigma^{k-k\rho} H\underline{\mathbb{F}_2})^{\mathcal{K}})$ is*

$$\frac{1}{x^{3k}}[(1+x+\cdots+x^{2k-2})(1+\cdots+x^{k-2})+x^{k-1}(1+\cdots+x^{k-1})^2]$$

3 Review of $G = C_2$

A Mackey functor for the group C_2 may be depicted by the Lewis diagram

$$\begin{tikzcd} M(G) \arrow[d, bend right] \\ M(e), \arrow[u, bend right] \end{tikzcd}$$

where we have omitted the C_2-action on $M(e)$.

3.1 The main players

Example 3.1 The constant Mackey functor is

$$\underline{\mathbb{F}_2} = \begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "1"'] \\ \mathbb{F}_2. \arrow[u, bend right, "0"'] \end{tikzcd}$$

Example 3.2 The geometric Mackey functor is

$$\underline{g} = \phi^*_{C_2}(\mathbb{F}_2) = \begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right] \\ 0. \arrow[u, bend right] \end{tikzcd}$$

Since $\underline{g}(e) = 0$, it follows that $(H\underline{g})^e \simeq *$. Smashing the cofiber sequence

$$(C_2)_+ \longrightarrow S^0 \longrightarrow S^\sigma$$

with $H\underline{g}$ implies that

$$\Sigma^{k\sigma} H\underline{g} \simeq H\underline{g} \qquad \text{and} \qquad \Sigma^{k\rho} H\underline{g} \simeq \Sigma^k H\underline{g}.$$

Thus, using either Proposition 2.16 or Proposition 2.15, it follows that $\Sigma^k H\underline{g}$ is a $2k$-slice.

Example 3.3 The free Mackey functor is

$$\underline{f} = \begin{tikzcd} 0 \arrow[d, bend right] \\ \mathbb{F}_2. \arrow[u, bend right] \end{tikzcd}$$

This is relevant because

$$\Sigma^{1-\sigma} H\underline{\mathbb{F}_2} \simeq H\underline{f}. \tag{3.4}$$

Note that these Mackey functors sit in an exact sequence

$$\underline{f} \hookrightarrow \underline{\mathbb{F}_2} \twoheadrightarrow \underline{g}.$$

The resulting cofiber sequence

$$H\underline{f} \longrightarrow H\underline{\mathbb{F}_2} \longrightarrow H\underline{g} \tag{3.5}$$

can be used to compute the homotopy of $\Sigma^{k\rho} H\underline{\mathbb{F}_2}$.

Proposition 3.6 *For $k \geq 0$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H_{C_2}\underline{\mathbb{F}_2}$ are*

$$\underline{\pi}_i\left(\Sigma^{k\rho} H_{C_2}\underline{\mathbb{F}_2}\right) \cong \begin{cases} \underline{\mathbb{F}_2} & i = 2k \\ \underline{g} & i \in [k, 2k-1] \end{cases}$$

Proof This follows by induction from repeated use of the cofiber sequence

$$\Sigma^{(j-1)\rho+2} H\underline{\mathbb{F}_2} \simeq \Sigma^{j\rho} H\underline{f} \longrightarrow \Sigma^{j\rho} H\underline{\mathbb{F}_2} \longrightarrow \Sigma^{j\rho} H\underline{g} \simeq \Sigma^{j} H\underline{g},$$

where $j \geq 1$. □

Example 3.7 The opposite to the constant Mackey functor is

$$\underline{\mathbb{F}_2}^* = \begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "0"'] \\ \mathbb{F}_2 \arrow[u, bend right, "1"'] \end{tikzcd} .$$

We again have a twisting

$$\Sigma^{1-\sigma} H\underline{f} \simeq H\underline{\mathbb{F}_2}^*. \tag{3.8}$$

We also have the exact sequence of Mackey functors

$$\underline{g} \hookrightarrow \underline{\mathbb{F}_2}^* \twoheadrightarrow \underline{f}.$$

The resulting cofiber sequence

$$H\underline{g} \longrightarrow H\underline{\mathbb{F}_2}^* \longrightarrow H\underline{f} \tag{3.9}$$

can be used to compute the homotopy of $\Sigma^{-k\rho} H\underline{\mathbb{F}_2}^*$ (which also follows from Proposition 3.6 by Proposition 2.9).

Proposition 3.10 *For $k \geq 0$, the nontrivial homotopy Mackey functors of $\Sigma^{-k\rho} H_{C_2}\underline{\mathbb{F}_2}^*$ are*

$$\underline{\pi}_{-i}\left(\Sigma^{-k\rho} H_{C_2}\underline{\mathbb{F}_2}^*\right) \cong \begin{cases} \underline{\mathbb{F}_2}^* & i = 2k \\ \underline{g} & i \in [k, 2k-1] \end{cases}$$

Together, Propositions 3.6 and 3.10 combine to give the $RO(C_2)$-graded homotopy Mackey functors of $H\underline{\mathbb{F}_2}$, which we display in Fig. 1.

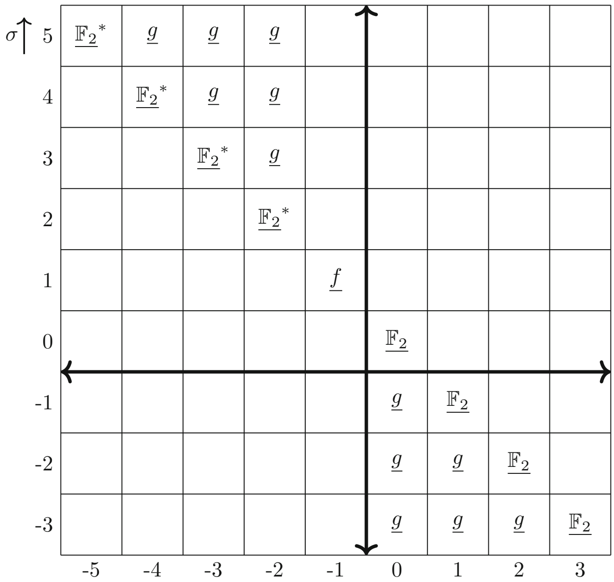

Fig. 1 $\underline{\pi}_{n+k\sigma} H_{C_2}\underline{\mathbb{F}_2}$

3.2 The slice tower for $\Sigma^n H_{C_2}\underline{\mathbb{F}_2}$

Example 3.12 The spectrum $H\underline{\mathbb{F}_2}$ is a 0-slice by Proposition 2.14.

Example 3.13 The spectrum $\Sigma^1 H\underline{\mathbb{F}_2}$ is a 1-slice by Proposition 2.14.

Example 3.14 The spectrum $\Sigma^2 H\underline{\mathbb{F}_2}$ is a 2-slice, since

$$\Sigma^2 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \left(\Sigma^{1-\sigma} H\underline{\mathbb{F}_2}\right) \simeq \Sigma^\rho H\underline{f}.$$

Example 3.15 The spectrum $\Sigma^3 H\underline{\mathbb{F}_2}$ is a 3-slice, since

$$\Sigma^3 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \left(\Sigma^{2-\sigma} H\underline{\mathbb{F}_2}\right) \simeq \Sigma^\rho \Sigma^1 H\underline{f}.$$

Since $\Sigma^1 H\underline{f}$ is a 1-slice, the claim follows.

Example 3.16 The spectrum $\Sigma^4 H\underline{\mathbb{F}_2}$ is a 4-slice, since

$$\Sigma^4 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \left(\Sigma^{3-\sigma} H\underline{\mathbb{F}_2}\right) \simeq \Sigma^\rho \Sigma^2 H\underline{f} \simeq \Sigma^{2\rho}\Sigma^{1-\sigma} H\underline{f} \simeq \Sigma^{2\rho} H\underline{\mathbb{F}_2}^*.$$

Since $H\underline{\mathbb{F}_2}^*$ is a 0-slice, the claim follows.

Example 3.17 The spectrum $\Sigma^5 H\underline{\mathbb{F}_2}$ has both a 5-slice and a 6-slice, since

$$\Sigma^5 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \left(\Sigma^{4-\sigma} H\underline{\mathbb{F}_2}\right) \simeq \Sigma^\rho \Sigma^3 H\underline{f} \simeq \Sigma^{2\rho}\Sigma^{2-\sigma} H\underline{f} \simeq \Sigma^{2\rho}\Sigma^1 H\underline{\mathbb{F}_2}^*.$$

Now the slice tower for $\Sigma^1 H\underline{\mathbb{F}_2}^*$ is the suspension of (3.9):

$$\Sigma^1 H\underline{g} \longrightarrow \Sigma^1 H\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^1 H\underline{f}.$$

The left spectrum is a 2-slice, while the right one is a 1-slice. Thus the slice tower for $\Sigma^5 H\underline{\mathbb{F}_2}$ is

$$P_6^6 = \Sigma^3 H\underline{g} \longrightarrow \Sigma^5 H\underline{\mathbb{F}_2} \longrightarrow \Sigma^{2\rho+1} H\underline{f} = P_5^5.$$

More generally, we have

Theorem 3.18 *The slice tower of $\Sigma^n H\underline{\mathbb{F}_2}$, for $n \geq 4$, is*

n even

$$\begin{tikzcd}
P_{2n-4}^{2n-4} = \Sigma^{n-2} H\underline{g} \arrow[r] & \Sigma^n H\underline{\mathbb{F}_2} \arrow[d] \\
P_{2n-6}^{2n-6} = \Sigma^{n-3} H\underline{g} \arrow[r] & \Sigma^{(n-1)+\sigma} H\underline{\mathbb{F}_2} \arrow[d] \\
& \vdots \arrow[d, dotted, no head] \\
P_{n+2}^{n+2} = \Sigma^{\frac{n}{2}+1} H\underline{g} \arrow[r] & \Sigma^{(\frac{n}{2}-3)\rho+6} H\underline{\mathbb{F}_2} \arrow[d] \\
& P_n^n = \Sigma^{(\frac{n}{2}-2)\rho+4} H\underline{\mathbb{F}_2} \simeq \Sigma^{\frac{n}{2}\rho} H\underline{\mathbb{F}_2}^*
\end{tikzcd}$$

n odd

$$\begin{tikzcd}
P_{2n-4}^{2n-4} = \Sigma^{n-2} H\underline{g} \arrow[r] & \Sigma^n H\underline{\mathbb{F}_2} \arrow[d] \\
P_{2n-6}^{2n-6} = \Sigma^{n-3} H\underline{g} \arrow[r] & \Sigma^{(n-1)+\sigma} H\underline{\mathbb{F}_2} \arrow[d] \\
& \vdots \arrow[d, dotted, no head] \\
P_{n+1}^{n+1} = \Sigma^{\frac{n+1}{2}} H\underline{g} \arrow[r] & \Sigma^{\left(\frac{n+1}{2}-3\right)\rho+5} H\underline{\mathbb{F}_2} \arrow[d] \\
& P_n^n = \Sigma^{\frac{n-3}{2}\rho+3} H\underline{\mathbb{F}_2} \simeq \Sigma^{\frac{n-1}{2}\rho+1} H\underline{f}
\end{tikzcd}$$

Proof The 2ρ-suspension of $H\underline{g} \longrightarrow H\underline{\mathbb{F}_2}^* \longrightarrow H\underline{f}$ is

$$\Sigma^2 H\underline{g} \longrightarrow \Sigma^4 H\underline{\mathbb{F}_2} \longrightarrow \Sigma^{\rho+2} H\underline{\mathbb{F}_2}.$$

The theorem is obtained by repeated application of suspensions of this cofiber sequence. □

Corollary 3.19 *The C_2-spectrum $\Sigma^n H\underline{f}$ is an n-slice for $n = 0, 1, 2$. If $n \geq 3$, then $n \leq \Sigma^n H\underline{f} \leq 2n-2$ and $P_{2n-2}^{2n-2}(\Sigma^n H\underline{f}) \simeq \Sigma^{n-1} H\underline{g}$.*

Proof This follows from the fiber sequence

$$\Sigma^{n-1} H\underline{g} \longrightarrow \Sigma^n H\underline{f} \longrightarrow \Sigma^n H\underline{\mathbb{F}_2}$$

and Theorem 3.18. Indeed, the slice tower is given by augmenting the slice tower for $\Sigma^n H\underline{\mathbb{F}_2}$ with the above fiber sequence. □

4 Mackey functors for $\mathcal{K} = C_2 \times C_2$

A Lewis diagram for a Mackey functor over the Klein-four group takes the shape

$$\begin{tikzcd}
& M(\mathcal{K}) \arrow[dl, bend right] \arrow[d, bend right] \arrow[dr, bend left] & \\
M(L) \arrow[ur, bend right] \arrow[dr, bend left] & M(D) \arrow[u, bend right] \arrow[d, bend right] & M(R) \arrow[ul, bend left] \arrow[dl, bend left] \\
& M(e) \arrow[ul, bend left] \arrow[u, bend right] \arrow[ur, bend right] &
\end{tikzcd}$$

We have not drawn in the C_2-actions on the intermediate groups or the $\mathcal{K}$-action on $M(e)$ (these actions are trivial in all of our examples). In the examples below, we only draw restriction or transfer maps that are nonzero.

Example 4.1 We have the constant Mackey functor

$$\underline{\mathbb{F}_2} = \begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[d, "1"] \arrow[dr, "1"] & \\ \mathbb{F}_2 \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[d, "1"] & \mathbb{F}_2, \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd}$$

as well as its dual

$$\underline{\mathbb{F}_2}^* = \begin{tikzcd} & \mathbb{F}_2 & \\ \mathbb{F}_2 \arrow[ur, "1"] & \mathbb{F}_2 \arrow[u, "1"] & \mathbb{F}_2, \arrow[ul, "1"'] \\ & \mathbb{F}_2 \arrow[ul, "1"] \arrow[u, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

Proposition 4.2 $\Sigma^{-\rho} H\underline{\mathbb{F}_2} \simeq \Sigma^{-4} H\underline{\mathbb{F}_2}^*$.

Proof Restricting to L, say, we have

$$\iota_L^*(\Sigma^{-\rho} H_{\mathcal{K}}\underline{\mathbb{F}_2}) \simeq \Sigma^{-2-2\sigma} H_{C_2}\underline{\mathbb{F}_2} \simeq \Sigma^{-4}\Sigma^{2-2\sigma} H_{C_2}\underline{\mathbb{F}_2} \simeq \Sigma^{-4} H_{C_2}\underline{\mathbb{F}_2}^*.$$

The same argument applies to the restriction to D and R. Theorem 2.22 gives that $(\Sigma^{-\rho} H\underline{\mathbb{F}_2})^{\mathcal{K}} \simeq \Sigma^{-4} H\mathbb{F}_2$.

The transfer map from R to $\mathcal{K}$ fits into a fiber sequence

$$(\Sigma^{-2\rho} H_R\underline{\mathbb{F}_2})^R \simeq (\mathcal{K}/R_+ \wedge \Sigma^{-\rho} H_{\mathcal{K}}\underline{\mathbb{F}_2})^{\mathcal{K}} \longrightarrow (\Sigma^{-\rho} H_{\mathcal{K}}\underline{\mathbb{F}_2})^{\mathcal{K}} \longrightarrow (\Sigma^{\alpha-\rho} H_{\mathcal{K}}\underline{\mathbb{F}_2})^{\mathcal{K}}.$$

By Theorem 2.18 this becomes a fiber sequence

$$\Sigma^{-4} H\mathbb{F}_2 \longrightarrow \Sigma^{-4} H\mathbb{F}_2 \longrightarrow *,$$

so that the transfer map is an equivalence. By symmetry, we find that the other transfer maps are equivalences as well.

Similarly, the restriction from $\mathcal{K}$ to R fits into a fiber sequence

$$(\Sigma^{-\rho-\alpha} H_{\mathcal{K}}\underline{\mathbb{F}_2})^{\mathcal{K}} \longrightarrow (\Sigma^{-\rho} H_{\mathcal{K}}\underline{\mathbb{F}_2})^{\mathcal{K}} \longrightarrow (\mathcal{K}/R_+ \wedge \Sigma^{-\rho} H_{\mathcal{K}}\underline{\mathbb{F}_2})^{\mathcal{K}} \simeq (\Sigma^{-2\rho} H_R\underline{\mathbb{F}_2})^R.$$

By Theorem 2.22, the spectrum $(\Sigma^{-\rho-\alpha} H_{\mathcal{K}}\underline{\mathbb{F}_2})^{\mathcal{K}}$ has $\pi_{-4} \cong \mathbb{F}_2 \cong \pi_{-5}$. It follows from the long exact sequence in homotopy that the restriction map must be zero. $\square$

Example 4.3 The geometric Mackey functor is

$$\underline{g} := \phi^*_{\mathcal{K}}(\mathbb{F}_2) = \begin{tikzcd} & \mathbb{F}_2 & \\ 0 & 0 & 0, \\ & 0 & \end{tikzcd}$$

We will later write $\underline{g}^n$ to denote $\underline{g}^{\oplus n} = \phi^*_{\mathcal{K}}(\bigoplus_n \mathbb{F}_2)$, the direct sum of n copies of $\underline{g}$.

Example 4.4 The free Mackey functor is

$$\underline{f} := \begin{tikzcd} & 0 & \\ 0 & 0 & 0, \\ & \mathbb{F}_2 & \end{tikzcd}$$

Unlike the case for $G = C_2$, the $\mathcal{K}$-spectrum $H_{\mathcal{K}}\underline{f}$ is *not* equivalent to $\Sigma^V H_{\mathcal{K}}\underline{\mathbb{F}_2}$ for any V.

Example 4.5

$$\underline{m} := \begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[d, "1"] \arrow[dr, "1"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2. \\ & 0 & \end{tikzcd}$$

and

$$\underline{m}^* := \begin{tikzcd} & \mathbb{F}_2 & \\ \mathbb{F}_2 \arrow[ur, "1"] & \mathbb{F}_2 \arrow[u, "1"] & \mathbb{F}_2. \arrow[ul, "1"'] \\ & 0 & \end{tikzcd}$$

Example 4.6

$$
\underline{w} :=
\begin{tikzcd}
 & 0 & \\
\mathbb{F}_2 \arrow[dr, "1"] & \mathbb{F}_2 \arrow[d, "1"] & \mathbb{F}_2 . \arrow[dl, "1"'] \\
 & \mathbb{F}_2 &
\end{tikzcd}
$$

and

$$
\underline{w}^* :=
\begin{tikzcd}
 & 0 & \\
\mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 . \\
 & \mathbb{F}_2 \arrow[ul, "1"] \arrow[u, "1"'] \arrow[ur, "1"'] &
\end{tikzcd}
$$

Example 4.7

$$
\underline{mg} :=
\begin{tikzcd}
 & \mathbb{F}_2 \oplus \mathbb{F}_2 \arrow[dl, "p_1"'] \arrow[d, "\nabla"'] \arrow[dr, "p_2"] & \\
\mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 . \\
 & 0 &
\end{tikzcd}
$$

and

$$
\underline{mg}^* :=
\begin{tikzcd}
 & \mathbb{F}_2 \oplus \mathbb{F}_2 & \\
\mathbb{F}_2 \arrow[ur, "i_1"] & \mathbb{F}_2 \arrow[u, "\Delta"] & \mathbb{F}_2 . \arrow[ul, "i_2"'] \\
 & 0 &
\end{tikzcd}
$$

Proposition 4.8 *There are equivalences*

(1) $\Sigma^{-\rho} H\underline{m} \simeq \Sigma^{-2} H\underline{mg}^*$
(2) $\Sigma^{\rho} H\underline{m}^* \simeq \Sigma^{2} H\underline{mg}$

Proof We prove the second statement. The first follows in a similar way, or by citing Proposition 2.9. Consider the (nonsplit) short exact sequence

$$\phi_L^* \underline{\mathbb{F}_2}^* \hookrightarrow \underline{m}^* \twoheadrightarrow \phi_{DR}^* \underline{f}.$$

This gives a nonsplit cofiber sequence

$$\Sigma^2 H\phi_L^* \underline{f} \simeq \Sigma^{\rho} H\phi_L^* \underline{\mathbb{F}_2}^* \longrightarrow \Sigma^{\rho} H\underline{m}^* \longrightarrow \Sigma^{\rho} H\phi_{DR}^* \underline{f} \simeq \Sigma^2 H\phi_{DR}^* \underline{\mathbb{F}_2}.$$

It follows that $\Sigma^{\rho} H\underline{m}^* \simeq \Sigma^2 H\underline{E}$, for some nonsplit extension $\underline{E}$ of $\phi^*_{DR}\underline{\mathbb{F}_2}$ by $\phi^*_L \underline{f}$. But we can similarly express $\underline{E}$ as an extension of $\phi^*_{LR}\underline{\mathbb{F}_2}$ by $\phi^*_D \underline{f}$. The only possibility is $\underline{E} \cong \underline{mg}$. $\square$

Example 4.9

$$\underline{W} := \begin{tikzcd} & \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & \\ \mathbb{F}_2 \arrow[ur, "i_1"] \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[u, "i_2"] \arrow[d, "1"] & \mathbb{F}_2. \arrow[ul, "i_3"'] \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd}$$

and

$$\underline{W}^* := \begin{tikzcd} & \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2. \\ & \mathbb{F}_2 \arrow[ul, "1"] \arrow[u, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

5 The slices of $\Sigma^n H_{\mathcal{K}}\underline{\mathbb{F}_2}$

Proposition 5.1 *For $n \geq 0$ and $\mathcal{K} = C_2 \times C_2$, the $\mathcal{K}$-spectrum $\Sigma^n H\underline{\mathbb{F}_2}$ satisfies $n \leq \Sigma^n H\underline{\mathbb{F}_2} \leq 4(n-3)$.*

Proof The lower bound follows by Corollary 2.13. For the upper bound, note that after restricting to the trivial subgroup, the spectrum is an n-slice. By Theorem 3.18, the restriction to a cyclic subgroup is bounded above by $2n-4$ if $n \geq 4$ (it is an n-slice if $n = 0, \ldots, 3$). It therefore remains to check that

$$\pi_{k\rho_{\mathcal{K}}+r}(\Sigma^n H\underline{\mathbb{F}_2}) = [S^{k\rho_{\mathcal{K}}+r}, \Sigma^n H\underline{\mathbb{F}_2}]^{\mathcal{K}} = 0$$

for $r \geq 0$ and $4k > 4(n-3)$. In other words, the homotopy groups $\pi_r^{\mathcal{K}}\big(\Sigma^{n-k\rho_{\mathcal{K}}} H\underline{\mathbb{F}_2}\big)$ must vanish if $r \geq 0$ and $k > n-3$. This follows from Corollary 2.23. $\square$

Moreover, we know a priori in which dimensions the slices appear.

Proposition 5.2 *For $n \geq 0$ and $\mathcal{K} = C_2 \times C_2$, all slices of the $\mathcal{K}$-spectrum $\Sigma^n H\underline{\mathbb{F}_2}$ above level n are even slices. Furthermore, if $n \geq 4$, then all slices above level $2n-4$ occur only in dimensions that are multiples of 4.*

Proof Since the restriction to any cyclic subgroup is bounded above by $2n-4$ by Theorem 3.18, it follows that all slices of $\Sigma^n H\underline{\mathbb{F}_2}$ above dimension $2n-4$ must be geometric. Thus, we know further that the only nontrivial slices of $\Sigma^n H\underline{\mathbb{F}_2}$ above dimension $2n-4$ are $4k$-slices.

Similarly, by Theorem 3.18, the restriction of $\Sigma^n H\underline{\mathbb{F}_2}$ to any cyclic subgroup has only even slices, except for possibly the n-slice. Thus any odd slices above level n must be geometric. But geometric $\mathcal{K}$-spectra only have slices in dimension a multiple of $|\mathcal{K}| = 4$. It follow that all odd slices above level n must be trivial. $\square$

5.1 The *n*-slice

We will use the following recursion to establish the bottom slices of $\Sigma^n H\underline{\mathbb{F}_2}$.

Proposition 5.3 *Let $n \geq 7$. Then*

$$P_k^k(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^\rho P_{k-4}^{k-4}(\Sigma^{n-4} H\underline{\mathbb{F}_2})$$

for $k \in [n, 2n-7]$.

Proof By Proposition 2.15, we have

$$P_k^k(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^\rho P_{k-4}^{k-4}(\Sigma^{n-\rho} H\underline{\mathbb{F}_2}) \simeq \Sigma^\rho P_{k-4}^{k-4}(\Sigma^{n-4} H\underline{\mathbb{F}_2}^*).$$

Thus it suffices to compare the $(k-4)$-slices of $\Sigma^{n-4} H\underline{\mathbb{F}_2}^*$ and $\Sigma^{n-4} H\underline{\mathbb{F}_2}$.

The short exact sequences of Mackey functors

$$0 \longrightarrow \underline{m}^* \longrightarrow \underline{\mathbb{F}_2}^* \longrightarrow \underline{f} \longrightarrow 0$$

and

$$0 \longrightarrow \underline{f} \longrightarrow \underline{\mathbb{F}_2} \longrightarrow \underline{m} \longrightarrow 0$$

give the following diagram of fiber sequences

$$\begin{tikzcd}
\Sigma^j H\underline{m}^* \arrow[r] \arrow[d] & \Sigma^j H\underline{\mathbb{F}_2}^* \arrow[r] \arrow[d, equal] & \Sigma^j H\underline{f} \arrow[d] \\
\mathrm{fib}(\lambda) \arrow[r] \arrow[d] & \Sigma^j H\underline{\mathbb{F}_2}^* \arrow[r, "\lambda"] \arrow[d] & \Sigma^j H\underline{\mathbb{F}_2} \arrow[d] \\
\Sigma^{j-1} H\underline{m} \arrow[r] & * \arrow[r] & \Sigma^j H\underline{m}
\end{tikzcd}$$

Then $\mathrm{fib}(\lambda)$ is $j-1$-connective, and the underlying spectrum of $\mathrm{fib}(\lambda)$ is contractible. By Theorem 2.12, it follows that $\mathrm{fib}(\lambda) \geq 2j-2$ as long as $j \geq 1$. Similarly, $\Sigma\,\mathrm{fib}(\lambda) \geq 2j$. By [5, Corollary 4.17], it follows that λ induces an equivalences of slices below $2j-2$. Taking $j = n-4$ gives the result. $\square$

Note that the above argument, using only the fiber sequence

$$\Sigma^j H\underline{m}^* \longrightarrow \Sigma^j H\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^j H\underline{f},$$

gives the following result that will be employed below.

Proposition 5.4 *Let $n \geq 7$. Then*

$$P_k^k(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^\rho P_{k-4}^{k-4}(\Sigma^{n-4} H\underline{f})$$

for $k \in [n, 2n-5]$.

Proposition 5.5 *For $n \geq 0$ and $\mathcal{K} = C_2 \times C_2$, the bottom slice of $\Sigma^n H\underline{\mathbb{F}_2}$ is*

$$P_n^n(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \begin{cases} \Sigma^n H\underline{\mathbb{F}_2} & n \in [0,4] \\ \Sigma^{\frac{n}{4}\rho} H\underline{\mathbb{F}_2}^* & n \equiv 0 \pmod 4, n \geq 4 \\ \Sigma^{\frac{n-1}{4}\rho+1} H\underline{f} & n \equiv 1 \pmod 4, n \geq 4 \\ \Sigma^{\frac{n-2}{4}\rho+2} H\underline{f} & n \equiv 2 \pmod 4, n \geq 4 \\ \Sigma^{\frac{n-3}{4}\rho+3} H\underline{\mathbb{F}_2} & n \equiv 3 \pmod 4, n \geq 4 \end{cases}$$

Proof By Proposition 5.3, it suffices to establish the base cases, in which $n \leq 7$. These are given in Sect. 6 below. □

5.2 The $4k$-slices

Theorem 5.6 *For all $n > 4$,*

$$P_{4k}^{4k}(\Sigma^n H\underline{\mathbb{F}_2}) = \begin{cases} \Sigma^k H\underline{g}^{2(n-k)-5} & 4k \in [2n-3, 4n-12] \\ \Sigma^{k\rho} H\left(\phi_{LDR}^*(\underline{\mathbb{F}_2}^*) \oplus \underline{g}^{4k-n-2}\right) & 4k \in [n+2, 2n-4] \\ \Sigma^{k\rho} H\underline{mg}^* & 4k = n+1 \\ \Sigma^{k\rho} H\underline{\mathbb{F}_2}^* & 4k = n \\ * & \textit{otherwise} \end{cases}$$

Proof The above formula agrees with Theorem 3.18 upon restriction to the cyclic subgroups. To determine the $\mathcal{K}$-fixed points, we use that

$$P_{4k}^{4k}(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^{k\rho} H\underline{\pi}_0 \Sigma^{n-k\rho} H\underline{\mathbb{F}_2}$$

by repeated application of Proposition 2.15. The fixed points are then given by Lemma 5.7.

It remains to consider the restriction and transfer maps if $4k \in [n, 2n-4]$. The restriction maps to the subgroup R, for instance, fit into fiber sequences

$$\left(\Sigma^{n-k\rho-\alpha} H\underline{\mathbb{F}_2}\right)^{\mathcal{K}} \longrightarrow \left(\Sigma^{n-k\rho} H\underline{\mathbb{F}_2}\right)^{\mathcal{K}} \xrightarrow{res} \left(\Sigma^{n-k\rho} H\underline{\mathbb{F}_2}\right)^{R}.$$

Fixing $k > 1$, we argue by induction on n that Lemma 5.8 implies that the long exact sequence in homotopy splits into a series of short exact sequences of $\mathbb{F}_2$-vector spaces

$$0 \to \underline{\pi}^R_{k\rho-n+1} H\underline{\mathbb{F}_2} \hookrightarrow \underline{\pi}^{\mathcal{K}}_{k\rho+\alpha-n} H\underline{\mathbb{F}_2} \twoheadrightarrow \underline{\pi}^{\mathcal{K}}_{k\rho-n} H\underline{\mathbb{F}_2} \to 0,$$

linked together by the null restriction map. Since $4k \in [n, 2n-4]$, it follows that $2k+2 \leq n \leq 4k$.

The base case for our induction argument is the case $n = 2k+2$, so that $4k$ is $2n-4$. In this base case, the left term $\underline{\pi}^R_{k\rho-2k-1} H\underline{\mathbb{F}_2}$ vanishes, and the other two terms are both of dimension $2k-1$. This establishes the base case.

For the induction step, we suppose that $\underline{\pi}^R_{k\rho-n+1} H\underline{\mathbb{F}_2} \hookrightarrow \underline{\pi}^{\mathcal{K}}_{k\rho+\alpha-n} H\underline{\mathbb{F}_2}$ is injective. It follows that we have an exact sequence

$$0 \longrightarrow \mathbb{F}_2 \hookrightarrow \mathbb{F}_2^{4k-n+2} \to \mathbb{F}_2^{4k-n+1},$$

which shows that the map on the right must be surjective.

A similar argument shows that the transfer map from the subgroup R, say, up to $\mathcal{K}$ is injective. We argue by (downward) induction on n that Lemma 5.9 implies that the long exact sequence in homotopy for the fiber sequence

$$\left(\Sigma^{n-k\rho} H\underline{\mathbb{F}_2}\right)^R \xrightarrow{tr} \left(\Sigma^{n-k\rho} H\underline{\mathbb{F}_2}\right)^{\mathcal{K}} \longrightarrow \left(\Sigma^{n-k\rho+\alpha} H\underline{\mathbb{F}_2}\right)^{\mathcal{K}}$$

splits into a series of short exact sequences

$$0 \to \underline{\pi}^R_{k\rho-n} H\underline{\mathbb{F}_2} \hookrightarrow \underline{\pi}^{\mathcal{K}}_{k\rho-n} H\underline{\mathbb{F}_2} \twoheadrightarrow \underline{\pi}^{\mathcal{K}}_{k\rho-\alpha-n} H\underline{\mathbb{F}_2} \to 0.$$

The base case is $n = 2k+1$, so that $4k = 2n-2$. In this case, $\underline{\pi}^R_{k\rho-2k-1} H\underline{\mathbb{F}_2} = 0$, and the other two terms are both of the same dimension (using Lemma 5.9).

For the induction step, we suppose that the transfer map $\underline{\pi}^R_{k\rho-n} H\underline{\mathbb{F}_2} \hookrightarrow \underline{\pi}^{\mathcal{K}}_{k\rho-n} H\underline{\mathbb{F}_2}$ is injective. It follows that we have an exact sequence

$$0 \longrightarrow \mathbb{F}_2 \hookrightarrow \mathbb{F}_2^{4k-n+1} \to \mathbb{F}_2^{4k-n},$$

which shows that the map on the right must be surjective.

It remains to show that the transfer maps are linearly independent if $4k \geq n+2$ and have distinct images if $4k = n+1$. In the case $4k = n+1$, consider the exact sequence of Mackey functors

$$\underline{\pi}_0\left(\mathcal{K}/R_+ \wedge \Sigma^{4k-1-k\rho} H\underline{\mathbb{F}_2}\right) \longrightarrow \underline{\pi}_0\left(\Sigma^{4k-1-k\rho} H\underline{\mathbb{F}_2}\right) \longrightarrow \underline{\pi}_0\left(\Sigma^{4k-1-k\rho+\alpha} H\underline{\mathbb{F}_2}\right)$$

The left Mackey functor vanishes at L and D, and the R-transfer map in the middle Mackey functor is in the image of the left Mackey functor. Thus to see that the L or D transfers in the middle Mackey functor have image distinct from that of the R transfer, it suffices to show that the L or D transfer in the right Mackey functor is nontrivial. But a similar argument to that for the transfer maps above shows that the Mackey functor on the right is $\underline{\mathbb{F}_2}^*$, so we are done. By symmetry, we similarly conclude that the images of the L and D transfers are distinct.

Finally, if $4k \geq n+2$, to show additionally that the three transfers are linearly independent, we consider the exact sequence

$$\pi_0\left(\mathcal{K}/D_+ \wedge \Sigma^{n-k\rho+\alpha+\beta} H\underline{\mathbb{F}_2}\right) \longrightarrow \pi_0\left(\Sigma^{n-k\rho+\alpha+\beta} H\underline{\mathbb{F}_2}\right) \longrightarrow \pi_0\left(\Sigma^{n-k\rho+\alpha+\beta+\gamma} H\underline{\mathbb{F}_2}\right)$$

This is an exact sequence of the form

$$\mathbb{F}_2 \longrightarrow \mathbb{F}_2^{4k-n-1} \longrightarrow \mathbb{F}_2^{4k-n-2},$$

so that the first map cannot be zero. We conclude that the D transfer map is nonzero after factoring out the L and R transfers, so that the three transfer maps are independent if $4k \geq n+2$. □

The following lemmas are direct consequences of Theorem 2.22.

Lemma 5.7 *For $n \geq 4$, we have*

$$\pi_0\left(\Sigma^{n-k\rho} H\underline{\mathbb{F}_2}\right)^{\mathcal{K}} \cong \begin{cases} \mathbb{F}_2^{2(n-k)-5} & 4k \in [2n-3, 4n-12] \\ \mathbb{F}_2^{4k-n+1} & 4k \in [n, 2n-4] \\ 0 & \textit{else} \end{cases}$$

Proof The dimension of the fixed points is given by the coefficient of x^{k-n} in the Poincaré series of Corollary 2.23. Equivalently, the dimension is given by the coefficient of x^{4k-n} in the polynomial

$$p(x) = (1+x+\cdots+x^{2k-2})(1+\cdots+x^{k-2}) + x^{k-1}(1+\cdots+x^{k-1})^2$$

This polynomial can be described as follows: The constant coefficient is 1. Then the coefficients increase by 1 until $(2k-1)x^{2k-2}$ and then decrease by 2 until $1 \cdot x^{3k-3}$. In other words, the coefficient of x^i is

$$\begin{cases} i+1 & 0 \leq i \leq 2k-2 \\ 6k-2i-5 & 2k-1 \leq i \leq 3k-3 \\ 0 & \text{else.} \end{cases}$$

Plugging in $i = 4k - n$ gives the result. □

Lemma 5.8 *For $n \geq 4$, we have*

$$\pi_0\big(\Sigma^{n-k\rho-\alpha} H\underline{\mathbb{F}_2}\big)^{\mathcal{K}} \cong \begin{cases} \mathbb{F}_2^{2(n-k)-5} & 4k \in [2n-4, 4n-12] \\ \mathbb{F}_2^{4k-n+2} & 4k \in [n-1, 2n-6] \\ 0 & \textit{else} \end{cases}$$

Proof The dimension of the fixed points here is given by the coefficient of x^{k-n} in the Poincaré series of Theorem 2.22. Equivalently, the dimension is given by the coefficient of x^{4k-n+1} in the polynomial

$$p(x) = (1 + x + \cdots + x^{2k-2})(1 + \cdots + x^{k-1}) + x^{k-1}(1 + \cdots + x^{k-1})^2$$

The polynomial is nearly the same as that from Lemma 5.7 and can be described as follows: The constant coefficient is 1. Then the coefficients increase by 1 until $(2k-1)x^{2k-2}$, remain constant for the term $(2k-1)x^{2k-1}$, and then decrease by 2 until $1 \cdot x^{3k-2}$. In other words, the coefficient of x^i is

$$\begin{cases} i+1 & 0 \leq i \leq 2k-2 \\ 6k-2i-3 & 2k-1 \leq i \leq 3k-2 \\ 0 & \text{else.} \end{cases}$$

Plugging in $i = 4k - n + 1$ gives the result. □

Lemma 5.9 *For $n > 4$, we have*

$$\pi_0\big(\Sigma^{n-k\rho+\alpha} H\underline{\mathbb{F}_2}\big)^{\mathcal{K}} \cong \begin{cases} \mathbb{F}_2^{2(n-k)-5} & 4k \in [2n-2, 4n-12] \\ \mathbb{F}_2^{4k-n} & 4k \in [n+1, 2n-4] \\ 0 & \textit{else} \end{cases}$$

Proof Since $n > 4$, $k > 1$ so the dimension of the fixed points in this case is still given by the coefficient of x^{k-n} in the Poincaré series of Theorem 2.22. Equivalently, the dimension is given by the coefficient of x^{4k-n-1} in the polynomial

$$p(x) = (1 + x + \cdots + x^{2k-2})(1 + \cdots + x^{k-3}) + x^{k-2}(1 + \cdots + x^{k-1})^2$$

The polynomial is similar to those in Lemmas 5.7 and 5.8 and can be described as follows: The constant coefficient is 1. Then the coefficients increase by 1 until $(2k-2)x^{2k-3}$, decrease by 1 for the single term $(2k-3)x^{2k-2}$, and then decrease by 2 until $1 \cdot x^{3k-4}$. In other words, the coefficient of x^i is

$$\begin{cases} i+1 & 0 \leq i \leq 2k-3 \\ 6k-2i-7 & 2k-2 \leq i \leq 3k-4 \\ 0 & \text{else.} \end{cases}$$

Plugging in $i = 4k - n - 1$ gives the result. □

5.3 The $(4k+2)$-slices

In this section, we obtain the $4k+2$-slices. We begin with the top such slices.

Proposition 5.10 *Let $n \geq 8$ be even. Then*

$$P^{2n-6}_{2n-6}(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^{(\frac{n}{2}-2)\rho+1} H\phi^*_{LDR}\underline{f}.$$

Proof. By Proposition 5.4, we have

$$P^{2n-6}_{2n-6}(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^\rho P^{2n-10}_{2n-10}(\Sigma^{n-4} H\underline{f}).$$

The short exact sequence

$$0 \longrightarrow \underline{f} \longrightarrow \underline{\mathbb{F}_2} \longrightarrow \underline{m} \longrightarrow 0$$

gives rise to the fiber sequence

$$\Sigma^{n-5} H\underline{m} \longrightarrow \Sigma^{n-4} H\underline{f} \longrightarrow \Sigma^{n-4} H\underline{\mathbb{F}_2}.$$

By Proposition 5.2, we have

$$P^{2n-10}_{2n-10}(\Sigma^{n-5} H\underline{\mathbb{F}_2}) \simeq * \simeq P^{2n-10}_{2n-10}(\Sigma^{n-4} H\underline{\mathbb{F}_2}).$$

It then follows from Proposition 2.17 that $\Sigma^{n-5} H\underline{m} \longrightarrow \Sigma^{n-4} H\underline{f}$ induces an isomorphism on $(2n-10)$-slices.

The short exact sequence

$$0 \longrightarrow \underline{m} \longrightarrow \phi^*_{LDR}\underline{\mathbb{F}_2} \longrightarrow \underline{g}^2 \longrightarrow 0$$

gives a fiber sequence

$$\Sigma^{n-6} H\underline{g}^2 \longrightarrow \Sigma^{n-5} H\underline{m} \longrightarrow \Sigma^{n-5} H\phi^*_{LDR}\underline{\mathbb{F}_2}.$$

If $n \geq 8$ is even, then by Theorem 3.18, we get that

$$P^{2n-10}_{2n-10}(\Sigma^{n-5} H\underline{m}) \simeq \Sigma^{(\frac{n}{2}-3)\rho+1} H\phi^*_{LDR}\underline{f}.$$

□

Proposition 5.11 *Let $n \geq 5$ be odd. Then*

$$P^{2n-4}_{2n-4}(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^{(\frac{n+1}{2}-2)\rho+1} H\phi^*_{LDR}\underline{f}.$$

Proof For $n = 5$, this is given in Example 6.6 below. We have

$$P^{2n-4}_{2n-4}(\Sigma^n H\underline{\mathbb{F}_2}) \simeq P^{2n-4}_{2n-4}(\Sigma^{\rho+n-4} H\underline{\mathbb{F}_2}^*) \simeq \Sigma^\rho P^{2n-8}_{2n-8}(\Sigma^{n-4} H\underline{\mathbb{F}_2}^*).$$

The short exact sequence

$$0 \longrightarrow \underline{g} \longrightarrow \underline{\mathbb{F}_2}^* \longrightarrow \underline{w}^* \longrightarrow 0$$

gives a fiber sequence

$$\Sigma^{n-4} H\underline{g} \longrightarrow \Sigma^{n-4} H\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^{n-4} H\underline{w}^*.$$

It follows that, if $n \geq 5$, then $\Sigma^{n-4} H\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^{n-4} H\underline{w}^*$ induces an equivalence on $(2n-8)$-slices.

Next, the short exact sequence

$$0 \longrightarrow \underline{w}^* \longrightarrow \underline{W}^* \longrightarrow \underline{g}^3 \longrightarrow 0$$

yields a fiber sequence

$$\Sigma^{n-5} H\underline{g}^3 \longrightarrow \Sigma^{n-4} H\underline{w}^* \longrightarrow \Sigma^{n-4} H\underline{W}^*.$$

It follows that, if $n \geq 7$, then $\Sigma^{n-4} H\underline{w}^* \longrightarrow \Sigma^{n-4} H\underline{W}^*$ induces an equivalence on $(2n-8)$-slices.

Finally, consider the short exact sequence

$$0 \longrightarrow \phi^*_{LDR}\underline{\mathbb{F}_2} \longrightarrow \underline{W}^* \longrightarrow \underline{f} \longrightarrow 0.$$

This gives a fiber sequence

$$\Sigma^{n-4} H\phi^*_{LDR}\underline{\mathbb{F}_2} \longrightarrow \Sigma^{n-4} H\underline{W}^* \longrightarrow \Sigma^{n-4} H\underline{f}.$$

The restriction of $\Sigma^{n-4} H\underline{f}$ to either L, D, or R has no slices above level $2n-12$, and similarly for $\Sigma^{n-5} H\underline{f}$. Thus the $(2n-8)$-slice of $\Sigma^{n-4} H\underline{f}$ (and $\Sigma^{n-5} H\underline{f}$) must be geometric. Since $2n-8$ is not a multiple of 4, we conclude that the $(2n-8)$-slices are trivial. By Proposition 2.17, we conclude that $\Sigma^{n-4} H\phi^*_{LDR}\underline{\mathbb{F}_2} \longrightarrow \Sigma^{n-4} H\underline{W}^*$ induces an equivalence on $(2n-8)$-slices. We are now done by Theorem 3.18. □

Proposition 5.12 *Let* $4k+2 \in (n, 2n-4]$. *Then*

$$P^{4k+2}_{4k+2}(\Sigma^n H\underline{\mathbb{F}_2}) \simeq \Sigma^{k\rho+1} H\phi^*_{LDR}\underline{f}.$$

Proof If $4k+2 \leq 2n-7$, this follows from Proposition 5.3 and the base cases discussed in Sect. 6. This leaves only the cases of $4k+2 = 2n-6$ if n is even, or $4k+2 = 2n-4$ if n is odd. These cases are handled in the two preceding propositions. □

6 The slice towers of $\Sigma^n H_{\mathcal{K}}\underline{\mathbb{F}_2}$

We determine the slice towers of $\Sigma^n H\underline{\mathbb{F}_2}$ for $0 \leq n \leq 8$.

Example 6.1 $H\underline{\mathbb{F}_2}$ is a zero-slice.

Example 6.2 $\Sigma^1 H\underline{\mathbb{F}_2}$ is a 1-slice since the restriction maps are injective.

Example 6.3 $\Sigma^2 H\underline{\mathbb{F}_2}$ is a 2-slice. Since this is true upon restriction to each of the proper subgroups, it suffices to show that

$$[S^{n\rho+r}, \Sigma^2 H\underline{\mathbb{F}_2}] = 0$$

for $n > 0$ and $r \geq 0$. This follows from Theorem 2.22.

Example 6.4 $\Sigma^3 H\underline{\mathbb{F}_2}$ is a 3-slice. Since this is true upon restriction to each of the proper subgroups, it suffices to show that

$$[S^{n\rho+r}, \Sigma^3 H\underline{\mathbb{F}_2}] = 0$$

for $n, r > 0$. This follows from Theorem 2.22. Alternatively, $\Sigma^{-\rho}\Sigma^3 H\underline{\mathbb{F}_2} \simeq \Sigma^{-1} H\underline{\mathbb{F}_2}^*$ is a (-1)-slice according to Proposition 2.14 since the transfer maps are surjective. It follows that $\Sigma^3 H\underline{\mathbb{F}_2}$ is a 3-slice.

Example 6.5 By Proposition 4.2, $\Sigma^4 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho H\underline{\mathbb{F}_2}^*$. It follows that $\Sigma^4 H\underline{\mathbb{F}_2}$ is a 4-slice.

Example 6.6 Consider the short exact sequences of Mackey functors

$$0 \longrightarrow \underline{g} \longrightarrow \underline{\mathbb{F}_2}^* \longrightarrow \underline{w}^* \longrightarrow 0$$

and

$$0 \longrightarrow \phi^*_{LDR}\underline{f} \longrightarrow \underline{w}^* \longrightarrow \underline{f} \longrightarrow 0,$$

where $\underline{w}^*$ is defined in Example 4.6. The resulting cofiber sequences produce the slice tower for $\Sigma^5 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \Sigma^1 H\underline{\mathbb{F}_2}^*$:

$$\begin{tikzcd}
P^8_8 = \Sigma^2 H\underline{g} \arrow[r] & \Sigma^{\rho+1} H\underline{\mathbb{F}_2}^* \simeq \Sigma^5 H\underline{\mathbb{F}_2} \arrow[d] \\
P^6_6 = \Sigma^{\rho+1} H\phi^*_{LDR}\underline{f} \arrow[r] & \Sigma^{\rho+1} H\underline{w}^* \arrow[d] \\
& P^5_5 = \Sigma^{\rho+1} H\underline{f},
\end{tikzcd}$$

Example 6.7 Suspending the slice tower for $\Sigma^5 H\underline{\mathbb{F}_2}$ gives the tower for $\Sigma^6 H\underline{\mathbb{F}_2}$:

$$\begin{tikzcd}
P^{12}_{12} = \Sigma^3 H\underline{g} \arrow[r] & \Sigma^{\rho+2} H\underline{\mathbb{F}_2}^* \simeq \Sigma^6 H\underline{\mathbb{F}_2} \arrow[d] \\
P^8_8 = \Sigma^{\rho+2} H\phi^*_{LDR}\underline{f} \arrow[r] & \Sigma^{\rho+2} H\underline{w}^* \arrow[d] \\
& P^6_6 = \Sigma^{\rho+2} H\underline{f}.
\end{tikzcd}$$

Lemma 6.8 $\Sigma^2 H\underline{f}$ *is a 2-slice.*

Proof The short exact sequence of Mackey functors

$$0 \longrightarrow \underline{f} \longrightarrow \underline{\mathbb{F}_2} \longrightarrow \underline{m} \longrightarrow 0,$$

where $\underline{m}$ is defined in Example 4.5, gives a cofiber sequence

$$\Sigma^1 H\underline{m} \longrightarrow \Sigma^2 H\underline{f} \longrightarrow \Sigma^2 H\underline{\mathbb{F}_2}.$$

The spectrum $\Sigma^2 H\underline{\mathbb{F}_2}$ is a 2-slice, and the cofiber sequence

$$\Sigma^1 H\phi^*_{L,D}\underline{f} \longrightarrow \Sigma^1 H\underline{m} \longrightarrow \Sigma^1 H\phi^*_R\underline{\mathbb{F}_2}$$

shows that $\Sigma^1 H\underline{m}$ is also a 2-slice. $\square$

Example 6.9 For $\Sigma^7 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \Sigma^3 H\underline{\mathbb{F}_2}^*$, we have fiber sequences as in

$$\begin{tikzcd}
P_{16}^{16} = \Sigma^4 H\underline{g} \arrow[r] & \Sigma^{\rho+3} H\underline{\mathbb{F}_2}^* \simeq \Sigma^7 H\underline{\mathbb{F}_2} \arrow[d] \\
P_{12}^{12} = \Sigma^3 H\underline{g}^3 \arrow[r] & \Sigma^{\rho+3} H\underline{w}^* \arrow[d] \\
P_{10}^{10} = \Sigma^{\rho+3} H\phi_{LDR}^* \underline{\mathbb{F}_2} \arrow[r] & \Sigma^{\rho+3} H\underline{W}^* \arrow[d] \\
P_8^8 = \Sigma^{\rho+2} H\underline{m} \arrow[r] & P_7^8 = \Sigma^{\rho+3} H\underline{f} \arrow[d] \\
 & P_7^7 = \Sigma^{\rho+3} H\underline{\mathbb{F}_2},
\end{tikzcd}$$

where $\underline{W}^*$ is defined in Example 4.9. Note that $\Sigma^2 H\underline{m}$ is a 4-slice, as $\Sigma^2 H\underline{m} \simeq \Sigma^\rho H\underline{mg}^*$ according to Proposition 4.8.

Example 6.10 For $\Sigma^8 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \Sigma^4 H\underline{\mathbb{F}_2}^*$, we have fiber sequences as in

$$\begin{tikzcd}
P_{20}^{20} = \Sigma^5 H\underline{g} \arrow[r] & \Sigma^{\rho+4} H\underline{\mathbb{F}_2}^* \simeq \Sigma^8 H\underline{\mathbb{F}_2} \arrow[d] \\
P_{16}^{16} = \Sigma^4 H\underline{g}^3 \arrow[r] & \Sigma^{\rho+4} H\underline{w}^* \arrow[d] \\
P_{12}^{12} = \Sigma^{3\rho} H\big(\phi_{LDR}^* \underline{\mathbb{F}_2}^* \oplus \underline{g}^2\big) \arrow[r] & \Sigma^{\rho+4} H\underline{W}^* \arrow[d] \\
 & P^{10}\Sigma^8 H\underline{\mathbb{F}_2}
\end{tikzcd}$$

The twelve slice is given by Theorem 5.6.

Proposition 6.11 *The slice section $P^{10}\Sigma^8 H\underline{\mathbb{F}_2}$ is equivalent to $\Sigma^{\rho+3}C$, where C is the cofiber of $H\underline{\mathbb{F}_2} \longrightarrow H\phi_{LDR}^* \underline{\mathbb{F}_2}$.*

Proof The cofiber C has homotopy Mackey functors $\underline{\pi}_1(C) \cong \underline{f}$ and $\underline{\pi}_0(C) \cong \underline{g}^2$. Thus the ρ-suspension of the Postnikov sequence is a fiber sequence

$$\Sigma^{\rho+2} H\underline{g}^2 \longrightarrow \Sigma^{\rho+4} H\underline{f} \longrightarrow \Sigma^{\rho+3} C.$$

On the other hand, the short exact sequence $\phi_{LDR}^* \underline{\mathbb{F}_2} \hookrightarrow \underline{W}^* \longrightarrow \underline{f}$ gives a fiber sequence

$$\Sigma^{3\rho} H\phi_{LDR}^* \underline{\mathbb{F}_2}^* \simeq \Sigma^{\rho+4} H\phi_{LDR}^* \underline{\mathbb{F}_2} \longrightarrow \Sigma^{\rho+4} H\underline{W}^* \longrightarrow \Sigma^{\rho+4} H\underline{f}.$$

Since the 12-slice is the sum of the left terms in these two sequences and $\Sigma^{\rho+4} H\underline{W}^*$ is $P^{12}\Sigma^8 H\underline{\mathbb{F}_2}$, the octahedral axiom gives a cofiber sequence

$$P_{12}^{12}(\Sigma^8 H\underline{\mathbb{F}_2}) \longrightarrow P^{12}\Sigma^8 H\underline{\mathbb{F}_2} \longrightarrow \Sigma^{\rho+3} C.$$

But $8 \leq \Sigma^{\rho+3} C \leq 10$, so we are done. □

Remark 6.12 The $\mathcal{K}$-spectrum $\Sigma^{\rho+3} C$ is the first example of a $\mathcal{K}$-spectrum that is not an $RO(\mathcal{K})$-graded suspension of an Eilenberg–MacLane spectrum and yet which occurs as a slice or slice section in the tower for $\Sigma^n H\underline{\mathbb{F}_2}$. Indeed, the restriction of C to each cyclic subgroup is $\Sigma^1 H\underline{f}$. But if $\Sigma^V H\underline{M}$ restricts to $\Sigma^1 H\underline{M}$ for each cyclic subgroup, it must be that $V = 1$. As C has a nontrivial $\underline{\pi}_0$, it cannot be of the form $\Sigma^V H\underline{M}$.

Thus the slice tower is given by

$$\begin{array}{ccc}
P_{20}^{20} = \Sigma^5 H\underline{g} & \longrightarrow & \Sigma^{\rho+4} H\underline{\mathbb{F}_2}^* \simeq \Sigma^8 H\underline{\mathbb{F}_2} \\
 & & \downarrow \\
P_{16}^{16} = \Sigma^4 H\underline{g}^3 & \longrightarrow & \Sigma^{\rho+4} H\underline{w}^* \\
 & & \downarrow \\
P_{12}^{12} = \Sigma^{3\rho} H\big(\phi_{LDR}^* \underline{\mathbb{F}_2}^* \oplus \underline{g}^2\big) & \longrightarrow & \Sigma^{\rho+4} H\underline{W}^* \\
 & & \downarrow \\
P_{10}^{10} = \Sigma^{\rho+3} H\phi_{LDR}^* \underline{\mathbb{F}_2} & \longrightarrow & \Sigma^{\rho+3} C \\
 & & \downarrow \\
 & & P_8^8 = \Sigma^{\rho+4} H\underline{\mathbb{F}_2},
\end{array}$$

For the higher suspensions, we do not know the slice tower explicitly. We give a diagram of fibrations which is close to the slice tower in the next two examples.

Example 6.13 For $\Sigma^9 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho \Sigma^5 H\underline{\mathbb{F}_2}^*$, we have fiber sequences as in

$$\begin{tikzcd}
P_{24}^{24} = \Sigma^6 H\underline{g} \arrow[rr] & & \Sigma^{\rho+5} H\underline{\mathbb{F}_2}^* \simeq \Sigma^9 H\underline{\mathbb{F}_2} \arrow[d] \\
P_{20}^{20} = \Sigma^5 H\underline{g}^3 \arrow[rr] & & \Sigma^{\rho+5} H\underline{w}^* \arrow[d] \\
\boxed{\mathrm{A}}\ \Sigma^4 H\underline{g}^3 \arrow[r] & \Sigma^{\rho+5} H\phi^*_{LDR}\underline{\mathbb{F}_2} \arrow[r] \arrow[d] & \Sigma^{\rho+5} H\underline{W}^* \arrow[dd] \\
 & P_{14}^{14} = \Sigma^{3\rho+1} H\phi^*_{LDR}\underline{f} & \\
\boxed{\mathrm{B}}\ \Sigma^{\rho+3} H\underline{g}^2 \arrow[rr] & & \Sigma^{\rho+5} H\underline{f} \arrow[d] \\
 & \boxed{\mathrm{C}}\ \Sigma^{\rho+4} H\phi^*_{LDR}\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{\rho+4} C \arrow[d] \\
 & \boxed{\mathrm{D}}\ \Sigma^{\rho+2}\underline{g} \arrow[r] & \Sigma^{\rho+5} H\underline{\mathbb{F}_2} \arrow[d] \\
 & P_{10}^{10} = \Sigma^{2\rho+1} H\phi^*_{LDR}\underline{f} \arrow[r] & \Sigma^{2\rho+1} H\underline{w}^* \arrow[d] \\
 & & P_9^9 = \Sigma^{2\rho+1} H\underline{f},
\end{tikzcd}$$

This is not quite the slice tower. According to Theorem 5.6, the 16-slice is the sum $\boxed{\mathrm{A}} \vee \boxed{\mathrm{B}}$. Similarly, the 12-slice is the sum $\boxed{\mathrm{C}} \vee \boxed{\mathrm{D}}$.

Example 6.14 For $\Sigma^{10} H\underline{\mathbb{F}_2} \simeq \Sigma^{\rho}\Sigma^{6} H\underline{\mathbb{F}_2}^*$, we have fiber sequences as in

$$\begin{tikzcd}
P^{28}_{28} = \Sigma^7 H\underline{g} \arrow[rr] & & \Sigma^{\rho+6} H\underline{\mathbb{F}_2}^* \simeq \Sigma^{10} H\underline{\mathbb{F}_2} \arrow[d] \\
P^{24}_{24} = \Sigma^6 H\underline{g}^3 \arrow[rr] & & \Sigma^{\rho+6} H\underline{w}^* \arrow[d] \\
\boxed{\mathrm{A}}\ \Sigma^5 H\underline{g}^3 \arrow[r] & \Sigma^{\rho+6} H\phi^*_{LDR}\,\underline{\mathbb{F}_2} \arrow[r] \arrow[d] & \Sigma^{\rho+6} H\underline{W}^* \arrow[dd] \\
& \boxed{\mathrm{C}}\ \Sigma^{3\rho+2} H\phi^*_{LDR}\,\underline{f} & \\
\boxed{\mathrm{B}}\ \Sigma^{\rho+4} H\underline{g}^2 \arrow[rr] & & \Sigma^{\rho+6} H\underline{f} \arrow[d] \\
\boxed{\mathrm{D}}\ \Sigma^{3\rho+1} H\underline{g}^3 \arrow[r] & \Sigma^{\rho+5} H\phi^*_{LDR}\,\underline{\mathbb{F}_2} \arrow[r] \arrow[d] & \Sigma^{\rho+5} C \arrow[dd] \\
& P^{14}_{14} = \Sigma^{3\rho+1} H\phi^*_{LDR}\underline{f} & \\
& \boxed{\mathrm{E}}\ \Sigma^{\rho+3}\underline{g} \arrow[r] & \Sigma^{\rho+6} H\underline{\mathbb{F}_2} \arrow[d] \\
& P^{12}_{12} = \Sigma^{2\rho+2} H\phi^*_{LDR}\underline{f} \arrow[r] & \Sigma^{2\rho+2} H\underline{w}^* \arrow[d] \\
& & P^{10}_{10} = \Sigma^{2\rho+2} H\underline{f},
\end{tikzcd}$$

According to Theorem 5.6, the 20-slice is the sum $\boxed{\mathrm{A}} \vee \boxed{\mathrm{B}}$ and the 16-slice is the sum $\boxed{\mathrm{C}} \vee \boxed{\mathrm{D}} \vee \boxed{\mathrm{E}}$.

7 Homotopy Mackey functor computations

Here we collect some computations of homotopy Mackey functors of various twisted Eilenberg–MacLane spectra.

Theorem 7.1 *For all $k \geq 1$, the nontrivial homotopy Mackey functors of $\Sigma^{-k\rho} H\underline{\mathbb{F}_2}^*$ are*

$$\underline{\pi}_{-i}(\Sigma^{-k\rho} H\underline{\mathbb{F}_2}^*) = \begin{cases} \underline{\mathbb{F}_2}^* & i = 4k \\ \underline{mg}^* & i = 4k-1 \\ \phi^*_{LDR}\underline{\mathbb{F}_2}^* \oplus \underline{g}^{4k-2-i} & i \in [2k, 4k-2] \\ \underline{g}^{2(i-k)+1} & i \in [k, 2k-1]. \end{cases}$$

Proof By repeated application of Proposition 2.15, Theorem 5.6 is equivalent to the statement that the only nontrivial homotopy Mackey functors of $\Sigma^{-k\rho} H\underline{\mathbb{F}_2}$ are

$$\underline{\pi}_{-n}(\Sigma^{-k\rho} H\underline{\mathbb{F}_2}) = \begin{cases} \underline{\mathbb{F}_2}^* & n = 4k \\ \underline{mg}^* & n = 4k-1 \\ \phi^*_{LDR}\underline{\mathbb{F}_2}^* \oplus \underline{g}^{4k-2-n} & n \in [2k+2, 4k-2] \\ \underline{g}^{2(n-k)-5} & n \in [k+3, 2k+1]. \end{cases}$$

We know that

$$\underline{\pi}_{-i}(\Sigma^{-k\rho} H\underline{\mathbb{F}_2}^*) = \underline{\pi}_{-(i+4)}(\Sigma^{-(k+1)\rho} H\underline{\mathbb{F}_2}).$$

The result follows by setting $n = i+4$ and replacing k by $k+1$. □

Proposition 2.9 then gives the following result, which gives the homotopy Mackey functors of the bottom slice in the case $n \equiv 0, 3 \pmod 4$.

Corollary 7.2 *For all $k \geq 1$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{\mathbb{F}_2}$ are*

$$\underline{\pi}_{i}(\Sigma^{k\rho} H\underline{\mathbb{F}_2}) = \begin{cases} \underline{\mathbb{F}_2} & i = 4k \\ \underline{mg} & i = 4k-1 \\ \phi^*_{LDR}\underline{\mathbb{F}_2} \oplus \underline{g}^{4k-2-i} & i \in [2k, 4k-2] \\ \underline{g}^{2(i-k)+1} & i \in [k, 2k-1]. \end{cases}$$

For the homotopy of the bottom slice in the remaining two cases, namely $n \equiv 1, 2 \pmod 4$, we need some auxiliary computations.

Proposition 7.3 *Let $k \geq 1$. The nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{m}$ are*

(1) $\underline{\pi}_{2k}(\Sigma^{k\rho} H\underline{m}) \cong \phi^*_{LDR}\underline{\mathbb{F}_2}$
(2) $\underline{\pi}_{i}(\Sigma^{k\rho} H\underline{m}) \cong \underline{g}^3$ *for* $i \in [k+1, 2k-1]$
(3) $\underline{\pi}_{k}(\Sigma^{k\rho} H\underline{m}) \cong \underline{g}$

Proof The short exact sequence

$$\phi^*_{LDR}\underline{f} \hookrightarrow \underline{m} \twoheadrightarrow \underline{g}$$

gives a cofiber sequence

$$\Sigma^{(k-1)\rho+2} H\phi^*_{LDR}\underline{\mathbb{F}_2} \simeq \Sigma^{k\rho} H\phi^*_{LDR}\underline{f} \longrightarrow \Sigma^{k\rho} H\underline{m} \longrightarrow \Sigma^{k\rho} H\underline{g} \simeq \Sigma^{k} H\underline{g}.$$

The result now follows from Proposition 3.6. □

The same argument, using instead the short exact sequence $\phi^*_{LDR}\underline{f} \hookrightarrow \underline{m} \twoheadrightarrow \underline{g}^2$, applies to show

Proposition 7.4 *Let $k \geq 1$. The nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{mg}$ are*

(1) $\underline{\pi}_{2k}(\Sigma^{k\rho} H\underline{mg}) \cong \phi^*_{LDR}\underline{\mathbb{F}_2}$
(2) $\underline{\pi}_i(\Sigma^{k\rho} H\underline{mg}) \cong \underline{g}^3$ *for* $i \in [k+1, 2k-1]$
(3) $\underline{\pi}_k(\Sigma^{k\rho} H\underline{mg}) \cong \underline{g}^2$

The homotopy of the bottom slice when $n \equiv 1, 2 \pmod 4$ is now given by the following result.

Corollary 7.5 *For all $k \geq 1$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho} H\underline{f}$ are*

$$\underline{\pi}_i(\Sigma^{k\rho} H\underline{f}) = \begin{cases} \underline{\mathbb{F}_2} & i = 4k \\ \underline{mg} & i = 4k-1 \\ \phi^*_{LDR}\underline{\mathbb{F}_2} \oplus \underline{g}^{4k-2-i} & i \in [2k+1, 4k-2] \\ \underline{g}^{2(i-k-1)} & i \in [k+2, 2k]. \end{cases}$$

Proof We have the cofiber sequence

$$\Sigma^{k\rho} H\underline{f} \longrightarrow \Sigma^{k\rho} H\underline{\mathbb{F}_2} \longrightarrow \Sigma^{k\rho} H\underline{m}$$

arising from the short exact sequence of Mackey functors. From the long exact sequence in homotopy we get the desired homotopy for $i \in [2k+1, 4k]$ because $\underline{\pi}_i(\Sigma^{k\rho} H\underline{m}) = 0$ for $i > 2k$ by Proposition 7.3.

When $i = 2k$, in the long exact sequence we have

$$\underline{\pi}_{2k}(\Sigma^{k\rho} H\underline{f}) \hookrightarrow \phi^*_{LDR}\underline{\mathbb{F}_2} \oplus \underline{g}^{2k-2} \to \phi^*_{LDR}\underline{\mathbb{F}_2}.$$

On subgroups of size 2, the map on the right is an isomorphism and thus maps $\phi^*_{LDR}\underline{\mathbb{F}_2}$ isomorphically to the target forcing $\underline{\pi}_{2k}(\Sigma^{k\rho} H\underline{f}) = \underline{g}^{2k-2}$.

For $i \in [k+2, 2k-1]$ we have

$$\underline{\pi}_i(\Sigma^{k\rho} H\underline{f}) \to \underline{g}^{1+2(i-k)} \to \underline{g}^3$$

and thus $\underline{\pi}_i(\Sigma^{k\rho} H\underline{f}) = \underline{g}^j$, where $j \geq 2(i-k-1)$.

To show that $j \leq 2(i-k-1)$, we use the cofiber sequence

$$\Sigma^{(k-1)\rho+4} H\underline{\mathbb{F}_2} \simeq \Sigma^{k\rho} H\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^{k\rho} H\underline{f} \longrightarrow \Sigma^{k\rho+1} H\underline{m}^* \simeq \Sigma^{(k-1)\rho+3} H\underline{mg}.$$

For $i \in [k+3, 2k-1]$ we have the following in the long exact sequence in homotopy:

$$\underline{g}^{2(i-k)-5} \to \underline{\pi}_i(\Sigma^{k\rho} H\underline{f}) \to \underline{g}^3$$

and thus, $\underline{\pi}_i(\Sigma^{k\rho} H\underline{f}) = \underline{g}^j$ where $j \leq 2(i-k-1)$. When $i = k+2$, we have $\underline{\pi}_i(\Sigma^{k\rho} H\underline{f}) = \underline{g}^2$ as desired since $\underline{\pi}_i(\Sigma^{(k-1)\rho+4} H\underline{\mathbb{F}_2}) = 0$ and $\underline{\pi}_i(\Sigma^{(k-1)\rho+3} H\underline{mg}) = \underline{g}^2$ by Proposition 7.4. For $i < k+2$ we can see from either long exact sequence that $\underline{\pi}_i(\Sigma^{k\rho} H\underline{f}) = 0$. □

The homotopy of the slices in dimension congruent to 2 modulo 4 is much simpler.

Proposition 7.6 *The nontrivial homotopy Mackey functors of* $\Sigma^{k\rho+1} H\phi^*_{LDR}\underline{f}$ *are*

$$\underline{\pi}_i(\Sigma^{k\rho+1} H\phi^*_{LDR}\underline{f}) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & i = 2k+1 \\ \underline{g}^3 & i \in [k+2, 2k] \end{cases}$$

Proof This follows directly from Proposition 3.6, given that $\Sigma^{\rho_{C_2}} H_{C_2}\underline{f} \simeq \Sigma^2 H_{C_2}\underline{\mathbb{F}_2}$. □

8 The slice spectral sequence

The Mackey functor-valued slice spectral sequence for $\Sigma^n H\underline{\mathbb{F}_2}$ must recover that the only nontrivial homotopy Mackey functor is $\underline{\pi}_n(\Sigma^n H\underline{\mathbb{F}_2}) \cong \underline{\mathbb{F}_2}$. This Mackey functor already occurs in the bottom slice, and all higher slices get wiped out by the spectral sequence. To a large extent, the answer forces many of the differentials. Furthermore, the slice spectral sequence must restrict to recover the slice spectral sequence on each cyclic subgroup, which further allows us to deduce many differentials. In practice, only a few differentials require further argument. We discuss this in several examples.

We remind the reader of the indexing convention for the slice spectral sequence chosen in [4, Section 4.4.2]: the Mackey functor $\underline{\pi}_n P^t_t(X)$ appears as $\underline{E}_2^{t-n,t}$ and the charts are displayed using the Adams convention, so that $\underline{\pi}_n P^t_t(X)$ appears in position $(n, t-n)$ in our charts. The differential $d_r : \underline{E}_r^{s,t} \longrightarrow \underline{E}_r^{s+r,t+r-1}$ therefore points left one and up r, as is customary in Adams spectral sequence charts.

Example 8.1 The first example which has a nontrivial slice tower is $\Sigma^5 H\underline{\mathbb{F}_2}$. The slices are

$$P^5_5(\Sigma^5 H\underline{\mathbb{F}_2}) = \Sigma^{\rho+1} H\underline{f}, \qquad P^6_6(\Sigma^5 H\underline{\mathbb{F}_2}) \simeq \Sigma^3 H\phi^*_{LDR}\underline{\mathbb{F}_2}, \qquad P^8_8(\Sigma^5 H\underline{\mathbb{F}_2}) = \Sigma^2 H\underline{g}.$$

Thus the 6 and 8-slices are Eilenberg–MacLane. The homotopy Mackey functors of the 5-slice are given in Corollary 7.5

In the slice spectral sequence, the only possibility is that we have differentials

$$d_2 : \underline{\pi}_4(P^5_5) \cong \underline{mg} \hookrightarrow \underline{\pi}_3(P^6_6) = \phi^*_{LDR}\,\underline{\mathbb{F}_2}.$$

and

$$d_3 : \underline{\pi}_3(P^6_6)/d_1 \xrightarrow{\cong} \underline{\pi}_2(P^8_8) \cong \underline{g}.$$

The slice spectral sequence for $\Sigma^6 H\underline{\mathbb{F}_2}$ is just the suspension of that for $\Sigma^5 H\underline{\mathbb{F}_2}$. Those for $\Sigma^7 H\underline{\mathbb{F}_2}$ and $\Sigma^8 H\underline{\mathbb{F}_2}$ are not much more complicated. There is only one possible pattern of differentials, which is displayed in Fig. 2.

Example 8.2 In the slice spectral sequence for $\Sigma^9 H\underline{\mathbb{F}_2}$, displayed in Fig. 3, almost all differentials are forced by the fact that only $\pi_9(P^9_9\Sigma^9 H\underline{\mathbb{F}_2}) \cong \underline{\mathbb{F}_2}$ can survive the spectral sequence. The sole exception is that the summand $\underline{g}$ of

$$\underline{\pi}_6(P^9_9\Sigma^9 H\underline{\mathbb{F}_2}) \cong \phi^*_{LDR}\underline{\mathbb{F}_2} \oplus \underline{g}$$

can support either a d_4 to $\underline{\pi}_5(P^{12}_{12}\Sigma^9 H\underline{\mathbb{F}_2}) \cong \underline{g}^3$ or a d_6 to $\underline{\pi}_5(P^{14}_{14}\Sigma^9 H\underline{\mathbb{F}_2}) \cong \underline{g}^3$.

To see that it must in fact support the shorter d_4, we use that the map

$$\Sigma^9 H\underline{\mathbb{F}_2} \simeq \Sigma^{\rho+5} H\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^{\rho+5}\underline{f}$$

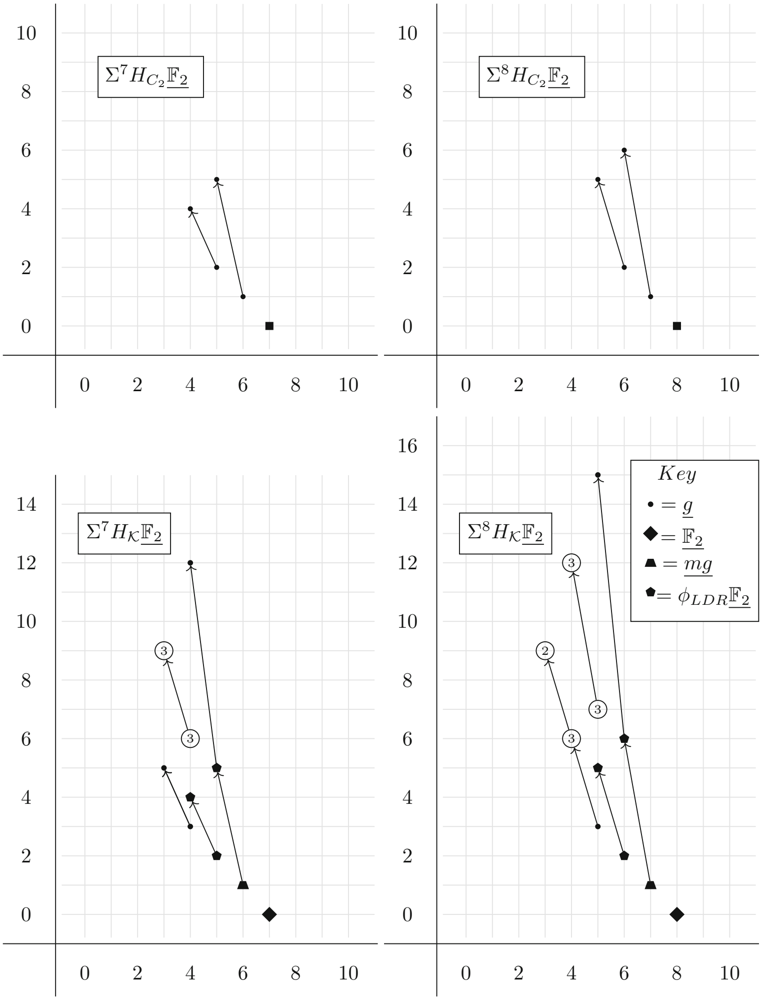

Fig. 2 The slice spectral sequence over C_2 and $C_2 \times C_2$, $n = 7, 8$

(see Example 6.13) induces an equivalence on 9, 10, and 12-slices. Since $\Sigma^{\rho+5} H\underline{f}$ only has nontrivial $\underline{\pi}_9$ and $\underline{\pi}_8$ by Corollary 7.5, there must be a d_4 in the slice spectral sequence for $\Sigma^{\rho+5} H\underline{f}$ in order to wipe out the $\underline{\pi}_6$ and $\underline{\pi}_5$.

Example 8.3 Most differentials in the slice spectral sequence for $\Sigma^{10} H\underline{\mathbb{F}_2}$, which is displayed in Fig. 3, are forced by the fact that only $\underline{\pi}_{10}(P^{10}_{10}\Sigma^{10} H\underline{\mathbb{F}_2})$ survives.

To see that $d_3 : \underline{\pi}_6(P^{10}_{10}) \longrightarrow \underline{\pi}_5(P^{12}_{12})$ is injective, we use that the map

$$\Sigma^{10} H\underline{\mathbb{F}_2} \simeq \Sigma^{\rho+6} H\underline{\mathbb{F}_2}^* \longrightarrow \Sigma^{2\rho+2} H\underline{w}^*$$

(see Example 6.14) induces an equivalence on 10 and 12 slices. The cofiber sequence

$$\Sigma^{\rho+4} H\underline{\mathbb{F}_2} \longrightarrow \Sigma^{2\rho} H\underline{w}^* \longrightarrow \Sigma^{2\rho+1} H\underline{g}$$

shows that $\underline{\pi}_6(\Sigma^{2\rho+2} H\underline{w}^*) = 0$, which forces the claimed d_3-differential.

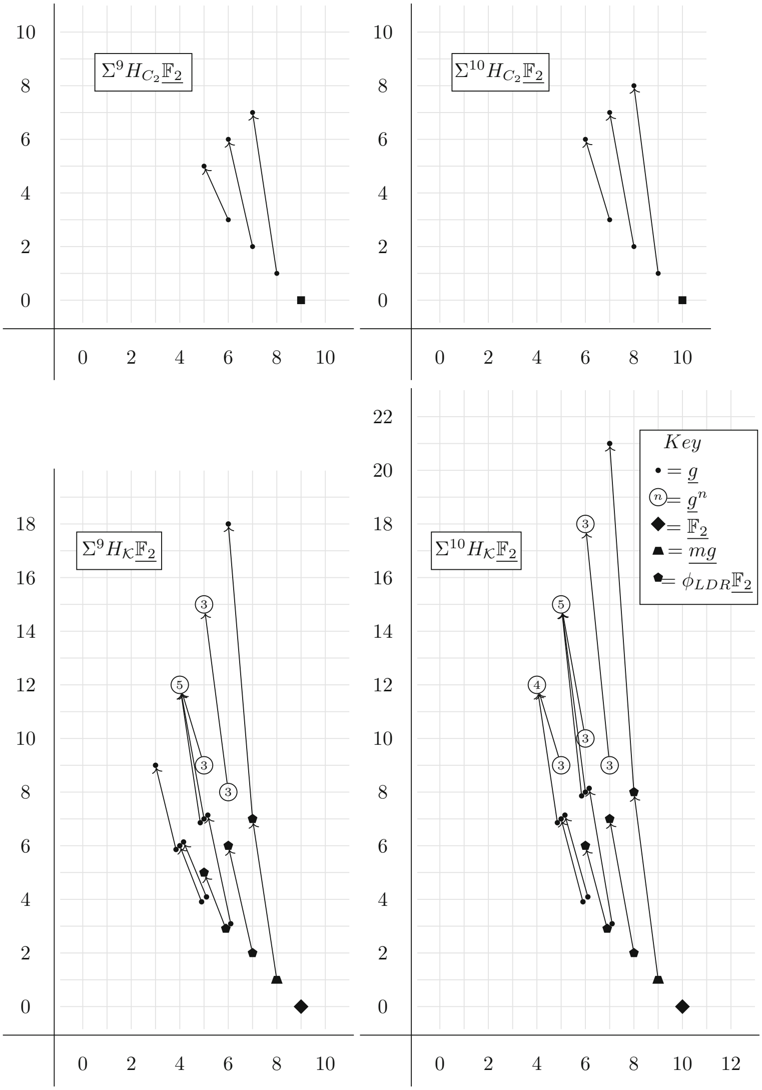

Fig. 3 The slice spectral sequence over C_2 and $C_2 \times C_2$, $n = 9, 10$

Similarly, the $\underline{g}$ summand of $\underline{\pi}_7(P_{10}^{10})$ supports a d_5 to the 14-slice. This can be seen by using the map to $\Sigma^{\rho+5}C$, which induces an equivalence of slices up to level 14. The cofiber sequence

$$\Sigma^{\rho+4} H\underline{f} \longrightarrow \Sigma^{\rho+3} C \longrightarrow \Sigma^{\rho+3} \underline{g}^2$$

shows that $\underline{\pi}_5(\Sigma^{\rho+3}C) = 0$ and $\underline{\pi}_6(\Sigma^{\rho+3}C) \cong \underline{g}^2$. This forces the claimed d_5-differential. Similar arguments produce the spectral sequences displayed in Figs. 4 and 5.

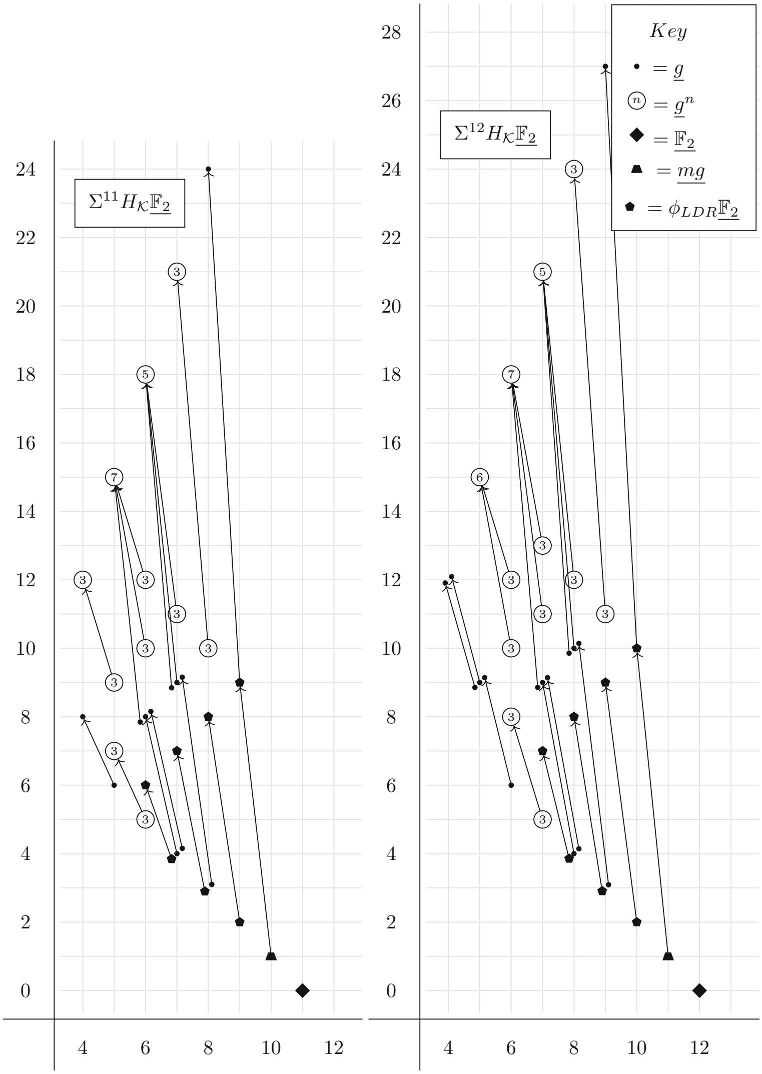

Fig. 4 The slice spectral sequence over $C_2 \times C_2$, $n = 11, 12$

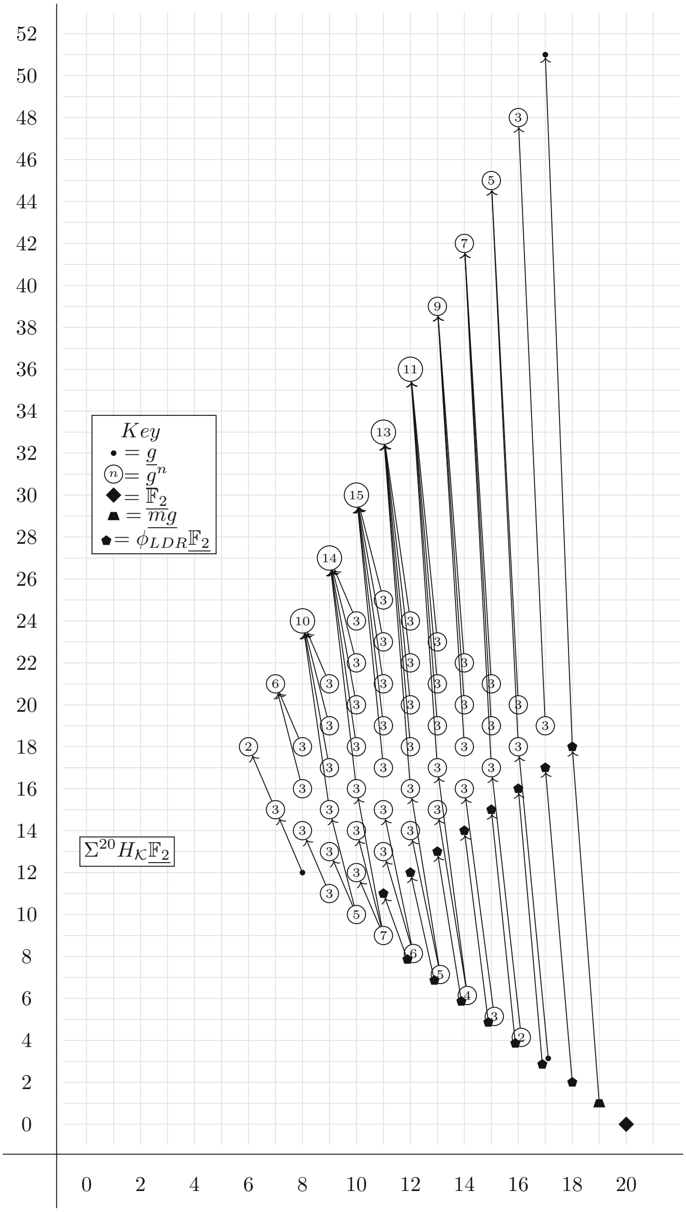

Fig. 5 The slice spectral sequence over $C_2 \times C_2$, $n = 20$

Appendix: Mackey functors

name	Mackey functor	description	reference
$\underline{\mathbb{F}_2} = \blacklozenge$	$\begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[d, "1"] \arrow[dr, "1"] & \\ \mathbb{F}_2 \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[d, "1"] & \mathbb{F}_2, \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd}$		
$\underline{\mathbb{F}_2}^*$	$\begin{tikzcd} & \mathbb{F}_2 & \\ \mathbb{F}_2 \arrow[ur, "1"] & \mathbb{F}_2 \arrow[u, "1"] & \mathbb{F}_2, \arrow[ul, "1"'] \\ & \mathbb{F}_2 \arrow[ul, "1"] \arrow[u, "1"] \arrow[ur, "1"'] & \end{tikzcd}$	$H\underline{\mathbb{F}_2}^* \simeq \Sigma^{4-\rho} H\underline{\mathbb{F}_2}$	Proposition 4.2
$\underline{g} = \bullet$	$\begin{tikzcd} & \mathbb{F}_2 & \\ 0 & 0 & 0, \\ & 0 & \end{tikzcd}$	$\underline{g} = \phi^*_{\mathcal{K}}(\mathbb{F}_2)$	
$\underline{f}$	$\begin{tikzcd} & 0 & \\ 0 & 0 & 0 \\ & \mathbb{F}_2 & \end{tikzcd}$		
$\phi^*_{LDR}\underline{f}$	$\begin{tikzcd} & 0 & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$		Notation 2.5
$\phi^*_{LDR}\underline{\mathbb{F}_2} = ⬟$	$\begin{tikzcd} & \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ & 0 & \end{tikzcd}$		Notation 2.5

name	Mackey functor	description	reference
$\underline{m}$	$\begin{tikzcd} & \mathbb{F}_2 \arrow[dl, "1"'] \arrow[d, "1"] \arrow[dr, "1"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2. \end{tikzcd}$	$\mathrm{coker}\left(\underline{f} \longrightarrow \underline{\mathbb{F}_2}\right)$	
$\underline{mg} =$ ⏢	$\begin{array}{c} 0 \\ \hline \begin{tikzcd} & \mathbb{F}_2 \oplus \mathbb{F}_2 \arrow[dl, "p_1"'] \arrow[d, "\nabla"'] \arrow[dr, "p_2"] & \\ \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2. \end{tikzcd} \end{array}$	$H\underline{mg} \simeq \Sigma^{\rho-2} H\underline{m}^*$	Proposition 4.8
	$\begin{array}{c} 0 \\ \hline 0 \end{array}$		
$\underline{w}$	$\begin{tikzcd} \mathbb{F}_2 \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[d, "1"] & \mathbb{F}_2. \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd}$	$\ker\left(\underline{\mathbb{F}_2} \longrightarrow \underline{g}\right)$	
$\underline{W}$	$\begin{array}{c} \hline \begin{tikzcd} & \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & \\ \mathbb{F}_2 \arrow[ur, "\iota_1"] \arrow[dr, "1"'] & \mathbb{F}_2 \arrow[u, "\iota_2"] \arrow[d, "1"] & \mathbb{F}_2. \arrow[ul, "\iota_3"'] \arrow[dl, "1"] \\ & \mathbb{F}_2 & \end{tikzcd} \end{array}$		

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