ON THE STEENROD MODULE STRUCTURE OF $\mathbb{R}$-MOTIVIC SPANIER-WHITEHEAD DUALS

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Abstract. The $\mathbb{R}$-motivic cohomology of an $\mathbb{R}$-motivic spectrum is a module over the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$. In this paper, we describe how to recover the $\mathbb{R}$-motivic cohomology of the Spanier-Whitehead dual DX of an $\mathbb{R}$-motivic finite complex X, as an $\mathcal{A}^{\mathbb{R}}$-module, given the $\mathcal{A}^{\mathbb{R}}$-module structure on the cohomology of X. As an application, we show that 16 out of 128 different $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1) := \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ are self-dual.

1. Introduction

Given a finite cell complex X, it is useful to determine its Spanier-Whitehead dual DX, which is the dual of the suspension spectrum $\Sigma^\infty X$ in the stable homotopy category of spectra. For instance, the mod 2 cohomology of DX, as a (left) module over the mod 2 Steenrod algebra $\mathcal{A}$, is an input of Adams spectral sequences computing homotopy class of maps out of X. An interesting case is when X is self-dual, as it leads to additional symmetries [MR] often useful for computational purposes.

In the classical case, the $\mathcal{A}$-module structure on $\mathrm{H}^*(\mathrm{DX}; \mathbb{F}_2) \cong \mathrm{H}_*(\mathrm{X}, \mathbb{F}_2)$ is determined easily by the standard formula (1.1) which involves the Kronecker pairing and the antiautomorphism $\chi\colon \mathcal{A} \longrightarrow \mathcal{A}$ of the Steenrod algebra. However, one should not expect an $\mathbb{R}$-motivic generalization of the standard formula because, first, the cohomology of the Spanier Whitehead dual of an $\mathbb{R}$-motivic finite complex X is not always the linear dual of the cohomlogy of X, as the coefficient ring $\mathbb{M}_2$ is not a field (see (2.1)). Second, the $\mathbb{R}$-motivic Steenrod algebra is not known to support an anti-automorphism (but see Appendix A).

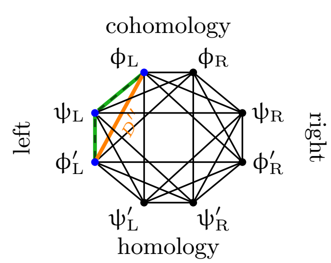

Figure 1.1. Boardman's mandala

The ring of stable operations $\mathrm{E}^*\mathrm{E}$ of a cohomology theory E is not typically equipped with an antiautomorphism [B, p. 204]. The natural conjugation on $\mathrm{E}_*\mathrm{E}$ is not E_*-linear and so does not pass to the dual $\mathrm{E}^*\mathrm{E}$. Even in the presence of an antiautomorphism, the usual formula (1.1) does not yield an $\mathrm{E}^*\mathrm{E}$-action on duals, (see Remark A.1).

This article is concerned with the case $\mathrm{E} = \mathbf{H}_{\mathbb{R}}\mathbb{F}_2$, $\mathbb{R}$-motivic cohomology with coefficients in $\mathbb{F}_2$. We rely on Boardman's mandala [B] (see Figure 1.1) to demonstrate a method that computes the action of

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the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ on the Spanier-Whitehead duals of those finite $\mathbb{R}$-motivic spectra whose cohomology is free over $\mathbb{M}_2$.

Let us pause to briefly discuss Boardman's mandala. Given a finite cell complex X there are eight ways in which its mod 2 homology and cohomology interact with the Steenrod algebra and its dual. They represent the vertices of the mandala. Boardman identified the relationships between them, which represent the edges. Each edge of the mandala corresponds to a formula. For example, the edge D'' in Figure 1.1 corresponds to the formula (see [B, p.190])

$$\langle(\mathrm{D}''\phi'_{\mathrm{L}})(\alpha\otimes\mathsf{f}),\mathsf{x}\rangle=\langle\mathsf{f},\phi'_{\mathrm{L}}(\chi(\alpha)\otimes\mathsf{x})\rangle \tag{1.1}$$

that relates the left $\mathcal{A}$-module structure on the cohomology $\mathrm{H}^*(\mathrm{X})$ with that of the left $\mathcal{A}$-module structure on the homology of X. However, not all edges of the Mandala exist for a general cohomology theory E ([B, Section 6]).

When $\mathrm{H}^\star(\mathrm{X}) := [\mathrm{X},\mathbf{H}_{\mathbb{R}}\mathbb{F}_2]^\star$ is free and finitely generated over $\mathbb{M}_2$, $\mathrm{H}_\star(\mathrm{X})$ is the $\mathbb{M}_2$-linear dual of $\mathrm{H}^\star(\mathrm{X})$, as the relevant universal coefficient spectral sequence collapses. Consequently, the work in [B] relates the left action of $\mathcal{A}^{\mathbb{R}}$ on $\mathrm{H}^\star(\mathrm{X})$ as well as the left action of $\mathcal{A}^{\mathbb{R}}$ on $\mathrm{H}_\star(\mathrm{X})$, to the $\mathcal{A}^{\mathbb{R}}_\star$-comodule structure on $\mathrm{H}^\star(\mathrm{X})$ (see Proposition 3.1, Proposition 3.3 and Proposition 3.4). These relations are the green dashed edges in Figure 1.1. As a result, one deduces the left $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}_\star(\mathrm{X})$ from that of $\mathrm{H}^\star(\mathrm{X})$ without resorting to an anti-automorphism (unlike (1.1)).

Our main application is concerned with identifying the $\mathbb{R}$-motivic spectra in the class $\mathcal{A}_1^{\mathbb{R}}$ introduced in [BGL2]. Each spectrum in $\mathcal{A}_1^{\mathbb{R}}$ is a realization of some $\mathcal{A}^{\mathbb{R}}$-module structure on the subalgebra $\mathcal{A}^{\mathbb{R}}(1) := \mathbb{M}_2^{\mathbb{R}}\langle \mathrm{Sq}^1,\mathrm{Sq}^2\rangle \subset \mathcal{A}^{\mathbb{R}}$ (see Figure 4.1). In the classical case, Davis and Mahowald [DM] showed that the subalgebra $\mathcal{A}(1)$ of the Steenrod algebra admits four different left $\mathcal{A}$-module structures, of which two are self-dual. In [BGL2], we showed that $\mathcal{A}^{\mathbb{R}}(1)$ admits 128 different $\mathcal{A}^{\mathbb{R}}$-module structures. In this paper, we show:

Theorem 1.1. *Among the 128 different $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1)$, only 16 are self-dual.*

Remark 1.2. In [BGL2] we showed that every $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ can be realized as a finite $\mathbb{R}$-motivic spectrum, but we do not know if they are unique. Hence, the spectra realizing a self-dual $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ may not be Spanier-Whitehead self-dual.

Davis and Mahowald also showed [DM] that each realization of $\mathcal{A}(1)$ is the cofiber of a self-map of the spectrum $\mathcal{Y} := \mathbb{S}/2\wedge\mathbb{S}/\eta$, where η is the first Hopf element in the stable stems. In the $\mathbb{R}$-motivic stable stems, both $2,\mathsf{h}\in\pi_{0,0}(\mathbb{S}_{\mathbb{R}})$ are lifts of $2\in\pi_0(\mathbb{S})$ in the classical stable stems, and $\eta_{1,1}\in\pi_{1,1}(\mathbb{S})$ is the unique lift of η in bidegree $(1,1)$. This results in two different $\mathbb{R}$-motivic lifts of $\mathcal{Y}$, namely

$$\mathcal{Y}^{\mathbb{R}}_{(2,1)}=\mathbb{S}_{\mathbb{R}}/2\wedge\mathbb{S}_{\mathbb{R}}/\eta_{1,1} \text{ and } \mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}=\mathbb{S}_{\mathbb{R}}/\mathsf{h}\wedge\mathbb{S}_{\mathbb{R}}/\eta_{1,1}.$$

We showed in [BGL2, Theorem 1.8] that each $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}}(1)$ can be realized as the cofiber of a map between these $\mathbb{R}$-motivic lifts of $\mathcal{Y}$. Here we show:

Theorem 1.3. *Of the self-dual $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1)$, 8 can be realized as the cofiber of a self-map on $\mathcal{Y}^{\mathbb{R}}_{(2,1)}$ and 8 as the cofiber of a self-map on $\mathcal{Y}^{\mathbb{R}}_{(\mathsf{h},1)}$.*

Notation 1.1. In all diagrams depicting modules over the Steenrod algebra, (i.e. in Figure 3.1, Figure 4.1, and Figure 4.2), a dot • represents a rank one free module over the coefficient ring, black vertical lines indicate the action of Sq^1, blue curved lines indicate the action of Sq^2, and red bracket-like lines represent the action of Sq^4. A label on an edge represents that the operation hits that multiple of the generator. For example, in Figure 3.1, $\mathrm{Sq}^2(\mathsf{x}_2)$ is $\tau \cdot \mathsf{x}_4$ and $\mathrm{Sq}^4(\mathsf{x}_2)$ is $\rho^2 \cdot \mathsf{x}_4$.

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2. A review of the $\mathbb{R}$-motivic Steenrod algebra and its dual

In [V], Voevodsky defined the motivic Steenrod operations Sq^n, for $n \geq 0$, and gave a complete description of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$. It is free as a left module over the $\mathbb{R}$-motivic homology of a point,

$$\mathbb{M}_2^{\mathbb{R}} := \pi_\star^{\mathbb{R}} \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \cong \mathbb{F}_2[\tau, \rho], \tag{2.1}$$

where the element τ is in bidegree $\star = (0, -1)$, and ρ is in bidegree $\star = (-1, -1)$.

The subalgebra $\mathbb{M}_2^{\mathbb{R}} \subset \mathcal{A}^{\mathbb{R}}$ is not central, and therefore $\mathcal{A}^{\mathbb{R}}$ has two $\mathbb{M}_2^{\mathbb{R}}$-module structures, one given by left multiplication and the other by right multiplication.

The $\mathbb{R}$-motivic dual Steenrod algebra $\mathcal{A}_\star^{\mathbb{R}}$ is defined to be the (left) $\mathbb{M}_2^{\mathbb{R}}$-linear dual of $\mathcal{A}^{\mathbb{R}}$; it inherits an $\mathbb{M}_2^{\mathbb{R}}$-module structure, which we call the left action. The right $\mathbb{M}_2^{\mathbb{R}}$-action on $\mathcal{A}^{\mathbb{R}}$ also induces an action of $\mathbb{M}_2^{\mathbb{R}}$ on $\mathcal{A}_\star^{\mathbb{R}}$, which we call the right action of $\mathbb{M}_2^{\mathbb{R}}$ on $\mathcal{A}_\star^{\mathbb{R}}$ (see [V, p. 48])[1]. These correspond to the left and the right unit

$$\eta_{\mathrm{L}}, \eta_{\mathrm{R}} \colon \mathbb{M}_2^{\mathbb{R}} \longrightarrow \mathcal{A}_\star^{\mathbb{R}}$$

of the Hopf algebroid $(\mathbb{M}_2^{\mathbb{R}}, \mathcal{A}_\star^{\mathbb{R}})$. Explicitly,

$$\mathcal{A}_\star^{\mathbb{R}} \cong \frac{\mathbb{M}_2^{\mathbb{R}}[\tau_0, \tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots]}{\tau_n^2 = \tau \xi_{n+1} + \rho \tau_0 \xi_{n+1} + \rho \tau_{n+1}} \tag{2.2}$$

with $\eta_{\mathrm{L}}(\rho) = \eta_{\mathrm{R}}(\rho) = \rho$, $\eta_{\mathrm{L}}(\tau) = \tau$ and $\eta_{\mathrm{R}}(\tau) = \tau + \rho\tau_0$. The comultiplication

$$\Delta : \mathcal{A}_\star^{\mathbb{R}} \longrightarrow \mathcal{A}_\star^{\mathbb{R}} \underset{\mathbb{M}_2^{\mathbb{R}}}{\otimes} \mathcal{A}_\star^{\mathbb{R}} \tag{2.3}$$

is given by

- $\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i$, and
- $\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \tau_{n-i}$,

for all $n \in \mathbb{N}$, where ξ_0 is the unit 1. The conjugation map $\mathsf{c} \colon \mathcal{A}_\star^{\mathbb{R}} \longrightarrow \mathcal{A}_\star^{\mathbb{R}}$ of the Hopf algebroid structure sends

- $\mathsf{c}(\rho) = \rho$,
- $\mathsf{c}(\tau) = \tau + \rho\tau_0$,
- $\mathsf{c}(\xi_n) = \displaystyle\sum_{i=0}^{n-1} \xi_{n-i}^{2^i} \mathsf{c}(\xi_i)$, and

[1]Since $\mathbb{M}_2^{\mathbb{R}}$ is commutative, there is no meaningful distinction between "left" and "right" actions. The adjectives are merely a bookkeeping device.

- $\mathsf{c}(\tau_n) = \tau_n + \sum_{i=0}^{n-1} \xi_{n-i}^{2^i}\mathsf{c}(\tau_i)$.

Remark 2.1. The coproduct Δ in (2.3) is an $\mathbb{M}_2^{\mathbb{R}}$-bimodule map.

Remark 2.2. The conjugation is not a map of left $\mathbb{M}_2^{\mathbb{R}}$-modules. In fact, it interchanges the left and right $\mathbb{M}_2^{\mathbb{R}}$-module structures on $\mathcal{A}_\star^{\mathbb{R}}$.

2.1. **Kronecker product.**

The $\mathbb{R}$-motivic Kronecker product is a natural pairing between $\mathbb{R}$-motivic homology and cohomology which is constructed as follows: If $\varphi : \mathrm{X} \longrightarrow \Sigma^{\mathsf{i},\mathsf{j}}\mathbf{H}_{\mathbb{R}}\mathbb{F}_2$ represents the class $[\varphi] \in \mathrm{H}^\star(\mathrm{X})$ and $\mathsf{x} : \Sigma^{\mathsf{m},\mathsf{n}}\mathbb{S}_{\mathbb{R}} \longrightarrow \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathrm{X}$ represents $[\mathsf{x}] \in \mathrm{H}_{\mathsf{m},\mathsf{n}}(\mathrm{X})$, then the composition

$$\Sigma^{\mathsf{m},\mathsf{n}}\mathbb{S}_{\mathbb{R}} \longrightarrow \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathrm{X} \longrightarrow \Sigma^{\mathsf{i},\mathsf{j}}\mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \longrightarrow \Sigma^{\mathsf{i},\mathsf{j}}\mathbf{H}_{\mathbb{R}}\mathbb{F}_2$$

is the element $\langle \mathsf{x}, \varphi \rangle \in \pi_\star(\mathbf{H}_{\mathbb{R}}\mathbb{F}_2) \cong \mathbb{M}_2^{\mathbb{R}}$.

The Kronecker pairing leads to a homomorphism

$$\mathfrak{n} : \mathrm{H}^\star(\mathrm{X}) \longrightarrow \mathrm{Hom}_{\mathbb{M}_2^{\mathbb{R}}}(\mathrm{H}_\star(\mathrm{X}), \mathbb{M}_2^{\mathbb{R}}), \tag{2.4}$$

where $\mathfrak{n}(\varphi)(\mathsf{x}) = \langle \mathsf{x}, \varphi \rangle$.

Remark 2.3. When $\mathrm{H}_\star(\mathrm{X})$ is free and finitely generated as an $\mathbb{M}_2^{\mathbb{R}}$-module, the map $\mathfrak{n}$ in (2.4) is an isomorphism. Consequently, elements in $\mathrm{H}^\star(\mathrm{X})$ can be identified with linear maps from $\mathrm{H}_\star(\mathrm{X})$, and the Kronecker product is simply the evaluation of functionals.

Notation 2.1. Since both $\mathcal{A}^{\mathbb{R}}$ and $\mathcal{A}_\star^{\mathbb{R}}$ have a left and a right action of $\mathbb{M}_2^{\mathbb{R}}$, let $\mathcal{A}^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}}^{\text{left}} \mathcal{A}_\star^{\mathbb{R}}$ (likewise $\mathcal{A}^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}}^{\text{right}} \mathcal{A}_\star^{\mathbb{R}}$) denote the tensor product of left (likewise right) $\mathbb{M}_2^{\mathbb{R}}$-modules.

Remark 2.4. When X is $\mathbf{H}_{\mathbb{R}}\mathbb{F}_2$, the Kronecker product is a map from $\mathcal{A}_\star^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}}^{\text{left}} \mathcal{A}^{\mathbb{R}}$ to $\mathbb{M}_2^{\mathbb{R}}$.

2.2. **The Milnor basis.**

The dual Steenrod algebra $\mathrm{H}_\star(\mathbf{H}_{\mathbb{R}}\mathbb{F}_2) \cong \mathcal{A}_\star^{\mathbb{R}}$ is free and degree-wise finitely generated as an $\mathbb{M}_2^{\mathbb{R}}$-module. Consequently, the natural map of (2.4) gives an isomorphism

$$\mathcal{A}^{\mathbb{R}} \cong \mathrm{Hom}_{\mathbb{M}_2^{\mathbb{R}}}(\mathcal{A}_\star^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) \tag{2.5}$$

of left $\mathbb{M}_2^{\mathbb{R}}$-modules. Taking advantage of the above isomorphism, Voevodsky [V, §13] defines the Milnor basis of the $\mathbb{R}$-motivic Steenrod algebra using the monomial basis of the dual Steenrod algebra (2.2).

For finite sequences $\mathrm{E} = (\mathsf{e}_0, \mathsf{e}_1, \ldots, \mathsf{e}_m)$ and $\mathrm{R} = (\mathsf{r}_1, \ldots, \mathsf{r}_n)$ of non-negative integers, let $\rho(\mathrm{E}, \mathrm{R})$ denote the element in $\mathcal{A}^{\mathbb{R}}$ dual to the monomial

$$\tau(\mathrm{E})\xi(\mathrm{R}) := \prod_{\mathsf{i}\geq 0} \tau_{\mathsf{i}}^{\mathsf{e}_{\mathsf{i}}} \prod_{\mathsf{j}\geq 1} \xi_{\mathsf{i}}^{\mathsf{r}_{\mathsf{i}}}$$

in $\mathcal{A}_\star^{\mathbb{R}}$. It is standard practice to set $\mathcal{P}^{\mathrm{R}} := \rho(\mathbf{0}, \mathrm{R})$ and $\mathcal{Q}^{\mathrm{E}} := \rho(\mathrm{E}, \mathbf{0})$. Moreover, $\mathcal{Q}_i$ is shorthand for the dual to τ_i.

degree	$\mathsf{x} \in \mathcal{A}^{\mathbb{R}}_\star$	$\mathsf{c}(\mathsf{x})$	$\mathsf{x}^* \in \mathcal{A}^{\mathbb{R}}$	x^* in terms of $\mathcal{G}$
$(0,0)$	1	1	1	1
$(1,0)$	τ_0	τ_0	$\mathcal{Q}_0$	Sq^1
$(2,1)$	ξ_1	ξ_1	$\mathcal{P}^1$	Sq^2
$(3,1)$	$\tau_0\xi_1$	$\tau_0\xi_1$	$\mathcal{Q}_0\mathcal{P}^1$	$\mathrm{Sq}^1\,\mathrm{Sq}^2$
$(3,1)$	τ_1	$\tau_1+\tau_0\xi_1$	$\mathcal{Q}_1$	$\mathrm{Sq}^1\,\mathrm{Sq}^2+\mathrm{Sq}^2\,\mathrm{Sq}^1$
$(4,1)$	$\tau_0\tau_1$	$\tau_0\tau_1+\tau\xi_1^2$ $+\rho\tau_0\xi_1^2+\rho\tau_1\xi_1$	$\mathcal{Q}_0\mathcal{Q}_1$	$\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1$
$(4,2)$	ξ_1^2	ξ_1^2	$\mathcal{P}^2$	Sq^4
$(5,2)$	$\tau_0\xi_1^2$	$\tau_0\xi_1^2$	$\mathcal{Q}_0\mathcal{P}^2$	$\mathrm{Sq}^1\,\mathrm{Sq}^4$
$(5,2)$	$\tau_1\xi_1$	$\tau_1\xi_1+\tau_0\xi_1^2$	$\mathcal{Q}_1\mathcal{P}^1$	$\mathrm{Sq}^1\,\mathrm{Sq}^4+\mathrm{Sq}^4\,\mathrm{Sq}^1$
$(6,2)$	$\tau_0\tau_1\xi_1$	$\tau_0\tau_1\xi_1+\tau\xi_1^3$ $+\rho\tau_0\xi_1^3+\rho\tau_1\xi_1^2$	$\mathcal{Q}_0\mathcal{Q}_1\mathcal{P}^1$	$\mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^1$
$(6,3)$	ξ_1^3	ξ_1^3	$\mathcal{P}^3$	$\mathrm{Sq}^2\,\mathrm{Sq}^4+\tau\,\mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^1$
$(6,3)$	ξ_2	$\xi_2+\xi_1^3$	$\mathcal{P}^{(0,1)}$	$\mathrm{Sq}^2\,\mathrm{Sq}^4+\mathrm{Sq}^4\,\mathrm{Sq}^2$
$(7,3)$	τ_2	$\tau_2+\tau_1\xi_1^2$ $+\tau_0\xi_2+\tau_0\xi_1^3$	$\mathcal{Q}_2$	$\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4+\mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^2$ $+\mathrm{Sq}^2\,\mathrm{Sq}^4\,\mathrm{Sq}^1+\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1$
$(7,3)$	$\tau_0\xi_1^3$	$\tau_0\xi_1^3$	$\mathcal{Q}_0\mathcal{P}^3$	$\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4+\rho\,\mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^1$
$(7,3)$	$\tau_0\xi_2$	$\tau_0\xi_2+\tau_0\xi_1^3$	$\mathcal{Q}_0\mathcal{P}^{(0,1)}$	$\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4+\mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^2$
$(7,3)$	$\tau_1\xi_1^2$	$\tau_1\xi_1^2+\tau_0\xi_1^3$	$\mathcal{Q}_1\mathcal{P}^2$	$\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4+\rho\,\mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^1$ $+\mathrm{Sq}^2\,\mathrm{Sq}^4\,\mathrm{Sq}^1$
$(8,4)$	ξ_1^4	ξ_1^4	$\mathcal{P}^4$	Sq^8
$(8,4)$	$\xi_1\xi_2$	$\xi_1\xi_2+\xi_1^4$	$\mathcal{P}^{(1,1)}$	$\mathrm{Sq}^2\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\tau\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4\,\mathrm{Sq}^1$

TABLE 2.1. The Milnor basis in low degrees

In Table 2.1, we record, for each monomial $\tau(\mathrm{E})\xi(\mathrm{R}) \in \mathcal{A}^{\mathbb{R}}_\star$ in low degree, its image under the conjugation c and its dual element in $\mathcal{A}^{\mathbb{R}}$, both in terms of the Milnor basis as well as in terms of the generators $\mathcal{G} := \{\mathrm{Sq}^{2^k} : k \geq 1\}$. The latter description will be used in Section 3.3 and Section 4.

A number of these descriptions in terms of $\mathcal{G}$ can be found in [V]. For example, see [V, Lemma 13.1 and Lemma 13.6]. The Adem relations (see [BGL2, Appendix A]) are another useful tool. For example, the Adem relation $\mathrm{Sq}^2\,\mathrm{Sq}^4 = \mathrm{Sq}^6 + \tau\,\mathrm{Sq}^5\,\mathrm{Sq}^1$ leads to the description for $P^3 = \mathrm{Sq}^6$. The formula for $\mathcal{P}^{(0,1)}$ follows from [K, (6)]. Finally, the formula for $\mathcal{P}^{(1,1)}$ can be deduced from expressing $\mathrm{Sq}^6\,\mathrm{Sq}^2$ in terms of the Milnor basis. This can be done by evaluating the formula [V, (12.9)]

$$\langle \mathsf{x}, \varphi\psi\rangle = \sum \left\langle \mathsf{x}', \varphi\eta_R\big(\langle \mathsf{x}'', \psi\rangle\big)\right\rangle, \qquad \Delta(\mathsf{x}) = \sum \mathsf{x}' \otimes \mathsf{x}''$$

at $\varphi = \mathrm{Sq}^6$, $\psi = \mathrm{Sq}^2$, and x monomials in low degree. This shows that $\mathrm{Sq}^6\,\mathrm{Sq}^2$ is the sum $\mathcal{P}^{(1,1)} + \tau\mathcal{Q}_0\mathcal{Q}_1\mathcal{P}^2$.

3. DUALIZING $\mathcal{A}^{\mathbb{R}}$-MODULES

For any $\mathbb{R}$-motivic spectrum X, its Spanier-Whitehead dual is the function spectrum $\mathrm{DX} := \mathrm{F}(\mathrm{X}, \mathbb{S}_{\mathbb{R}})$. The goal of this section is to identify the $\mathcal{A}^{\mathbb{R}}$-module structure $\mathrm{H}^\star(\mathrm{DX})$ given the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^\star(\mathrm{X})$ under the following assumption.

Assumption 3.1. Let X be a finite $\mathbb{R}$-motivic spectrum such that its homology $\mathrm{H}_\star(\mathrm{X})$ is free over $\mathbb{M}_2^{\mathbb{R}}$.

Notation 3.1. For an $\mathbb{M}_2^{\mathbb{R}}$-module $\mathbf{N}$ let

$$\mathbf{N}^\vee := \mathrm{Hom}_{\mathbb{M}_2^{\mathbb{R}}}(\mathbf{N}, \mathbb{M}_2^{\mathbb{R}})$$

be the set of $\mathbb{M}_2^{\mathbb{R}}$-linear functionals.

3.1. **From ψ_{L} to ϕ'_{L}.** Recall that $\mathrm{H}^\star(\mathrm{X})$ is naturally a left $\mathcal{A}^{\mathbb{R}}$-module. We will also use an $\mathcal{A}^{\mathbb{R}}_\star$-comodule structure on $\mathrm{H}^\star(\mathrm{X})$

$$\psi_{\mathrm{L}} \colon \mathrm{H}^\star(\mathrm{X}) \longrightarrow \mathcal{A}^{\mathbb{R}}_\star \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathrm{H}^\star(\mathrm{X}) \tag{3.1}$$

which can be constructed as follows.

First, note that $\mathcal{A}^{\mathbb{R}}_\star$ is free as a right $\mathbb{M}_2^{\mathbb{R}}$-module with basis $\mathcal{B}$ given by the conjugate of any left $\mathbb{M}_2^{\mathbb{R}}$-module basis. Then we have a splitting

$$\mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \simeq \bigvee_{\mathcal{B}} \mathbf{H}_{\mathbb{R}}\mathbb{F}_2$$

as right $\mathbf{H}_{\mathbb{R}}\mathbb{F}_2$-modules. Define a map of motivic spectra ψ as the composite

$$\psi \colon \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \cong \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbb{S}_{\mathbb{R}} \xrightarrow{\mathrm{id}\wedge\iota} \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \simeq \textstyle\bigvee_{\mathcal{B}} \mathbf{H}_{\mathbb{R}}\mathbb{F}_2,$$

where ι is the unit map of $\mathbf{H}_{\mathbb{R}}\mathbb{F}_2$. For any finite motivic spectrum, the map ψ induces the map ψ_{L} (see [B, Theorem 2.9(b)]) giving $\mathrm{H}^\star(\mathrm{X})$ the structure of an $\mathcal{A}^{\mathbb{R}}_\star$-comodule as explained in [B, Section 6]. Further, Boardman showed that:

Proposition 3.1. [B, Lemma 3.4] *Let $\mathbf{N}$ be a left $\mathcal{A}^{\mathbb{R}}_\star$-comodule. Then $\mathbf{N}^\vee$ inherits a left $\mathcal{A}^{\mathbb{R}}$-module structure*

$$\phi_{\mathrm{L}} \colon \mathcal{A}^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbf{N}^\vee \longrightarrow \mathbf{N}^\vee$$

via the formula

$$(\varphi \cdot \lambda)(n) = (\varphi \otimes \lambda)\psi_{\mathrm{L}}(n) \tag{3.2}$$

for $\varphi \in \mathcal{A}^{\mathbb{R}}$, $\lambda \in \mathbf{N}^\vee$, and $n \in \mathbf{N}$.

Remark 3.2. If $\psi_{\mathrm{L}}(n) = \sum_i a_i \otimes n_i$, for $a_i \in \mathcal{A}^{\mathbb{R}}_\star$ and $n_i \in \mathbf{N}$, then (3.2) can be rewritten as

$$(\varphi \cdot \lambda)(n) = \sum_i \varphi\Big(a_i \eta_{\mathrm{R}}\big(\lambda(n_i)\big)\Big). \tag{3.3}$$

Combining Proposition 3.1 with the following result, one can deduce the left $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^\star(\mathrm{DX})$ (ϕ'_{L} in Figure 1.1) from the left $\mathcal{A}^{\mathbb{R}}_\star$-comodule structure on $\mathrm{H}^\star(\mathrm{X})$ (ψ_{L} in Figure 1.1).

Proposition 3.2. *Suppose* X *satisfies Assumption 3.1. There are isomorphisms of left $\mathcal{A}^{\mathbb{R}}$-modules* $\mathrm{H}^\star(\mathrm{DX}) \cong (\mathrm{H}_\star(\mathrm{DX}))^\vee \cong (\mathrm{H}^\star(\mathrm{X}))^\vee$.

Proof. Under Assumption 3.1 the map $\mathfrak{n} : \mathrm{H}^{\star}(\mathrm{DX}) \longrightarrow (\mathrm{H}_{\star}(\mathrm{DX}))^{\vee}$ defined in (2.4), is not just an isomorphism of $\mathbb{M}_2^{\mathbb{R}}$-modules (see Remark 2.3), but also an isomorphism of left $\mathcal{A}^{\mathbb{R}}$-modules according to [B, Lemma 6.2].

For the second isomorphism, first note that Assumption 3.1 implies that there exists an isomorphism

$$\mathrm{H}_{\star}(\mathrm{DX}) \cong \mathrm{H}^{\star}(\mathrm{X}) \tag{3.4}$$

of $\mathbb{M}_2^{\mathbb{R}}$-modules. By Proposition 3.1, it is enough to lift (3.4) to an isomorphism of $\mathcal{A}_{\star}^{\mathbb{R}}$-comodules. To this end, we first observe that the comodule structure on $\mathrm{H}^{\star}(\mathrm{X})$ is induced by the map

$$\mathrm{F}(\mathrm{X}, \mathbf{H}_{\mathbb{R}}\mathbb{F}_2) \cong \mathrm{F}(\mathrm{X}, \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbb{S}_{\mathbb{R}}) \xrightarrow{\mathrm{F}(\mathrm{X}, \mathrm{id}\wedge\iota)} \mathrm{F}(\mathrm{X}, \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbf{H}_{\mathbb{R}}\mathbb{F}_2).$$

(see (3.1) or [B, Theorem 5.4])). The result then follows from the commutativity of the diagram

$$\begin{array}{ccc}
\mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathrm{F}(\mathrm{X}, \mathbb{S}_{\mathbb{R}}) & \longrightarrow & \mathrm{F}(\mathrm{X}, \mathbf{H}_{\mathbb{R}}\mathbb{F}_2) \\
\| & & \| \\
\mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbb{S}_{\mathbb{R}} \wedge \mathrm{F}(\mathrm{X}, \mathbb{S}_{\mathbb{R}}) & \longrightarrow & \mathrm{F}(\mathrm{X}, \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbb{S}_{\mathbb{R}}) \\
\big\downarrow{\scriptstyle \mathrm{id}\wedge\iota\wedge\mathrm{id}} & & \big\downarrow{\scriptstyle (\mathrm{id}\wedge\iota)_*} \\
\mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathrm{F}(\mathrm{X}, \mathbb{S}_{\mathbb{R}}) & \longrightarrow & \mathrm{F}(\mathrm{X}, \mathbf{H}_{\mathbb{R}}\mathbb{F}_2 \wedge \mathbf{H}_{\mathbb{R}}\mathbb{F}_2),
\end{array}$$

where the horizontal maps are evaluation at X. ■

3.2. **From ϕ_{L} to ψ_{L}.** For any $\varphi \in \mathcal{A}^{\mathbb{R}} \cong \mathrm{Hom}_{\mathbb{M}_2^{\mathbb{R}}}(\mathcal{A}_{\star}^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$, let $\varphi\mathsf{c}$ denote the composition

$$\varphi\mathsf{c} : \mathcal{A}_{\star}^{\mathbb{R}} \xrightarrow{\mathsf{c}} \mathcal{A}_{\star}^{\mathbb{R}} \xrightarrow{\varphi} \mathbb{M}_2^{\mathbb{R}},$$

which is a right $\mathbb{M}_2^{\mathbb{R}}$-module map as the conjugation c is an isomorphism from the right $\mathbb{M}_2^{\mathbb{R}}$-module structure to the left $\mathbb{M}_2^{\mathbb{R}}$-module structure of $\mathcal{A}_{\star}^{\mathbb{R}}$.

Proposition 3.3. *Let* $\mathbf{N}$ *be a left* $\mathcal{A}_{\star}^{\mathbb{R}}$*-comodule with coproduct* ψ_{L}*. Then, for* $n \in \mathbf{N}$ *and* $\varphi \in \mathcal{A}^{\mathbb{R}}$*, the formula*

$$\varphi \cdot n = (\varphi\mathsf{c} \otimes \mathrm{id})\psi_{\mathrm{L}}(n)$$

defines a left $\mathcal{A}^{\mathbb{R}}$*-module structure on* $\mathbf{N}$*.*

Proof. Using the coassociativity of the coaction, the statement reduces to checking that

$$(\varphi\psi)(\mathsf{c}(a)) = \sum \varphi\Big(\mathsf{c}\big(\eta_{\mathrm{L}}(\psi(\mathsf{c}(a_i')))a_i''\big)\Big), \tag{3.5}$$

for $\varphi, \psi \in \mathcal{A}^{\mathbb{R}}$ and $a \in \mathcal{A}_{\star}^{\mathbb{R}}$. The formula (3.5) follows from combining [B, Lemma 3.3(a)] with $\mathsf{c} \circ \eta_{\mathrm{L}} = \eta_{\mathrm{R}}$ and

$$\Delta(\mathsf{c}(a)) = \sum_i \mathsf{c}(a_i'') \otimes \mathsf{c}(a_i')$$

whenever $\Delta(a) = \sum_i a_i' \otimes a_i''$. ■

Remark 3.3. The right $\mathbb{M}_2^{\mathbb{R}}$-module structure on $\mathcal{A}_\star^{\mathbb{R}}$ is defined [V, Section 12] such that

$$a \cdot \eta_{\mathrm{R}}(m)(\varphi) = a(\varphi \cdot m)$$

for $m \in \mathbb{M}_2^{\mathbb{R}}$, $a \in \mathcal{A}_\star^{\mathbb{R}}$ and $\varphi \in \mathcal{A}^{\mathbb{R}}$. This shows that evaluation pairing defines a map

$$\mathcal{A}^{\mathbb{R}} \otimes^{\mathrm{right}}_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_\star^{\mathbb{R}} \longrightarrow \mathbb{M}_2^{\mathbb{R}}$$

of $\mathbb{M}_2^{\mathbb{R}}$-bimodules, where the left $\mathbb{M}_2^{\mathbb{R}}$-module structure on $\mathcal{A}^{\mathbb{R}} \otimes^{\mathrm{right}}_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_\star^{\mathbb{R}}$ is obtained via the left action on $\mathcal{A}^{\mathbb{R}}$, and the right $\mathbb{M}_2^{\mathbb{R}}$-module structure via the left action on $\mathcal{A}_\star^{\mathbb{R}}$. Consequently, the left action constructed in Proposition 3.3 can be described as the composition ϕ_{L} in the diagram

$$\begin{array}{ccccc}
\mathcal{A}^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbf{N} & \xrightarrow{\mathrm{id} \otimes \psi_{\mathrm{L}}} & \mathcal{A}^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} (\mathcal{A}_\star^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbf{N}) & \cong & (\mathcal{A}^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_\star^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbf{N} \\
& & & & \downarrow{\scriptstyle \mathrm{id} \otimes \mathsf{c} \otimes \mathrm{id}} \\
& & & & (\mathcal{A}^{\mathbb{R}} \otimes^{\mathrm{right}}_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_\star^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbf{N} \\
& & & & \downarrow{\scriptstyle \mathrm{eval} \otimes \mathrm{id}} \\
& \overset{\phi_{\mathrm{L}}}{\dashrightarrow} & & & \mathbf{N} \cong \mathbb{M}_2^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbf{N}.
\end{array}$$

Note that while c is not a right $\mathbb{M}_2^{\mathbb{R}}$-module map, the composition

$$\mathcal{A}^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_\star^{\mathbb{R}} \xrightarrow{\mathrm{id} \otimes \mathsf{c}} \mathcal{A}^{\mathbb{R}} \otimes^{\mathrm{right}}_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_\star^{\mathbb{R}} \xrightarrow{\mathrm{eval}} \mathbb{M}_2^{\mathbb{R}}$$

is a map of $\mathbb{M}_2^{\mathbb{R}}$-bimodules.

If we set $\mathbf{N} = \mathrm{H}^\star(\mathrm{X})$, i.e. the cohomology of a finite spectrum X with the $\mathcal{A}_\star$-comodule structure of (3.1), Proposition 3.3 recovers the usual $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^\star(\mathrm{X})$ (see [B, Lemma 6.3]). Our next result reverse-engineers Proposition 3.3 to obtain a formula that calculates the $\mathcal{A}_\star^{\mathbb{R}}$-comodule on $\mathrm{H}^\star(\mathrm{X})$ (ψ_{L} in Figure 1.1) from the $\mathcal{A}^{\mathbb{R}}$-module on $\mathrm{H}^\star(\mathrm{X})$ (ϕ_{L} in Figure 1.1).

Let $\mathscr{B}$ be the monomial basis of the left $\mathbb{M}_2^{\mathbb{R}}$-module structure on $\mathcal{A}_\star^{\mathbb{R}}$ (as in Section 2.2). For simplicity, let $\mathbf{b}_i$ denote the elements of $\mathscr{B}$, and let $\mathbf{B}^i \in \mathcal{A}^{\mathbb{R}}$ be the dual basis in the following result.

Proposition 3.4. *Let $\mathbf{N}$ be a left $\mathcal{A}_\star^{\mathbb{R}}$-comodule with coaction map ψ_{L}. Then ψ_{L} is related to ϕ_{L} using the formula*

$$\psi_{\mathrm{L}}(n) = \sum_i c(\mathbf{b}_i) \otimes (\mathbf{B}^i \cdot n),$$

where $\cdot$ is the action of $\mathcal{A}^{\mathbb{R}}$ on $\mathbf{N}$ constructed using Proposition 3.3.

Proof. Since $\{c(\mathbf{b}_i)\}$ is a basis for $\mathcal{A}_\star^{\mathbb{R}}$ as a free right $\mathbb{M}_2^{\mathbb{R}}$-module, it follows that there is a unique expression $\psi_{\mathrm{L}}(n) = \sum_i c(\mathbf{b}_i) \otimes n_i$ for appropriate elements n_i.

On the other hand,

$$\begin{aligned} \mathbf{B}^k \cdot n &= (\mathbf{B}^k \mathsf{c} \otimes \mathrm{id})\psi_{\mathrm{L}}(n) \\ &= \sum_i \mathbf{B}^k \mathsf{c}(\mathsf{c}(\mathbf{b}_i)) \otimes n_i \\ &= \sum_i \mathbf{B}^k(\mathbf{b}_i) \otimes n_i \\ &= n_k \end{aligned}$$

by Proposition 3.3. ∎

3.3. **Preliminary examples.** We now demonstrate the usefulness of Proposition 3.1, Proposition 3.3, and Proposition 3.4 by identifying the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^\star(\mathrm{DX})$, for a few well-known finite $\mathbb{R}$-motivic finite complexes X.

Notation 3.2. In the following examples, the $\mathbb{R}$-motivic spectrum X will satisfy Assumption 3.1. In particular, $\mathrm{H}^\star(\mathrm{X})$ will be a free $\mathbb{M}_2^{\mathbb{R}}$-module. By $\mathsf{x}_{\mathsf{i},\mathsf{j}}$, we will denote an element of its $\mathbb{M}_2^{\mathbb{R}}$-basis which lives in cohomological bidegree (i,j). By $\hat{\mathsf{x}}_{\mathsf{i},\mathsf{j}}$, we will denote an element of $(\mathrm{H}^\star(\mathrm{X}))^\vee$ dual to $\mathsf{x}_{\mathsf{i},\mathsf{j}}$. Note that the bidegree of $\hat{\mathsf{x}}_{\mathsf{i},\mathsf{j}}$ is $(-\mathsf{i},-\mathsf{j})$ under the isomorphism $(\mathrm{H}^\star(\mathrm{X}))^\vee \cong \mathrm{H}^\star(\mathrm{DX})$.

Example 3.1 (The $\mathbb{R}$-motivic mod 2 Moore spectrum)**.** As an $\mathbb{M}_2^{\mathbb{R}}$-module, $\mathrm{H}^\star(\mathbb{S}_{\mathbb{R}}/2)$ has generators $\mathsf{x}_{0,0}$ and $\mathsf{x}_{1,0}$. The $\mathcal{A}^{\mathbb{R}}$-module structure is then determined by the relations

$$\mathrm{Sq}^1(\mathsf{x}_{0,0}) = \mathsf{x}_{1,0},\ \mathrm{Sq}^2(\mathsf{x}_{0,0}) = \rho\mathsf{x}_{1,0}.$$

By Proposition 3.4, we get

$$\psi_{\mathrm{L}}(\mathsf{x}_{1,1}) = 1 \otimes \mathsf{x}_{1,1},\ \psi_{\mathrm{L}}(\mathsf{x}_0) = 1 \otimes \mathsf{x}_{0,0} + \tau_0 \otimes \mathsf{x}_{1,0} + \rho\xi_{1,0} \otimes \mathsf{x}_{1,1},$$

which determines the $\mathcal{A}_\star^{\mathbb{R}}$-comodule structure on $\mathrm{H}^\star(\mathbb{S}_{\mathbb{R}}/2)$. Then we apply Proposition 3.1, in particular (3.3), to obtain

$$\mathrm{Sq}^1(\hat{\mathsf{x}}_{1,0}) = \hat{\mathsf{x}}_{0,0},\ \mathrm{Sq}^2(\hat{\mathsf{x}}_{1,0}) = \rho\hat{\mathsf{x}}_{0,0},$$

which shows $\left(\mathrm{H}^\star(\mathbb{S}_{\mathbb{R}}/2)\right)^\vee \cong \Sigma^{-1}\mathrm{H}^\star(\mathbb{S}_{\mathbb{R}}/2)$ as $\mathcal{A}^{\mathbb{R}}$-modules. This aligns with the fact that $D(\mathbb{S}_{\mathbb{R}}/2)$ is equivalent to $\Sigma^{-1}\mathbb{S}_{\mathbb{R}}/2$.

Example 3.2 ($\mathbb{R}$-motivic mod h Moore spectrum)**.** As a graded $\mathbb{M}_2^{\mathbb{R}}$-module, $\mathrm{H}^\star(\mathbb{S}/\mathsf{h})$ is isomorphic to $\mathrm{H}^\star(\mathbb{S}/2)$. However, they differ in their $\mathcal{A}^{\mathbb{R}}$-module structures in that

$$\mathrm{Sq}^1(\mathsf{x}_{0,0}) = \mathsf{x}_{1,0},\ \mathrm{Sq}^2(\mathsf{x}_{0,0}) = 0$$

determines the $\mathcal{A}^{\mathbb{R}}$-module structure on $\mathrm{H}^\star(\mathbb{S}/\mathsf{h})$. By Proposition 3.4

$$\psi_{\mathrm{L}}(\mathsf{x}_{1,1}) = 1 \otimes \mathsf{x}_{1,1},\ \psi_{\mathrm{L}}(\mathsf{x}_{0,0}) = 1 \otimes \mathsf{x}_{0,0} + \tau_0 \otimes \mathsf{x}_{1,0},$$

and using (3.3) we see that $\left(\mathrm{H}^\star(\mathbb{S}_{\mathbb{R}}/\mathsf{h})\right)^\vee \cong \Sigma^{-1}\mathrm{H}^\star(\mathbb{S}_{\mathbb{R}}/\mathsf{h})$. This aligns with the fact that $D(\mathbb{S}_{\mathbb{R}}/\mathsf{h})$ is equivalent to $\Sigma^{-1}\mathbb{S}_{\mathbb{R}}/\mathsf{h}$.

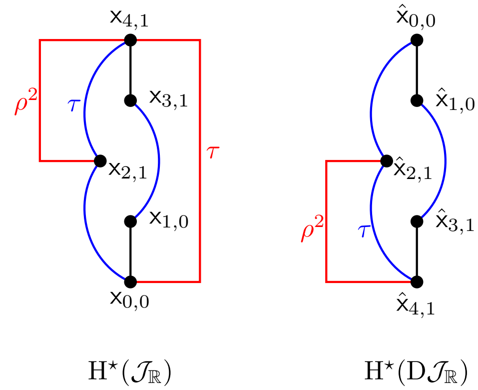

FIGURE 3.1. The $\mathcal{A}^{\mathbb{R}}$-module structures on the $\mathbb{R}$-motivic $\mathfrak{J}$oker and its dual.

Example 3.3. (The $\mathbb{R}$-motivic $\mathfrak{J}$oker) The $\mathcal{A}^{\mathbb{R}}(1)$-module of the $\mathbb{R}$-motivic $\mathfrak{J}$oker $\mathcal{J}_{\mathbb{R}}$ (discussed in [GL]) is the quotient $\mathcal{A}^{\mathbb{R}}(1)/\operatorname{Sq}^3$. In Figure 3.1, we have displayed a particular $\mathcal{A}^{\mathbb{R}}$-module extension of $\mathcal{A}^{\mathbb{R}}(1)/\operatorname{Sq}^3$ obtained using Theorem 4.1. In fact, this $\mathcal{A}$-module can be realized as an $\mathbb{R}$-motivic 5-cell complex using the fact that the Toda bracket $\langle \mathsf{h}, \eta, \mathsf{h}\rangle$ contains the product $\eta\cdot\tau\eta$ (this follows from a Massey product in $\mathrm{Ext}_{\mathbb{R}}$ using the Moss Convergence Theorem [BI, Theorem 8.2]). Using Proposition 3.4, in conjunction with Table 2.1, we notice that

$$\begin{aligned}
\psi_L(x_{4,2}) &= 1\otimes x_{4,2}\\
\psi_L(x_{3,1}) &= 1\otimes x_{3,1} + \tau_0\otimes x_{4,2}\\
\psi_L(x_{2,1}) &= 1\otimes x_{2,1} + (\tau\xi_1 + \rho\tau_0\xi_1 + \rho\tau_1 + \rho^2\xi_1^2)\otimes x_{4,2}\\
\psi_L(x_{1,0}) &= 1\otimes x_{1,0} + \xi_1\otimes x_{3,1} + \tau_1\otimes x_{4,2}\\
\psi_L(x_{0,0}) &= 1\otimes x_{0,0} + \tau_0\otimes x_{1,0} + \xi_1\otimes x_{2,1} + (\tau_0\xi_1+\tau_1)\otimes x_{3,1}\\
&\quad +(\tau_0\tau_1 + \rho^2\xi_2 + \rho^2\xi_1^3)\otimes x_{4,2}
\end{aligned}$$

determines the $\mathcal{A}^{\mathbb{R}}_\star$-comodule structure of $\mathrm{H}^\star(\mathcal{J}_{\mathbb{R}})$. Then (3.3) produces the $\mathcal{A}^{\mathbb{R}}$-module structure on the dual displayed in Figure 3.1.

4. SELF-DUAL $\mathcal{A}^{\mathbb{R}}$-MODULE STRUCTURES ON $\mathcal{A}^{\mathbb{R}}(1)$

Let $x_{i,j}$ and $y_{i,j}$ denote the elements of the $\mathbb{M}_2^{\mathbb{R}}$-basis of $\mathcal{A}^{\mathbb{R}}(1)$ introduced in [BGL2, Notation 1.5] in bidegree (i,j).

Theorem 4.1. [BGL2, Theorem 1.6] *For every vector*

$$\overline{v} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathbb{F}_2^7,$$

there exists a unique isomorphism class of $\mathcal{A}^{\mathbb{R}}$-module structures on $\mathcal{A}^{\mathbb{R}}(1)$, which we denote by $\mathcal{A}^{\mathbb{R}}_{\mathrm{v}}(1)$, determined by the formulas

$$\begin{aligned}
\mathrm{Sq}^4(\mathsf{x}_{0,0}) &= \beta_{03}(\rho \cdot \mathsf{y}_{3,1}) + (1+\beta_{03}+\beta_{14})(\tau \cdot \mathsf{y}_{4,1}) + \alpha_{03}(\rho \cdot \mathsf{x}_{3,1}) \\
\mathrm{Sq}^4(\mathsf{x}_{1,0}) &= \mathsf{y}_{5,2} + \beta_{14}(\rho \cdot \mathsf{y}_{4,1}) \\
\mathrm{Sq}^4(\mathsf{x}_{2,1}) &= \beta_{26}(\tau \cdot \mathsf{y}_{6,2}) + \beta_{25}(\rho \cdot \mathsf{y}_{5,2}) + \jmath_{24}(\rho^2 \cdot \mathsf{y}_{4,1}) \\
\mathrm{Sq}^4(\mathsf{x}_{3,1}) &= (\beta_{25}+\beta_{26})(\rho \cdot \mathsf{y}_{6,2}) \\
\mathrm{Sq}^4(\mathsf{y}_{3,1}) &= \gamma_{36}(\rho \cdot \mathsf{y}_{6,2}) \\
\mathrm{Sq}^8(\mathsf{x}_{0,0}) &= \beta_{06}(\rho^2 \cdot \mathsf{y}_{6,2}),
\end{aligned}$$

where $\jmath_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25}+\beta_{26})$. Further, any $\mathcal{A}^{\mathbb{R}}$-module whose underlying $\mathcal{A}^{\mathbb{R}}(1)$-module is free on one generator is isomorphic to one listed above.

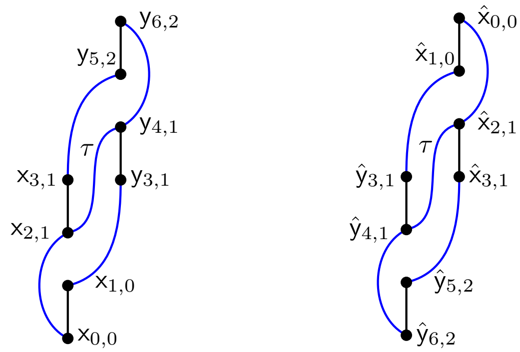

FIGURE 4.1. A singly-generated free $\mathcal{A}^{\mathbb{R}}(1)$-module (on the left), and its dual (on the right).

Using Proposition 3.4, we calculate the $\mathcal{A}^{\mathbb{R}}_{\star}$-comodule structure ψ_{L} on $\mathcal{A}^{\mathbb{R}}_{\mathrm{v}}(1)$:

$$\begin{aligned}
\psi_{\mathrm{L}}(\mathsf{y}_{6,2}) &= 1 \otimes \mathsf{y}_{6,2} \\
\psi_{\mathrm{L}}(\mathsf{y}_{5,2}) &= 1 \otimes \mathsf{y}_{5,2} + \tau_0 \otimes \mathsf{y}_{6,2} \\
\psi_{\mathrm{L}}(\mathsf{y}_{4,1}) &= 1 \otimes \mathsf{y}_{4,1} + \xi_1 \otimes \mathsf{y}_{6,2} \\
\psi_{\mathrm{L}}(\mathsf{y}_{3,1}) &= 1 \otimes \mathsf{y}_{3,1} + \tau_0 \otimes \mathsf{y}_{4,1} + (\tau_1 + \tau_0\xi_1 + \gamma_{36}\rho\xi_1^2) \otimes \mathsf{y}_{6,2} \\
\psi_{\mathrm{L}}(\mathsf{x}_{3,1}) &= 1 \otimes \mathsf{x}_{3,1} + \xi_1 \otimes \mathsf{y}_{5,2} + (\tau_1 + (\beta_{25}+\beta_{26})\rho\xi_1^2) \otimes \mathsf{y}_{6,2} \\
\psi_{\mathrm{L}}(\mathsf{x}_{2,1}) &= 1 \otimes \mathsf{x}_{2,1} + \tau_0 \otimes \mathsf{x}_{3,1} + (\tau\xi_1 + \rho\tau_1 + \rho\tau_0\xi_1 + \jmath_{24}\rho^2\xi_1^2) \otimes \mathsf{y}_{4,1} \\
&\quad + (\tau_1 + \tau_0\xi_1 + \beta_{25}\rho\xi_1^2) \otimes \mathsf{y}_{5,2} + (\tau_0\tau_1 + (1+\beta_{26})\tau\xi_1^2) \otimes \mathsf{y}_{6,2} \\
&\quad + ((1+\beta_{25})\rho\tau_0\xi_1^2 + \rho\tau_1\xi_1 + \jmath_{24}\rho^2\xi_2) \otimes \mathsf{y}_{6,2} \\
\psi_{\mathrm{L}}(\mathsf{x}_{1,0}) &= 1 \otimes \mathsf{x}_{1,0} + \xi_1 \otimes \mathsf{y}_{3,1} + (\tau_1 + \beta_{14}\rho\xi_1^2) \otimes \mathsf{y}_{4,1} + \xi_1^2 \otimes \mathsf{y}_{5,2} \\
&\quad + (\tau_1\xi_1 + \gamma_{36}\rho\xi_1^3 + (\beta_{14}+\gamma_{36})\rho\xi_2) \otimes \mathsf{y}_{6,2}
\end{aligned}$$

$$\begin{aligned}\psi_L(\mathsf{x}_{0,0}) &= 1 \otimes \mathsf{x}_{0,0} + \tau_0 \otimes \mathsf{x}_{1,0} + \xi_1 \otimes \mathsf{x}_{2,1} + (\tau_1 + \alpha_{03}\rho\xi_1^2) \otimes \mathsf{x}_{3,1} \\ &+(\tau_1 + \tau_0\xi_1 + \beta_{03}\rho\xi_1^2) \otimes \mathsf{y}_{3,1} \\ &+(\tau_0\tau_1 + (\beta_{03}+\beta_{14})\tau\xi_1^2 + (\beta_{03})\rho\tau_0\xi_1^2 + \jmath_{24}\rho^2\xi_2 + \jmath_{24}\rho^2\xi_1^3) \otimes \mathsf{y}_{4,1} \\ &+(\tau_1\xi_1 + \tau_0\xi_1^2 + \beta_{25}\rho\xi_1^3 + (\alpha_{03}+\beta_{25})\rho\xi_2) \otimes \mathsf{y}_{5,2} \\ &+(\beta_{26}\tau\xi_1^3 + (\beta_{26}+\gamma_{36})\rho\tau_0\xi_1^3 + (\beta_{25}+\beta_{26}+\gamma_{36})\rho\tau_1\xi_1^2) \otimes \mathsf{y}_{6,2} \\ &+((1+\beta_{03}+\beta_{14}+\beta_{26})\tau\xi_2 + (1+\beta_{03}+\beta_{26}+\gamma_{36})\rho\tau_0\xi_2) \otimes \mathsf{y}_{6,2} \\ &+((1+\alpha_{03}+\beta_{03}+\beta_{25}+\beta_{26}+\gamma_{36})\rho\tau_2 + \jmath_{24}\rho^2\xi_1\xi_2) \otimes \mathsf{y}_{6,2} \\ &+(\tau_0\tau_1\xi_1 + (\jmath_{24}+\beta_{06})\rho^2\xi_1^4) \otimes \mathsf{y}_{6,2}.\end{aligned}$$

Using (3.3), we get the following result, where $\hat{\mathsf{x}}_{i,j}$ and $\hat{\mathsf{y}}_{i,j}$ are the elements in $(\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1))^\vee$ dual to $\mathsf{x}_{i,j}$ and $\mathsf{y}_{i,j}$, respectively.

Theorem 4.2. *The $\mathcal{A}^{\mathbb{R}}(1)$-module structure on the dual $(\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1))^\vee$ is as displayed in the right of Figure 4.1. Moreover, its $\mathcal{A}^{\mathbb{R}}$-module structure is determined by*

$$\begin{aligned}\mathrm{Sq}^4(\hat{\mathsf{y}}_{6,2}) &= (\beta_{25}+\beta_{26})(\rho\cdot\hat{\mathsf{x}}_{3,1}) + (1+\beta_{26})(\tau\cdot\hat{\mathsf{x}}_{2,1}) + \gamma_{36}(\rho\cdot\hat{\mathsf{y}}_{3,1}) \\ \mathrm{Sq}^4(\hat{\mathsf{y}}_{5,2}) &= \hat{\mathsf{x}}_{1,0} + \beta_{25}(\rho\cdot\hat{\mathsf{x}}_{2,1}) \\ \mathrm{Sq}^4(\hat{\mathsf{y}}_{4,1}) &= (\beta_{03}+\beta_{14})(\tau\cdot\hat{\mathsf{x}}_{0,0}) + \beta_{14}(\rho\cdot\hat{\mathsf{x}}_{1,0}) + \jmath_{24}(\rho^2\cdot\hat{\mathsf{x}}_{2,1}) \\ \mathrm{Sq}^4(\hat{\mathsf{y}}_{3,1}) &= \beta_{03}(\rho\cdot\hat{\mathsf{x}}_{0,0}) \\ \mathrm{Sq}^4(\hat{\mathsf{x}}_{3,1}) &= \alpha_{03}(\rho\cdot\hat{\mathsf{x}}_{0,0}) \\ \mathrm{Sq}^8(\hat{\mathsf{y}}_{6,2}) &= (\jmath_{24}+\beta_{06})(\rho^2\cdot\hat{\mathsf{x}}_{0,0}).\end{aligned}$$

Corollary 4.1. *For the $\mathcal{A}^{\mathbb{R}}$-module $\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1)$, its dual is isomorphic to*

$$\Sigma^{6,2}(\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1))^\vee \cong \mathcal{A}^{\mathbb{R}}_{\delta(\overline{v})}(1),$$

where $\delta(\overline{v}) = (\gamma_{36}, \beta_{25}+\beta_{26}, \beta_{25}, \jmath_{24}+\beta_{06}, \beta_{14}, \beta_{03}+\beta_{14}, \alpha_{03})$. Thus, $\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1)$ is self dual if and only if

(1) $\alpha_{03} = \gamma_{36}$,
(2) $\beta_{03} = \beta_{25}+\beta_{26}$, and
(3) $\beta_{14} = \beta_{25}$.

Remark 4.3. The constant $\jmath_{24}$ has a geometric significance noted in [BGL2, Remark 1.21]. It follows from Corollary 4.1 that $\jmath_{24} = 0$ whenever $\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1)$ is self-dual.

Remark 4.4. The underlying classical $\mathcal{A}$-module structure on $\mathcal{A}(1)$ is self-dual if and only if $\beta_{26} = \beta_{03}+\beta_{14}$. In the presence of (3), this is equivalent to (2). Thus the conditions of Corollary 4.1 can be thought of as the classical condition, plus conditions (1) and (3).

In [BGL2], we showed that the $\mathcal{A}^{\mathbb{R}}$-modules $\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1)$ can be realized as the cohomology of an $\mathbb{R}$-motivic spectrum for all values of $\overline{v}$.

Corollary 4.2. *Suppose $\mathcal{A}_1^{\mathbb{R}}[\overline{v}]$ is an $\mathbb{R}$-motivic spectrum realizing $\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1)$, and suppose that $\mathcal{A}^{\mathbb{R}}_{\overline{v}}(1)$ is a self-dual $\mathcal{A}^{\mathbb{R}}$-module .Then $\mathcal{A}_1^{\mathbb{R}}[\overline{v}]$ is the cofiber of a v_1-self-map on either $\mathcal{Y}^{\mathbb{R}}_{2,1}$ or $\mathcal{Y}^{\mathbb{R}}_{\mathsf{h},1}$.*

Proof. By [BGL2, Theorem 1.8], the $\mathbb{R}$-motivic spectrum $\mathcal{A}_1^{\mathbb{R}}[\overline{v}]$ is the cofiber of a v_1-self map on $\mathcal{Y}^{\mathbb{R}}_{2,1}$ if $\beta_{25}+\beta_{26}+\gamma_{36} = 1$ and $\alpha_{03}+\beta_{03} = 1$, whereas it is the cofiber of a v_1-self-map on $\mathcal{Y}^{\mathbb{R}}_{\mathsf{h},1}$ if $\beta_{25}+\beta_{26}+\gamma_{36} = 0$ and $\alpha_{03}+\beta_{03} = 0$. But conditions (1) and (2) of Corollary 4.1 imply that $\beta_{25}+\beta_{26}+\gamma_{36}$ is equal to $\alpha_{03}+\beta_{03}$. ∎

Our main results Theorem 1.1 and Theorem 1.3 follows from Corollary 4.1 and Corollary 4.2 respectively.

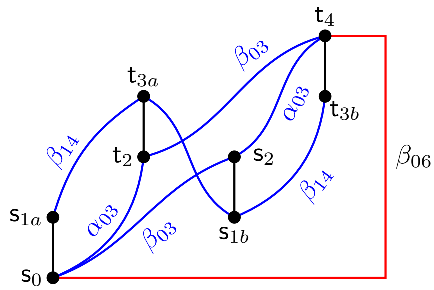

FIGURE 4.2. The $\mathcal{A}$-module structure of a self-dual $\mathrm{H}^*(\Phi(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]))$.

Remark 4.5. Using the Betti realization functor, [BGL2] produced C_2-equivariant realizations of analogous $\mathcal{A}^{\mathrm{C}_2}$-modules $\mathcal{A}_{\overline{\mathrm{v}}}^{\mathrm{C}_2}(1)$. Using the comparison result [BGL2, Theorem 1.19], the $\mathcal{A}$-module structures on $\Phi(\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}])$, the geometric fixed points of $\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]$, was identified in [BGL2, Figure 4.12]. In Figure 4.2, we record the $\mathcal{A}$-module structure on the geometric fixed points of a self-dual $\mathcal{A}_1^{\mathrm{C}_2}[\overline{\mathrm{v}}]$.

APPENDIX A. ON THE ANTIAUTOMORPHISM OF $\mathcal{A}^{\mathbb{R}}$

Although Boardman [B, §6] pointed out that the set of E-cohomology operations $[\mathrm{E}, \mathrm{E}]^*$ may not necessarily have an antiautomorphism for a cohomology theory E, we find the case of $\mathrm{E} = \mathbf{H}_{\mathbb{R}}\mathbb{F}_2$ a rather curious one.

The case of $\mathrm{E} = \mathbf{H}\mathbb{F}_2$ is exceptional; the Steenrod algebra $\mathcal{A} := [\mathbf{H}\mathbb{F}_2, \mathbf{H}\mathbb{F}_2]_*$ is well-known to be a Hopf algebra and, therefore, equipped with an antiautomorphism $\chi : \mathcal{A} \longrightarrow \mathcal{A}$. The composition of extension of scalars and Betti realization,

$$\mathbf{Sp}^{\mathbb{R}} \xrightarrow{\mathbb{C}\otimes_{\mathbb{R}}-} \mathbf{Sp}^{\mathbb{C}} \xrightarrow{\beta} \mathbf{Sp},$$

induces maps of Steenrod algebras

$$\mathcal{A}^{\mathbb{R}} \xrightarrow{\pi_1} \mathcal{A}^{\mathbb{C}} \cong \mathcal{A}^{\mathbb{R}}/(\rho) \xrightarrow{\pi_2} \mathcal{A} \cong \mathcal{A}^{\mathbb{R}}/(\tau, \rho).$$

The algebra map π_1 is a quotient map with kernel (ρ). The map $\pi_{(2)}$ is also a quotient map with kernel (τ).

The antiautomorphism χ of the classical Steenrod algebra is known to lift along π_2,

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{C}} & \xrightarrow{\chi^{\mathbb{C}}} & \mathcal{A}^{\mathbb{C}} \\ \pi_2 \downarrow & & \downarrow \pi_2 \\ \mathcal{A} & \xrightarrow{\chi} & \mathcal{A} \end{array}$$

as the $\mathbb{C}$-motivic Steenrod algebra is a connected bialgebra. However, lifting $\chi^{\mathbb{C}}$ along π_1 is less straightforward. The dual $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}_\star^{\mathbb{R}}$ is a Hopf *algebroid*, rather than a Hopf algebra, so that its dual is not a Hopf algebra.

One feature that distinguishes $\mathcal{A}^{\mathbb{R}}$ from $\mathcal{A}^{\mathbb{C}}$ is the fact that τ is not central in $\mathcal{A}^{\mathbb{R}}$. In the following result, we use the commutators $[\tau, \mathrm{Sq}^{2^n}]$ in $\mathcal{A}^{\mathbb{R}}$ (computed using the Cartan formula [V, Proposition 9.7]) to compute the values of a hypothetical antiautomorphism in low degrees.

Proposition A.1. *Suppose that $\chi^{\mathbb{R}} \colon \mathcal{A}^{\mathbb{R}} \longrightarrow \mathcal{A}^{\mathbb{R}}$ is a ring antihomomorphism and an involution. Then*

$$\begin{aligned} \chi^{\mathbb{R}}(\tau) &= \tau \\ \chi^{\mathbb{R}}(\rho) &= \rho \\ \chi^{\mathbb{R}}(\mathrm{Sq}^1) &= \mathrm{Sq}^1 \\ \chi^{\mathbb{R}}(\mathrm{Sq}^2) &= \mathrm{Sq}^2 + \rho \, \mathrm{Sq}^1 \\ \chi^{\mathbb{R}}(\mathrm{Sq}^4) &= \mathrm{Sq}^4 + \rho \, \mathrm{Sq}^2 \mathrm{Sq}^1 + \tau \, \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 . \end{aligned}$$

Proof. If $\chi^{\mathbb{R}}$ is a ring antihomomorphism then

$$\chi^{\mathbb{R}}[r,s] = [\chi^{\mathbb{R}} r, \chi^{\mathbb{R}} s] \tag{A.1}$$

in characteristic 2. Since τ and Sq^1 are unique $\mathbb{F}_2$-generators in their bidegree and $\chi^{\mathbb{R}}$ is an automorphism, it follows that

$$\chi^{\mathbb{R}}(\tau) = \tau \qquad \text{and} \qquad \chi^{\mathbb{R}}(\mathrm{Sq}^1) = \mathrm{Sq}^1 .$$

For degree reasons, $\chi^{\mathbb{R}}(\mathrm{Sq}^2)$ must be $\mathrm{Sq}^2 + \varepsilon \rho \, \mathrm{Sq}^1$, where ε is either 0 or 1. But the commutator $[\tau, \mathrm{Sq}^2]$ is equal to $\rho\tau \, \mathrm{Sq}^1$. Applying (A.1), we see that

$$\begin{aligned} \chi^{\mathbb{R}}(\rho\tau \, \mathrm{Sq}^1) &= [\chi^{\mathbb{R}}(\tau), \chi^{\mathbb{R}}(\mathrm{Sq}^2)] \\ \Rightarrow \qquad \mathrm{Sq}^1 \tau\rho &= [\tau, \mathrm{Sq}^2 + \varepsilon\rho \, \mathrm{Sq}^1] \\ \Rightarrow \qquad \rho\tau \, \mathrm{Sq}^1 + \rho^2 &= \rho\tau \, \mathrm{Sq}^1 + \varepsilon\rho^2, \end{aligned}$$

and therefore, ε must be 1.

Similarly, degree considerations imply that $\chi^{\mathbb{R}}(\mathrm{Sq}^4)$ must be of the form $\mathrm{Sq}^4 + \delta\rho \, \mathrm{Sq}^1 \mathrm{Sq}^2 + \varepsilon\rho \, \mathrm{Sq}^2 \mathrm{Sq}^1 + \lambda\tau \, \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1$. The commutator $[\tau, \mathrm{Sq}^4]$ is $\rho\tau \, \mathrm{Sq}^1 \mathrm{Sq}^2$, so we conclude that

$$\begin{aligned} [\chi^{\mathbb{R}}\tau, \chi^{\mathbb{R}} \mathrm{Sq}^4] &= [\tau, \mathrm{Sq}^4 + \delta\rho \, \mathrm{Sq}^1 \mathrm{Sq}^2 + \varepsilon\rho \, \mathrm{Sq}^2 \mathrm{Sq}^1 + \lambda\tau \, \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1] \\ &= (1+\lambda)\rho\tau \, \mathrm{Sq}^1 \mathrm{Sq}^2 + \lambda\rho\tau \, \mathrm{Sq}^2 \mathrm{Sq}^1 + (\delta+\varepsilon)\rho^2 \, \mathrm{Sq}^2 + \delta\rho^3 \, \mathrm{Sq}^1 \end{aligned}$$

must agree with

$$\begin{aligned} \chi^{\mathbb{R}}(\rho\tau \, \mathrm{Sq}^1 \mathrm{Sq}^2) &= (\mathrm{Sq}^2 + \rho \, \mathrm{Sq}^1) \, \mathrm{Sq}^1 \, \tau\rho \\ &= \rho\tau \, \mathrm{Sq}^2 \mathrm{Sq}^1 + \rho^2 \, \mathrm{Sq}^2, \end{aligned}$$

and therefore, $\delta = 0$, $\varepsilon = 1$, and $\lambda = 1$ as desired. ■

Proposition A.1 suggests there might be an $\mathbb{R}$-motivic antiautomorphism on the subalgebra $\mathcal{A}^{\mathbb{R}}(2) := \mathbb{M}_2^{\mathbb{R}}\langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle \subset \mathcal{A}^{\mathbb{R}}$. It seems likely that the method above can be extended to produce an antiautomorphism on all of $\mathcal{A}^{\mathbb{R}}$. However, we leave open the question of whether or not this is possible.

On the other hand, the following remark shows that an antihomomorphism on $\mathcal{A}^{\mathbb{R}}$ may not be directly of use in dualizing $\mathcal{A}^{\mathbb{R}}$-modules.

Remark A.1. Even if $\mathcal{A}^{\mathbb{R}}$ were to be equipped with an antiautomorphism $\chi^{\mathbb{R}}$, this may not be so useful for the purpose of dualization. The reason is that the classical formula (1.1) does not work in this setting. More precisely, let $\mathbf{N}$ be an $\mathcal{A}^{\mathbb{R}}$-module, let $\lambda \in \mathbf{N}^{\vee}$, $\varphi \in \mathcal{A}^{\mathbb{R}}$, and $n \in \mathbf{N}$. Then defining a new action $\varphi \odot \lambda$ by

$$(\varphi \odot \lambda)(n) = \lambda(\chi^{\mathbb{R}}\varphi \cdot n)$$

does not produce an $\mathbb{M}_2^{\mathbb{R}}$-linear function. For instance, consider the case $\mathbf{N} = \mathrm{H}^{\star}(\mathbb{S}_{\mathbb{R}}/\mathsf{h})$ from Example 3.2. Then $(\mathrm{Sq}^2 \odot \hat{\mathsf{x}}_{1,0})(\tau \mathsf{x}_{0,0})$ vanishes, whereas $(\mathrm{Sq}^2 \odot \hat{\mathsf{x}}_{1,0})(\mathsf{x}_{0,0})$ is equal to ρ. It follows that the formula for $\mathrm{Sq}^2 \odot \hat{\mathsf{x}}_{1,0}$ is not $\mathbb{M}_2^{\mathbb{R}}$-linear and is therefore not a valid element of $\mathbf{N}^{\vee}$.

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