INERTIA GROUPS IN THE METASTABLE RANGE

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Abstract. We prove that the inertia groups of all sufficiently-connected, high-dimensional $(2n)$-manifolds are trivial. Specifically, for $m \gg 0$ and $k > 5/12$, suppose M is a (km)-connected, smooth, closed, oriented m-manifold and Σ is an exotic m-sphere. We prove that, if $M\sharp\Sigma$ is diffeomorphic to M, then Σ bounds a parallelizable manifold. Our proof is an application of higher algebra in Pstragowski's category of synthetic spectra, and builds on previous work of the authors.

Contents

1. Introduction

An m-dimensional *exotic sphere* Σ is a smooth, oriented manifold that is homeomorphic, but not necessarily diffeomorphic, to S^m. When $m \geq 5$, exotic spheres up to orientation-preserving diffeomorphism constitute the Kervaire–Milnor group Θ_m, with group operation given by connected sum. Kervaire and Milnor proved that there is an exact sequence

$$0 \to \mathrm{bP}_{m+1} \to \Theta_m \to \mathrm{coker}(J)_m,$$

where bP_{m+1} is the subgroup of exotic spheres that bound parallelizable $(m+1)$-manifolds. When m is even, bP_{m+1} contains only the standard sphere S^m.

If M is any smooth, oriented m-manifold, and $\Sigma \in \Theta_m$, then we may form the connected sum $M\sharp\Sigma$. Often, this construction will change the diffeomorphism type of M, but this need not always be the case.

Definition 1.1. Suppose M is a smooth, closed, oriented m-manifold. Then the *inertia group* of M, denoted $I(M)$, is the subgroup of Θ_m consisting of all exotic spheres Σ such that $M\sharp\Sigma$ is diffeomorphic to M via an orientation-preserving diffeomorphism.

Remark 1.2. One is often also interested in specific subgroups of $I(M)$. For example, there is the *homotopy inertia group* $I_h(M) \subseteq I(M)$ of spheres Σ such that $M\sharp\Sigma \simeq M$ via a diffeomorphism lifting the standard homeomorphism. There is also the *concordance inertia group* $I_c(M) \subseteq I(M)$ of spheres such that $M\sharp\Sigma$ is concordant to M [Mun70, Lev70].

Essentially by definition, the inertia group of any exotic sphere is trivial. On the other hand, Winkelnkemper constructed manifolds of every dimension with $I(M) = \Theta_m$ [Win75], and calculations of either $I(M)$ or the subgroups of Remark 1.2 are known only by hard work in special circumstances. There have been explicit calculations of inertia groups of low dimensional complex and quaternionic projective spaces [Kaw68, BK17, BK18], as well as of certain products of spheres with one another and with low dimensional complex projective spaces [Sch71, BKS15]. The inertia groups of hyperbolic manifolds are the subject of striking results of Farrell and Jones [FJ89, FJ94], and a sample of additional results may be found in [Nao73, Fra84].

The most general class of manifolds for which inertia groups are reasonably understood are the *highly connected manifolds.* A $(2n)$-dimensional manifold is said to be highly connected if it is $(n-1)$-connected, meaning it is simply connected and its first $(n-1)$ integral homology groups vanish. In the 1960s, Wall [Wal62] [Wal67, §16] and Kosinski [Kos67] proved the following:

Theorem 1.3 (Wall, Kosinski)**.** *Suppose that M is a stably frameable $(n-1)$-connected, smooth, closed $(2n)$-manifold, with $n \geq 3$. Then $I(M) = 0$.*

The assumption that M be stably frameable is essential in the above result, as the following examples show:

Theorem 1.4 (Kramer–Stolz [KS07])**.** *The inertia group of $\mathbb{HP}^2$ is all of Θ_8, and the inertia group of $\mathbb{OP}^2$ is all of Θ_{16}. Neither of these groups are trivial.*

The projective planes $\mathbb{HP}^2$ and $\mathbb{OP}^2$ are far from generic highly connected manifolds. Indeed, the solution [Ada66] of the Hopf Invariant 1 question precludes any similar objects in larger dimensions. In [BHS19], the authors proved that this is no accident. As part of that work, we showed that any $(n-1)$-connected $(2n)$-manifold of dimension larger than 464 has trivial inertia group, whether or not it is stably frameable. This result is an enhancement of previous work of Stephan Stolz [Sto85, Theorem D], which relies on unpublished work of Mahowald [Mah75] (see [Cha20]) and resolved the cases where $n \not\equiv 1$ modulo 8.

In this paper, we extend the results of [BHS19] deeper into the metastable range. Specifically, we study (km)-connected m-manifolds for real numbers $k > 5/12$. We find, for each such real number k, an integer N_k such that the following theorem is true:

Theorem 1.5. *Suppose M is a (km)-connected, smooth, closed, oriented m-manifold of dimension $m \geq N_k$. Then any exotic sphere $\Sigma \in I(M)$ must bound a parallelizable manifold. In particular, if M is even-dimensional then $I(M) = 0$, and if M is odd-dimensional then $I(M)$ is a subgroup of the cyclic group bP_{m+1}.* [1]

To the best of our knowledge, Theorem 1.5 is the first general result on the inertia groups of (km)-connected m-manifolds for any $k < 1/2$. The closest precursors to this result dealt with restricted classes of $(n-1)$-connected $(2n+1)$,$(2n+2)$, and $(2n+3)$-manifolds [Kos67] [Wil75] [Sto85, Theorem D].

Remark 1.6. An explicit value for N_k can be extracted from our proof whenever $k > 13/30$. For example, Theorem 3.2 shows that one may take $N_{0.45}$ to be 600, or even a slightly smaller integer, when interpreting $(0.45m)$-connectivity as $\lfloor 0.45m \rfloor$-connectivity.

For $13/30 \geq k > 5/12$, we express N_k in terms of the unknown (and therefore inexplicit) vanishing curve on the E_∞-page of the Adams-Novikov spectral sequence for the homotopy groups of spheres. That such a vanishing curve exists, and is sublinear, is the main technical result powering the Nilpotence Theorem of Devinatz–Hopkins–Smith [DHS88].

As we recall in Section 2, a classic geometric argument allows us to deduce Theorem 1.5 as a consequence of the following theorem:

[1] The order of bP_{m+1} is known except when $m = 125$, where the last remaining case of the Kervaire Invariant one problem remains unresolved.

Theorem 1.7. *Suppose M is a (km)-connected smooth $(m+1)$-manifold with boundary homeomorphic to a sphere. Then, if $m \geq N_k$, ∂M also bounds a parallelizable manifold.*

Theorem 1.7 has the additional geometric consequence that, for $m \gg 0$, any $(2km)$-connected topological $(2m+1)$-manifold that is smoothable away from a disk must be smoothable.

As we explain in Section 2, the Pontryagin–Thom construction allows us to deduce Theorem 1.7 from the following purely homotopy-theoretic result:

Theorem 1.8. *There exists a sublinear function $\epsilon(n)$ such that, if $0 \leq d \leq \frac{2}{5}n - \epsilon(n)$, then the kernel of the unit map*

$$\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle$$

is equal to the degree $2n+d$ part of the image of J.

A slightly more precise version of Theorem 1.8 is stated as Theorem 3.2, and its proof occupies all of the paper subsequent to Section 2.

Remark 1.9. Our results were anticipated in a 1985 book by Stephan Stolz [Sto85], who sketched a proof that Theorem 1.8 would follow from certain classic conjectures about the mod 2 Adams spectral sequence for the 2-completed sphere [Sto85, pp. XX-XXI]. Our overarching proof strategy is similar in spirit to Stolz's sketch, but an application of the key new technique introduced in [BHS19] allows us to avoid assuming these still out-of-reach conjectures. Specifically, we manipulate a Toda bracket, a priori defined in terms of $\mathbb{E}_\infty$-ring structures, into a form that may be lifted to Pstrągowski's category of $\mathbb{F}_2$-synthetic spectra [Pst18].

We are hopeful that one day Stolz's original sketch may be realized, and specifically that the following will be proved:

Conjecture 1.10. *The E_∞-page of the $\mathbb{F}_2$-Adams spectral sequence for the sphere admits a line of slope $1/6$ above which every class is v_1-periodic.*

The reader may note that the existence of v_2-periodic families precludes any such line of slope less than $1/6$. The direct odd primary analog of Conjecture 1.10 is proved in forthcoming work [Bur21] of the first author, but the prime 2 appears substantially more subtle.

Assuming Conjecture 1.10, one follows [Sto85, loc. cit.] to obtain the following improvement on Theorem 1.8:

Conjecture 1.11. *There exists a sublinear function $\epsilon(n)$ such that, if $0 \leq d \leq n-\epsilon(n)$, then the kernel of the unit map*

$$\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle$$

is equal to the degree $2n+d$ part of the image of J.

Remark 1.12. The distinction between the unproven Conjecture 1.11 and our main Theorem 1.8 is the improvement of a number from $2/5$ to 1. We believe that the analogous statement is false for any real number larger than 1, and this seems to be an interesting open problem. For comments, see Question 2.5.

Remark 1.13. Stolz's arguments are substantially more elementary than the ones in this paper, avoiding use of both higher algebra and synthetic spectra. However, even assuming Conjecture 1.10, more sophisticated arguments seem necessary to obtain optimal values for the integers N_k.

It would be particularly desirable to obtain sharp results on the inertia groups of highly connected manifolds. This is because, if $n \geq 3$ and $n \neq 63$, then the only remaining obstacle to a full classification of $(n-1)$-connected $(2n)$-manifolds is the determination of certain inertia groups [BS20]. These inertia groups must be 0 in dimensions above 464, and we expect many of the lower dimensions to be accessible by careful combination of the techniques of this paper with those of [BS20]. A version of Crowley's Q-form conjecture, proved in the forthcoming PhD thesis of Nagy [Nag], should provide the geometric input necessary to reduce the study of inertia groups of highly connected manifolds entirely to problems of homotopy theory. Indeed, Crowley and Nagy have announced a complete determination of the inertia groups of 3-connected 8-manifolds [CN20], and Crowley, Teichner, and Olbermann have work in progress determining the inertia groups of all 7-connected 16-manifolds.

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Conventions: To keep this paper reasonably succinct, we assume the reader to be comfortable with both higher algebra in the language of [Lur17] and with the $\mathbb{F}_p$-synthetic spectra of [Pst18]. For an introduction to synthetic spectra using our notational conventions, the reader is referred to [BHS19, §9]. We shall also have occasion to reference results from Sections 5 and 10, as well as Appendix B, of [BHS19].

2. The geometry of inertia groups

For any integer $n \geq 3$, we define $\mathrm{MO}\langle \mathrm{n} \rangle$ to be the Thom spectrum associated to the canonical map $\tau_{\geq n}\mathrm{BO} \to \mathrm{BO}$. In Section 3 - Section 6, we will find conditions on integers $n \geq 3$ and $d \geq 0$ such that the following hypothesis holds:

Hypothesis 2.1. The kernel of the unit map

$$\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n} \rangle$$

is exactly the degree $2n+d$ component of the image of J.

In this section, we explain, for fixed n and d, the geometric consequences of Hypothesis 2.1. In terms of the results stated in the Introduction, we prove that Theorem 1.8 implies Theorem 1.7, which in turn implies Theorem 1.5.

First, let us note the following straightforward result, first spelled out explicitly by Stolz.

Proposition 2.2 (Stolz)**.** *Suppose Hypothesis 2.1, and let M be an $(n-1)$-connected, smooth, compact, oriented $(2n+d)$-manifold with boundary an exotic sphere Σ. Then Σ must bound a parallelizable manifold.*

Proof. This is [Sto85, Satz 1.7]. The basic idea is to consider the long exact sequence associated to the unit map of $\mathrm{MO}\langle \mathrm{n} \rangle$, which is of the form

$$\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n} \rangle \to \pi_{2n+d}\,(\mathrm{MO}\langle \mathrm{n} \rangle\,/\,\mathbb{S}) \to \pi_{2n+d-1}\,\mathbb{S}\,.$$

By the Pontryagin-Thom construction, classes in $\pi_{2n+d}\,(\mathrm{MO}\langle \mathrm{n} \rangle\,/\,\mathbb{S})$ may be interpreted as bordism classes of $(n-1)$-connected, smooth $(2n+d)$-manifolds that have boundary an exotic sphere equipped with some framing. The map to $\pi_{2n+d-1}\,\mathbb{S}$ then records the class of that exotic sphere in framed bordism.

We may arbitrarily equip the boundary Σ of the manifold M in question with a framing, thus obtaining a class in $\pi_{2n+d}\,(\mathrm{MO}\langle \mathrm{n} \rangle\,/\,\mathbb{S})$. Hypothesis 2.1 ensures that the image in $\pi_{2n+d-1}\,\mathbb{S}$ must be in the image of J, which means it is a standard sphere equipped with some framing. $\square$

Next, we recall a classic geometric argument using the construction of a *modified mapping torus*. To the best of our knowledge, an argument of this form first appeared in [Wal62] (cf. [Sto85, 15.5–15.10]).

Proposition 2.3. *Suppose that M is an $(n-1)$-connected, smooth, closed, oriented $(2n+d-1)$-manifold, and that $\Sigma \in \Theta_{2n+d-1}$ is a class in the inertia group of M. Then Σ is the boundary of an $(n-1)$-connected, smooth, oriented $(2n+d)$-manifold.*

Proof. Consider Σ as obtained by gluing two $(2n+d-1)$-dimensional disks along a diffeomorphism f of S^{2n+d-2}. We may assume that f is the identity when restricted to the upper hemisphere $D_+^{2n+d-2} \subset S^{2n+d-2}$.

Let N denote M with a disk removed–specifically, we remove the disk used to form the connected sum $M\sharp\Sigma$. Then ∂N is diffeomorphic to S^{2n+d-2}, and the assumption that Σ is in the inertia group of M may be rephrased as the existence of a diffeomorphism $g : N \to N$ such that $g|_{\partial N} = f$.

Now, we may form a new manifold T from $N \times I$ by identifying $(n,1)$ with $(g(n),0)$. By our assumption on f, $D_+^{2n+d-2} \times S^1 \subset \partial T$. We may thus attach $D^{2n+d-2} \times D^2$ along that part of the boundary, obtaining an $(n-1)$-connected $(2n+d)$-manifold with boundary diffeomorphic to Σ.

For more details, see either [Wal62, pp. 1-2] or [Sto85, §15] □

The results of this section motivate the following definition and question:

Definition 2.4. For each integer $n \geq 3$, let $\alpha(n)$ denote the smallest integer k such that the kernel

$$\ker\left(\pi_k \mathbb{S} \to \pi_k \mathrm{MO\langle n\rangle}\right)$$

contains a class not in the image of J.

For a given prime p, we use $\alpha_p(n)$ to denote the least d such that

$$\ker\left(\pi_k \mathbb{S}_{(p)} \to \pi_k \mathrm{MO\langle n\rangle}_{(p)}\right)$$

contains a class not in the p-localized image of J.

Question 2.5. What is the asymptotic behavior of $\alpha(n)$? Is there an infinite increasing sequence of dimensions, $n_1 \leq n_2 \leq \cdots$, such that

$$\lim_{k\to\infty} \frac{\alpha(n_k)}{n_k} = 3?$$

Conjecture 1.11 predicts that $\liminf_{n\to\infty} \frac{\alpha(n)}{n} \geq 3$, but there does not appear to be any known estimate of limsup. It may be enlightening, and still non-trivial, to determine the asymptotic behavior of $\alpha_p(n)$ for an odd prime p. It seems likely that the asymptotics of $\alpha_p(n)$ are controlled by the v_1-banded vanishing line in the mod p Adams spectral sequence for the p-completed sphere, which is of slope $\frac{1}{|v_2|}$ [Bur21]. Specifically, we conjecture that the liminf of $\frac{\alpha_p(n)}{n}$ is equal to $\frac{|v_2|}{|v_1|}$.

Remark 2.6. The arguments of this paper, and those of [Sto85, p. XX], suggest that any element of in the krenel of the unit map for $\mathrm{MO\langle n\rangle}$ not in the image of J must be of relatively large Adams filtration. However, it is not the case that a generic element of large Adams filtration must die in the homotopy of $\mathrm{MO\langle n\rangle}$.

Indeed, many well-understood elements of large Adams filtration are detected by the topological modular forms spectrum tmf. The Ando–Hopkins–Rezk string orientation $\mathrm{MO\langle 8\rangle} \to \mathrm{tmf}$ ensures that classes detected in tmf must also be detected by $\mathrm{MO\langle n\rangle}$ whenever $n \geq 8$ [AHR10].

3. The bar spectral sequence approach to the unit of $\mathrm{MO\langle n\rangle}$

The remaining sections are devoted to a homotopy theoretic proof of the following theorem, which is our main result.

Theorem 3.1. *There exists a sublinear function $\epsilon(n)$ such that if $0 \leq d \leq \frac{2}{5}n - \epsilon(n)$, then the kernel of the unit map*

$$\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle$$

is equal to the degree $2n+d$ part of the image of J.

Our argument is a direct generalization of techniques in [BHS19]. A quick outline follows:

(1) At the end of this Section 3, we recall how [BHS19, Theorem 5.2] reduces Theorem 3.1 to a question about the suspension spectrum $\Sigma^\infty \mathrm{O}\langle n-1\rangle$.
(2) In Section 4 we recall a bit of additional background. Specifically, we first recall how Goodwillie calculus provides a canonical filtration of $\Sigma^\infty \mathrm{O}\langle n-1\rangle$. We then recall [BHS19, Lemma 10.18 and Lemma 10.19], which concern the $\mathbb{F}_p$-Adams filtrations of a map of spectra $J : \Sigma^\infty \mathrm{O}\langle n-1\rangle \to \mathbb{S}$.
(3) Section 5 forms the technical heart of the paper. We lift a diagram of spectra to a diagram of synthetic spectra, generalizing the main ideas of [BHS19, §10]. The results of our arguments are lower bounds on the $\mathbb{F}_p$-Adams filtrations of elements in the kernel of $\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle$.
(4) In the final Section 6, we apply known vanishing lines in the $\mathbb{F}_p$-Adams spectral sequences for spheres to complete the proof.

In addition to our brief recollections of the main results of [BHS19, §5,§10], we will assume the reader has a general familiarity with both Goodwillie calculus and synthetic spectra. For background we suggest the reader consult [Lur17, Chapter 6], [Kuh06], and [BHS19, §9].

We will prove Theorem 3.1 as a corollary of the following more comprehensive statement, which treats the p-local components of the problem separately:

Theorem 3.2. *Let K denote the kernel of the unit map*

$$\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle,$$

and $K_{(p)}$ its p-local summand. These groups satisfy the following conditions:

(1) *If $0 \leq d \leq n-2$, then $\mathrm{Im}(J) \subseteq K$.*
(2) *There exists a sublinear function $\epsilon(n)$ such that if $2 \leq d \leq \frac{2}{5}n - \epsilon(n)$, then*
$$K_{(2)} = \mathrm{Im}(J)_{(2)}.$$
(3) *If $p = 2$ and $2 \leq d < \frac{4}{13}n - \frac{50}{13} - \frac{30}{13}\log_2(3n)$, then $K_{(2)} = \mathrm{Im}(J)_{(2)}$.*
(4) *If $p = 3$, $n \geq 60$ and $2 \leq d \leq n-2$, then $K_{(3)} = \mathrm{Im}(J)_{(3)}$.*
(5) *If $p = 5$, $n \geq 192$ and $2 \leq d \leq n-2$, then $K_{(5)} = \mathrm{Im}(J)_{(5)}$.*
(6) *If $p = 7$, $n \geq 144$ and $2 \leq d \leq n-2$, then $K_{(7)} = \mathrm{Im}(J)_{(7)}$.*
(7) *If $p = 11$, $n \geq 100$ and $2 \leq d \leq n-2$, then $K_{(11)} = \mathrm{Im}(J)_{(11)}$.*
(8) *If $p = 13$, $n \geq 120$ and $2 \leq d \leq n-2$, then $K_{(13)} = \mathrm{Im}(J)_{(13)}$.*
(9) *If $p \geq 17$ and $2 \leq d \leq n-2$, then $K_{(p)} = \mathrm{Im}(J)_{(p)}$.*

Remark 3.3. Theorem 3.2 does not treat the cases $d = 0$ and $d = 1$ of Theorem 3.1. However, these cases were analyzed in detail in [BHS19, Theorem 1.1]. In particular, both $\ker(\pi_{2n}\,\mathbb{S} \to \pi_{2n}\,\mathrm{MO}\langle \mathrm{n}\rangle)$ and $\ker(\pi_{2n+1}\,\mathbb{S} \to \pi_{2n+1}\,\mathrm{MO}\langle \mathrm{n}\rangle)$ are known to agree with $\mathrm{Im}(J)$ when $n > 232$.

The main theorem of [BS20] is a complete analysis of $\ker(\pi_{2n-1}\,\mathbb{S} \to \pi_{2n-1}\,\mathrm{MO}\langle \mathrm{n}\rangle)$. When $d < -1$, the analysis of $\ker(\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle)$ is comparatively straightforward.

Remark 3.4. Theorem 3.2 implies that, if $2 \leq d \leq n-2$ and $n \geq 192$, then the cokernel of $\mathrm{Im}(J)$ in $\ker(\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle)$ must be 2-local.

Notation 3.5. Throughout the remainder of the paper we establish the following notation, where an integer $n \geq 3$ is considered fixed:

- $B \coloneqq \tau_{\geq n-1}\Sigma^{-1}\mathrm{bo}$,
- $\mathrm{O}\langle n-1\rangle \coloneqq \Omega^\infty B$,
- $R \coloneqq \Sigma^\infty \mathrm{O}\langle n-1\rangle = \Sigma^\infty\Omega^\infty B$.

3.1. **Manipulating the bar spectral sequence.** Recall that $\mathrm{MO}\langle \mathrm{n}\rangle$ is by definition the Thom spectrum of the map $\tau_{\geq n}\mathrm{BO} \to \mathrm{BO}$. Looping this map once, one obtains a map $\mathrm{O}\langle n-1\rangle \to \mathrm{O}$. We may compose with the classical J homomorphism $\mathrm{O} \to \Omega^\infty \mathbb{S}$ and apply the $(\Sigma^\infty, \Omega^\infty)$ adjunction to obtain a map $R = \Sigma^\infty \mathrm{O}\langle n-1\rangle \to \mathbb{S}$.

Notation 3.6. We denote by J the above map $R \to \mathbb{S}$.

The map J is naturally one of non-unital $\mathbb{E}_\infty$-ring spectra, and there is an associated unital $\mathbb{E}_\infty$-ring map $J_+ : \Sigma^\infty_+ \mathrm{O}\langle n-1\rangle \to \mathbb{S}$.

According to [ABG+14], the Thom spectrum $\mathrm{MO}\langle \mathrm{n}\rangle$ can be presented as a relative tensor product $\mathbb{S} \otimes_{\Sigma^\infty_+\mathrm{O}\langle \mathrm{n}-1\rangle} \mathbb{S}$, where the action on the left is given by the augmentation and the action on the right by J_+. In [BHS19], the authors manipulated this tensor product expression to prove the following:

Proposition 3.7 ([BHS19, Theorem 5.2])**.** *After applying $\tau_{\leq 3n-2}$, there is an equivalence of $\mathbb{E}_0$-algebras between $\mathrm{MO}\langle \mathrm{n}\rangle$ and the three term complex produced from*

$$\begin{tikzcd} R^{\otimes 2} \arrow[r, "m-1\otimes J"] \arrow[rr, bend left, "0"] & R \arrow[u, phantom, "\Updownarrow \mathrm{can}"] \arrow[r, "J"] & \mathbb{S}, \end{tikzcd}$$

where the nullhomotopy "can" comes from the canonical filling of the square

$$\begin{tikzcd} R^{\otimes 2} \arrow[r, "1\otimes J"] \arrow[d, "m"] & R \arrow[d, "J"] \arrow[dl, Rightarrow, "\mathrm{can}"] \\ R \arrow[r, "J"] & \mathbb{S}, \end{tikzcd}$$

which is part of the data making J a map of non-unital $\mathbb{E}_\infty$-algebras.[2]

Using this proposition, we learn through the range of interest that there are two sorts of elements in the kernel of the unit map for $\mathrm{MO}\langle \mathrm{n}\rangle$. Specifically, elements of $\ker(\pi_{2n+d}\,\mathbb{S} \to \pi_{2n+d}\,\mathrm{MO}\langle \mathrm{n}\rangle)$ consist of

(1) Elements in the image of the map $\pi_{2n+d}J$.
(2) Elements in the image of the Toda bracket

$$\begin{tikzcd} \mathbb{S}^{2n+d-1} \arrow[r, "x"] \arrow[rr, bend left, "0"] & R^{\otimes 2} \arrow[u, phantom, "\Updownarrow \mathrm{arbitrary}"] \arrow[r, "m-1\otimes J"] \arrow[rr, bend right, "0"'] & R \arrow[d, phantom, "\Updownarrow \mathrm{can}"] \arrow[r, "J"] & \mathbb{S}, \end{tikzcd}$$

[2]In [BHS19] this proposition was stated with n congruent to -1 mod 4, however that assumption was not used in the proof.

where x and the nullhomotopy of $x(m-1\otimes J)$ are arbitrary. [3] We will sometimes refer to this Toda bracket as $d_2(x)$ and the condition that $x(m-1\otimes J)=0$ as $d_1(x)=0$. [4]

Warning 3.8. The use of J to denote the stable map $\Sigma^\infty \mathrm{O}\langle n-1\rangle \to \mathbb{S}$ introduces an unfortunate clash of notation. Namely, classes in the image of π_* of this stable J map need not always be in the image of the unstable J homomorphism $\mathrm{O} \to \Omega^\infty \mathbb{S}$. It is the image of the unstable map that it is classically referred to as $\mathrm{Im}(J)$.

We will be careful to use the phrases $\mathrm{Im}(J)$ and 'the image of J' only in accordance with their classical meaning. However, as in item (1) above, we will need to also refer to classes in the image of the stable J map. We are comfortable with this clash of notation because we will shortly prove that, in the range of degrees of Theorem 3.2, there is no difference between the image of the stable J map and the classical image of J. We hope this does not cause the reader any undue confusion.

4. Additional background

Before proceeding with the proof of Theorem 3.2, we recall a few useful results.

4.1. Background on non-unital algebras.

We begin with some very general statements about non-unital $\mathbb{E}_\infty$-algebras. Our setting will be that of an arbitrary stable and presentably symmetric monoidal category $\mathcal{C}$. In particular, the reader is encouraged to keep the examples $\mathcal{C} = \mathrm{Sp}$ and $\mathcal{C} = \mathrm{Syn}_{\mathbb{F}_p}$ in mind.

Notation 4.1. We establish notation for some of the basic structure present on any stable presentably symmetric monoidal category $\mathcal{C}$. There is:

- For each $k \geq 0$, an endofunctor $D_k : \mathcal{C} \to \mathcal{C}$, which is given by $X \mapsto (X^{\otimes k})_{h\Sigma_k}$,
- a natural tranformation $c : (-)^{\otimes 2} \to D_2$,
- an underlying object functor $U : \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \to \mathcal{C}$,
- a multiplication natural transformation $m : (U(-))^{\otimes 2} \to U$,
- a squaring natural transformation $\hat{m} : D_2(U(-)) \to U(-)$,
- a natural homotopy between m and $c\hat{m}$.

Specializing all of the above to a map of non-unital $\mathbb{E}_\infty$-algebras $f : A \to B$, we obtain the following triangular prism:

$$\begin{tikzcd}
A^{\otimes 2} \arrow[rr, "f\otimes f"] \arrow[dr, "c"] \arrow[dd, "m"'] & & B^{\otimes 2} \arrow[dr, "c"] \arrow[dd, "m"] & \\
& D_2(A) \arrow[rr, "D_2(f)"'] \arrow[dl, "\hat{m}"] & & D_2(B) \arrow[dl, "\hat{m}"] \\
A \arrow[rr, "f"] & & B &
\end{tikzcd}$$

The relevance of this is that the homotopy "can" used in Section 3 is the composite

[3] Note that the indeterminacy in the given bracket is exactly elements of the first kind, so this gives an unambigous expression for the kernel of the unit.

[4] Our terminology of d_1 and d_2 comes from the spectral sequence associated to the three term complex above.

$$\begin{tikzcd}
 & A \otimes B \arrow[dr, "f\otimes 1"] & \\
A^{\otimes 2} \arrow[ur, "1\otimes f"] \arrow[rr, "f\otimes f"] \arrow[d, "m"] \arrow[rr, phantom, bend left, "(\otimes)"] & & B^{\otimes 2} \arrow[d, "m"] \\
A \arrow[rr, "f"] \arrow[urr, phantom, "(m)"] & & B
\end{tikzcd}$$

where the homotopy filling the triangle comes from the monoidal structure and the homotopy filling the square is the same one which appeared in the back face of the prism above. Thus, we can break "can" up into a composite of five different homotopies. This factorization will lie at the heart of our ability to analyze $d_2(x)$.

4.2. **Background from Goodwillie calculus.**

In this subsection we specialize to the case where $\mathcal{C} = \mathrm{Sp}$. Here, we can use Goodwillie calculus to gain a better understanding of the functor U and the natural transformation $\hat{m}$.

Let us begin with an analysis of the U functor. Using the equivalence between non-unital $\mathbb{E}_\infty$-algebras and augmented $\mathbb{E}_\infty$-algebras we may specialize [Kuh06, Theorem 3.10] to obtain a tower of functors and natural transformations.[5]

$$\begin{tikzcd}
 & & D_n(\mathrm{TAQ}(\mathbb{S}\oplus A;\mathbb{S})) \arrow[d] & & D_2(\mathrm{TAQ}(\mathbb{S}\oplus A;\mathbb{S})) \arrow[d] & \mathrm{TAQ}(\mathbb{S}\oplus A;\mathbb{S}) \arrow[d] & \\
U(A) \arrow[r] & \cdots \arrow[r] & \mathcal{P}_n(U)(A) \arrow[r] & \cdots \arrow[r] & \mathcal{P}_2(U)(A) \arrow[r] & \mathcal{P}_1(U)(A) \arrow[r] & 0
\end{tikzcd}$$

Notation 4.2. For the sake of brevity, we let T denote the functor $\mathrm{TAQ}(\mathbb{S}\oplus(-);\mathbb{S})$ and $\pi : U \to T$ the natural transformation given by

$$U(A) \to P_1(U)(A) \simeq T(A).$$

Next, we recall how calculus interacts with the natural transformation $\hat{m}$. In short, we are interested in the diagram

$$\begin{tikzcd}
 & \mathcal{D}_2(D_2U) \arrow[d] \arrow[dr] & & \mathcal{D}_1(D_2U) \arrow[d] \arrow[dr, "\simeq"] & \\
D_2U \arrow[r] \arrow[d, "\hat{m}"] & \cdots \arrow[r] & \mathcal{P}_2(D_2U) \arrow[rr] \arrow[dd] & & \mathcal{P}_1(D_2U) \arrow[dd] \\
 & D_2T \arrow[dr] & & T \arrow[dr, "\simeq"] & \\
U \arrow[r] & \cdots \arrow[r] & \mathcal{P}_2(U) \arrow[rr] & & \mathcal{P}_1(U).
\end{tikzcd}$$

Given the lemma below, the diagram simplifies to

[5] We have displayed the tower evaluated at a object A. Note that as long as A is 1-connective the tower is convergent.

$$
\begin{tikzcd}
 & D_2T \arrow[dd, "\simeq"] \arrow[dr, "\simeq"] & & 0 \arrow[dd] \arrow[dr] & \\
D_2U \arrow[r] \arrow[dd, "\hat{m}"] & \cdot \arrow[r] & D_2T \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
 & D_2T \arrow[dr] & & T \arrow[dr, "\simeq"] & \\
U \arrow[r] & \cdots \arrow[r] & \mathcal{P}_2(U) \arrow[rr] & & \mathcal{P}_1(U)
\end{tikzcd}
$$

Lemma 4.3. *$\mathcal{D}_1(D_2U) = 0$ and the vertical map $\mathcal{D}_2(D_2U) \to D_2T$ is an equivalence. Moreover, the map $D_2U \to \mathcal{P}_2(D_2U) \simeq D_2(T)$ is $D_2(\pi)$.*

The lemma follows easily from the expected calculus chain rule. Unfortunately, we could not find the appropriate version of the chain rule in the literature, so we provide a less conceptual proof below.

Proof. Using that Sp is stable and D_2 commutes with filtered colimits, we have

$$
\begin{aligned}
\mathcal{D}_1(D_2U) &\simeq \operatorname*{colim}_{n\to\infty} \Omega^n D_2(U(\Sigma^n-)) \simeq \operatorname*{colim}_{n\to\infty} \Omega^n D_2(\Sigma^n\Omega^n U(\Sigma^n-)) \\
&\simeq \operatorname*{colim}_{a,b\to\infty} \Omega^a D_2(\Sigma^a\Omega^b U(\Sigma^b-)) \simeq \operatorname*{colim}_{a\to\infty} \Omega^a D_2(\Sigma^a \operatorname*{colim}_{b\to\infty} \Omega^b U(\Sigma^b-)) \\
&\simeq \mathcal{D}_1(D_2) \circ \mathcal{D}_1(U).
\end{aligned}
$$

Since D_2 is homogeneous of degree 2, $\mathcal{D}_1(D_2) = 0$.

In order to show that the vertical map $\mathcal{D}_2(D_2U) \to D_2T$ is an equivalence it will suffice to show that the linearization of the second cross-effect functor is an equivalence. In non-unital $\mathbb{E}_\infty$-algebras, $U(A \coprod B) \simeq U(A) \oplus (U(A) \otimes U(B)) \oplus U(B)$, and thus $cr_2(U)(A,B) \simeq U(A) \otimes U(B)$.

Similarly, after splitting $D_2(U(A))$ and $D_2(U(B))$ off of $D_2(U(A \coprod B))$ we see that $cr_2(D_2U)(A,B)$ is the direct sum of four terms:

$$
\begin{aligned}
cr_2(D_2U)(A,B) \simeq \quad & D_2(U(A) \otimes U(B)) && \oplus \; U(A) \otimes (U(A) \otimes U(B)) \\
\oplus\; & (U(A) \otimes U(B)) \otimes U(B) && \oplus \; U(A) \otimes U(B).
\end{aligned}
$$

When restricted to the final summand, $cr_2(\hat{m})$ is the identity map. Since the other three terms of $cr_2(D_2U)$ have connectivity which increases by 2 for every suspension (on at least one input), these terms have trivial linearizations. Thus, the linearization of the second cross effects is an equivalence, as desired.

In order to identify the map $D_2U \to D_2T$, we proceed as follows. Since $\mathcal{P}_2(D_2U)$ is the universal quadratic approximation of D_2U, the natural transformation $D_2(\pi)$ factors as

$$D_2U \to \mathcal{P}_2(D_2U) \to \mathcal{P}_2(D_2T) \simeq D_2T.$$

Thus, it will suffice to show that the span $U \leftarrow D_2U \to D_2T$ induces the identity automorphism on the linearization of second cross effects. We must calculate the result of applying D_2 to the map

$$U(A) \oplus (U(A) \otimes U(B)) \oplus U(B) \xrightarrow{\begin{bmatrix} \pi & 0 & 0 \\ 0 & 0 & \pi \end{bmatrix}} T(A) \oplus T(B),$$

splitting off the parts only coming from A and B, and then linearizing. Our previous computations identify the above span as $A \otimes B \xleftarrow{1} A \otimes B \xrightarrow{1} A \otimes B$. $\square$

Example 4.4. Specializing to the case of $R = \Sigma^\infty \mathrm{O}\langle n-1 \rangle$ we obtain a diagram

$$\begin{tikzcd} D_2(R) \arrow[r, "\hat{m}"] \arrow[d, "D_2(\pi)"] & R \arrow[d] \arrow[dr, "\pi"] & \\ D_2(B) \arrow[r] & \mathcal{P}_2(U)(R) \arrow[r] & B, \end{tikzcd}$$

where the vertical maps are equivalences through degree $3n-2$ for connectivity reasons and the bottom row is a cofiber sequence. The map π can easily be identified with the mate of the identity map on $\Omega^\infty B = \mathrm{O}\langle n-1\rangle$.

We can take this lemma further by noting that the bottom row is split exact on homotopy groups.

Lemma 4.5. *The cofiber sequence*

$$D_2(B) \to \mathcal{P}_2(U)(R) \to B$$

is split exact on homotopy groups in degrees $\leq 3n-2$.

Proof. It will suffice to show that the map π admits a section after applying Ω^∞. Since π is the mate of the identity on $\Omega^\infty B$, the unit of the $(\Sigma^\infty, \Omega^\infty)$-adjunction provides a section of $\Omega^\infty \pi$. ☐

4.3. **Remarks on the synthetic** D_2**.** For p a prime, we note some useful features of D_2 on the presentably symmetric monoidal category $\mathrm{Syn}_{\mathbb{F}_p}$ of $\mathbb{F}_p$-synthetic spectra.

Definition 4.6. For any integer n, the category of *n-effective* $\mathbb{F}_p$-synthetic spectra is the smallest stable, full subcategory of $\mathrm{Syn}_{\mathbb{F}_p}$, closed under colimits, that contains $\nu(X)$ for every n-connective spectrum X.

This definition is useful because of the following lemma:

Lemma 4.7. *Suppose a synthetic spectrum K is n-effective. Then, the invert τ map*

$$\pi_{t-s,t}(K) \to \pi_{t-s}(\tau^{-1}K)$$

is an isomorphism whenever $t \leq n$.

Proof. This property is preserved under cofiber sequences and filtered colimits of synthetic spectra, so it suffices to check it when K is $\nu(X)$ for X an n-connective spectrum. Here, the result follows from the definition of ν as a connective cover. ☐

Since $\tau^{-1} : \mathrm{Syn}_{\mathbb{F}_p} \to \mathrm{Sp}$ is a symmetric monoidal left adjoint, it sends the D_2 functor on $\mathrm{Syn}_{\mathbb{F}_p}$ to the D_2 functor on Sp. Furthermore, we have the following comparison between these functors:

Lemma 4.8. *If X is an n-connective spectrum, and a and b are integers, then $D_2(\Sigma^{a,b}\nu X)$ is $(2n+2b)$-effective. In particular, the map*

$$\pi_{t-s,t}(D_2(\Sigma^{a,b}\nu X)) \to \pi_{t-s}(D_2(\Sigma^a X))$$

is an isomorphism for $t \leq 2n+2b$.

Proof. By definition $D_2(\Sigma^{a,b}\nu X)$ is the colimit, indexed over BC_2, of a diagram valued at $(\Sigma^{a,b}\nu X)^{\otimes 2}$. Since $(2n+2b)$-effectivity is preserved under colimits, it will suffice to check that $(\Sigma^{a,b}\nu X)^{\otimes 2}$ is $(2n+2b)$-effective. Since ν is symmetric monoidal, $(\Sigma^{a,b}\nu X)^{\otimes 2} \simeq \Sigma^{2a,2b}\nu(X \otimes X)$. Since $X \otimes X$ is $(2n)$-connective, $\nu(X \otimes X)$ is $(2n)$-effective, and the result follows. ☐

Lemma 4.9. *Suppose $p \neq 2$. Then, for any integers a and b and any spectrum X, inverting τ yields an isomorphism*

$$\pi_{t-s,t}(D_2(\Sigma^{a,b}\nu X)) \to \pi_{t-s}(D_2(\Sigma^a X))$$

whenever $s \leq 2b-2a$

Proof. Since p is odd, $D_2(\Sigma^{a,b}\nu X)$ is a summand of $(\Sigma^{a,b}\nu X)^{\otimes 2}$, in a manner compatible with the splitting of $D_2(\Sigma^a X)$ off of $(\Sigma^a X)^{\otimes 2}$. Thus, it will suffice to check that inverting τ yields an isomorphism

$$\pi_{t-s,t}((\Sigma^{a,b}\nu X)^{\otimes 2}) \to \pi_{t-s}((\Sigma^a X)^{\otimes 2})$$

whenever $s \leq 2b - 2a$

We compute, using the fact that ν is symmetric monoidal, that

$$(\Sigma^{a,b}\nu X)^{\otimes 2} \simeq \Sigma^{2a,2b}(\nu X^{\otimes 2}) \simeq \Sigma^{0,2b-2a}\nu((\Sigma^a X)^{\otimes 2}).$$

For any spectrum Y, inverting τ gives an isomorphism $\pi_{t-s,t}(\nu Y) \to \pi_{t-s}(Y)$ when $s \leq 0$, and the result follows by setting $Y = \Sigma^a X$. □

4.4. **Background on J.** Our primary technique for proving that an element of the homotopy groups of spheres lies in $\mathrm{Im}(J)$ is to show that it has sufficiently high $\mathbb{F}_p$-Adams filtration for each prime p. All of our Adams filtration bounds have essentially one source, which is a bound on the $\mathbb{F}_p$-Adams filtration of the stable J map. However, there is a technical complication. Specifically, the map J does not necessarily have high Adams filtration as a map from R, but it does upon restricting to a finite skeleton of R. For this reason, in [BHS19, Section 10] we systematically worked with finite skeleta. Here, to take what we believe is a cleaner approach, we work directly in categories of truncated objects.

Definition 4.10. Let $\mathrm{Sp}^{[0,3n-2]}$ be the full subcategory of spectra whose homotopy groups lie between 0 and $3n-2$. Let $\mathrm{Syn}_{\mathbb{F}_p}^{[0,3n-2]}$ be the full subcategory of $\mathbb{F}_p$-synthetic spectra whose bigraded homotopy groups $\pi_{a,b}(X)$ are nonzero only for $0 \leq a \leq 3n-2$. In the language of [Bur21] these are the objects concentrated between 0 and $3n-2$ in the vertical t-structure.

Remark 4.11. The truncation of an object X to live within $\mathrm{Syn}_{\mathbb{F}_p}^{[0,3n-2]}$ does not modify the bigraded homotopy groups $\pi_{a,b}X$ with $0 \leq a \leq 3n-2$. This means that we can draw conclusions about the Adams filtration on the homotopy of X from considering the truncation of νX (as long as we are in the appropriate range).

We now establish the following convention, in force throughout the remainder of the paper:

Convention 4.12. All objects are implicitly truncated to live in $\mathrm{Sp}^{[0,3n-2]}$ or $\mathrm{Syn}_{\mathbb{F}_p}^{[0,3n-2]}$, and all colimits and limits are taken in these categories.

In order to ensure that the truncation functors $\mathrm{Sp}^{\geq 0} \to \mathrm{Sp}^{[0,3n-2]}$ and $\mathrm{Syn}_{\mathbb{F}_p}^{\geq 0} \to \mathrm{Syn}_{\mathbb{F}_p}^{[0,3n-2]}$ are symmetric monoidal left adjoints, compatible with inverting τ, we also enforce the following convention.

Convention 4.13. All objects in $\mathrm{Syn}_{\mathbb{F}_p}^{[0,3n-2]}$ will be implicitly $\tilde{p}$-completed, where $\tilde{p}$ is the unique class such that $\tau\tilde{p} = p$. Similarly, all spectra will be implicitly p-completed, where p is understood from context. All tensor products will be taken in these p-complete categories.

With these conventions in place we can now recall the map $\widetilde{J}$ produced in [BHS19, Construction 10.5].

Notation 4.14. Let $h(k)$ denote the number of integers $0 < s \leq k$ which are congruent to $0, 1, 2$ of 4 mod 8. We set

$$M := \begin{cases} h(n-1) - \lfloor \log_2(3n) \rfloor + 1 & \text{if } p = 2 \\ \max\left(\left\lfloor \frac{n}{2p-2} \right\rfloor - \left\lfloor \log_p\left(\frac{3n}{2}\right) \right\rfloor, 0 \right) & \text{if } p \neq 2 \end{cases}.$$

Note that this notation suppresses the dependence of M on p and n.

Lemma 4.15 ([BHS19]). *In the truncated category of $\mathbb{F}_p$-synthetic spectra there exists a map*

$$\widetilde{J} : \Sigma^{0,M}\nu R \to \mathbb{S}$$

which becomes J upon inverting τ.

Proof. Using [BHS19, Lemmas 10.18 and 10.19], after restricting to the $(3n-1)$-skeleton on the source the map J has $\mathbb{F}_p$-Adams filtration at least M[6]. Using [BHS19, Lemma 9.15], after applying ν to the restricted map it becomes divisible by τ^M. Now, applying our convention that everything is truncated, the difference between ν of a skeleton of R and νR disappears and we just obtain a map

$$\widetilde{J} : \Sigma^{0,M}\nu R \to \mathbb{S}\,.$$

□

Example 4.16. Using our new conventions, Example 4.4 simplifies to a cofiber sequence

$$D_2(B) \xrightarrow{\hat{m}} R \xrightarrow{\pi} B.$$

5. Bounding Adams filtrations with synthetic lifts

In this section we will show that elements in the kernel of the unit of $\mathrm{MO}\langle \mathrm{n}\rangle$ have relatively high $\mathbb{F}_p$-Adams filtrations for each prime p.

Notation 5.1. Given an integer $-1 \leq d \leq n-2$, we let m denote $2n+d$. Recall also our standing definition of M from Notation 4.14.

Specifically, we will prove the following four propositions.

Proposition 5.2. *If $0 \leq d \leq n-2$ and $x \in \pi_m R$, then either $J(x) \in \mathrm{Im}(J)$ or $J(x)$ is detected in $\mathbb{F}_2$-Adams filtration at least $2M-d-2$.*

Proposition 5.3. *For $p \neq 2$, if $0 \leq d \leq n-2$ and $x \in \pi_m R$, then either $J(x) \in \mathrm{Im}(J)$ or $J(x)$ is detected in $\mathbb{F}_p$-Adams filtration at least $2M$.*

Proposition 5.4. *Given a class $x \in \pi_m\left(R^{\otimes 2}\right)$ where $-1 \leq d \leq 2(M-6)$ and such that $x(m-1 \otimes J) = 0$, the associated class $d_2(x) \in \pi_{m+1}(\mathbb{S})$ is detected in $\mathbb{F}_2$-Adams filtration at least $2M-d-2$.*

Proposition 5.5. *For $p \neq 2$, given a class $x \in \pi_m\left(R^{\otimes 2}\right)$ where $0 \leq d \leq n-2$ and such that $x(m-1 \otimes J) = 0$, the associated class $d_2(x) \in \pi_{m+1}(\mathbb{S})$ is detected in $\mathbb{F}_p$-Adams filtration at least $2M-2$.*

Proof of Proposition 5.2, Proposition 5.3 and Theorem 3.2(1). Using the splitting of $\pi_m(R)$ given by Lemma 4.5, it will suffice to compute $J(x)$ separately for $x \in \pi_m(B)$ and $x \in \pi_m(D_2(B))$.

We begin by handling the elements from B. The composite of the splitting $\pi_m B \to \pi_m R$ with $\pi_m J$ is given by applying π_m to the sequence of maps of spaces

$$\mathrm{O}\langle n-1\rangle \longrightarrow \Omega^\infty\Sigma^\infty\mathrm{O}\langle n-1\rangle \xrightarrow{\Omega^\infty J} \Omega^\infty\,\mathbb{S},$$

where the first map is the unit of the $(\Sigma^\infty, \Omega^\infty)$-adjunction. As such, the composite is just the classical unstable J homomorphism

$$\mathrm{O}\langle n-1\rangle \to \mathrm{O} \to \Omega^\infty\,\mathbb{S}\,.$$

This identification proves Theorem 3.2(1).

Now we handle the elements from $D_2(B)$. This means understanding what the composite $D_2(R) \xrightarrow{\hat{m}} R \xrightarrow{J} \mathbb{S}$ does on homotopy groups. Since J is a map of non-unital $\mathbb{E}_\infty$-algebras we have a commuting square

[6] At odd primes we have implicitly used that $2p-2$ is divisible by 4 here. Also, we must look at the proof of 10.19 and not just the statement.

$$\begin{tikzcd} D_2(R) \arrow[r, "\hat{m}"] \arrow[d, "D_2(J)"] & R \arrow[d, "J"] \\ D_2(\mathbb{S}) \arrow[r, "\hat{m}"] & \mathbb{S} \end{tikzcd}$$

We will bound the $\mathbb{F}_p$-Adams filtration of the composite $xD_2(J)\hat{m}$. Let $k = d+2$ if $p = 2$ and $k = 0$ if $p \neq 2$. Using the map $\widetilde{J}$ we can construct the following sequence,

$$\mathbb{S}^{m,m+2M-k} \xrightarrow{\widetilde{x}} D_2(\Sigma^{0,2M}\nu R) \xrightarrow{D_2(\widetilde{J})} D_2(\nu\,\mathbb{S}) \xrightarrow{\hat{m}} \nu\,\mathbb{S},$$

where $\widetilde{x}$ is a lift of x along the isomorphism from Lemma 4.8 (or Lemma 4.9 if p is odd). Applying [BHS19, Corollary 9.21] to this diagram finishes the proof. $\square$

Proof (of Proposition 5.4 and Proposition 5.5). In [BHS19, Theorem 5.2], which was recalled in Section 3, we identified $d_2(x)$ with $\langle x, m - 1\otimes J, J\rangle$, where the null-homotopy on the right is the homotopy "can" (also discussed in Section 3). We will accomplish our goal by first manipulating this Toda bracket expression into a form that does not rely on the fact that J is a ring map, and then lifting it to the synthetic category using $\widetilde{J}$. The reason we need to remove the dependence on the ring structure is that the synthetic map $\widetilde{J}$ is not obviously any kind of ring map. In order to streamline our presentation we will defer the verification of several key inputs to a sequence of lemmas after the main body of the proof. The first of these is the following:

$$\text{Both } xm \text{ and } x(1\otimes J) \text{ are nullhomotopic.} \tag{1}$$

Using this, we can expand $\langle x, m - 1\otimes J, J\rangle$ into the matric form below.

$$\begin{tikzcd} \mathbb{S}^m \arrow[r, "x"] \arrow[rr, bend left, "0"] \arrow[dr, bend right, "0"'] & R^{\otimes 2} \arrow[r, "1\otimes J"] \arrow[d, "m"] & R \arrow[d, "J"] \arrow[dl, Rightarrow, "\text{can}"'] \\ & R \arrow[r, "J"] & \mathbb{S} \end{tikzcd}$$

Specializing the triangular prism produced in Section 4.1 to the map J will allow us to remove dependence on ring structures from the above diagram. Specifically, in the language of that prism we will prove the following fact, which is strictly stronger than (1):

$$\text{Both } xc \text{ and } x(1\otimes J) \text{ are nullhomotopic.} \tag{2}$$

Assuming (2), the diagram above can now be refined to the following. Note that the size of the indeterminacy remains unchanged.

$$\begin{tikzcd}
\mathbb{S}^m \arrow[rr, "0"] \arrow[dr, "x"] \arrow[ddr, bend right] & & R \otimes \mathbb{S} \arrow[dr, "J\otimes 1"] \arrow[d, phantom, "(\otimes)"] & \\
0 \arrow[r, Leftrightarrow, "(2)"] & R^{\otimes 2} \arrow[u, phantom, "\Updownarrow (2)"] \arrow[ur, "1\otimes J"] \arrow[rr, "J\otimes J"] \arrow[d, "c"] & \arrow[d, phantom, "(c)"] & \mathbb{S}^{\otimes 2} \arrow[dl, "c"] \arrow[dd, "\simeq"] \\
& D_2(R) \arrow[r, "D_2(J)"] & D_2(\mathbb{S}) \arrow[dr, "\hat{m}"] & \\
& & & \mathbb{S}
\end{tikzcd}$$

Now we lift this diagram to the synthetic category. Let $k = d + 2$ if $p = 2$ and $k = 1$ if $p \neq 2$. The key point will be to prove the existence of synthetic lifts of the nullhomotopies from (2):

$$\text{Both } \tau^k(\Sigma^{0,2M}\nu x)c \text{ and } \tau(\Sigma^{0,2M}\nu x)(1 \otimes \widetilde{J}) \text{ are nullhomotopic.} \tag{3}$$

Assume that (3) holds. Since the diagram above has no dependence on the ring structure on J, we can use $\widetilde{J}$, the nullhomotopies from (3) and the natural transformation c in the category of synthetic spectra to produce the following diagram.

$$\begin{tikzcd}
\mathbb{S}^{m,m+2M-k} \arrow[rr, "0"] \arrow[dr, "\tau^k\Sigma^{0,2M}\nu x"] \arrow[ddr, bend right] & & \Sigma^{0,M}\nu R \arrow[dr, "\widetilde{J}\otimes 1"] \arrow[d, phantom, "(\otimes)"] & \\
0 \arrow[r, Leftrightarrow, "(3)"] & (\Sigma^{0,M}\nu R)^{\otimes 2} \arrow[u, phantom, "\Updownarrow (3)"] \arrow[ur, "1\otimes\widetilde{J}"] \arrow[rr, "\widetilde{J}\otimes\widetilde{J}"] \arrow[d, "c"] & \arrow[d, phantom, "(c)"] & (\mathbb{S}^{0,0})^{\otimes 2} \arrow[dl, "c"] \arrow[dd, "\simeq"] \\
& D_2(\Sigma^{0,M}\nu R) \arrow[r, "D_2(\widetilde{J})"] & D_2(\mathbb{S}^{0,0}) \arrow[dr, "\hat{m}"] & \\
& & & \mathbb{S}^{0,0}
\end{tikzcd}$$

The compatibility of the natural transformation c with the symmetric monoidal functor that inverts τ implies that the element of $\pi_{m+1,m+2M-k}(\mathbb{S}^{0,0})$ associated to this diagram maps to $d_2(x)$ upon inverting τ. Using [BHS19, Corollary 9.21] to relate bigrading to Adams filtration completes the proof. □

In the remainder of this section, we prove the existence of the nullhomotopies (1), (2) and (3). Their existence is immediate from the following lemma.

Lemma 5.6. *Let $k = d+2$ if $p = 2$ and $k = 1$ if $p \neq 2$. In the situation of Proposition 5.4 or Proposition 5.5, the following homotopy classes are trivial:*
(a) xc, (b) $x(1 \otimes J)$, (c) $\tau^k(\Sigma^{0,2M}\nu x)c$ and (d) $\tau(\nu x)(1 \otimes \widetilde{J})$.

Proof (reduction to (d)). Clearly (d) implies (b), upon inverting τ. By hypothesis $x(c\hat{m} - (1 \otimes J)) = 0$, and by Lemma 4.5 the map $\hat{m}$ is injective on homotopy groups; therefore, (a) and (b) are equivalent. From Lemma 4.8 (or Lemma 4.9) we know that the map

$$\pi_{m,m+2M-k}(D_2(\Sigma^{0,M}\nu B)) \to \pi_m(D_2(B)),$$

is an isomorphism, and so (a) and (c) are equivalent. □

In order to prove (d), the following two lemmas are helpful.

Lemma 5.7. *For $s \geq \frac{1}{2}(t-s) - n + 5$, the bigraded homotopy groups $\pi_{t-s,t}(\nu D_2(B))$ are 2-torsion free.*

Proof. Using the fact that $D_2(B)$ is $2(n-1)$-connective and [MM81], the bigraded homotopy groups in this region are determined by the 2-Bockstein spectral sequence converging to the integral homology of $D_2(B)$. The lemma is now reduced to the claim that (the implicitly 2-completed) $\mathbb{Z} \otimes D_2(B)$ is a direct sum of shifts of $\mathbb{Z}$ and of $\mathbb{F}_2$, with no non-simple 2-torsion.

Since D_2 is quadratic and $\mathbb{Z} \otimes B$ is a sum of copies of $\mathbb{Z}$ and $\mathbb{F}_2$, the equivalence $\mathbb{Z} \otimes D_2(B) \simeq D_2^{\mathbb{Z}}(\mathbb{Z} \otimes B)$ reduces the problem to showing that, for any integer ℓ, each of $D_2^{\mathbb{Z}}(\Sigma^\ell \mathbb{Z})$, $D_2^{\mathbb{Z}}(\Sigma^\ell \mathbb{F}_2)$, $\Sigma^\ell \mathbb{Z} \otimes_{\mathbb{Z}} \Sigma^\ell \mathbb{Z}$, $\Sigma^\ell \mathbb{Z} \otimes_{\mathbb{Z}} \Sigma^\ell \mathbb{F}_2$ and $\Sigma^\ell \mathbb{F}_2 \otimes_{\mathbb{Z}} \Sigma^\ell \mathbb{F}_2$ is a sum of shifts of $\mathbb{Z}$ and $\mathbb{F}_2$. The third object is just a shift of $\mathbb{Z}$. The second, fourth and fifth objects are all $\mathbb{F}_2$-modules. The first object is the cohomology of C_2 with $\mathbb{Z}$ coefficients for ℓ even and $\mathbb{Z}^{\mathrm{sgn}}$ coefficients for ℓ odd. □

Lemma 5.8. *If $p \neq 2$, then $\pi_{t-s,t}(\nu D_2(B))$ is τ-torsion free. If $s > 0$, then it is p-torsion free as well.*

Proof. At odd primes $D_2(B)$ is a summand of $B \otimes B$, and B is a retract of a suspension of ku. This lemma thus reduces to the claim that the Adams spectral sequence for $\mathrm{ku} \otimes \mathrm{ku}$ degenerates at E_2 and is v_0-torsion free for $s > 0$. These claims are proved in [Ada95, Part III §17]. □

Proof (of lemma 5.6(d)). The proof of (d) will come down to an analysis of the cofiber sequence

$$D_2(B) \to R \to B$$

from Example 4.4.

Let $y := (\nu x)(1 \otimes \widetilde{J})$. We have a diagram

$$\begin{tikzcd} & & \mathbb{S}^{m,m+M} \arrow[d, "y"] & \\ \nu D_2(B) \arrow[r] & F \arrow[r] \arrow[d] & \nu R \arrow[r] & \nu B \\ & E & & \end{tikzcd}$$

where F is the fiber the of the right map and E is the cofiber of the left map. In [BHS19, Lemma 11.15] the authors showed that E is a $C\tau$-module. We also know that the bigraded homotopy of νB is τ-torsion free, since the classical Adams spectral sequence for bo degenerates.

By hypothesis we have an equality $x(1 \otimes J) = xc\hat{m}$. Then, since $\hat{m}\pi = 0$ we learn that $x(1 \otimes J)\pi = 0$. Using the fact that the homotopy of νB is τ-torsion free we can conclude that $y\nu\pi = 0$ and y lifts to F. Since E is a $C\tau$-module we can conclude that τy lifts to the homotopy of $\pi_{m,m+M-1}\nu D_2(B)$. Let z be such a lift.

Finally, we may apply the previous Lemma 5.7 and 5.8 to z. If $p = 2$, then we use Lemma 5.7 which applies since[7]

$$M - 1 \geq \frac{1}{2}(2n + d) - n + 5 = \frac{d}{2} + 5.$$

If $p \neq 2$, then we use Lemma 5.8, which applies when $M \geq 2$. If $M < 2$, then the filtration bound of Proposition 5.5 that we are proving is vacuous.

[7]This is where the condition on d in Proposition 5.4 comes from.

Either way, we learn that z is either zero or non-p-torsion and non-τ-torsion. In particular, to show that $\tau y = 0$ it will suffice to show that z is torsion after inverting τ. Again using that the map $\hat{m}$ is injective on homotopy groups, it will suffice to show that $x(1 \otimes J)$ is torsion. For this we may note that J becomes, upon restriction to any finite skeleton of R, torsion as a map of spectra. □

6. Applications of vanishing lines

In this section we complete the proof of Theorem 3.2 by showing that the Adams filtration bounds from the previous section are sufficient to conclude that $J(x)$ and $d_2(x)$ are in the image of J.

Definition 6.1. Let $\Gamma_p(k)$ denote the minimal m such that every $\alpha \in \pi_k \mathbb{S}_{(p)}$ with $\mathbb{F}_p$-Adams filtration strictly greater than m is detected $K(1)$-locally.

Remark 6.2. At odd primes, all classes in $\pi_k \mathbb{S}_{(p)}$ that are detected $K(1)$-locally are in $\mathrm{Im}(J)$. At the prime 2, a class detected $K(1)$-locally may be the sum of a class in $\mathrm{Im}(J)$ with a class in the μ-family. As explained in [BHS19, p. 30], no μ-family class is killed by the unit of $\mathrm{MO}\langle \mathrm{n}\rangle$. Specifically, composing this unit with the Atiyah–Bott–Shapiro orientation yields a sequence

$$\mathbb{S} \to \mathrm{MO}\langle \mathrm{n}\rangle \to \mathrm{MO}\langle 3\rangle = \mathrm{MSpin} \to \mathrm{bo}$$

that on homotopy groups has the effect of killing $\mathrm{Im}(J)$ while not killing any of the μ-family. As such, no sum of an $\mathrm{Im}(J)$ class and μ-family class may be in the kernel of the unit map to $\pi_*(\mathrm{MO}\langle \mathrm{n}\rangle)$.

As a consequence of the above remark, we will conclude Theorem 3.2 by comparing the lower bounds of the previous sections with upper bounds on Γ_p. Such upper bounds were the main subject of [BHS19, Appendix B], and we recall the relevant results below. First, we set up some notation.

Notation 6.3.

(1) $q \coloneqq 2p-2$,
(2) $v_p(k)$ will denote the p-adic valuation of an integer $k \in \mathbb{Z}$,
(3) $\ell(k) \coloneqq \begin{cases} v_2(k+1)+v_2(k+2) & \text{if } p=2 \\ v_p(k+2) & \text{if } p \neq 2 \text{ and } k+2 \equiv 0 \pmod{q} \\ 0 & \text{otherwise} \end{cases}$,

We will sometimes use that $\ell(k) \in O(\log(k))$.

Davis and Mahowald proved the bound

$$\Gamma_2(k) \leq \frac{3}{10}k + 3 + v_2(k+2) + v_2(k+1)$$

in [DM89, Corollary 1.3]. This is the best known explicit bound at $p = 2$. In [Bur21, Theorem 20.2], the first author proved that

$$\Gamma_3(k) \leq \frac{1}{16}k + 7 + \frac{1}{4},$$

$$\Gamma_p(k) \leq \frac{1}{2p^2-2}k + 2p^2 - 4p + 4 + \frac{1}{2p^2-2},$$

where the second line is for $p \geq 5$. These are explicit bounds with the best linear term that are curently known at odd primes. For our purposes we will also need an older bound with better intercept due to Gonazalez [Gon00]. For $p \geq 5$,

$$\Gamma_p(k) \leq \frac{(2p-1)}{(2p-2)(p^2-p-1)}k + 4 + \ell(k).$$

Sharper bounds are known if we allow them to depend on the following function which quantifies the vanishing curve of the Adams–Novikov spectral sequence.

Definition 6.4. Let $f_{\mathrm{BP}}(k)$ denote the minimal m such that for every connective p-local spectrum X, $i < k$, and $\alpha \in \pi_i(X)$, if α has BP-Adams filtration at least m, then $\alpha = 0$.

Hopkins and Smith observed that another formulation of the Nilpotence theorem from [DHS88] is the fact that

$$f_{\mathrm{BP}}(k) = o(k).$$

As such, adding $f_{\mathrm{BP}}(k)$ as an "error term" does not affect the leading order behavior of a linear bound.

In terms of f_{BP}, [BHS19, Theorem B.7] provides the following sharper bound,

$$\Gamma_p(k) \leq \frac{(q+1)}{q|v_2|}k + \frac{(q+1)(|v_2|+1)}{q|v_2|} f_{\mathrm{BP}}(k) + \ell(k).$$

Proof (of Theorem 3.2(2)). Using Proposition 5.2 and Proposition 5.4, the problem is reduced to knowing when $2M - d - 2 > \Gamma_2(m)$ and $d \leq 2(M-6)$. Using the bound above it will suffices to show that

$$2\left(h(n-1) - \lfloor \log_2(3n) \rfloor + 1\right) - d - 2 > \frac{1}{4}m + \frac{21}{12} f_{\mathrm{BP}}(m) + \ell(m)$$

Since f_{BP} is sublinear, $h(n)$ is $n/2$ up to a constant error term, ℓ is at most logarithmic, and $d \leq n$ this can be simplified to,

$$n - d > \frac{1}{4}(m) + \epsilon'(n)$$

where $\epsilon'(n)$ is a sublinear error term. Simplifying further we get

$$\frac{2}{5}n - \frac{4}{5}\epsilon'(n) > d,$$

which is the desired conclusion. □

Proof (of Theorem 3.2(3)). Using Proposition 5.2 and Proposition 5.4, the problem is reduced to knowing when

$$2M - d - 2 > \Gamma_2(m)$$

and $d \leq 2(M-6)$. Using the bound on Γ_2 proved by Davis–Mahowald it will suffice to show that

$$2\left(h(n-1) - \lfloor \log_2(3n) \rfloor + 1\right) - d - 2 > \frac{3}{10}(2n+d) + 3 + \log_2(2n+d+2).$$

Using elementary manipulations one can show that it suffices to have,

$$4n - 50 - 30\log_2(3n) > 13d.$$

□

Proof (of Theorem 3.2(4)). Using Proposition 5.3 and Proposition 5.5, the problem is reduced to knowing when

$$2M > \Gamma_3(m) \text{ and } 2M - 2 > \Gamma_3(m)$$

Using the bound on Γ_3 above it will suffice to show that

$$2\left\lfloor \frac{n}{2p-2} \right\rfloor - 2\left\lfloor \log_p\left(\frac{3n}{2}\right) \right\rfloor - 2 > \frac{1}{16}(2n+d) + 7 + \frac{1}{4}$$

Using that $d \leq n-2$ and rearranging it will suffice to know that

$$\frac{5}{16}n > 11 + \frac{1}{8} + 2\log_3\left(\frac{3n}{2}\right).$$

Elementary arguments now suffice to conclude that this inequality holds for $n \geq 60$. □

Proof (of Theorem 3.2(5-9)). Using Proposition 5.3 and Proposition 5.5, the problem is reduced to knowing when

$$2M > \Gamma_3(m) \text{ and } 2M - 2 > \Gamma_3(m)$$

Using Gonzalez' bound on Γ_p it it will suffice to show that

$$2\left\lfloor \frac{n}{2p-2} \right\rfloor - 2\left\lfloor \log_p\left(\frac{3n}{2}\right)\right\rfloor - 2 > \frac{(2p-1)}{(2p-2)(p^2-p-1)}(2n+d) + 4 + \ell(2n+d).$$

Using that $d \leq n-2$ and rearranging it will suffice to show that,

$$\frac{2p^2-8p+1}{(2p-2)(p^2-p-1)}n > 8 + 3\log_p(3n) - 2\log_p(2) - \log_p(2p-2).$$

Elementary arguments now suffice to conclude that this holds in the following cases:

- $p = 5$ and $n \geq 192$,
- $p = 7$ and $n \geq 144$,
- $p = 11$ and $n \geq 160$,
- $p = 13$ and $n \geq 168$,
- $p = 17$ and $n \geq 224$,
- $p = 19$ and $n \geq 252$,
- $p \geq 23$ and $n \geq \frac{2p^2-2p-2}{3}$.

Since $2p^2 - 2p - 2$ is the first degree in which the cokernel of J is nontrivial we now only need to consider the primes less than 23. Using specific knowledge of the homotopy groups of spheres in low degrees from [NO76, NO77], we can improve the bounds for $p \geq 11$ to the following:

- $p = 11$ and $n \geq 100$,
- $p = 13$ and $n \geq 120$,
- $p = 17$ and all n,
- $p = 19$ and all n.

The bounds at $p = 5, 7$ could be likely be improved by making use of Ravenel's computations of the 5-complete stable stems through 1000 [Rav86] and the Oka–Nakamura results [NO76, NO77]. However, this is less straightforward than for larger primes. □

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