TRANSCHROMATIC EXTENSIONS IN MOTIVIC AND REAL BORDISM

AGNÈS BEAUDRY, MICHAEL A. HILL, XIAOLIN DANNY SHI, AND MINGCONG ZENG

Abstract. We show a number of Toda brackets in the homotopy of the motivic bordism spectrum MGL and of the Real bordism spectrum $MU_{\mathbb{R}}$. These brackets are "red-shifting" in the sense that while the terms in the bracket will be of some chromatic height n, the bracket itself will be of chromatic height $(n+1)$. Using these, we deduce a family of exotic multiplications in the $\pi_{(*,*)}MGL$-module structure of the motivic Morava K-theories, including non-trivial multiplications by 2. These in turn imply the analogous family of exotic multiplications in the $\pi_{\star}MU_{\mathbb{R}}$-module structure on the Real Morava K-theories.

1. Introduction

Complex bordism has played a fundamental role in stable homotopy since the 1960s. Work of Quillen connected complex bordism to formal groups, and this gives rise to the chromatic approach to stable homotopy theory. Building on Atiyah's Real K-theory [2], which can be viewed as Galois descent in families, Fujii and Landweber defined Real bordism [7, 16]. This theory plays an analogous role in C_2-equivariant homotopy theory that ordinary complex bordism does classically, and a detailed exploration of it was carried out by Hu–Kriz [13].

The Real bordism spectrum has proven central to understanding classical chromatic phenomena. By the Goerss–Hopkins–Miller theorem, the Lubin–Tate spectra E_n are acted upon by the Morava stabilizer group, and in particular, they can be viewed as genuine G-equivariant spectra for any finite subgroup G. Work of Hahn and the third author proved that at the prime 2, there is a Real orientation of all of the Lubin–Tate spectra, and for a finite subgroup G of the Morava stabilizer group that contains C_2, this extends to a G-equivariant map

$$MU^{((G))} = N_{C_2}^{G} MU_{\mathbb{R}} \to E_n,$$

where $MU^{((G))}$ is the norm of $MU_{\mathbb{R}}$ [8], introduced by Hill–Hopkins–Ravenel in the solution to the Kervaire invariant one problem [10]. This has turned questions about computations with the Lubin–Tate theories into questions about computations with the norms of $MU_{\mathbb{R}}$ and its quotients (see, for example [3]).

Motivically, there is a beautifully parallel story. The role of complex bordism is played by the spectrum MGL. Working over Spec($\mathbb{R}$), Galois descent is built in. Just as in classical and equivariant homotopy theory, MGL is a fundamental object of study in motivic homotopy, providing not only a chromatic filtration but also a way to understand the motivic homotopy sheaves of the sphere spectrum via Voevodsky's slice filtration.

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Voevodsky's slice filtration is an analogue of the Postnikov tower, where instead of killing all maps of spheres of a particular degree, we kill off all maps out of sufficiently many smash powers of $\mathbb{P}^1$. Applied to the algebraic K-theory spectrum KGL (an MGL-module spectrum), this yields the motivic cohomology to algebraic K-theory spectral sequence considered by Friedlander–Suslin, Voevodsky, and others (See [6, 25]). Hopkins and Morel generalized this, showing that the slices of MGL are suspensions of the spectrum representing motivic cohomology, and Hoyois provided a careful treatment and generalization of this result [12]. Work of Levine further connected the slice filtration to MGL, showing that the slice filtration for the sphere can be built out of the slice filtrations of the Adams–Novikov resolution based on MGL [17], and for the latter, the Hopkins–Hoyois–Morel result describes all of the starting pieces.

In this short paper, we produce a surprising family of Toda brackets in the homotopy groups of MGL which describe unexpected trans-chromatic phenomena. Recall that there is a canonical map

$$\pi_{2*}MU \to \pi_{(2*,*)}MGL$$

classifying the canonical group law for the motivic orientation (see [4] or [22]), and hence we have associated to the chromatic classes $v_n \in \pi_{2^{n+1}-2}MU$ the motivic classes $\bar{v}_n \in \pi_{(2^{n+1}-2,2^n-1)}MGL$. Recall also that we have a canonical element $\rho \in \pi_{(-1,-1)}S^0$ corresponding to the unit $-1 \in \mathbb{R}^\times$, by Morel's computation of the motivic zero stem [21].

Theorem. *For all $n \geqslant 0$, in the motivic homotopy of MGL over $\mathbb{R}$, we have an inclusion*

$$\rho^{2^{n+1}}\bar{v}_{n+1} \in \langle \bar{v}_n, \rho^{2^{n+1}-1}, \bar{v}_n\rangle,$$

and for all $k \geqslant 1$, we have an equality

$$\rho^{2^{n+1}+k}\bar{v}_{n+1} = \rho^k\langle \bar{v}_n, \rho^{2^{n+1}-1}, \bar{v}_n\rangle.$$

If we also consider the shifts of $\bar{v}_n$ to other weights, using some of the other motivic lifts $\bar{v}_n(b)$ of the chromatic classes, then we have even longer transchromatic connections.

Theorem. *For all $n \geqslant 0$, in the motivic homotopy of MGL over $\mathbb{R}$, we have an inclusion*

$$\rho^{2^{n+j+2}-2^{n+1}}\bar{v}_{n+j+1} \in \langle \bar{v}_n(2^j-1), \rho^{2^{n+1}-1}, \bar{v}_n\rangle,$$

and for all $k \geqslant 1$, we have an equality

$$\rho^{2^{n+j+2}-2^{n+1}+k}\bar{v}_{n+j+1} = \rho^k\langle \bar{v}_n(2^j-1), \rho^{2^{n+1}-1}, \bar{v}_n\rangle.$$

These transchromatic shifts imply a surprising number of hidden extensions in very naturally occurring quotients like the motivic Morava K-theories. Put in a pithy way, killing $\bar{v}_n$ without also killing $\bar{v}_{n+1}$ does not fully kill $\bar{v}_n$:

Theorem. *For all $n \geqslant 1$, for all $0 \leqslant k \leqslant n$ and for all $b \geqslant 0$, in the homotopy of the motivic Morava K-theory spectrum $K_{GL}(n)$, there are nontrivial multiplications by $\bar{v}_k(b)$.*

As a further example, we deduce exotic multiplications by topological Hopf maps. Work of Li–Shi–Wang–Xu on the Hurewicz image in $MU_{\mathbb{R}}$ goes through without change motivically [19], and we deduce that the classical h_k-family in the homotopy groups of spheres are detected motivically. Consideration of the slice spectral sequence for MGL and $k_{GL}(n)$ by Kylling allows us to identify these classes, and we see that h_n and h_n^2 are also detected in $k_{GL}(n)$.

Corollary 1.1. *For all $k < n$, the class h_k is zero in the homotopy of $k_{GL}(n)$. However, the class h_n^2 in $k_{GL}(n)$ is divisible by h_k for $0 \leqslant k \leqslant n$.*

We should think of this as an extremely harsh obstruction to a ring structure existing on these motivic and Real Morava K-theories.

There is a natural functor from motivic spectra over $\mathbb{R}$ to C_2-equivariant spectra, extending the functor "take complex points of a variety defined over $\mathbb{R}$", and this takes MGL to the Real bordism spectrum $MU_{\mathbb{R}}$ [14]. Moreover, as studied by Hu–Kriz, Hill, and Heard, this connects the motivic slice filtration to Dugger's C_2-equivariant slice filtration [14], [11], [9], [5].

This functor takes ρ to the Euler class $a_\sigma \in \pi_{-\sigma}^{C_2} MU_{\mathbb{R}}$ and the copy of the Lazard ring in the homotopy of MGL to the copy of the Lazard ring described by Araki in the homotopy of $MU_{\mathbb{R}}$ [1]. In particular, the motivic classes $\bar{v}_n$ are sent to the equivariant classes $\bar{v}_n \in \pi_{(2^n-1)\rho_2}^{C_2} MU_{\mathbb{R}}$, where $\rho_2 = 1 + \sigma$ is the regular representation of C_2. The classes $\bar{v}_n(b)$ are sent to the classes $\bar{v}_n u_{2\sigma}^{2^n b}$, using the notation of [10]. We therefore deduce the equivariant versions of these transchromatic phenomena. We spell these out for those more familiar with the equivariant literature.

Corollary. For all $n \geqslant 0$, in the $RO(C_2)$-graded homotopy of $MU_{\mathbb{R}}$, we have an inclusion

$$a_\sigma^{2^{n+j+2}-2^{n+1}} \bar{v}_{n+j+1} \in \langle \bar{v}_n(2^j - 1), a_\sigma^{2^{n+1}-1}, \bar{v}_n \rangle,$$

In general, the indeterminacy of these brackets may be larger in the C_2-equivariant context. We do not address this here.

We also have similar extensions in the Real Morava K-theories introduced by Hu–Kriz [13].

Theorem. *For all $n \geqslant 1$ and for all $0 \leqslant k \leqslant n$ and $b \geqslant 0$, in the $RO(C_2)$-graded homotopy of the Real Morava K-theory spectrum $K_{\mathbb{R}}(n)$, there are nontrivial multiplications by $\bar{v}_k(b)$.*

For $n = 1$, this recovers the classical observation that multiplication by 2 is not identically zero in $KO/2$. For larger n and k, these seems unknown.

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2. A non-trivial bracket

2.1. **The homotopy of MGL.** Just as classically, 2-locally we have a splitting of MGL into various suspensions of a motivic ring spectrum $BPGL$, and since this is smaller, we will mainly work with it. The bigraded homotopy ring of $BPGL$ has been completely determined [11], [15]. We very quickly recall the answer here, using the description from [11].

As an algebra, it is generated by the classes

$$\bar{v}_n(b) = \left[\bar{v}_n \tau^{2^{n+1}b}\right] \in \pi_{(2^{n+1}-2, 2^n-1-2^{n+1}b)} BPGL,$$

for all $n \geqslant 0$ and $b \geqslant 0$, together with the class

$$\rho \in \pi_{(-1,-1)} BPGL.$$

Here, we use the names which arise from the ρ-Bockstein spectral sequence. We will also normally use $\bar{v}_n$ for $\bar{v}_n(0)$.

There are relations which reflect the underlying products with τ: if $n \geqslant m$, then we have

$$\bar{v}_m(b) \cdot \bar{v}_n(c) = \bar{v}_m(b + 2^{n-m}c) \cdot \bar{v}_n.$$

We also have relations involving ρ: for all n and b,

$$\rho^{2^{n+1}-1} \bar{v}_n(b) = 0.$$

The homotopy is actually largely concentrated in a particular bidegree sector. With the exception of the subalgebra generated by ρ, the first (i.e. "topological") dimension is positive.

Proposition 2.1. *Outside of the subalgebra spanned by ρ, all elements have non-negative first coordinate.*

The only generators with first coordinate zero are

$$\rho^{2^{n+1}-2} \bar{v}_n(b),$$

with b arbitrary. The products of any of these is zero.

Proof. Of the listed algebra generators, only ρ has a negative first coordinate, and the bidegree of $\rho^j \bar{v}_n(b)$ is

$$|\rho^j \bar{v}_n(b)| = \big((2^{n+1} - 2) - j, (2^n - 1) - j - 2^{n+1}b\big).$$

Since the only non-zero values correspond to $0 \leqslant j \leqslant 2^{n+1} - 2$, we see that the first coordinate is always non-negative. This gives the first part.

For the second part, note that it is the last ρ power that gives a zero first coordinate. Since ρ times this is zero, we deduce that the product of any of these elements with first coordinate zero is zero. □

2.2. **An Adams spectral sequence.** Just as classically, the homology of $BPGL$ is cotensored up along a quotient Hopf algebroid. Voevodsky computed the dual Steenrod algebra over $\mathbb{R}$, showing that as a Hopf algebroid over

$$\mathbb{M}_2 = \mathbb{F}_2[\rho, \tau],$$

the motivic homology of a point [26], we have

$$\mathcal{A}_{*,*} = \mathbb{M}_2[\xi_1, \dots][\tau_0, \dots]/\big(\tau_i^2 + \rho\tau_{i+1} + (\tau + \rho\tau_0)\xi_{i+1}\big).$$

The element ρ is primitive, the left unit on τ is the obvious inclusion, and the right unit is $\tau + \rho\tau_0$. The coproducts on the ξ_i and the τ_i are the classical ones [27].

Definition 2.2. Let

$$\mathcal{E}(\infty) = \mathbb{M}_2[\tau_0, \tau_1, \dots]/(\tau_i^2 - \rho\tau_{i+1})$$

be the quotient of the motivic dual Steenrod algebra by the ideal generated by the ξ_is.

This is a Hopf algebroid under the motivic dual Steenrod algebra, and the generators τ_i are now primitive. In particular, all of the interesting behavior in Ext is determined by the left and right units on τ (since ρ, being in the Hurewicz image, is necessarily primitive). Just as classically, this Hopf algebroid is related to the homology of $BPGL$.

Theorem 2.3 ([4], [23])**.** *We have an isomorphism of $\mathcal{A}_\star$-comodule algebras*

$$H_*(BPGL;\mathbb{F}_2) \cong \mathcal{A}_\star \underset{\mathcal{E}(\infty)}{\square} \mathbb{M}_2.$$

Note that there are no ρ-torsion elements in either $\mathbb{M}_2$ or $\mathcal{E}(\infty)$. We will make heavy use of the ability to divide uniquely by ρ various ρ-divisible elements.

Theorem 2.4 ([11, Cor 5.2 and Thm 5.3])**.** *The E_2-term of the Adams spectral sequence computing the homotopy of $BPGL$ is*

$$E_2^{s,\star} = \mathrm{Ext}^{s,\star}_{(\mathbb{M}_2,\mathcal{E}(\infty))}(\mathbb{M}_2,\mathbb{M}_2).$$

The elements $\bar{v}_n(b)$ are detected by

$$\frac{d(\tau^{(1+2b)2^n})}{\rho^{2^{n+1}-1}} = \frac{\tau^{(1+2b)2^n} + (\tau+\rho\tau_0)^{(1+2b)2^n}}{\rho^{2^{n+1}-1}} \in \mathrm{Ext}^{1,(2^{n+1}-1,2^n-1-2^{n+1}b)}$$

and ρ by itself in $\mathrm{Ext}^{0,(-1,-1)}$. The spectral sequence collapses with no exotic extensions.

Remark 2.5. We can avoid the change-of-rings by instead working in the category of $BPGL$ or MGL-modules. The bigraded homotopy of $H\mathbb{F}_2 \underset{BPGL}{\wedge} H\mathbb{F}_2$ is isomorphic to $\mathcal{E}(\infty)$ as a Hopf algebroid over $\mathbb{M}_2$. Working over MGL instead introduces infinitely more generators with analogous relations.

2.3. **Connections between the $\bar{v}_n$s.** The Hopf algebroid $\big(\mathbb{M}_2,\mathcal{E}(\infty)\big)$ is computationally very simple: we have a primitive polynomial generator ρ and a second, non-primitive element τ. The comorphism ring $\mathcal{E}(\infty)$ is not polynomial on

$$\tau_0 = \frac{\eta_L(\tau)-\eta_R(\tau)}{\rho}.$$

Instead, we have a kind of "ρ-divided power algebra". The real power of Theorem 2.4 is that all of the generators of Ext^1 can be realized as ρ-fractional multiples of the usual cobar differential on powers of τ. That will allow us to easily compute Massey products.

It is helpful in what follows to blur the chromatic heights of the elements, focusing instead on the powers of τ and their differentials.

Notation 2.6. For an integer k, let $\nu_2(k)$ be its 2-adic valuation. For each $k \geqslant 1$, let $m_k = 2^{\nu_2(k)+1}-1$ and let

$$\bar{v}(k) = \frac{\eta_L(\tau)^k - \eta_R(\tau)^k}{\rho^{m_k}}.$$

Note that $\bar{v}_n(b)$ is detected by the element $\bar{v}\big((1+2b)2^n\big)$.

It is not hard to show that ρ^{m_k} is the largest power of ρ which divides the cobar differential on τ^k.

Theorem 2.7. *Let k and ℓ be non-negative, and let r, s, and t be natural numbers such that*

$$s + r \geqslant m_k \text{ and } s + t \geqslant m_\ell.$$

Then we have an inclusion

$$\rho^{m_{k+\ell} - m_k - m_\ell + r + s + t} \bar{v}(k + \ell) \in \left\langle \rho^r \bar{v}(k), \rho^s, \rho^t \bar{v}(\ell) \right\rangle.$$

Proof. There are preferred null-homotopies of $\bar{v}(k)\rho^{r+s}$ and $\bar{v}(\ell)\rho^{s+t}$:

$$\rho^{r+s-m_k} \tau^k \mapsto \rho^{r+s} \bar{v}(k) \text{ and } \rho^{s+t-m_\ell} \tau^\ell \mapsto \rho^{s+t} \bar{v}(\ell).$$

The bracket in question then contains

$$\rho^{s+t+r-m_\ell} \bar{v}(k) \eta_R(\tau^\ell) + \rho^{r+s+t-m_k} \tau^k \bar{v}(\ell).$$

Unpacking the ρ-fractions giving $\bar{v}(k)$ and $\bar{v}(\ell)$ and recalling that the cobar differential is a bimodule derivation, we see that this particular element is

$$\rho^{m_{k+\ell} - m_k - m_\ell + r + s + t} \bar{v}(k + \ell). \qquad \square$$

Remark 2.8. We have written the statement and proof in a suggestive, but somewhat general way, to emphasize that these same relations will hold for any similar Hopf algebroid framework with a primitive element playing the role of ρ and the differential on a class analogous to τ involving only ρ divisibility. A similar story plays out with the Miller–Ravenel–Wilson Chromatic Spectral Sequence approach to understanding the classical Adams–Novikov E_2-term [20]. For example, we see the same brackets involving the α-family.

When k and ℓ have different 2-adic valuations, then this is just recovering the description of elements like $\bar{v}_n(b)$ as various brackets given by the ρ-Bockstein spectral sequence (or equivalently, given by the slice differentials). In both the ρ-Bockstein and slice spectral sequences, the differentials on the powers of τ depend on the 2-adic valuation of the exponent, and the targets of later differentials annihilate the lower chromatic classes. This gives the usual rewriting of the $\bar{v}_n(b)$ as Massey products.

Example 2.9. We have ρ-Bockstein differentials, $d_3(\tau^2) = \rho^3 \bar{v}_1$ and $d_7(\tau^4) = \rho^7 \bar{v}_2$, and hence $\bar{v}_1 \tau^4$ is a d_7-cycle which actually represents $\bar{v}_1(1)$. In this formulation, however, we see that it also represents the bracket

$$\langle \bar{v}_1, \rho^7, \bar{v}_2 \rangle = \langle \bar{v}(2), \rho^{m_4}, \bar{v}(4) \rangle.$$

When k and ℓ have the same 2-adic valuation, then we get the exotic new brackets. We rewrite these relations in the usual chromatic names. Although this is an immediate corollary of Theorem 2.7, we write it as a theorem to stress the central role in this paper.

Theorem 2.10. *Let b and c be nonegative numbers, and let r, s, and t be such that*

$$r + s, s + t \geqslant m_n = (2^{n+1} - 1).$$

Let j and d be such that we have

$$1 + b + c = (1 + 2d)2^j.$$

Then we have an inclusion

$$\rho^{(2^{n+j+2} - 2^{n+2} + r + s + t + 1)} \bar{v}_{n+j+1}(d) \in \left\langle \rho^r \bar{v}_n(b), \rho^s, \rho^t \bar{v}_n(c) \right\rangle.$$

In particular, Theorem 2.10 recovers the classical bracket involving the Hopf classes
$$h_n = \rho^{2^n-1}\bar{v}_n.$$

Corollary 2.11. *For each n, we have an inclusion*
$$h_{n+1}^2 \in \langle h_n, h_{n+1}, h_n\rangle.$$

Proof. Taking $r = t = (2^n - 1)$ and $s = (2^{n+1} - 1)$ in Theorem 2.10 gives an inclusion
$$\rho^{2^{n+1}-1}h_{n+1} = \rho^{2^{n+2}-2}\bar{v}_{n+1} \in \langle \rho^{2^n-1}\bar{v}_n, \rho^{2^{n+1}-1}, \rho^{2^n-1}\bar{v}_n\rangle.$$
Multiplying by $\bar{v}_{n+1}$ and applying the standard shuffle inclusions gives
$$h_{n+1}^2 \in \bar{v}_{n+1}\langle h_n, \rho^{2^{n+1}-1}, h_n\rangle \subset \langle \bar{v}_{n+1}h_n, \rho^{2^{n+1}-1}, h_n\rangle \subset \langle h_n, h_{n+1}, h_n\rangle. \qquad \square$$

Remark 2.12. When $n = 0$, Theorem 2.10 gives the formula
$$\rho\eta \in \langle 2, \rho, 2\rangle,$$
since $\rho\bar{v}_1$ is the ordinary, topological η. Since $2\rho = 0$, this is recovering a universal formula in an A_∞-ring spectrum that forming the balanced bracket with 2 gives η multiplication [24].

2.4. **Indeterminacy.** One of the most useful parts of these brackets is that the indeterminacy is easily controlled.

Theorem 2.13. *Fix non-negative numbers m, n, b, and c. If r, s, and t are non-negative integers such that*
$$r + s \geqslant (2^{m+1} - 1) \text{ and } s + t \geqslant (2^{n+1} - 1),$$
then the indeterminacy of
$$\langle \rho^r\bar{v}_m(b), \rho^s, \rho^t\bar{v}_n(c)\rangle$$
is nonzero in only two cases:

(1) *when $t = 0$, $r + s = 2^{m+1} - 1$, $m \geqslant n$, and $m > 0$, where the indeterminacy is*
$$\mathbb{Z}_{(2)} \cdot \bar{v}_0\big((1 + 2b)2^{m-1}\big)\bar{v}_n(c),$$
(2) *or when $r = 0$, $s+t = 2^{n+1}-1$, $n \geqslant m$, and $n > 0$, where the indeterminacy is*
$$\mathbb{Z}_{(2)} \cdot \bar{v}_0\big((1 + 2c)2^{n-1}\big)\bar{v}_m(b).$$

Proof. The indeterminacy of the bracket $\langle \rho^r\bar{v}_m(b), \rho^s, \rho^t\bar{v}_n(c)\rangle$ is the subgroup
$$\rho^r\bar{v}_m(b)\pi_{(x_t,y_t)}BPGL + \rho^t\bar{v}_n(c)\pi_{(x_r,y_r)}BPGL,$$
where
$$(x_t, y_t) = |\rho^{s+t}\bar{v}_n(c)| + (1, 0) = \big(2^{n+1} - 1 - s - t, 2^n - 1 - s - t - 2^{n+1}c\big)$$
is the degree of the choices of null-homotopy of $\rho^{s+t}\bar{v}_n(c)$, and similarly for (x_r, y_r).

By our assumptions on r, s, and t, the first coordinate is always non-positive. This gives a huge number of brackets with trivial indeterminacy immediately: if
$$r + s > (2^{m+1} - 1) \text{ and } s + t > (2^{n+1} - 1),$$
then we have
$$\pi_{(x_r,y_r)}BPGL = \pi_{(x_t,y_t)}BPGL = 0,$$

since the first coordinate is negative and the second does not equal the first. In particular, this means we have no indeterminacy. We need only consider the cases that $r+s=(2^{m+1}-1)$ or $s+t=(2^{n+1}-1)$.

There is an obvious symmetry here in m and n, so it suffices to understand the case $r+s=(2^{m+1}-1)$. In this case,

$$(x_r, y_r) = \big(0, -2^m - 2^{m+1}b\big).$$

Proposition 2.1 shows that the only classes with zero first coordinate are the classes $\rho^{2^{k+1}-2}\bar{v}_k(a)$, which is in bidegree

$$|\rho^{2^{k+1}-2}\bar{v}_k(a)| = \big(0, 1-2^k(1+2a)\big).$$

We are therefore looking for all pairs (k,a) such that

$$1-2^k(1+2a) = -2^m(1+2b).$$

If $m>0$, then we must have $k=0$ and $a=2^{m-1}(1+2b)$, and this corresponds to the class

$$\bar{v}_0\big((1+2b)2^{m-1}\big).$$

This generates a $\mathbb{Z}_{(2)}$, and hence the contribution to the indeterminacy is the subgroup generated by

$$\bar{v}_0\big((1+2b)2^{m-1}\big)\cdot\rho^t\bar{v}_n(c).$$

If $t>0$, then this is automatically zero (since $\rho\bar{v}_0(a)=0$), so if $r+s=(2^{m+1}-1)$ and $t>0$, then we have no contribution to the indeterminacy. On the other hand, if $t=0$, then we have a contribution:

$$\mathbb{Z}_{(2)}\cdot\bar{v}_0\big((1+2b)2^{m-1}\big)\bar{v}_n(c) = \mathbb{Z}_{(2)}\cdot\bar{v}_0\big(2^{m-1}+2^m b+2^n c\big)\bar{v}_n.$$

Note that if $t=0$, then the condition that $s+t\geqslant(2^{n+1}-1)$ implies that $s\geqslant(2^{n+1}-1)$. This with the condition that $r+s=(2^{m+1}-1)$ in particular shows the condition $m\geqslant n$.

Now let $m=0$. We find $k=\big(\nu_2(1+b)+1\big)$, and

$$a = \frac{2^{1-k}(1+b)-1}{2}.$$

Note also that $k\geqslant 1$, so the contribution to the indeterminacy is

$$\mathbb{Z}/2\cdot\rho^{2^{k+1}-2}\bar{v}_k(a)\rho^t\bar{v}_n(c).$$

Again, if $t>0$, then this is automatically zero, since

$$\rho^{2^{k+1}-1}\bar{v}_k(a)=0.$$

If $t=0$, then the conditions $r+s=2^{0+1}-1=1$ and $s+t=(2^{n+1}-1)$ imply that in fact, $n=0$ as well. Since $k\geqslant 1$, $(2^{k+1}-2)\geqslant 1$, and hence

$$\rho^{2^{k+1}-2}\bar{v}_k(a)\bar{v}_0(c)=0.$$

Thus in all cases, the contribution to indeterminacy here is zero. □

Remark 2.14. Note that the two cases we have indeterminacy actually overlap: $n=m>0$, $r=t=0$, and $s=(2^{n+1}-1)$. Viewing this as the first case, we see that the indeterminacy is generated by

$$\bar{v}_0\big((1+2b)2^{n-1}\big)\bar{v}_n(c) = \bar{v}_0\big((1+2b+2c)2^{n-1}\big)\bar{v}_n = \bar{v}_0\big((1+2c)2^{n-1}\big)\bar{v}_n(b),$$

which is the generator of indeterminacy we see viewing it as the second case.

Corollary 2.15. *Let b and c be nonegative numbers, and let r, s, and t be such that*

$$r+s, s+t \geqslant m_n = (2^{n+1}-1).$$

Let j and d be such that we have

$$1+b+c = (1+2d)2^j.$$

Finally, let $k \geqslant 1$. Then we have an equality

$$\rho^{2^{n+j+2}-2^{n+2}+r+s+t+1+k}\bar{v}_{n+j+1}(d) = \rho^k \big\langle \rho^r \bar{v}_n(b), \rho^s, \rho^t \bar{v}_n(c) \big\rangle.$$

Proof. Multiplication by ρ kills the class $\bar{v}_0\big((1+2b)2^{n-1}\big)$, and hence

$$\bar{v}_0\big((1+2b)2^{n-1}\big)\bar{v}_n(c).$$

This generated the only indeterminacy for any of the brackets. □

3. Application to Morava K-theories

The transchromatic brackets give us some surprising consequences for the action of $\pi_{(*,*)}MGL$ on various quotients. Since MGL is a commutative monoid, we have a good category of MGL-modules. Working here, we can form various iterated quotients, killing elements in homotopy groups. For a single element, we define the quotient via the cofiber sequence

$$\Sigma^{|x_i|}MGL \xrightarrow{x_i} MGL \to MGL/x_i,$$

and for a family of elements $x_1, x_2 \dots$, we form

$$MGL/(x_1, x_2, \dots) = MGL/x_1 \underset{MGL}{\wedge} MGL/x_2 \underset{MGL}{\wedge} \cdots.$$

Definition 3.1. For each $n \geqslant 0$, let

$$k_{GL}(n) = BPGL/(\bar{v}_0, \dots, \bar{v}_{n-1}, \bar{v}_{n+1}, \dots)$$

be the nth connective motivic Morava K-theory.

Let $K_{GL}(n) = \bar{v}_n^{-1}k_{GL}(n)$ be motivic Morava K-theory.

Using the Hopkins–Hoyois–Morel determination of the slices of MGL [12], Levine–Tripathi determined the slices for $k_{GL}(n)$ (and related quotients).

Theorem 3.2 ([18, Cor. 4.6])**.** *The slice associated graded for $k_{GL}(n)$ is*

$$Gr(k_{GL}(n)) = \bigvee_{m \geqslant 0} \Sigma^{(2^{n+1}-2)m,(2^n-1)m} H\mathbb{F}_2.$$

Moreover, the associated slice spectral sequence is a spectral sequence of modules over the slice spectral sequence for $BPGL$, which has been spelled out explicitly in [15].

Theorem 3.3 ([15])**.** *The motivic slice E_2 term for $BPGL$ is given by*

$$\mathbb{Z}_{(2)}[\rho, \tau^2, \bar{v}_1, \bar{v}_2, \dots],$$

where $|\rho| = (-1,-1)$, where $|\tau^2| = (0,-2)$, and where $|\bar{v}_i| = (2^{i+1}-2, 2^i-1)$.

The differentials are given by

$$d_{2^{n+1}-1}(\tau^{2^n}) = \rho^{2^{n+1}-1}\bar{v}_n.$$

Remark 3.4. These differentials are the same as the differentials in the equivariant slice spectral sequence for $BP_{\mathbb{R}}$ [10]. Under the map from motivic to C_2-equivariant spectra, the element τ^2 is taken to $u_{2\sigma}$. This is essential in Kylling's analysis, building on work of Hu–Kriz [14] and the second author [11].

For degree reasons, the spectral sequence for $k_{GL}(n)$ is even simpler; we have thrown away many of the classes which supported or could have supported differentials.

Theorem 3.5 ([15, Theorem 9.6], [28]). *The slice spectral sequence for $k_{GL}(n)$ has E_2-term*

$$\mathbb{F}_2[\rho, \tau, \bar{v}_n]\{\iota\}.$$

As an $\mathbb{F}_2[\rho, \bar{v}_n, \tau^{2^{n+1}}]$-module, the non-trivial differentials are generated by

$$d_{2^{n+1}-1}(\tau^{a+2^n}\iota) = \rho^{2^{n+1}-1}\bar{v}_n\tau^a\iota,$$

where $0 \leqslant a \leqslant 2^n - 1$.

Away from the subalgebra $\mathbb{F}_2[\rho]$, the localization map

$$k_{GL}(n) \to K_{GL}(n)$$

is injective. The latter is an $MGL[\bar{v}_n^{-1}]$-module. In $MGL[\bar{v}_n^{-1}]$, the class $\tau^{2^{n+1}}$ is a permanent cycle and multiplication by this is well-defined in $K_{GL}(n)$. By injectivity of the localization, we may therefore view everything as being a module over

$$\mathbb{Z}_{(2)}[\rho, \bar{v}_n, \tau^{2^{n+1}}].$$

Corollary 3.6. *As a module over $\mathbb{Z}_{(2)}[\rho, \bar{v}_n, \tau^{2^{n+1}}]$, the homotopy of $k_{GL}(n)$ is generated by the classes $\tau^a\iota$ for $0 \leqslant a \leqslant (2^n - 1)$.*

Although the spectral sequence collapses here, we have a surprising number of non-trivial multiplications by the $\bar{v}_k$-generators that we killed to form $k_{GL}(n)$, including non-trivial multiplications by 2. These are all detected by our brackets, using the sparseness of the spectral sequence (and hence sparseness of the bigraded homotopy groups).

We have a simple consequence of the module structure and the presence of $\tau^{2^{n+1}}$: we need only check a small number of hidden extensions.

Corollary 3.7. *We need only determine hidden extensions of the form $\bar{v}_k(b)\tau^a\iota$ for $0 \leqslant b \leqslant (2^{n-k} - 1)$.*

Proof. Since there is a class $\tau^{2^{n+1}}$, we have

$$\bar{v}_k(b + 2^{n-k}) = \bar{v}_k(b)\tau^{2^{n+1}}.$$

□

Remark 3.8. While going through our analysis of extensions, the reader is encouraged to consult Figure 1 which shows the slice associated graded for the bigraded homotopy groups of $k_{GL}(3)$. The grading is the usual motivic bigrading. The circled classes are the generators as a module over $\mathbb{Z}_{(2)}[\rho, \bar{v}_n, \tau^{2^{n+1}}]$, and the notation is as follows:

- The dotted lines indicate multiplication by ρ.
- The dashed blue lines indicate exotic multiplication by $\bar{v}_2$, and the solid red lines indicate exotic multiplication by $\bar{v}_1$.

- The bullets • and solid black squares ■ both indicate copies of $\mathbb{F}_2$; bullets complexify to zero while black squares complexify to the corresponding generator.
- The open circles ○ are copies of $\mathbb{Z}/4$ which complexify to $\mathbb{Z}/2$.

To avoid clutter, we do not draw the exotic multiplications by $\bar{v}_k(b)$ for $b > 0$ and $k < 3$.

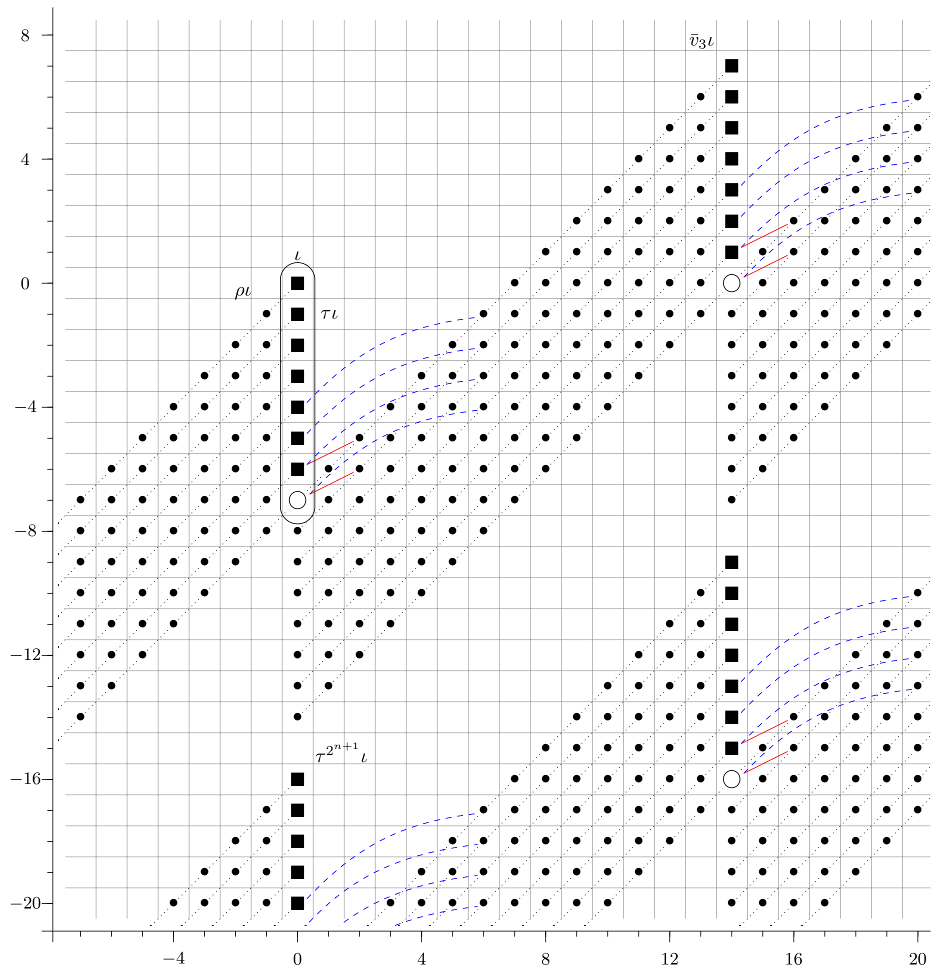

FIGURE 1. The slice E_∞ term for $k_{GL}(3)$, together with extensions. See Remark 3.8 for the notation.

Lemma 3.9. *Let a and b be non-negative integers and $0 \leqslant k < n$. Then there is at most one non-zero class that could be $\bar{v}_k(b)\tau^a\iota$.*

If

$$a + 2^k(1 + 2b) < 2^n \text{ or } 2^{n+1} - 1 < a + 2^k(1 + 2b),$$

then we have $\bar{v}_k(b)\tau^a\iota = 0$.

Proof. The E_∞-page of the slice spectral sequence looks like a quilt. Applying the degree sheer $(x', y') = (x, y - x)$, the homotopy groups are built out of rectangles of size $(2^{n+1}-1)$ by (2^n), with a single non-zero class in each degree. The generator of each rectangle appears as the top right corner and is given by $\tau^{2^{n+1}m}\bar{v}_n^\ell \iota$ for $m \geqslant 0$ and $\ell \geqslant 0$.

Since $\bar{v}_n$ has sheered bidegree $\big((2^{n+1}-2), -(2^n-1)\big)$, we see that the rectangle for α overlaps with the one for $\bar{v}_n\alpha$ in exactly one corner: the bottom right for α and the top left for $\bar{v}_n\alpha$. Multiplication by $\tau^{2^{n+1}}$ adds in a rectangle with generator in degree $|\alpha| + (0, -2^{n+1})$. This does not intersect either the original rectangle or any of its $\bar{v}_n$-multiples. In particular, this proves the first claim, since almost all degrees have a single non-zero class in them, and the degrees with overlap have a single class in slice filtration larger than zero (where any hidden extensions must land).

For the second claim, recall that the degree of $\bar{v}_k(b)\tau^a\iota$ is

$$|\bar{v}_k(b)\tau^a\iota| = \big(2^{k+1}-2, 2^k - 1 - 2^{k+1}b - a\big).$$

Since the classes $\tau^a\iota$ are at the rightmost edge of their rectangle, these extensions must show up in the rectangle from $\bar{v}_n\iota$. The top edge of the rectangle for $\bar{v}_n\iota$ is given by the classes $\rho^m\bar{v}_n\iota$. The only class on that edge with first coordinate $(2^{k+1}-2)$ is $\rho^{2^{n+1}-2^{k+1}}\bar{v}_n\iota$, which is in bidegree

$$|\rho^{2^{n+1}-2^{k+1}}\bar{v}_n\iota| = \big(2^{k+1}-2, 1 - 2^n + 2^{k+1} - 2\big).$$

Similarly, the bottom edge of the rectangle on $\bar{v}_n\iota$ is given by the classes $\rho^m\bar{v}_n\tau^{2^n-1}\iota$. Rearranging the second coordinates gives the desired vanishing region. $\square$

Theorem 3.10. *In the homotopy of $k_{GL}(n)$, we have the following hidden $\pi_{(*,*)}BPGL$-multiplications for $0 \leqslant k \leqslant (n-1)$:*

- *for $0 \leqslant b \leqslant (2^{n-k-1}-1)$ and $0 \leqslant a \leqslant \big(2^k(1+2b)-1\big)$,*
$$\bar{v}_k(b)\tau^{2^n - 2^k(1+2b)+a}\iota = \rho^{2^{n+1}-2^{k+1}}\bar{v}_n\tau^a\iota,$$
- *and for $2^{n-k-1} \leqslant b \leqslant (2^{n-k}-1)$, and $0 \leqslant a \leqslant \big(2^{n+1}-1-2^k(1+2b)\big)$,*
$$\bar{v}_k(b)\tau^a\iota = \rho^{2^{n+1}-2^{k+1}}\bar{v}_n\tau^{a+2^k(1+2b)-2^n}\iota.$$

Proof. These follow surprisingly quickly from Theorem 2.10. We note that since multiplication by ρ is injective on both the source and target of any extension involving $\bar{v}_k(b)$ for $k > 0$, there is no ambiguity caused by the indeterminacy. There was no indeterminacy anyway for brackets involving $\bar{v}_0(b)$.

For $0 \leqslant b \leqslant (2^{n-k-1}-1)$, Theorem 2.10 shows that we have a bracket

$$\rho^{2^{n+1}-2^{k+1}}\bar{v}_n = \big\langle \bar{v}_k(b), \rho^{2^{k+1}-1}, \bar{v}_k(2^{n-k-1}-1-b)\big\rangle,$$

(where again, we ignore indeterminacy since it does not contribute). This gives us

$$\rho^{2^{n+1}-2^{k+1}}\bar{v}_n\tau^a\iota = \big\langle \bar{v}_k(b), \rho^{2^{k+1}-1}, \bar{v}_k(2^{n-k-1}-1-b)\big\rangle\tau^a\iota.$$

Lemma 3.9 shows that for $a < 2^k(1+2b)$, we have

$$\bar{v}_k(2^{n-k-1}-1-b)\tau^a\iota = 0,$$

so we can shuffle the bracket, giving:

$$\rho^{2^{n+1}-2^{k+1}}\bar{v}_n\tau^a\iota = \bar{v}_n(b)\big\langle \rho^{2^{k+1}-1}, \bar{v}_k(2^{n-k-1}-1-b), \tau^a\iota\big\rangle.$$

A degree check shows that the only possible value for the bracket is

$$\left\langle \rho^{2^{k+1}-1}, \bar{v}_k(2^{n-k-1}-1-b), \tau^a\iota \right\rangle = \tau^{2^n-2^k(1+2b)+a}\iota.$$

For $2^{n-k-1} \leqslant b \leqslant 2^{n-k}-1$, we instead will use brackets with $\bar{v}_n(1) = \bar{v}_n\tau^{2^{n+1}}$, arguing somewhat indirectly. If we let

$$c = 2^{n-k} + 2^{n-k-1} - b - 1,$$

then we have

$$\rho^{2^{n+1}-2^{k+1}}\bar{v}_n(1) = \left\langle \bar{v}_k(c), \rho^{2^{k+1}-1}, \bar{v}_k(b) \right\rangle.$$

If $\bar{v}_k(b)\tau^a\iota = 0$, then we can multiply $\tau^a\iota$ by both sides and shuffle the bracket, getting

$$\rho^{2^{n+1}-2^{k+1}}\bar{v}_n(1)\tau^a\iota = \bar{v}_k(c)\left\langle \rho^{2^{k+1}-1}, \bar{v}_k(b), \tau^a\iota \right\rangle.$$

The degree of the bracket is $\big(0, -a-2^k(1+2b)\big)$. The bounds

$$0 \leqslant a \leqslant 2^{n+1}-1-2^k(1+2b)$$

and $b \geqslant 2^{n-k-1}$ show that we have bounds

$$-2^k - 2^n \geqslant -a - 2^k(1+2b) \geqslant 1 - 2^{n+1}.$$

We are therefore at the leftmost edge of the "quilt rectangle" generated by $\bar{v}_n\iota$. All of the elements with first coordinate zero here are annihilated by ρ, so we reach a contradiction:

$$0 \neq \rho^{2^{n+1}-2^{k+1}+1}\bar{v}_n(1)\tau^a\iota = \bar{v}_k(c)\rho\left\langle \rho^{2^{k+1}-1}, \bar{v}_k(b), \tau^a\iota \right\rangle = 0.$$

As always, there is a unique possibility for the value of $\bar{v}_k(b)\tau^a\iota$, and checking degrees, we see it must be the listed one. □

Specializing to the usual chromatic classes $\bar{v}_k$, we have a series of non-trivial extensions.

Corollary 3.11. *In the homotopy of $k_{GL}(n)$, we have the following hidden $\bar{v}_k$-multiplications for $0 \leqslant k \leqslant (n-1)$ and $0 \leqslant a \leqslant (2^k-1)$:*

$$\bar{v}_k\tau^{2^n-2^k+a}\iota = \rho^{2^{n+1}-2^{k+1}}\bar{v}_n\tau^a\iota.$$

Since $\bar{v}_0$ detects multiplication by 2 in MGL-modules, we see that we have non-trivial additive extensions. This resolves the question described in [15, Remark 9.8].

Remark 3.12. These extensions show just how far the quotient is from being a ring. One way to parse Theorem 3.10 is that although we cone-off $\bar{v}_k$ for $0 \leqslant k \leqslant n-1$, and hence we seem to kill all of the classes $\bar{v}_k(b)$, we actually see non-trivial multiplications by all of the generators of $\pi_{*,*}BPGL/(\bar{v}_{n+1}, \dots)$.

Even though the indeterminacy for brackets may grow in the passage from motivic homotopy over $\mathbb{R}$ to C_2-equivariant homotopy, the extensions we found are visible just in the homotopy. They will in particular go through without change. We indicate the result using the usual equivariant names.

Corollary 3.13. *In the homotopy groups of $K_{\mathbb{R}}(n)$, we have exotic $\pi_\star MU_{\mathbb{R}}$ multiplications for $0 \leqslant k \leqslant (n-1)$:*

- *for $0 \leqslant b \leqslant (2^{n-k-1}-1)$, and $0 \leqslant a \leqslant \big(2^k(1+2b)-1\big)$,*

$$\bar{v}_k(b)u_\sigma^{2^n-2^k(1+2b)+a}\iota = a_\sigma^{2^{n+1}-2^{k+1}}\bar{v}_n u_\sigma^a\iota,$$

- *and for* $2^{n-k-1} \leqslant b \leqslant (2^{n-k}-1)$, *and* $0 \leqslant a \leqslant \big(2^{n+1}-1-2^k(1+2b)\big)$,

$$\bar{v}_k(b)u_\sigma^a\iota = a_\sigma^{2^{n+1}-2^{k+1}}\bar{v}_n u_\sigma^{a+2^k(1+2b)-2^n}\iota.$$

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University of Colorado Boulder, Boulder, CO, 80309
E-mail address: agnes.beaudry@colorado.edu

University of California Los Angeles, Los Angeles, CA 90095
E-mail address: mikehill@math.ucla.edu

University of Chicago, Chicago, IL 60637
E-mail address: dannyshixl@gmail.com

Mathematical Institute, Utrecht University, Utrecht, 3584 CD, the Netherlands
E-mail address: m.zeng@uu.nl