CHAPTER II

MISCELLANEOUS APPLICATIONS IN STABLE HOMOTOPY THEORY

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A number of important results in stable homotopy theory are very easy consequences of quite superficial properties of extended powers of spectra. We give several such applications here.

The preservation properties of equivariant half-smash products (e.g. in I.1.2) do not directly imply such properties for extended powers since the jth power functor from spectra to $E_j$-spectra tends not to enjoy such properties. We illustrate the point in section 1 by analyzing the structure of extended powers of wedges and deriving useful consequences about extended powers of sums of maps. These results are largely spectrum level analogs of results of Nishida [90] about extended powers of spaces, but the connection with transfer was suggested by ideas of Segal [96].

Reinterpreting Nishida's proof [90], we show in section 2 that the nilpotency of the ring $\pi_* S$ of stable homotopy groups of spheres (or "stable stems") is an immediate consequence of the Kahn-Priddy theorem and our analysis of extended powers of wedges. The implication depends only on the fact that the sphere spectrum is an $H_\infty$ ring spectrum. This proof gives a very poor estimate of the order of nilpotency. Nishida also gave a different proof [90] which applies only to elements of order $p$ but gives a much better estimate of the order of nilpotency. In section 6, we show that this too results by specialization to $S$ of a result valid for general $H_\infty$ ring spectra. Here the key step is an application of a splitting theorem that Steinberger will prove by use of homology operations in the next chapter. His theorem will make clear to what extent this method of proof applies to elements of order $p^i$ with $i > 1$.

The material discussed so far dates to 1976-77 (and was described in [72]). The material of sections 3-5 is much more recent, dating from 1982-83. The ideas here are entirely due to Miller, Jones, and Wegmann, who saw applications of extended powers that we had not envisaged. (However, all of the information about extended powers needed to carry out their ideas was already explicit or implicit in [72] and the 1977 theses [23, 101] of Bruner and Steinberger.) Jones and Wegmann [44] constructed new homology and cohomology theories from old ones by use of systems of extended powers and showed that theorems of Lin [53] and Gunawardena [38] imply that these theories specialize to give exotic descriptions of stable homotopy
and stable cohomotopy. Jones [43] later gave a remarkably ingenious proof of the
Kahn-Priddy theorem in terms of these theories. The papers [43, 44] only treated
the case $p = 2$, and we give the details for all primes in sections 3 and 4. (In
fact, much of the work goes through for non-prime integers.) The idea for the
Jones-Wegmann theories grew out of Haynes Miller's unpublished observation that
systems of extended powers can be used to realize cohomologically a basic algebraic
construction introduced by Singer [52, 98]. We explain this fact and its
relationship to the cited theorems of Lin and Gunawardena in section 5.

§1. Extended powers of wedges and transfer maps

Fix positive integers $j$ and $k$ and spectra $Y_i$ for $1 \leq i \leq k$. Let
$Y = Y_1 \vee \cdots \vee Y_k$ and let $v_i : Y_i \to Y$ be the inclusion. For a partition
$J = (j_1, \ldots, j_k)$ of $j$, $j_1 \geq 0$ and $j_1 + \cdots + j_k = j$, write $a_J = a_{j_1}, \ldots, a_{j_k}$ and let $f_J$
denote the composite

$$D_j Y \to D_{j_1} Y_1 \wedge \cdots \wedge D_{j_k} Y_k.$$ 

For later use, note that permutations $\sigma \in \Sigma_k$ act on partitions and that 1.2.8
implies the equivariance formula $f_J = f_{\sigma J} \circ \sigma$. Note too that, for maps $h_i : Y_i \to E$
with wedge sum $h : Y \to E$, the following diagram commutes by the naturality of $a_J$.

Theorem 1.1. Let $Y = Y_1 \vee \cdots \vee Y_k$. Then the wedge sum

$$f_J : \bigvee_{j_1} D_{j_1} Y_1 \wedge \cdots \wedge D_{j_k} Y_k \to D_j Y$$

of the maps $f_J$ is an equivalence of spectra.

Proof. By the distributivity of smash products over wedges,

$$Y^{(j)} = \bigvee_{j_1} Y_1^{(j_1)} \wedge \cdots \wedge Y_1^{(j_k)}.$$
where $I$ runs over all sequences $(i_1, \ldots, i_j)$ such that $1 \leq i_r \leq k$. Say that $I \in J$ if there are exactly $j_s$ entries $i_r$ equal to $s$ for each $s$ from 1 to $k$. For each partition $J$ of $J$, let $\Sigma_J = \Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ and define

$$Y_J = \bigvee_{I \in J} Y_{i_1} \wedge \cdots \wedge Y_{i_j} = \Sigma_J \Sigma_J (Y_{j_1} \wedge \cdots \wedge Y_{j_k}).$$

(Here the isomorphism would be obvious on the space level and holds on the spectrum level by direct inspection of the definitions in [Equiv. II §§3–4].) Then $Y_J$ is a $\Sigma_J$-subspace of $Y(J)$ and $Y(J) = \bigvee_J Y_J$. Now

$$D_J Y = \bigvee_J \Sigma_J \Sigma_J Y_J \text{ and } \Sigma_J \Sigma_J Y_J = \Sigma_J \Sigma_J (Y_{j_1} \wedge \cdots \wedge Y_{j_k})$$

by I.1.2(i) and I.1.4. Clearly $f_J$ has image in $\Sigma_J \Sigma_J Y_J$ and factors as the composite

$$\begin{align*}
\left(\Sigma_J \Sigma_J Y_{j_1} \wedge \cdots \wedge \Sigma_J \Sigma_J Y_{j_k}\right) \\
\Downarrow \alpha \\
\left(\Sigma_J \Sigma_J Y_{j_1} \times \cdots \times \Sigma_J \Sigma_J Y_{j_k}\right) \\
\Downarrow i \bowtie 1 \\
\Sigma_J \Sigma_J (Y_{j_1} \wedge \cdots \wedge Y_{j_k}).
\end{align*}$$

Here $\alpha$ is an isomorphism. (Technically, the smash product in its domain is "internal" while that in its range is "external"; see [Equiv, II§3].) The map

$$i: \Sigma_J \Sigma_J \times \cdots \times \Sigma_J \Sigma_J \longrightarrow \Sigma_J \Sigma_J$$

is given by the commutation with products and naturality of the functor $E$ and is a $E_J$-equivalence. Therefore $i \bowtie 1$ is an equivalence (by [Equiv, VI.1.15]). The conclusion follows.

Our interest is mainly in finite wedges, but precisely the same argument applies to give an analog for infinite wedges.

**Theorem 1.2.** Let $\{Y_i\}$ be a set of spectra indexed on a totally ordered set of indices and let $Y = \bigvee_i Y_i$. For a strictly increasing sequence $I = (i_1, \ldots, i_k)$ of indices and a partition $J = (j_1, \ldots, j_k)$ of $J$ with each $j_1 > 0$ (hence $k \leq j$), let

$$f_{J, I}: D_J Y_{i_1} \wedge \cdots \wedge D_J Y_{i_k} \rightarrow D_J Y$$

be the composite of $f_J$ and the evident inclusion. Then the wedge sum
of the maps $f_{j,i}$ is an equivalence of spectra.

Parenthetically, this leads to an attractive alternative version of the definition, I.4.3, of an $H_d$ ring spectrum.

**Proposition 1.3.** An $H_d$ ring structure on $E$ determines and is determined by an $H_\infty$ ring structure on the wedge $\bigvee_I E$.  

**Proof.** If $\bigvee_i E$ is an $H_\infty$ ring spectrum with structural maps $\xi_j$, then the evident composites

$$
\xi_{j,i} : D_j E \longrightarrow D_j (\bigvee_h \Sigma^d h E) \longrightarrow \bigvee_h \Sigma^d h E \longrightarrow \Sigma^d E
$$

give $E$ an $H_\infty$ ring structure. If $E$ is an $H_\infty$ ring spectrum with structural maps $\xi_{j,i}$, then the maps

$$
f^{-1}_j : D_j (\bigvee_i \Sigma^d E) \longrightarrow \bigvee_i D_j E \longrightarrow \Sigma^d E
$$

determined by the composites

$$
D_j E \longrightarrow D_j E \longrightarrow D_j E \longrightarrow \Sigma^d E
$$

give $\bigvee_i E$ on $H_\infty$ ring structure. These correspondences are inverse to one another.

Returning to the context of Theorem 1.1, let

$$
g_j : D_j (Y_1 \vee \cdots \vee Y_k) \longrightarrow D_j Y_1 \vee \cdots \vee D_j Y_k
$$

denote the $j$th component of $f_{j}^{-1}$. Thus $g_j$ is the composite of the projection to $E_j \Sigma \Sigma_j Y_j$ and the inverse of the equivalence $(i \times 1)_a$ in the proof of the theorem. The theorem is of particular interest when $Y_1 = \cdots = Y_k$, hence we change notations and consider a spectrum $Y$ and its $k$-fold wedge sum, which we denote by $(k)Y$. Recall that finite wedges are finite products in the stable category and let

$$
\Delta : Y \longrightarrow (k)Y \quad \text{and} \quad \nu : (k)Y \longrightarrow Y
$$

denote the diagonal and folding maps.
Definition 1.4. Define \( \tau_j : D_j Y \rightarrow D_j Y \wedge \cdots \wedge D_j Y \) to be the composite
\[
D_j Y \xrightarrow{D_j \Delta} D_j (k) Y \xrightarrow{g_j} D_j Y \wedge \cdots \wedge D_j Y.
\]

Explicitly, let \( \pi_j : (k) Y \rightarrow \bigvee I \in J \) be the projection and let \( \tau_j \) also denote the map
\[
\pi_j : \pi_j(D_j Y) \rightarrow \bigvee I \in J \pi_j Y = \bigvee I \in J \pi_j Y.
\]

Our original map \( \tau_j \) is the composite of this map and the equivalence \([i \wedge 1] a^{-1}\). We write \( \tau_j \) for \( D_j Y \rightarrow Y(J) \) when \( k = J \) and each \( J_s = 1 \).

We think of \( \tau_j \) as a kind of spectrum level transfer map. When \( Y = \Sigma^\omega X^+ \) for a space \( X \) and \( \pi \subset \Sigma_j \), we have
\[
\pi \tau_j Y(J) = \Sigma^\omega (\pi \tau_j Y)(J) = \Sigma^\omega (\pi \tau_j Y)(J) = \Sigma^\omega (\pi \tau_j Y)(J).
\]

by I.1.1. We shall prove the following result in the sequel.

Theorem 1.5. When \( Y = \Sigma^\omega X^+ \), the map
\[
\tau_j : \tau_j Y(J) \rightarrow \tau_j Y(J)
\]
is the transfer associated to the natural cover
\[
\tau_j X^j + \tau_j X^j.
\]

We do not wish to overemphasize this result. As we shall see, the spectrum level maps \( \tau_j \), for general \( Y \), are quite easily studied directly.

The importance of these maps is that they measure the deviation from additivity of the functor \( D_j Y \).

For maps \( h_1 : Y \rightarrow E \), \( h_2 \cdots h_k \) is defined to be \( \vee(h_1 \vee \cdots \vee h_k) \Delta \). Thinking now in cohomological terms, consider the \( h_i \) as elements of the Abelian group \( E Y = [Y, E] \) of maps \( Y \rightarrow E \) in \( \bar{h} \).

Corollary 1.6. \( D_j(h_1 \cdots h) = \bigvee \tau_j^{(s)}(a_j(D_j h \wedge \cdots \wedge D_j h)) \). Moreover, the following equivariance formula holds for \( \sigma \in \Sigma_k \):
\[
\tau_j^{(s)}(a_j(D_j h \wedge \cdots \wedge D_j h)) = \tau_j^{(s)}(a_j(D_j h \wedge \cdots \wedge D_j h)).
\]
Proof. By Theorem 1.1 and the naturality diagram preceding it, the following diagram commutes.

\[
\begin{array}{cccccc}
D_j Y & \xrightarrow{D_j \Delta} & D_j (k) Y & \xrightarrow{D_j (h_1 \vee \cdots \vee h_k)} & D_j (k) E & \xrightarrow{D_j \psi} & D_j E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
V D_j Y & \xrightarrow{V \tau_j} & V D_j Y \wedge \cdots \wedge D_j Y & \xrightarrow{V D_j h_1 \wedge \cdots \wedge D_j h_k} & V D_j E \wedge \cdots \wedge D_j E & \xrightarrow{\alpha_j} & V D_j E \\
\end{array}
\]

The equivariance follows from 1.2.8, the formula \( \tau'_j = f_{j^*} \circ \sigma \), and the fact that \( \sigma \Delta = \Delta \).

Taking each \( h_i \) to be the identity map, we obtain the following special case.

**Corollary 1.7.** \( D_j (k) = \sum_j \tau_j^*(a_j) \), and \( \tau_j^*(a_j) \) depends only on the conjugacy class of \( J \) under the action of \( \Sigma_j \).

When \( J \) is a prime number \( p \) and \( k = p^i q \) with \( i \geq 1 \) and \( q \) prime to \( p \), a simple combinatorial argument demonstrates that every conjugacy class of partitions has \( p^i \) elements for some \( s \geq 1 \) except for the conjugacy class of the partition \( J(k) = (1, \ldots, 1, 0, \ldots, 0) \), which has \( (p, k-p) \) elements. Of course, \( p^{-1} \) but not \( p^i \) divides this binomial coefficient. A trivial diagram chase based on use of the projection \( (k)Y + (p)Y \) shows that \( \tau_j^*(k) \) coincides with \( \tau_j^*(p) = \tau_p^* D_p Y + Y(p) \). Also, by 1.2.7 and 1.2.11, \( a_j(p) = \tau_p^* E(p)^* + D_p E \). Putting these observations together, we obtain the following result.

**Corollary 1.8.** If \( k = p^i q \) with \( p \) prime, \( i \geq 1 \), and \( q \) prime to \( p \), then \( D_p k : D_p Y + D_p Y \) can be expressed in the form \( p^i \lambda + (p, k-p) \tau_p^* p \) for some map \( \lambda \).

In favorable cases, the following three lemmas will lead to a more precise calculation of \( D_p \) on general sums.

**Lemma 1.9.** The following diagram commutes for all \( Y, J, k \) and all partitions \( J \) of \( j \).

\[
\begin{array}{cccccc}
D_j Y & \xrightarrow{\tau_j} & D_j Y \wedge \cdots \wedge D_j Y & \xrightarrow{\tau_{j_1} \wedge \cdots \wedge \tau_{j_k}} & Y(j) \\
\downarrow & & \downarrow & & \downarrow \\
Y(J) & \xrightarrow{(j_1) \wedge \cdots \wedge (j_k)} & Y(j_1) \wedge \cdots \wedge Y(j_k) \\
\end{array}
\]
Proof. This follows from a straightforward diagram chase which boils down to the factorization of $\Delta: Y \to (J) Y$ as the composite
$$Y \xrightarrow{\Delta} (k) Y \xrightarrow{\Delta \vee \cdots \vee \Delta} (J_1) Y \vee \cdots \vee (J_k) Y$$
(where $\Delta: Y \to (0) Y = S$ is interpreted as the zero map if any $J_i = 0$).

**Lemma 1.10.** The composite $\tau_j: Y^{(j)} \to Y^{(j)}$ is the sum over $\sigma \in \Sigma_j$ of the permutation maps $\sigma: Y^{(j)} \to Y^{(j)}$.

**Proof.** This is an easy direct inspection of definitions and may be viewed as a particularly trivial case of the double coset formula.

**Lemma 1.11** For any ordinary homology theory $H_*$, the composite
$$H_* D_j Y \xrightarrow{\tau_j \star} H_* (D_j Y \times \cdots \times D_j Y) \xrightarrow{\alpha_j \star} H_* D_j Y$$
is multiplication by the multinomial coefficient $\left(\begin{array}{c} j_1, \ldots, j_k \end{array}\right)$. In particular, $\tau_j \star \tau_j \star$ is multiplication by $j!$.

**Proof.** We may assume that $Y$ is a CW-spectrum and exploit 1.2.1. Since $\pi^0 \Delta = 1: Y + Y$, where $\pi^0: (k) Y + Y$ is the $i$th projection, $\Delta_*: C_* Y + C_* (k) Y = C_* Y \otimes \cdots \otimes C_* Y$ is chain homotopy equivalent to the algebraic diagonal. With $Y_1 = \cdots = Y_k = Y$, the composite $(i \otimes 1)_* \alpha$ in the proof of Theorem 1.1 induces $\alpha_j$ upon passage to orbits over $\Sigma_j$ (rather than over $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$). Therefore $\alpha_j \circ \tau_j$ is just the composite
$$W_j \xrightarrow{\nu_j Y^{(j)}} \xrightarrow{\nu_j (k) Y^{(j)}} \xrightarrow{\nu_j Y^{(j)}} Y^{(j)} \xrightarrow{\nu_j Y^{(j)}} W_j \nu_j Y^{(j)}.$$

Since there are $(j_1, \ldots, j_k)$ sequences $I \subset J$ and thus $(j_1, \ldots, j_k)$ wedge summands here, the conclusion clearly holds on the level of cellular chains.

§2. **Power operations and Nishida's nilpotency theorem**

Let $E$ be an $H_*$ ring spectrum and $Y$ be any spectrum. Recall from I.4.1 that we have power operations $\tau_j: E^O Y + E^O D_j Y$ specified by $\tau_j (h) = \xi_j D_j (h)$. We use the results of section 1 to derive additivity formulas for these operations and apply these formulas to derive the nilpotency of $\tau_j S$. 
Lemma 2.1. For $h_i \in E^0(Y)$, $\prod J_j (h_1 + \cdots + h_k) = \prod J_j (h_1) \wedge \cdots \wedge \prod J_j (h_k)$, where the product $\wedge$ is the external product in $E$-cohomology and the sum extends over all partitions $J = (J_1, \ldots, J_k)$ of $j$.

Proof. This is immediate from Corollary 1.6 and the commutative diagram

Here the terms with one $J_i = j$ and the rest zero give the sum of the $\prod J_j (h_i)$. When $j$ is a prime number $p$, the remaining error term simplifies. The full generality of the following result is due to McClure.

Proposition 2.2. Let $h_i \in E^0(Y)$. If $p = 2$, then

$$\prod J_j (h_1 + \cdots + h_k) = \prod J_j (h_1) + \cdots + \prod J_j (h_k).$$

If $p$ is an odd prime and $Y$ and $E$ are $p$-local, then

$$\prod J_j (h_1 + \cdots + h_k) = \prod J_j (h_1) + \cdots + \prod J_j (h_k) + \sum_{1 \leq i < j < k} \tau_p (h_i \wedge h_j).$$

In particular, $\prod J_j (h_1 + \cdots + h_k) = \prod J_j (h_1) + \cdots + \prod J_j (h_k) + \tau_p (h_1 \wedge h_2)$ in both cases.

Proof. We must show that

$$j_1! \cdots j_k! \tau_J (\prod J_j (h_1) \wedge \cdots \wedge \prod J_j (h_k)) = \tau_J (j_1! \cdots j_k! \prod J_j (h_1) \wedge \cdots \wedge \prod J_j (h_k))$$

for a partition $J = (J_1, \ldots, J_k)$ of $p$ with no $J_i = p$. By Lemma 1.9,

$$\tau_J (h) = \tau_J (h_1) \wedge \cdots \wedge \tau_J (h_k).$$

Thus it suffices to show that

$$j! \prod J_j (h) = \tau_J (h^j)$$

for any $j \geq 0$ and $h \in E^0(Y)$. If $j = 0$, $h^{(0)}$ and $D_0 (h)$ are to be interpreted as the identity map of $S$ and the conclusion is trivial. If $j = 1$, the conclusion is also trivial. There are no more cases if $p = 2$, so assume that $p > 2$ and $1 < j < p$. By Lemma 1.11, the composite

$$D_j Y \xrightarrow{\tau_J} Y^j \xrightarrow{i_j} D_j Y$$
induces multiplication by $j!$ in ordinary homology. It is thus an equivalence since $Y$ and hence also $D_jY$ is $p$-local. Therefore $i_j^*: E^*(D_jY) \rightarrow E^*(Y(j))$ is a monomorphism and we need only check that

$$j! i_j^* \circ \tilde{P}_j(h) = i_j^* \tilde{P}_j(h^j).$$

The left side is $j! \tilde{h}$. By Lemma 1.10, the right side is the sum over $\sigma \in \mathcal{L}_p$ of $\sigma_*(h^j)$. The commutativity of $E$ implies that $\sigma_*(h^j) = h^j$ for all $\sigma$, $j$, and $h$, and the conclusion follows.

Now recall from 1.4.2 that elements $\alpha \in E_*(D_pS^q)$ determine homotopy operations $\tilde{a}: \pi_q E \rightarrow \pi_r E$ via the formula $\tilde{a}(h) = \alpha/\tilde{P}_p(h)$.
These relations specialize to give nilpotency assertions, the sharpest estimate being as follows.

**Corollary 2.6.** Let \( x \in \pi_q^*E \) satisfy \( p^ix = 0 \), where \( i > 0 \) and \( q \) is even if \( p > 2 \). Suppose that \( x = \tau_{-pq}^*\tau_{pq}^*(a) \) for some \( a \in \pi_{pq+q}(D_pS^q) \). Then \( p^{i-1}xP+2 = 0 \). Moreover, if \( p^ia = 0 \), then \( p^{i-1}xP+1 = 0 \).

The problem, of course, is to study \( \pi_p^*(D_pS^q) \) and \( \tau_{pq}^* \). Everything above applies to an arbitrary \( H \) ring spectrum \( E \), but to compute \( \tau_{pq}^* \) we must specialize. If \( E = MO \), for example, then every element of \( \pi_q^*E \) has order 2 and no element is nilpotent, hence \( \tau_{pq}^*:MO_*(D_2S^q) \to MO_*(S^q) \) must be the zero homomorphism for all \( q \). This does not contradict the following assertion.

**Conjecture 2.7.** Any element of finite order in the kernel of the (integral) Hurewicz homomorphism \( \pi_q^*E \to H_*E \) is nilpotent.

We shall prove the conjecture for elements of order exactly \( p \) in section 6, but the methods there fail for general elements of order \( p^i \) with \( i > 1 \).

When we specialize to \( E = S \), we find that the Kahn-Priddy theorem gives appropriate input for application of the results above.

**Theorem 2.8.** If \( p = 2 \), let \( \#(k) \) be the number of integers \( j \) such that \( 0 < j \leq k \) and \( j \equiv 0,1,2, \) or \( 4 \mod 8 \). If \( p > 2 \), let \( \#(k) = \lfloor k/(p-1) \rfloor \). Let \( q \) be an integer such that \( q \equiv 0 \mod \#(k) \), where \( q \) is even if \( p > 2 \). Then \( \tau_{pq}^*:\pi_p^*D_pS^q + \pi_p^*S^q \) is a (split) epimorphism for \( pq < r < pq+k(p-1) \).

We shall prove this in section 4. Actually, the purely stable methods we use will give surjectivity without giving a splitting. For this reason, we are really only entitled to use Corollary 2.4, rather than Corollary 2.5. This doesn't change the heuristic picture, but to give the correct estimate of the order of nilpotency, we assume the splitting (from \([46, 95, \) or \(27\)) in the discussion to follow.

**Theorem 2.9.** Let \( x \in \pi_nS \) satisfy \( p^ix = 0 \), where \( i > 0 \) and \( n \) is even if \( p > 2 \). Let \( m \) be minimal such that \( mn \equiv 0 \mod p^i\lfloor n/(p-1) \rfloor \). Then \( p^{i-1}xmp+1 = 0 \). Inductively, some power of \( x \) is zero.

**Proof.** Let \( q = mn \). Since \( n < \lfloor n/(p-1) \rfloor \), there exists \( a \in \pi_{pq+q}D_pS^q \) such that \( \tau_{-pq}^*\tau_{pq}^*(a) = x \). With \( h = x^m \), Corollary 2.4 gives \( p^{i-1}xmp+2 = 0 \). Using \( p^ia = 0 \), Corollary 2.5 gives \( p^{i-1}xmp+1 = 0 \).

Unfortunately, \( m \) increases rapidly with \( n \) (although our estimate for \( p > 2 \) is sharper than Nishida's since he only knew Theorem 2.8 for \( r < pq+k \)). For example,
the first stem in which an interesting element $x$ of order 2 occurs is the 14-stem
("interesting" meaning that $x$ is neither in $\pi_j$ nor a product of Hopf maps). Here
$m = 64$ and we can only conclude that $x^{129} = 0$, a truly stratospheric estimate. So
far, and granting that our stemwise calculations still extend through only a very
small range, we have no reason to disbelieve that $x^4 = 0$ if $2x = 0$. Corollary 2.6
seems to suggest that this answer might be correct. However, as pointed out to me
by Bruner, $\tau_2: \pi_2 D_0 S^4 \to \pi_2 S^2$ is not always an epimorphism and thus
Corollary 2.6 cannot be used to prove this answer.

 §3. The Jones-Wegmann homology and cohomology theories

The next three sections will all make heavy use of certain twisted diagonal
maps implicit in the general properties of extended powers.

Definition 3.1. Let $\pi$ be a subgroup of $\Sigma_j$ and let $W$ be a free $\pi$-CW complex. For a
based CW complex $X$ and a CW spectrum $Y$, define a map of spectra

$$\Delta: (W \llbrace \pi) \llbracket (j) \llbracket X \to W \llbracket (Y \llbracket X)(j)$$

by passage to orbits over $\pi$ from the $\pi$-map

$$(W \llbracket \pi) \llbracket (Y)(j) \llbracket X \leadsto (W \llbracket \pi) \llbracket X(j) \cong W \llbracket (Y \llbracket X)(j).$$

Here the isomorphism is given by I.1.2(ii) and the shuffle $\pi$-isomorphism
$\gamma(j) \llbracket X(j) \cong (Y \llbracket X)(j)$. Note that $\Delta$ is the identity map when $X = S^0$ and
that the following transitivity and commutativity diagram commutes, where $X'$ is
another based CW complex.

\[\begin{array}{ccc}
(W \llbracket \pi) \llbracket (Y)(j) \llbracket X \llbracket X' \llbracket X & \xrightarrow{\Delta \llbracket X} & \Delta \llbracket (W \llbracket \pi) \llbracket (Y)(j) \llbracket X \llbracket X' \\
W \llbracket (Y \llbracket X)(j) \llbracket X' \llbracket X' & \xrightarrow{\Delta} & \Delta \\
W \llbracket (Y \llbracket X)(j) \llbracket X \llbracket (Y \llbracket X')(j) \llbracket X' \llbracket X' & \xrightarrow{\Delta \llbracket (Y \llbracket X')(j)} & \Delta \llbracket (W \llbracket \pi) \llbracket \llbracket (Y \llbracket X')(j) \llbracket X \llbracket X' \llbracket X' \\
\end{array}\]

With $\pi = \Sigma_j$ and $W = E\Sigma_j$, we obtain

$$\Delta: (D_j Y) \llbracket X \to D_j (Y \llbracket X).$$

Although not strictly relevant to the business at hand, we record the relationship
between these maps and the maps $\iota_j$, $\alpha_{j,k}$, $\beta_{j,k}$, and $\delta_j$ of II2 and use them to construct new examples of $\mathcal{H}_\omega$ ring spectra.

**Lemma 3.2.** The following diagrams commute for spectra $Y$ and $Z$ and spaces $X$. The unlabeled arrows are obvious composites of shuffle maps and the diagonal on $X$.

![Diagram](image)

I learned the following lemma from Miller and McClure.

**Lemma 3.3.** Let $X$ be an unbased space and $E$ be an $\mathcal{H}_\omega$ ring spectrum. Then the function spectrum $F(X^+,E)$ is an $\mathcal{H}_\omega$ ring spectrum with structural maps the adjoints of the composites

$$D_j F(X^+,E)^X \xrightarrow{\Delta} D_j F(X^+,E)^{X^+} \xrightarrow{D_j \varepsilon} D_j E \xrightarrow{\varepsilon} E,$$

where $\varepsilon$ is the evaluation map. In particular, the dual $F(X^+,S)$ of $\mathcal{E}^X$ is an $\mathcal{H}_\omega$ ring spectrum.

**Proof.** If $j = 0$, $\Delta : S^X \times S^X \to S^{X^+} = S$ is to be interpreted as $\mathcal{E}^X$, where $\delta : X^+ + S^0$ is the discretization map sending $X$ to the non-basepoint. The diagrams of I.3.1 are easily checked to commute by use of the diagrams of the previous lemma.

Returning to the business at hand, observe that, with $X = S^1$, we obtain a natural map $\Delta : D_j Y + D_j Y$. Thus, for any integer $n$ (positive or negative), we have the map

$$\mathcal{E}^n \Delta : \mathcal{E}^{n+1} D_j \mathcal{E}^{-n-1} Y = \mathcal{E}^n D_j \mathcal{E}^{-n} Y \to \mathcal{E}^n D_j \mathcal{E}^{-n} Y.$$
We shall be interested in the resulting inverse system

\[ \cdots \rightarrow \mathbb{Z}_nD_z^nY \rightarrow \cdots \rightarrow \mathbb{Z}_nD_y^nY \rightarrow \cdots \rightarrow \mathbb{Z}_nD_z^nY \rightarrow \cdots \]

(\textit{where } n \geq 0). By the diagram in Definition 3.1, the maps

\[
\mathbb{Z}_n^A : (\mathbb{Z}_nD_z^nS^{-n}) \wedge X \cong \mathbb{Z}_n(D_z^nS^{-n} \wedge X) \rightarrow \mathbb{Z}_nD_z^n(S^{-n} \wedge X) \cong \mathbb{Z}_nD_z^n\mathbb{Z}^mX
\]

specify a morphism of systems, again denoted \( A \),

\[
\{(\mathbb{Z}_nD_z^nS^{-n}) \wedge X \} \rightarrow \{(\mathbb{Z}_nD_z^n\mathbb{Z}^mX) \}.
\]

We shall study the homological and homotopical properties of these systems. In this section, we consider any \( j \geq 2 \). We shall obtain calculational results when \( j \) is a prime in the following two sections.

Let \( E_* \) and \( E^* \) denote the homology and cohomology theories represented by a spectrum \( E \). For spectra \( Y \), define

\[
E_*^{(j)} Y = \lim E_*(\mathbb{Z}_nD_z^nS^{-n}Y) \quad \text{and} \quad E^*^{(j)} Y = \colim E^*(\mathbb{Z}_nD_z^nS^{-n}Y)
\]

\[
F_*^{(j)} Y = \lim F_*(\mathbb{Z}_nD_z^nS^{-n}Y) \quad \text{and} \quad F^*^{(j)} Y = \colim F^*(\mathbb{Z}_nD_z^nS^{-n}Y).
\]

Upon restriction to spaces (that is, to \( Y = S^\infty \)), we obtain induced natural transformations

\[
\Delta_* : F_*^{(j)} X \rightarrow E_*^{(j)} X \quad \text{and} \quad \Delta^* : E^*(j)X \rightarrow F^*(j)X,
\]

and these reduce to identity homomorphisms when \( X = S^0 \). It is clear that \( F_*^{(j)} \) is a homology theory and \( F^*^{(j)} \) is a cohomology theory on finite CW spectra. Passage to colimits from the homomorphisms

\[
(\mathbb{Z}_n^{-1})_* : E_*^{i+1}(\mathbb{Z}_nD_z^nS^{-n}Y) \rightarrow E_*^i(\mathbb{Z}_n^{-1}D_z^nS^{-n+1}Y) \rightarrow E_*^{i+1}(\mathbb{Z}_nD_z^nS^{-n}Y)
\]

yields suspension isomorphisms

\[
E_*^{i+1}Y \rightarrow E_*^iY,
\]

and \( \Delta^* \) is easily seen to commute with suspension. The analogous assertions hold for \( F_*^{(j)} \). With these notations, the main theorems of Jones and Wegmann [44] read as follows (although they only consider primes \( j \) and only provide proofs when \( j = 2 \)).

Theorem 3.4. The functor \( F_*^{(j)} \) is a cohomology theory on finite CW spectra, hence

\[
\Delta : E_*^{(j)}X + F_*^{(j)}X \quad \text{is an isomorphism for all finite CW complexes } X.
\]
Theorem 3.5. Let $E$ be connective and $j$-adically complete, with $\pi_* E$ of finite type over the $j$-adic integers $\hat{\mathbb{Z}}_j = \prod_p \hat{\mathbb{Z}}_p$. Then $E_j^*(X)$ is a homology theory on finite CW spectra, hence $\Delta_* : F^*_j X \times E^*_j X$ is an isomorphism for all finite CW complexes $X$.

We defer the proofs for a moment. As Jones and Wegmann point out, these results are no longer valid for infinite CW complexes.

Recall that $D_j S^0 = \mathbb{Z}^\infty \mathbb{E}_j^+$ and the discretization map $\mathbb{E}_j^+ \to S^0$ induces $\xi_j : D_j S^0 \to S^0$. Upon smashing with $Y$, the composites

$$
E_n D_j S^{-n} \xrightarrow{\Delta} D_j S^0 \xrightarrow{\xi_j} S^0
$$

give a morphism from the system $\{E_n D_j S^{-n} \wedge Y\}$ to the constant system at $Y$. We call this map of systems $\xi_j$ and obtain a map of cohomology theories

$$
\xi_j : E^* Y \to F^*_j Y,
$$

commutation with the suspension isomorphisms being easily checked. We shall shortly prove a complement to this observation.

Proposition 3.6. Let $E$ be an $H$-ring spectrum. Then the composites of the functions

$$
\mathcal{P}_j : E^* Y = [E^{-n} Y, E] \longrightarrow [D_j E^{-n} Y, E] = E^n(D_j E^{-n} Y)
$$

and the natural homomorphisms $E^n(D_j E^{-n} Y) \to E^*_j Y$ specify a map of cohomology theories

$$
\mathcal{P}_j : E^* Y \to E^*_j Y.
$$

We thus have the triangle of cohomology theories

$$
\begin{array}{ccc}
E^* X & \xrightarrow{\xi_j^*} & F^*_j X \\
\downarrow \Delta^* & & \downarrow \Delta^* \\
E^*_j X & \xrightarrow{\mathcal{P}_j^*} & E^*_j X
\end{array}
$$

on finite CW complexes $X$. Since $\mathcal{P}_j^*(x) = \xi_j^* \circ D_j (x)$, we see immediately that $\Delta^* \mathcal{P}_j^*(1) = \xi_j^* (1)$, where $1 \in E^0(S^0)$ is the identity element. It does not follow that $\Delta^* \mathcal{P}_j^* = \xi_j^*$ in general. As we shall see in the next section, this fails, for example when $E = MO$. However, as observed by Jones and Wegmann [44], this implication does hold for $E = S$. 

Proposition 3.7. The following diagram commutes for any finite CW complex $X$.

\[
\begin{array}{ccc}
\prod_j \xi_j^* & \cong & \lim_\Rightarrow \pi^*(L^nD_j_e^{-n}X) \\
\xi_j^* & \cong & \lim_\Rightarrow \pi^*(L^nD_jS^{-n}X)
\end{array}
\]

Proof. Since $\xi_j^*$ and $\xi_j^*$ are morphisms of cohomology theories, they are equal for all $X$ if they are equal for $X = S^0$. Any morphism $\phi: E^*X \to F^*X$ of cohomology theories is given by morphisms of $\pi^*S^0$-modules. When $E^* = \pi^*$ and $X = S^0$, $\phi(x) = \phi(1 \cdot x) = \phi(1)^*x$, so that $\phi$ is determined by its behavior on the unit $1 \in \pi^0(S^0)$.

For general $E$ and $X = S^0$, it is obvious that $\xi_j^*(x) = \xi_j^*(1)x$. It is not at all obvious that $(\Delta^* \xi_j^*)(x) = \Delta^* \xi_j^*(1) \cdot x$. We now have this relation for $E = S$, and we shall use it to prove the Kahn-Priddy theorem in the next section. As we shall explain in section 5, theorems of Lin when $p = 2$ and of Gunawardena when $p > 2$ imply that $\xi_p^*$ and thus $\xi_j^*$ in Proposition 3.7 are actually isomorphisms. We complete this section by giving the deferred proofs, starting with that of Proposition 3.6.

We need two lemmas.

Lemma 3.8. The following diagram commutes for any partition $J = (j_1, \ldots, j_k)$ of $j$.

\[
\begin{array}{ccc}
D_j(Y \wedge X) & \xrightarrow{\tau_j^*} & D_j(Y \wedge \Delta) \\
\downarrow \Delta & & \downarrow \Delta \wedge \cdots \wedge \Delta \\
D_j(Y \wedge X) & \xrightarrow{\tau_j^*} & D_j(Y \wedge X)(\text{shuffle})(1 \wedge \Delta)
\end{array}
\]

Proof. The "transfer" $\tau_j^*$ is specified in Definition 1.4, and the proof is an easy naturality argument.

Lemma 3.9. For an $H$-ring spectrum $E$, the composite

\[
[Y,E] \xrightarrow{\prod_j} [D_jY,E] \xrightarrow{\Delta^*} [L \Sigma D_j\Sigma^{-1}Y,E]
\]

is a homomorphism.

Proof. By Lemma 2.1, we have the formula

\[
\prod_j(x + y) = \prod_j(x) + \prod_j(y) + \sum_{i=1}^{p-1} \tau_j^{i,p-1}(\prod_i x \wedge \prod_{j-i} y).
\]
With $X = S^1$, Lemma 3.8 and the fact that $\Delta : S^1 \to S^1 \wedge S^1$ is null homotopic imply that $\tau_{i,j} = \tau_{i,j}^{-1} \Delta$ is null homotopic.

Thus $\mathcal{P}_j$ in Proposition 3.6 is a natural homomorphism. It is easily checked that $\mathcal{P}_j$ commutes with suspension and this proves the proposition.

Finally, we turn to the proofs of Theorems 3.4 and 3.5. Clearly it only remains to show that $E^k(j)$ and $E^k(j)$ satisfy the exactness axiom on finite CW pairs $(Y,B)$. Although not strictly necessary, we insert a general observation which helps explain the idea and will be used later.

**Lemma 3.10.** Let $f : B \to Y$ be a map of CW spectra with cofibre $C_f$. There is a map $\psi : CDjf \to D_jCf$, natural in $f$, such that the diagram

$$
\begin{array}{ccc}
D_Y & \xrightarrow{i} & CDf \\
\downarrow & & \downarrow \psi \\
D_jY & \xrightarrow{j} & D_jCf \\
\end{array}
$$

commutes, where $i : Y \to C_f$ and $j : C_f \to ZB$ are the canonical maps. If $f$ is the inclusion of a subcomplex in a CW spectrum, then the diagram

$$
\begin{array}{ccc}
CDf & \xrightarrow{\psi} & D_jCf \\
\downarrow \pi & & \downarrow \pi \\
D_j(Y/B) & & \end{array}
$$

also commutes, where the maps $\pi$ are the canonical (quotient) equivalences and the bottom map $\psi$ is induced by the quotient map $Y \to Y/B$.

**Proof.** $CDjf = D_jY \cup_{D_jf} D JB$ and $D_jCf = D_j(Y \cup_f CB)$; $\psi$ is induced by the inclusion $D_jY \to D_jCf$ and the composite of $\Delta : CDjB \to D_jCB$ and the inclusion $D_jCB \to D_jCf$. The diagrams are easily checked.

Of course, the bottom row in the first diagram is not a cofibre sequence and $\psi$ is not an equivalence. Now let $(Y,B)$ be a finite CW pair. For notational simplicity, set

$$
D_j(Y,B) = D_jY/D_jB \quad \text{and} \quad Z = Y/B.
$$

As $n$ varies, the maps

$$
E^n\psi : E^nD_j(E^{-n}Y, E^{-n}B) \to E^nD_jE^{-n}Z
$$


specify a map of inverse systems, again denoted \( \psi \), and we shall prove the following result.

**Proposition 3.11.** For any pair \((Y, B)\) of finite CW spectra,

\[
\psi : E(j) \mathcal{Z} \longrightarrow \text{Colim } E \mathcal{L} D_j(Y^-n, Y^-n B)
\]

and, under the hypotheses of Theorem 3.5,

\[
\psi_* : \text{lim } E_* \mathcal{L} D_j(Y^-n, Y^-n B) \longrightarrow E(j) \mathcal{Z}
\]

are isomorphisms.

Note that the assumptions on \( E \) in Theorem 3.5 imply that all groups in sight are finitely generated \( Z_j \)-modules and thus that all inverse limits in sight preserve exact sequences. Given the proposition, the required \( E^\ast(j) \) and \( E_\ast(j) \) exact sequences of the pair \((Y, B)\) are obtained by passage to colimits and limits from the \( E^\ast \) and \( E_\ast \) exact sequences of the pairs \((Y^-n, Y^-n B)\).

Following ideas of Bruner (which he uses in a much deeper way in chapters V and VI), we prove Proposition 3.11 by filtering \( Y(J) \). For \( 0 \leq s \leq j \), define

\[
r_s = r_s(Y, B) = \bigcup Y_1 \wedge \cdots \wedge Y_j,
\]

where \( Y_r = Y \) or \( Y_r = B \) and \( s \) of the \( Y_r \) are equal to \( B \). We have

\[
B(j) = \Gamma_j \subset \Gamma_{j-1} \subset \cdots \subset \Gamma_0 = Y(J).
\]

Each inclusion is a \( \Sigma_j \)-equivariant cofibration, and we define

\[
\Pi_s = \Pi_s(Y, B) = r_s(Y, B)/r_{s+1}(Y, B).
\]

Then \( \Pi_0 = Z(j) \) and, for \( 0 < s < j \), \( \Pi_s \) breaks up as the wedge of its \((s, j-s)\) distinct subspectra of the form \( \Sigma_1 \wedge \cdots \wedge \Sigma_j \), where \( Z_r = Z \) or \( Z_r = B \) and \( s \) of the \( Z_r \) are equal to \( B \). It follows that \( \Pi_s \) is the free \( \Sigma_j \)-spectrum generated by the \((\Sigma_s \times \Sigma_{j-s})\)-spectrum \( B(s) \wedge Z(j-s) \). That is,

\[
\Pi_s \cong \Sigma_j \wedge \Sigma_s \times \Sigma_{j-s} B(s) \wedge Z(j-s).
\]

The functor \( E(j) \wedge \Sigma_j \) converts \( \Sigma_j \)-cofibrations to cofibrations and commutes with quotients, hence we have cofibre sequences

\[
(*) \quad E(j) \wedge \Sigma_j \Gamma_s/\Gamma_t \longrightarrow E(j) \wedge \Sigma_j \Gamma_r/\Gamma_t \longrightarrow E(j) \wedge \Sigma_j \Gamma_r/\Gamma_s
\]

for \( 0 \leq r < s < t \leq j \). For a based space \( X \), the map \( \Delta : D_j Y \wedge X \to D_j(Y \wedge X) \) induces
compatible maps
\[ \Delta : [\Sigma_j \kappa_j \Gamma_s(Y, B)] \wedge X \to \Sigma_j \kappa_j \Gamma_s(Y \wedge X, B \wedge X) \]
and similarly for \( \Pi_s \) on passage to quotients. The following simple observation is
the crux of the matter.

**Lemma 3.12.** For \( 0 < s < j \), there is a natural equivalence
\[ \alpha : D_s B \wedge D_j - s Z \to \Sigma_j \kappa_j \Pi_s(Y, B) \]
such that the following diagram commutes for any \( X \).

\[
\begin{array}{ccc}
(D_s B \wedge D_j - s Z) \wedge X & \xrightarrow{(1, 1, 1)} & D_s B \wedge X \wedge D_j - s Z \wedge X \\
\downarrow \Delta & & \downarrow \Delta \\
[\Sigma_j \kappa_j \Pi_s(Y, B)] \wedge X & \to & \Sigma_j \kappa_j \Pi_s(Y \wedge X, B \wedge X)
\end{array}
\]

In particular, the bottom map \( \Delta \) is null homotopic when \( X = S^1 \).

**Proof.** By 1.1.4 and the description of \( \Pi_s(Y, B) \) above, we have
\[ \Sigma_j \kappa_j \Pi_s(Y, B) = (\Sigma_j \kappa_j \Pi_s(Y) \wedge B^z) \wedge Z^{j - s}. \]

As in the proof of Theorem 1.1, we may replace \( \Sigma_j \kappa_j \Pi_s(Y, B) \) on the right side, and it then becomes isomorphic to \( D_s B \wedge D_j - s Z \). The diagram is easily checked.

Now apply \( \pi_n \) to the cofibre sequence \( (*) \) for the pair \((\pi_n Y, \pi_n B)\) with quotient \( \pi_n Z \). We obtain an inverse system of cofibre sequences for \( 0 \leq r < s < t \leq j \). On passage to \( \pi_n \) and then to colimits (or to \( \pi_\ast \) and then to limits), there results a long exact sequence. For \( 0 < s < j \), the maps between terms of the system
\[ \{ \pi_n \Sigma_j \kappa_j \Pi_s(\pi_n Y, \pi_n B) \} \]
are null homotopic, hence its colimit of cohomologies is zero. Inductively, we conclude from the long exact sequences that the colimits of cohomologies associated to the quotients \( \Gamma_s / \Gamma_t \) with \( s > 0 \) are all zero and that the maps of colimits of cohomologies associated to the quotient maps \( \Gamma_0 / \Gamma_t + \Gamma_0 / \Gamma_s \) are all isomorphisms.

With \( s = 1 \) and \( t = p \), this proves Proposition 3.11.
§4. Jones' proof of the Kahn-Priddy theorem

We prove Theorem 2.8 here. The proof for $p = 2$ is due to Jones [43] and we have adapted his idea to the case $p > 2$. We begin more generally than necessary by relating the cofibre sequences (*) above Lemma 3.12 to the maps $\tau_j : D_j Y \to Y^{(j)}$ of Definition 1.4. The idea here is again due to Bruner. Thus let $(Y, B)$ be a pair of finite CW spectra with quotient $Z = Y/B$. The map $\tau_j$ is obtained by applying the functor $E \Sigma_j \kappa_j(?)$ to the composite

$$Y^{(j)} \xrightarrow{\Delta} (\Sigma_j Y)^{(j)} \xrightarrow{\pi_j} \Sigma_j \kappa_j Y^{(j)},$$

$J = (1, \ldots, 1)$, and using the equivalence $E \Sigma_j \kappa_j Y^{(j)} = Y^{(j)}$ of nonequivariant spectra (where, technically, the smash product is external on the left and internal on the right; see [Equiv. II §3]). The spectrum $\Sigma_j \kappa_j Y^{(j)}$ is a wedge of isomorphic copies of $Y^{(j)}$ indexed on the elements of $\Sigma_j$, and $\pi_j \Delta^{(j)}$ is just the sum of the $j!$ permutation maps. It follows that $\tau_j \Delta^{(j)}$ restricts to a $\Sigma_j$-equivariant map $\Gamma_0 + \Sigma_j \times \Gamma_s$ for $0 \leq s \leq j$. Upon passage to subquotients and application of the functor $E \Sigma_j \kappa_j(?)$, we obtain maps of cofibre sequences

$$E \Sigma_j \kappa_j \Gamma_0 \xrightarrow{\Gamma_j} E \Sigma_j \kappa_j \Gamma_s \xrightarrow{\Gamma_j} E \Sigma_j \kappa_j \Gamma_t$$

for $0 \leq r < s < t < j$. With $t = s+1$, the left map $\tau_j$ is nicely related to the equivalence $\alpha$ of Lemma 3.12, as can easily be checked by inspection of definitions.

Lemma 4.1. The following diagram commutes for $0 < s < j$, where $p$ is the projection onto the unpermuted wedge summand.

$$
\begin{array}{c}
D_j \xrightarrow{\alpha} E \Sigma_j \kappa_j \Pi_s (Y, B) \\
\downarrow \tau_j \\
\Pi_s \xrightarrow{\tau_j} \Pi_s
\end{array}
$$

When $j = 2$, there is only one map of cofibre sequences above, and we obtain the following conclusion.

Proposition 4.2. For CW pair $(Y, B)$ with quotient $Z = Y/B$,

$$
B \wedge Z \xrightarrow{\tau_1} D_2 Y \wedge D_2 B \xrightarrow{\tau_2} D_2 Z \xrightarrow{\tau_1} \Sigma B \wedge Z
$$
is a cofibre sequence, where $\psi$ is induced by the quotient map $Y \to Z$, $\tau_2$ is the composite
\[ B \cdot Z = (B \cdot Y)/(B \cdot B) \to (Y \cdot Y)/(B \cdot B) \xrightarrow{\tau_2} D_2 Y/D_2 B, \]
and $\tau'_2$ is the composite
\[ D_2 Z \xrightarrow{\tau_2} Z \cdot Z = (Y \cup CB) \cdot Z \xrightarrow{\tau'_2} \Sigma B \cdot Z. \]

Proof. Combine the cofibre sequence
\[ E \mathbb{Z} \xleftarrow{\alpha} \mathcal{P}_2(Y,B) \to D_2 Y/D_2 B \to D_2 Z \to \Sigma E \mathbb{Z} \xleftarrow{\beta} \mathcal{P}_2(Y,B) \]
with the equivalence $\alpha : B \cdot Z \to E \mathbb{Z} \xleftarrow{\alpha} \mathcal{P}_2(Y,B)$ and check that the resulting maps are those specified.

Our main interest is in the pair $(CY,Y)$.

Corollary 4.3. The following is a cofibre sequence.
\[ E(Y \cdot Y) \xrightarrow{\Sigma \tau_2} E D_2 Y \xleftarrow{\Lambda} D_2 Y \xrightarrow{\tau_2} \Sigma Y \cdot E Y. \]

Proof. Use the evident equivalence $D_2 CY/D_2 Y = \Sigma D_2 Y$ and check the maps, using Lemma 3.10 for the middle one.

For $j > 2$, we have too many cofibre sequences in sight. Henceforward, let $p$ be a prime and localize all spaces and spectra at $p$ without change of notation. We shall show that, for odd primes $p$ and pairs $(CS^q,S^q)$, our system of cofibre sequences collapses to a single one like that in the previous corollary. Recall from Lemma 1.10 that $\tau_r^i : Y^{(r)} \to Y^{(r)}$ is the sum of permutations map and $\tau_r^i : D_r Y \to D_r Y$ induces multiplication by $r!$ on ordinary homology. In particular, for $1 < r < p$, $D_r Y$ is a wedge summand of $Y^{(r)}$.

Lemma 4.4. For $1 < r < p$, $D_r S^{2q+1}$ is equivalent to the trivial spectrum and $\tau_r : S^{2q} \to D_r S^{2q}$ is an equivalence with inverse $\frac{1}{r!} \tau_r$.

Proof. When $Y = S^{2q}$, $\tau_r$ induces multiplication by $r!$ on homology; when $Y = S^{2q+1}$, it induces zero. The conclusions follow.

Thus, when $Y$ is a sphere spectrum, most of the spectra
\[ E_p \mathcal{P}_2(CY,Y) = D_2 Y \wedge D_2 S^p E Y \]
are trivial.
Corollary 4.5. Let $p > 2$ and let $q$ be an even integer. Then there are cofibre sequences

$$S^{pq-1} \to \varepsilon D_p S^{q-1} \to D_p S^q \xrightarrow{\Delta} D_p S^{q+1} \to S^{pq}$$

and

$$S^{pq+1} \to \varepsilon D_p S^q \to D_p S^{q+1} \to S^{pq+2}.$$

Proof. Let $\Gamma_s = \Gamma_s(\chi, Y)$ and $\Pi_s = \Gamma_s/\Gamma_{s+1}$. If $Y = S^{q-1}$, then $E_0 \kappa_{\Pi_s} \Pi_s$ is trivial for $2 \leq s < p$, hence $E_0 \kappa_{\Pi_s} \Gamma_r/\Gamma_s$ is trivial for $2 \leq r < s \leq p$. Thus $\Gamma_1/\Gamma_p + \Pi_1$ and $\Gamma_0/\Gamma_p + \Gamma_0/\Gamma_2$ induce equivalences upon application of $E_0 \kappa_{\Pi_s}(?)$ and there results a cofibre sequence

$$E_0 \kappa_{\Pi_s} \Pi_s \to E_0 \kappa_{\Pi_s} \Gamma_0/\Gamma_p \to E_0 \kappa_{\Pi_s} \Pi_0 \to E_0 \kappa_{\Pi_s} \Pi_1.$$

This gives the first sequence upon interpreting the terms and maps (by use of Lemmas 3.10, 3.12, 4.1, and 4.4). Similarly, if $Y = S^q$, then $E_0 \kappa_{\Pi_s} \Pi_s$ is trivial for $1 \leq s < p-1$, hence $E_0 \kappa_{\Pi_s} \Gamma_r/\Gamma_s$ is trivial for $1 \leq r < s \leq p-1$. Thus $\Gamma_0/\Gamma_{p-1} + \Pi_0$ and $\Pi_{p-1} + \Gamma_1/\Gamma_p$ induces equivalences upon application of $E_0 \kappa_{\Pi_s}(?)$ and there results a cofibre sequence

$$E_0 \kappa_{\Pi_s} \Pi_{p-1} \to E_0 \kappa_{\Pi_s} \Gamma_0/\Gamma_p \to E_0 \kappa_{\Pi_s} \Pi_0 \to E_0 \kappa_{\Pi_s} \Pi_{p-1}.$$

This gives the second sequence.

One can also check these cofibre sequences by direct homological calculation; compare Lemma 5.6 below. We need some further information about the spectra $\Sigma^D_p S^{-n}$ in order to use these sequences to prove Theorem 2.8. Proofs of the claims to follow will be given by Bruner in V§2.

If $p = 2$, let $L = \Sigma^m \mathbb{R}^m$ with its standard cell structure. (We write $L$ rather than the usual $P$ for uniformity with the case $p > 2$.) If $p > 2$, let $L$ be a CW spectrum of the $p$-local homotopy type of $\Sigma^m \mathbb{R}^m$ such that $L$ has one cell in each positive dimension $q \equiv 0$ or $-1 \mod 2(p-1)$. The existence and essential uniqueness of such an $L$ was pointed out by Adams [7, 2.2]. Let $I^k$ be the $k$-skeleton of $L$ and let $I^k_m = L/I^{m-1}$ and $I^{n+k}_m = L^{n+k}/L^{n-1}$ for $k \geq 0$. Let $\phi(k)$ be as in Theorem 2.8 (and recall that it depends on $p$). If $p = 2$, then

$$I^{n+k}_m = L^{n-m}L^{m+k}_m \quad \text{for } m \equiv n \mod 2^\phi(k).$$

If $p > 2$, $\epsilon = 0$ or 1, and $k \geq \epsilon$, then

$$I^{2n+k}_m = L^{2(n-m)}L^{2m+k}_m \quad \text{for } m \equiv n \mod p^\phi(k).$$
We use this periodicity to define spectra $L_n^{n+k}$ for non-positive $n$, so that these equivalences hold for all integers $m$ and $n$. We then have that

$$L_n^{n+k} \text{ is } (-1)-\text{dual to } L_{-n-1-k}.$$  

Our interest in these spectra comes from the following result (proven by Bruner in §82).

**Theorem 4.6.** For any integer $n$, $z^{-n}D_pS^n$ is $p$-locally equivalent to $L_n(p-1)$.

We define $D_pS^n = z^{-n}S_{n(p-1)+k}$. If $p = 2$, we may view $D_pS^n$ as $S^k \times S^{2n}$. If $p > 2$, no model for $E_p$ has few enough cells to give as convenient a filtration of $D_pS^n$. We shall shortly prove the following result.

**Proposition 4.7.** If $\rho:L_{-k}^0 + S^0$ is the projection onto the top cell, then

$$\rho^*:\pi_q(S^0) + \pi_q(L_{-k}^0)$$

is zero for $0 < q < k(p-1)$.

Since $\rho$ is $(-1)$-dual to the inclusion $i:S^{-1} + L_{-1}^{k-1}$ of the bottom cell, $i_\#:\pi_q(S^{-1}) + \pi_q(L_{-1}^{k-1})$ is zero for $0 \leq q < k(p-1)-1$. The cofibre sequences of Corollaries 4.3 and 4.5 restrict to give cofibre sequences

$$S^{-1} \rightarrow L_{-1}^{k-1} \rightarrow L_0^{k-1} \rightarrow \tau_p S^0.$$  

Thus, $\tau_p^*:\pi_q(S^0)$ is an epimorphism for $0 < q < k(p-1)$. Now let $k$ go to infinity. Of course, $L_0 = \Sigma^\infty E_p$ splits as the wedge $\Sigma^\infty E_p \vee S^0$. Since $\tau_p^*:S^0 + S^0$ has degree $p!$, the finiteness of $\pi_*S^0$ allows us to deduce the following version of the Kahn-Priddy Theorem.

**Theorem 4.8.** The restriction $\tau_p:\Sigma^\infty E_p + S^0$ induces an epimorphism

$$\pi_q(\Sigma^\infty E_p) + \pi_q(S^0) \otimes \mathbb{Z}(p) \text{ for } q > 0.$$

To prove Theorem 2.8, consider the following diagram, where $q \equiv 0 \mod p^\ell(k)$ and $q$ is even if $p > 2$.  

$$
\begin{array}{cccccccc}
S^{pq-1} & \xrightarrow{\varepsilon^p\tau_q} & S^{pq} & \xrightarrow{\varepsilon^p\tau_q} & S^{pq} & \xrightarrow{\varepsilon^p\tau_q} & S^{pq} \\
\downarrow \quad \downarrow & \quad & \downarrow \quad \downarrow & \quad & \downarrow \quad \downarrow & \quad & \downarrow \quad \downarrow \\
S^{pq-1} & \xrightarrow{\varepsilon^{pq+(p-1)q-k-1}} & S^{pq} & \xrightarrow{\varepsilon^{pq+(p-1)q-1}} & S^{pq} & \xrightarrow{\varepsilon^{pq+(p-1)q-k-1}} & S^{pq} \\
\end{array}
$$
The bottom cofibre sequence is obtained by restriction from sequences in Corollaries 4.3 and 4.5. Periodicity gives an equivalence \( v \) such that the left square commutes. Standard cofibration sequence arguments then give an equivalence \( \omega \) such that the remaining squares commute. The bottom map \( \tau_\nu \) factors through \( \tau_\nu : D_p S^q \to S^q \) and is an epimorphism in the range stated in Theorem 2.8.

It remains to prove Proposition 4.7. For amusement, we proceed a bit more generally. Recall the not necessarily commutative diagram

\[
\begin{array}{ccc}
E^* X & \xrightarrow{j} & F^* X \\
\downarrow{k/j} & & \downarrow{\Delta/k} \\
E^* (j) X & \xrightarrow{\Delta} & F^* (j) X
\end{array}
\]

below Proposition 3.6, where \( E \) is an \( H_\infty \) ring spectrum. With \( E = S \) and \( X = S^0 \), the following result is Proposition 4.7.

**Proposition 4.9.** Let \( X \) be a finite CW complex of dimension less than \( k(p-1) - q \), where \( 0 < q < k(p-1) \). Then

\[
(\rho \wedge 1)^*: E_{-q}^*(S^0 \wedge X) \to E_{-q}^*(S^0 \wedge X)
\]

is zero if \( E \) is a connective \( H_\infty \) ring spectrum such that \( \Delta^* \rho_{\nu} = \xi_{\nu}^* \).

**Proof.** For \( n \geq k \), the cofibre of \( \Delta : L^{n+1}_p S^{-n-1}_p + L^n D S^{-n}_p \) has dimension at most \( -k(p-1) \), and it follows that the colimit \( F(p) \) is attained as \( E_{-q}^*(S^0 \wedge X) \). Let \( i : L^0_{-k} \to L_{-k} \) be the inclusion and consider the following diagram, where \( x \) is any map \( X \to \Sigma^{-q} E \).
Since $\Delta^* P_p = \xi_p^*$, the bottom part commutes. We have

$$\xi_p \Delta = \rho: L_{-k}^0 + S^0$$

since the composite is obviously null homotopic on $L_{-k}^{-1}$ and of degree one on the top cell. We have

$$\Delta = 0: L_{-k}^0 + \Sigma^{-\infty} P S^q$$

since $\Sigma^{-\infty} P S^q$ is $O$-connected. The conclusion follows.

Replacing $S$ by $E$ in the deductions from Proposition 4.7 and using the results of section 2, we conclude that, for $q > 0$, all $p$-torsion elements of $\pi_p E$ are nilpotent if $\Delta^* P_p = \xi_p^*$. This implies our earlier claim that $\Delta^* P_2 \neq \xi_2^*$ when $E = MO$.

§5. The Singer construction and theorems of Lin and Gunawardena

Singer introduced a remarkable algebraic functor $R_+$ from $A$-modules to $A$-modules, where $A$ is the mod $p$ Steenrod algebra, and Miller began the study of the cohomology theories in section 3 by making the following basic observation. All homology and cohomology is to be taken with mod $p$ coefficients.

Theorem 5.1. Let $Y$ be a spectrum such that $H_* Y$ is bounded below and of finite type. Then $\text{colim} \ H^*(L^n D_p)_{-n}^* Y$ is isomorphic to $\xi_1 R_+ H_+ Y$.

We shall prove this and some related observations after explaining its relationship to the following theorems of Lin [53, 54] and Gunawardena [38, 39]. Let $\hat{\pi}^*$ and $\hat{\xi}_*$ denote the $p$-adic completions of stable cohomotopy and stable homotopy.

Theorem 5.2. The map $\xi_p^*: \hat{\pi}^* Y \to \text{colim} \ \hat{\pi}^* (\Sigma^n D_p S^{-n}) \wedge Y$ is an isomorphism for all finite CW spectra $Y$.

As we shall explain shortly, $\lim \hat{\pi}^* (\Sigma^n D_p S^{-n}) = \mathbb{Z}_p$. Realizing the unit by a compatible system of maps $\xi P: S^{-1} \to \Sigma^n D_p S^{-n}$ and smashing with $Y$, we obtain a compatible system of maps

$$\xi P: S^{-1} Y \to \Sigma^n D_p S^{-n} \wedge Y.$$
Since $\xi_p^*$ is a map of cohomology theories and $\xi_p^p$ is a map of homology theories, it suffices to prove these isomorphisms for $Y = S^0$. Since

$$z_nD_p^{k(p-1)-1S-n}$$

is $(-1)$-dual to $z_nD_p^{k(p-1)-1S-n-k}$, the theorems are essentially dual to one another. Indeed, using the $\text{lim}^1$ exact sequence and waving one's hands at certain compatibility questions, one finds the following chain of isomorphisms, where $m(p-1) > q$.

$$\text{colim}_n \hat{\omega}^q(z_nD_p^{S-n}) = \hat{\omega}^q(z_{m(p-1)}D_p^{S-m})$$

$$= \lim_k \hat{\omega}^q(z_{m(k(p-1)-1S-m)}$$

$$= \lim_k \hat{\omega}^q(z_{m+k(p-1)-1S-m-k})$$

$$= \lim_n \hat{\omega}^q(z_nD_p^{S-n})$$

There is a map of $A$-modules $\varepsilon: R_p + Z_p \to Z_p$, and the main point of the work of Lin and Gunawardena can be reformulated as follows; see Adams, Gunawardena, and Miller [9].

Theorem 5.4. $\xi^*: \text{Ext}_A(Z_p, Z_p) \to \text{Ext}_A(R_p, Z_p, Z_p)$ is an isomorphism.

An inverse system $\{Y_n\}$ of bounded below spectra $Y_n$ of finite type gives rise to an inverse limit

$$\{E_r\} = \text{lim} \{E_rY_n\}$$

of Adams spectral sequences, where $\{E_rY\}$ denotes the classical Adams spectral sequence for the computation of $\hat{\omega}_*Y$. Clearly

$$E_2 \cong \text{Ext}_A(\text{colim} H_*Y_n, Z_p).$$

As pointed out in [74], $\{E_r\}$ converges strongly to $\text{lim} \hat{\omega}_*Y_n$. We apply this with $Y_n = z_nD_p^{S-n}$. Here Theorems 5.1 and 5.4 give

$$E_2 \cong \text{Ext}_A(z^{-1}Z_p, Z_p).$$

From this and convergence, it is easy to check that $\text{lim} \hat{\omega}^{-1}(z_nD_p^{S-n}) = Z_p$. The compatible system of maps $z_p: S^{-1} \to z_nD_p^{S-n}$ then induces a map of spectral sequences

$$\{E_rz_p^p\}: \{E_rS^{-1}\} \to \{E_r\}.$$

By Theorem 5.4 again, $E_2z_p^p$ is an isomorphism, and Theorem 5.3 follows by
convergence. Theorem 5.2 can be obtained by a similar Adams spectral sequence argument (as in Lin [53] and Gunawardena [38]) or by dualization.

The crux of the proof of Theorem 5.1 is the following result of Steinberger, which is proven in VIII.3.2 of the sequel. For spaces, it is due to Nishida [89]; see also [68, 9.4]. Let $\mathbb{Z}/p$ be the cyclic group of order $p$. We assume familiarity with the mod $p$ homology $H_*D_nY$, its determination being a standard exercise in the homology of groups in view of 1.2.3 (see e.g. [68, §1]). Suffice it to say that $H_*D_nY$ has a basis consisting of elements of the form $e_0 \otimes x_1 \otimes \cdots \otimes x_p$ and $e_i \otimes x^p$, $i \geq 0$. Here the $x_i$ and $x$ run through basis elements of $H_*Y$, the $x_i$ are not all equal, and the $x_1 \otimes \cdots \otimes x_p$ and $x^p$ together run through a set of $\pi$-generators for $(H_*Y)^p$. Restricting to those $i$ of the form $(2s-q)(p-1)-\varepsilon$, where $q = \deg(x)$ and $\varepsilon = 0$ or 1, and to a set of $\pi_p$-generators for $(H_*Y)^p$, we obtain a basis for $H_*D_nY$. At least if $H_*Y$ is bounded below and of finite type, we have analogous dual bases for $H^*D_nY$ and $H^*D_pY$ with typical elements denoted $w_0 \otimes y_1 \otimes \cdots \otimes y_p$ and $w_1 \otimes y^p$.

**Theorem 5.5.** Assume that $H_*Y$ is bounded below and of finite type. The subspace of $H^*D_nY$ spanned by $\{w_0 \otimes y_1 \otimes \cdots \otimes y_p\}$ is closed under Steenrod operations and, modulo this subspace, the following relations hold for $y \in H^*Y$.

(i) For $p = 2$,

$$Sq^s(w_j \otimes y^2) = \sum_{s-2i} j+q-i \frac{j^q-i}{i} w_j \otimes (Sq^s y)^2.$$ 

(ii) For $p > 2$, let $\delta(2n+\varepsilon) = \varepsilon$, $m = \frac{1}{2}(p-1)$, and $a(q) = (-1)^mqm!$; then

$$P^s(w_j \otimes y^p) = \sum_{s-pi} \left[\frac{j}{2}\right]^q m-(p-1)1 \frac{j+2(s-pi)(p-1)}{s-pi} \otimes (P^i y)^p + \delta(j-1)a(q) \sum_{s-pi-1} \left[\frac{j}{2}\right]^q m-(p-1)1-1 \frac{j+2(s-pi)(p-1)}{s-pi-1} \otimes (P^i y)^p.$$ 

(iii) For $p > 2$, $\Delta(w_{2j-1} \otimes y^p) = w_{2j} \otimes y^p$.

We also need to know $\Delta^* : H^*D_nY \rightarrow H^*(\Sigma^n D_nY)$. Let $\varepsilon^n : H^q(Y) \rightarrow H^{q+n}(\Sigma^n Y)$ denote the iterated suspension isomorphism for any integer $n$.

**Lemma 5.6.** For $y \in H^qY$,

$$\Delta^* (w_j \otimes y^p) = (-1)^{j+1} a(q) \varepsilon^j(w_{j+p-1} \otimes (\Sigma^{-1} y)^p).$$

**Proof.** We first compute $\Delta_* : H_*(\Sigma^n D_nY) + H_*(D_n\Sigma Y)$. Take $f$ to be the identity map of $Y$ and replace $D_p$ by $D_n$ in Lemma 3.10. We find that the composite of $\Delta_*$ and the homology suspension $\Sigma_*$ is the suspension associated to the zero sequence.
By I.2.3 and [68,§1], we may instead use the zero sequence

$$W \otimes C_*(Y)^P \rightarrow W \otimes C_*(CY)^P \rightarrow W \otimes C_*(ZY)^P,$$

where $W$ is the standard $\pi$-free resolution of $Z_p$. A direct chain level computation, details of which are in [68, p. 166-167], gives the formula

$$\Delta_*E_0^{*+p-1} \otimes x^P = (-1)^{j+1}a(q)e_j \otimes (E_*x)^P$$

for $x \in H_{q-1}(Y)$. Clearly $\Delta_*E_0^{*0} (x_1 \otimes \cdots \otimes x_p) = 0$ for all $x_i$. The conclusion follows upon dualization (and a careful check of signs).

The results above determine colim $H^*(\mathbb{Z}^nD_z^{-n}Y)$ as an $A$-module, and similarly with $D_z$ replaced by $D_p$. To compare the answer to the Singer construction, we must first recall the definition of the latter [98,52]. When $p = 2$, $\mathbb{Z}^{-1}R^+M$ is additively isomorphic to $A \otimes M$, where $A$ is the Laurent series ring $Z_2[v,v^{-1}]$, deg $v = 1$. Its Steenrod operations are specified by

$$Sq^r(v^r \otimes x) = \sum_i (r-1)^{i-1}v^{r+s-i} \otimes Sq^i x.$$

When $p > 2$, $\mathbb{Z}^{-1}R^+M$ is additively isomorphic to $A \otimes M$, where $A = E\{u\} \times Z_p[v,v^{-1}]$, deg $u = 2p-3$ and deg $v = 2p-2$. Its Steenrod operations are specified by

$$p^s(u^sv^r-x \otimes x) = \sum_i (-1)^{i-1}(p-1)(r-1)-i \cdot u^vr+s-i-\epsilon \otimes p^ix$$

and

$$\beta(u^sv^r-x \otimes x) = \epsilon(v^r \otimes x).$$

We can now prove Theorem 5.1. We define an isomorphism

$$\omega: \text{colim } H^*(\mathbb{Z}^nD_z^{-n}Y) \rightarrow \mathbb{Z}^{-1}R^+H^*Y$$

as follows. For $p = 2$ and $y \in H^q(Y)$, let

$$\omega(\mathbb{Z}^n(w_{r-q+n} \otimes (\mathbb{Z}^{-n}y)^2 = v^r \otimes y).$$

For $p > 2$ and $y \in H^q(Y)$, let

$$\omega(\mathbb{Z}^n(w_{2r+n-q}(p-1)-\epsilon \otimes (\mathbb{Z}^{-n}y)^p) = (-1)^{r+q}(x+1)n_{v(q-n)-1}u^vr-x \otimes y,$$
where \( v(2j + \varepsilon) = (-1)^j (m!)^\varepsilon \). Note that

\[
\alpha(q)v(q-1)^{-1} = v(q)^{-1} \quad \text{and} \quad (-1)^q v(q)^{-1} = (-1)^{mq} v(q).
\]

By Lemma 5.6, these \( \omega \) induce a well-defined isomorphism on passage to colimits. By Theorem 5.5, we see that our constants have been so chosen that \( \omega \) is an isomorphism of \( A \)-modules.

Remark 5.7. When \( p > 2 \), there are two variants of the Singer construction. We are using the smaller one appropriate to \( D_\ast \). This is a summand of the larger variant, for which Theorem 5.1 is true with \( D_\ast \) replaced by \( D_\ast \). See Gunawardena [39,9] for details (but note that his signs don't quite agree with ours).

With \( Y = S^0 \), Theorem 5.1 specializes to an isomorphism

\[
\Lambda = \Sigma_{-1} R_\ast Z_\ast \cong \text{colim} \: H^\ast (\Sigma_{D_\ast} c^{-n}).
\]

Since \( \Lambda \) is an \( A \)-module, \( \Lambda \otimes M \) admits the diagonal \( A \) action, which is evidently quite different from that originally specified on \( \Sigma_{-1} R_\ast M \). For finite CW complexes \( X \), we have the isomorphism

\[
\Delta^* : \text{colim} \: H^\ast (\Sigma_{D_\ast} c^{-n} X) \longrightarrow \text{colim} \: H^\ast (\Sigma_{D_\ast} S^{-n} \wedge X)
\]

of Theorem 3.2. We next obtain an explicit description of the resulting isomorphism

\[
\Delta^* : \Sigma_{-1} R_\ast \wedge \wedge X + A \otimes \wedge^X.
\]

Thus consider \( \Delta : D_\ast Y \wedge X + D_\ast (Y \wedge X) \). When \( X = S^1 \), we computed \( \Delta^* \) in the proof of Lemma 5.6. When \( Y = S, D_\ast Y = \mathbb{Z} Bw \) and the effect of \( \Delta^* \) is implicit in the definition of the Steenrod operations; see Steenrod and Epstein [100] (or, for correct signs, [68, 9.1]). The following result is a common generalization of these calculations.

Proposition 5.8. Let \( x \in \wedge^k (X) \) and \( y \in H_q (Y) \). If \( p = 2 \),

\[
\Delta^* (e_r \otimes y^2 \otimes x) = \sum_i e_{r+2i-k} \otimes (y \otimes S^q x)^2.
\]

If \( p > 2 \), let \( v(2j+1) = (-1)^j (m!)^\varepsilon \) and \( c(2j+\varepsilon) = \varepsilon \); then

\[
\Delta^* (e_r \otimes y^p \otimes x) = (-1)^{mq} v(k) \sum_i (-1)^i e_{r+(p-1)(p-1)} \otimes (y \otimes P^i_k x)^p
\]

\[
- (-1)^{q+m(k-1)q_8(r)(k-1)} \sum_i (-1)^i e_{r+p+(2p-1)(p-1)} \otimes (y \otimes P^i_k x)^p.
\]
Proof. Modulo shuffling in $C_\pi(Y)^P$, which introduces the signs depending on $q$ when
$p > 2$, $\Delta_\pi$ is computable from the map obtained by quotienting out the action of $\pi$
from the $\pi$-map
$$\phi \otimes 1 : C_\pi(W) \otimes C_\pi(X)^P \rightarrow C_\pi(W) \otimes C_\pi(X)^P \otimes C_\pi(Y)^P$$
induced by a $\pi$-equivariant approximation $\phi$ of $1 \otimes \Delta$, where $\Delta'$ is a cellular
approximation of the diagonal $X \times X^P$; see e.g. [100, V3] or [68, 7.1]. The
essential point is that $Y$ acts like a dummy variable, so that the standard
calculation for $Y = S^0$ of [68, 9.1] implies the general result.

Dualizing, and paying careful attention to signs, we obtain the following
version in cohomology.

Proposition 5.9. Assume that $H_\pi X$ and $H_\pi Y$ are of finite type and that $H_\pi Y$ is bounded
below. Let $x \in H_\pi^i(X)$ and $y \in H_\pi^q(Y)$. If $p = 2$,
$$\Delta^*(w_j \otimes (y \otimes x)^\varphi) = \sum_i w_{j+k-i} \otimes y^{2i} \otimes Sq^i x.$$  
If $p > 2$,
$$\Delta^*(w_j \otimes (y \otimes x)^P) = (-1)^{mk(q+1)} v(k) \sum_i (-1)^i w_{j+(k-2i)(p-1)} \otimes y^P \otimes P^i x$$
$$-(-1)^{q+mk(q+1)} \delta(j-l) v(k) \sum_i (-1)^i w_{j+(k-2i)(p-1)-1} \otimes y^P \otimes P^i x.$$  
A check of constants gives the following consequence.

Corollary 5.10. For $M = H_\pi^i X$, the formula
$$\Delta^*(v^P \otimes x) = \sum_i v^{P-1} \otimes Sq^i x$$
if $p = 2$ and
$$\Delta^*(v \otimes v^{P-1} \otimes x) = \sum_i v \otimes v^{P-1-\varepsilon} \otimes P^i x - (1-\varepsilon) \sum_i v \otimes v^{P-1} \otimes P^i x$$
if $p > 2$ specifies a morphism of $A$-modules $\Delta^* : \Sigma^{-1} R \pi M + A \otimes M$.

The same formulae give a morphism of $A$-modules for all $A$-modules $M$ which are
either unstable or bounded above, either assumption ensuring that the relevant sums
are finite. In the bounded above case, but not in general in the unstable case,
this morphism is an isomorphism. See [98, 52, 82].
Define \( \varepsilon : R \times M \to M \) by the formulas

\[
\varepsilon \varepsilon (v^{p-1} \otimes x) = S^{p} x
\]

if \( p = 2 \) (where \( S^{r}(x) = 0 \) if \( r < 0 \)) and

\[
\varepsilon \varepsilon (uv^{p-1} \otimes x) = p^{r} x \quad \text{and} \quad \varepsilon \varepsilon (v^{p} \otimes x) = -p^{p} x
\]

if \( p > 2 \). By [98.3.4] and [52.3.5], \( \varepsilon \) is a well-defined morphism of \( A \)-modules. When \( \Delta^{*} \) is defined, \( \varepsilon \) is the composite

\[
R \times M \xrightarrow{\varepsilon \Delta^{*}} \varepsilon (E^{-1} R \times Z_{p} \otimes M) \xrightarrow{\varepsilon (e \otimes 1)} \varepsilon (E^{-1} Z_{p} \otimes M) = M.
\]

Generalizing Theorem 5.4, Adams, Gunawardena, and Miller [9] proved that \( \varepsilon \) is an \( \text{Ext} \)-isomorphism for any \( M \). This leads to a generalization of Theorem 5.3 to a version appropriate to \((Z_{p})^{k}\) for any \( k \geq 1 \), and this generalization is the heart of the proof of the Segal conjecture for elementary Abelian \( p \)-groups. See [9,74].


If \( x \in \pi_{n}E \) has order \( p \), then \( x \) extends over the Moore spectrum \( M^{p} = S^{n} \cup_{p} CS^{n} \). The idea of Nishida's second nilpotency theorem is to exploit this extension by showing that \( D_{j}M^{p} \) splits as a wedge of Eilenberg-MacLane spectra in a range of dimensions. The relevant splitting is a special case of the following result which, as we shall explain shortly, is in turn a special case of the general splitting theorem to be proven by Steinberger in the next chapter.

**Theorem 6.1.** Let \( Y \) be a spectrum obtained from \( S^{n} \) by attaching cells of dimension greater than \( n \). Assume that \( \pi_{n}Y \) is \( Z \) or \( Z_{p} \) and let \( v \in H^{r}(Y;Z_{p}) \) be a generator. Assume one of the following further hypotheses.

(a) \( p = 2 \) and either \( n \) is odd or \( S(v) \neq 0 \).
(b) \( p > 2, n \) is even, and \( S(v) \neq 0 \).
(c) \( p = 2 \) and \( S^{2}(v) \neq 0 \).
(d) \( p > 2, n \) is even, and \( S^{p^{2}}(v) \neq 0 \).

Then \( D_{j}Y \) splits \( p \)-locally as a wedge of suspensions of Eilenberg-MacLane spectra through dimensions \( r < n j + \frac{1}{p} (2p-3)(j+1)-1 \). In cases (a) and (b), only suspensions of \( H_{p}Z_{p} \) are needed.

Before discussing the proof, we explain how to use these splittings to obtain relations in the homotopy groups of \( H_{p} \)-ring spectra. Let \( Y \) and \( v \) be as in the theorem above and localize all spectra at \( p \).
Theorem 6.2. Let $E$ be an $H_\infty$ ring spectrum, let $F$ be a connective spectrum, and let $\phi: E \to F$ be any map (for example, the product when $F = E$ or the identity when $F = S$). Let $x \in \pi_n E$ and assume one of the following hypotheses.

(a) $p = 2$ and $n$ is odd; here let $Y = S^R$.
(b) $p > 2$, $n$ is even, and $x$ has order 2; here let $Y = M^n$.
(c) $p = 2$, $n$ is even, and $x$ extends over some $Y$ with $Sq^3(v) \neq 0$.
(d) $p > 2$, $n$ is even, and $x$ extends over some $Y$ with $gP^1(v) \neq 0$.

Let $R = \mathbb{Z}_p$ in cases (a) and (b) and $R = \mathbb{M}^n$ in cases (c) and (d) and let $y \in \pi_n F$ be in the kernel of the Hurewicz homomorphism $\pi_q F \to H_q(F; R)$. Then $x^j y = 0$ if $q < \frac{1}{p} (2p-3)(j+1)-1$.

Proof. Our hypotheses ensure that $H_n^E(D_j Y; R) \cong R$. We can choose a generator $\mu$ such that the composite

$$S^n \xrightarrow{\mu} D_j S^n \xrightarrow{D_j \phi} D_j Y \xrightarrow{x^j} E^{j} \xrightarrow{\phi} E$$

is $S^n e$, where $f: S^n \to Y$ is the inclusion of the bottom cell and $e: S \to HR$ is the unit. Choose $x: Y \to E$ such that $\tilde{x} f = x$. Then the solid arrow part of the following diagram commutes and the top composite is $x^j y$.

Here $r = n j + q$, $\omega: D_j Y \to (D_j Y)_r$ is the $r$th stage of a Postnikov decomposition of $D_j Y$, and $\rho: (D_j Y)_r \to E_n HR$ is the unique cohomology class such that $\rho \omega = \mu$. The previous theorem gives $\kappa: E_n HR \to (D_j Y)_r$ such that $\kappa \omega = 1$. The complementary wedge summand of $E_n HR$ in $(D_j Y)_r$ is $(nj)$-connected, and it follows that $\kappa \ast E_n^j e = \omega \ast D_j \phi^j$. Since $F$ is connective, $\omega \ast 1$ induces an isomorphism on $\pi_{nj} F$. Since $y$ is in the kernel of
the Hurewicz homomorphism and the latter is induced by $e \wedge 1 : F = SF \to HRF$, 
$\pi_0 e \wedge y = 0$. Chasing the diagram, we conclude that $x^j y = 0$.

In particular, with $F = E$, $q = n$, and $y = x$, we obtain $x^{j+1} = 0$. With $E = S$ and $n > 0$, case (b) applies to any even degree element of order $p$. As observed by Steinberger, when $p = 2$ case (a) applies to any odd degree element and gives a better estimate of the order of nilpotency than that obtained by applying case (b) to $x^2$. While this result gives a much better estimate of the order of nilpotency of elements of order $p$ in $\pi_1 S$ than does Theorem 2.9, the estimate is presumably still far from best possible. For example, if $p = 2$ and $n = 14$, the estimate is now $x^{30} = 0$. Cases (c) and (d) apply to some elements of order $p^i$ with $i > 1$. The idea is to add further cells to $S^n$, or to $S^n \cup_{p^i} CS^n$, so as to obtain a spectrum $Y$ for which the relevant Steenrod operation is non-zero. However, a given element $x$ need not extend over any such $Y$. (Conceivably some power of $x$ must so extend.) This explains why Nishida's second method fails to give the full nilpotency theorem and why we cannot yet prove Conjecture 2.7.

We must still explain how to prove Theorem 6.1. The idea is to approximate $D_n$ through the specified range by a spectrum with additional structure and then use homology operations to split the latter. The approximation is based on the following observation about mod $p$ homology.

**Proposition 6.3.** Let $Y$ be an $(n-1)$-connected spectrum with $H_n Y = \mathbb{Z}_p$, where $n$ is even if $p > 2$. Let $f : S^n \to Y$ induce an isomorphism on $H_n$. Then the homomorphism $H_i \pi_{n+1} D_n Y \to H_i \pi_{n+1} D_{n+1} Y$ induced by the composite

$$D_n S^n \xrightarrow{(1 \wedge f)} D_n Y \xrightarrow{q \cdot 1} D_{n+1} Y$$

is a monomorphism for all $i$ and is an isomorphism if $i < n(q+1) + \frac{1}{p} (2p-3)(q+1)$.

For spaces $X$, a self-contained calculation of $H_q D_n X$ for all $q$ is given in [28, I§4-5]. The generalization to spectra is given by McClure in Chapter IX, and the conclusion is easily read off from these calculations.

With the proposition as a hint, we construct the approximating spectra as follows.

**Definition 6.4.** Let $(Y,f)$ be a spectrum together with a map $f : S^n \to Y$ for some integer $n$ and define $D(Y,f) = \text{tel } \varepsilon^{-n} D_n Y$, where the $n$th map of the system is obtained by applying $\varepsilon^{-n(q+1)}$ to the composite

$$D_n S^n \xrightarrow{(1 \wedge f)} D_n Y \xrightarrow{q \cdot 1} D_{n+1} Y.$$
Corollary 6.5. With Y and f as in the proposition, assume further that Y is p-local of finite type. Then the natural map $D(Y,f)$ is an equivalence through dimensions less than $n_j + \frac{1}{p}(2p-3)(j+1) - 1$.

Proof. By the proposition, the maps $\mathbb{Z}^{n(q+1)}(a_q, 1 \Lambda f)$ used to construct $D(Y,f)$ induce isomorphisms in mod p homology and thus in p-local homology in degrees less than $\frac{1}{p}(2p-3)(q+1)$. This fact for $q \geq j$ implies the conclusion (with the usual loss of a dimension as one passes from homology to homotopy).

Thus, to prove Theorem 6.1, we need only split $D(Y,f)$.

The following ad hoc definition, which generalizes Nishida's notion of a $\Gamma$-spectrum [90,1.5], allows us to describe the structure present on the spectra $D(Y,f)$. In the rest of this section we shall refer to weak maps and weakly commutative diagrams when the domain is a telescope and phantom maps are to be ignored.

Definition 6.6. A spectrum $E$ is a pseudo $H_\infty$ ring spectrum if

(i) $E$ is the telescope of a sequence of connective spectra $E_q$, $q \geq 0$;

(ii) $E$ is a weak ring spectrum with unit induced from a map $S + E_0$ and product induced from a unital, associative, and commutative system of compatible maps $E_q \Lambda E_r + E_{q+r}$; and

(iii) For each $j \geq 0$ and $q \geq 0$, there exists an integer $d = d(j,q)$ and a map

$E^{d/2}_{q} + E^{d/2}_{q}$

whose composite with $E^{d/2}_{q}$ is $E^{d/2}_{q}$, the $(d/2)$th suspension of the iterated product $E_q + E_{q+r}$.

Examples 6.7. (i) With each $E_q = E$ and each $d(j,q) = 0$, a connective $H_\infty$ ring spectrum may be viewed as a pseudo $H_\infty$ ring spectrum.

(ii) With each $E_q = E$ and each $d(j,q) = d$, a connective $H_\infty$ ring spectrum may be viewed as a pseudo $H_\infty$ ring spectrum; since $E$ has structural maps $\xi_j$ for all $q$, negative as well as positive, we could obtain a different pseudo structure with each $d(j,q) = -d$.

(iii) For an $(n-1)$-connected spectrum $Y$ and map $f:S^n + Y$ such that either $2 = 0:Y + Y$ or $n$ is even, $D(Y,f)$ is a pseudo $H_\infty$ ring spectrum with $q$th term $\mathbb{Z}^{n(q+r)}D_qY$. Its product is induced by the maps

$\mathbb{Z}^{n(q+r)}D_qY \mathbb{Z}^{n(q+r)}D_rY \mathbb{Z}^{n(q+r)}D_qY$,

these forming a unital, associative, commutative, and compatible system by 1.2.6 and 1.2.8 and our added hypothesis, which serves to eliminate signs coming from permuta-
tions of spheres. With all $d(j,q) = n$, its structural maps are

$$
\xi_j = \beta_{j,q} : D_j \mathbb{L}^q(L^{-nq}D_j Y) = D_j D_j Y + D_j Y = \mathbb{L}^q(L^{-nq}D_j Y).
$$

The following analog of 1.3.6 and 1.4.5 admits precisely the same simple cohomological proof.

Proposition 6.8. Let $E$ be a pseudo $H_\infty$ ring spectrum with $\text{char } \pi_0 E = 2$ or all $d(j,q)$ even. Assume that $\pi_0 E = \pi_0 E_q$ for all $q \geq q_0$ and, for such $q$, let $i:E_q \rightarrow H(\pi_0 E)$ be the unique map which induces the identity homomorphism on $\pi_0$. Then the following diagrams commute, where $d = d(j,q)$:

$$
\begin{align*}
D_j \mathbb{L}^d q E & \xrightarrow{D_j \xi d q i} D_j \mathbb{L}^d q H(\pi_0 E) \\
\xi_j & \downarrow \quad \downarrow \xi_j \\
\mathbb{L}^d j q E & \xrightarrow{\mathbb{L}^d j q i} \mathbb{L}^d j q H(\pi_0 E)
\end{align*}
$$

In the next chapter, Steinberger will use a computation of the homology operations of the $H_\infty$ ring spectrum $\bigvee q \mathbb{L}^d q HZ_p$ to prove the following generalization of Nishida's result [90,3.2].

Theorem 6.9. Let $E$ be a $p$-local pseudo $H_\infty$ ring spectrum. If $\pi_0 E = \pi$, then $E$ splits as a wedge of suspensions of $HZ_p$. If $\pi_0 E = \mathbb{Z}_r$, $r > 1$, or $\pi_0 E = \mathbb{Z}(p)$ and if $p = 2$ and $Sq^2 i \neq 0$ or $p > 2$ and $p^{1} i \neq 0$, where $i$ generates $H^0(E;Z_p)$, then $E$ splits as a wedge of suspensions of $HZ_p^s$, $s \geq 1$, and $HZ(p)$.

Considering the natural map $\mathbb{L}^{-n} Y \rightarrow D(Y,f)$, and using the formula $\beta(\pi_0 \otimes \nu^2) = n\nu_1 \otimes \nu^2$ of Theorem 5.5 for case (a), we easily check that the theorem applies to split $D(Y,f)$ for $Y$ as in Theorem 6.1.

We complete this section with some remarks about the role played by Definition 6.4 in the general theory of $H_\infty$ ring spectra.

Remarks 6.10. Let $(E,e)$ be a spectrum with unit $e:S + E$. Let $DE = D(E,e)$ and let $\eta:E = D_1 E + DE$ be the natural inclusion. By I.2.7, I.2.9, and I.2.13, the maps $\beta_{j,k}:D_j D_k E - D_j k E$ induce a natural weak map $\mu_k : D_k E + DE$ such that the following diagrams (weakly) commute:
If $E$ is an $H_\infty$ ring spectrum, then, by Proposition 1.3, the maps $\xi_j : D_j E \to E$ determine a weak map $\xi : D E \to E$ such that the following diagrams (weakly) commute.

Conversely, by the same result, if $\psi : D E \to E$ makes these diagrams weakly commute, then its restrictions $\xi_j : D_j E \to E$ give $E$ a structure of $H_\infty$ ring spectrum. These assertions are analogous to, but weaker than, the assertions that $D$ is a monad and that an $H_\infty$ ring spectrum is an algebra over this monad (compare [69, §2]). The point is that the $\mu_k$ fail to satisfy the requisite compatibility to determine a weak map $\mu : DDE \to DE$. By 1.2.11 and 1.2.15, the compatibility they do have is described by the weakly commutative diagram

where $\nu_k$ is induced by the composites

and $\delta : D F \to D F \land D S$ is induced by the maps $\delta_j : D_j(F \land S) \to D_j F \land D_j S$. 