CHAPTER V
THE HOMOTOPY GROUPS OF $H_\infty$ RING SPECTRA

By Robert R. Bruner

§1. Explicit homotopy operations and relations

This section contains statements of our results on homotopy operations as well as some applications of these results. The proofs depend on material in §2 and will be given in §3.

Note that, aside from the computations in $\pi_\infty S$ at the end of this section, all the results here apply to the homotopy of any $H_\infty$ ring spectrum $Y$. Let $\xi: D_p Y \to Y$ denote the structure map.

The order of results in this section is:

- relation to other operations,
- particular operations and relations,
- Cartan formulas,
- computations in $\pi_\infty S$,
- remarks.

In order not to interrupt the main flow of ideas, we have deferred a number of remarks until the end of the section.

Throughout this section let $E_r(X,Y)$ be the ordinary mod p Adams spectral sequence converging to $[X,Y]_*$, and let $E_r(S,\mathcal{D})$ be the spectral sequence of IV §6 based on ordinary mod p homology. Let $\mathcal{D}$ be the sequence

$$\mathcal{D} = \{D_p S^n \to D_p^{1} S^n \to \cdots \to D_p^{j} S^n \to D_p^{0} S^n \}.$$

From the spectral sequence $E_r(S,\mathcal{D})$ we obtain an isomorphism between an associated graded of $\pi_\infty D_p S^n$ and $E_\infty(S,\mathcal{D})$:

$$E^0(\pi_\infty D_p S^n) \cong E_\infty(S,\mathcal{D}).$$

Write $E^0(a)$ for the image in $E^0_\infty(S,\mathcal{D})$ of an element $a \in \pi_\infty D_p S^n$ of filtration $s$. By IV.7.5, $E_2(S,\mathcal{D})$ is free over $E_2(S,S)$ on generators $e_i$ corresponding to the cells of $D_p S^n$. By 2.9 below, a more convenient basis over $E_2(S,S)$ is given by the elements
\[ g^{\epsilon p_j} = (-1)^{jv(n)}e^{jq-\epsilon n(p-1)} \]

where \( \epsilon = 0 \) or 1 (\( \epsilon = 0 \) if \( p = 2 \)), \( q = 2(p-1) \) (\( q = 1 \) if \( p = 2 \)), \( jq-\epsilon \geq n(p-1) \) and \( v \) is the function defined in IV.2.4 (\( v = 1 \) if \( p = 2 \)). Thus, \( E^0(a) \) can be written as a linear combination of the \( g^{\epsilon p_j} \) with coefficients in \( E_2(S,S) \). Recall the operation \( \alpha : \pi_n Y \to \pi_n Y \) associated to each element \( \alpha \in \pi_N D_p S^n \).

Relation of the \( \alpha^* \) to other operations

**Proposition 1.1.** If \( i: S^{np} \to D_p S^n \) is the natural map then

\[ i^*(x) = x^p \]

and

\[ E^0(i) = \begin{cases} p^n & p = 2 \\ p^j & p > 2 \text{ and } n = 2j \\ 0 & p > 2 \text{ and } n \text{ odd} \end{cases} \]

**Proposition 1.2.** Let \( h: \pi_n \to H_n \) be the Hurewicz homomorphism. If \( E^0(a) = g^{\epsilon p_j} \) then \( h \circ \alpha = g^{\epsilon Q^j} \circ h \), where \( g^{\epsilon Q^j} \) is the Dyer-Lashof operation defined in III.1.

If \( E^0(a) = \sum a_j, g^{\epsilon p_j} \), with each \( a_j, \epsilon E_2(S,S) \) and \( x \in E_2(S,Y) \), we let

\[ E^0(a)(x) = \sum a_j, g^{\epsilon p_j}(x). \]

**Proposition 1.3.** (Kahn, Milgram) If \( x \in \pi_n Y \) is detected by \( x \in E_2(S,Y) \), then \( \alpha^*(x) \) is detected by \( E^0(a)(x) \).

To see the relation to Toda brackets, suppose we have compressed \( \alpha \) into the \( np+1 \) skeleton \( D^i_p S^n \) and that it projects to \( \tilde{\alpha} \) on the top cell \( S^{np+i} \). Let \( D^{i-1}_p(x) = D^i_p(x)|D^{i-1}_p S^n \) and let \( c_i, \xi_{np+i-1} D_p S^n \) be the attaching map of the \( np+i \) cell.

**Proposition 1.4.** \( \alpha^*(x) \in C \subset C_{i} D^{i-1}_p S^n \). The set of all such \( \alpha^*(x) \) is a coset of \( D^{i-1}_p(x) \circ \pi^i_{np+i-1} S^n \).

**Note:** We will frequently find further that \( E^0(a) = as^{\epsilon p_j} \) where \( i = jq-\epsilon n(p-1) \) and \( (-1)^iv(n)a \) detects \( \tilde{\alpha} \). Then

\[ E^0(a)(x) = \sum E^0(a^*(x)) = as^{\epsilon p_j}(x), \]

so that \( \alpha^* \) is detected by Toda brackets in essentially the same fashion as by Steenrod operations in \( E_2(S,Y) \).
Particular operations and relations

Hereafter, if \( e \in \pi_nY \) and \( x \in \pi_nY \), let \( \Phi(x) = \{ \alpha^*(x) \mid E^0(\alpha) = e \} \). Clearly, the indeterminacy in \( \Phi(x) \), defined to be

\[
\text{Ind}(\Phi(x)) = \{ \alpha^*(x) - \beta^*(x) \mid \alpha \in \pi_nY, \beta \in E(S,S) \},
\]

is the set of values of all homotopy operations on \( x \) whose corresponding element in \( E(S,S) \) has higher filtration than does \( e \).

**Proposition 1.5** (Kahn, Milgram): The following are equivalent:

(i) \( \beta^p_j \) acts on \( \pi_nY \)

(ii) \( e_i \in E(S,S), i = jq - n(p-1) \)

(iii) \( D^p_{S^n} \) is reducible

(iv) if \( p = 2 \) then \( n \equiv -i-1 \) (2\( \Phi(i) \));

if \( p > 2 \) then \( e = 0 \) and \( n = 2j \),

or \( e = 1 \) and \( j \equiv 0 \) (p\( \Phi(i) \)).

The functions \( \Phi \) and \( \psi \) are defined in 2.5 and 2.11 below.

**Definition 1.6.** If \( p = 2 \), let \( \beta_0 = 2, \beta_1 = n, \beta_2 = v \) and let \( \beta_j \) be a generator of \( \text{Im} J \) in dimension \( 8a + 2b - 1 \), where \( j = 4a + b \) and \( 0 \leq b < 3 \). If \( p > 2 \), let \( a_0 = p \), and let \( \alpha_j \) be a generator of \( \text{Im} J \) in dimension \( jq - 1 \).

**Theorem 1.7** (Toda, Barratt, Mahowald, Cooley): Let \( p = 2 \). If \( x \in \pi_nY \) and \( j = 4a + b, 0 \leq b < 3 \), then

\[
\beta_j \circ x^2 = 0 \text{ if } n \equiv 2^j - 8a - 2^b - 1 \quad (2^{j+1})
\]

and \( \beta_j \circ P^{n+1}(x) = ax^2 \) for some \( a \in \pi_{8a + 2b - 1} \) if \( n \equiv 0 \) (2) and \( n \equiv 2^j - 8a - 2^b - 2 \) (2\( j+1 \)).

**Theorem 1.8.** Let \( p > 2 \) and \( x \in \pi_nY \). Let \( \epsilon_p(a) \) denote the exponent of \( p \) in the prime factorization of \( a \). If \( n = 2k - 1 \) then

\[
a_j \circ \beta^{k-1}x = 0 \quad \text{if } j = 0
\]

or \( j > 0 \) and \( \epsilon_p(k+j) = j-1 \).

If \( n = 2k \) then

\[
a_j \circ \beta^{k-1}x = a^p \quad \text{for some } a \in \pi_{(j+1)q-2} \text{ if } j = 0
\]

or \( j > 0 \) and \( \epsilon_p(k+j+1) = j-1 \).
Theorem 1.9. The operations listed in Tables 1.1 and 1.3 exist on $\pi_n$ and satisfy the relations listed in Tables 1.2 and 1.4. In Tables 1.1 and 1.3 the columns labelled "indeterminacy" list generators for the indeterminacy of each operation, and the columns labelled "$\tau_{p^k}$" list the values of

$$\tau_{p^k}: \pi_n \cdot p^k \Pi_n S_n + \pi_n^{Np^k} \equiv \pi_{N-np^k}$$

thereby indicating the deviation from additivity of the given operation (by IV.7.4).

### TABLE 1.1

**Operations on $\pi_n$ for $p > 2$**

<table>
<thead>
<tr>
<th>$n$</th>
<th>operations</th>
<th>indeterminacy</th>
<th>$\tau_{p^k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2k-1$</td>
<td>$g_{p^k}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_{0p^k}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$gl_{p^k}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

| $n = 2k-1$ | $g_{p^{k+1}}$ | $h_{0p^k}$ | 0 |
| $k = -1$ (p) |             |            |   |

| $n = 2k-1$ | $h_{0g_{p^{k+1}}}$ | $a_{1g_{p^k}}$ | 0 |
| $k = -2$ (p) | $g_{p^{k+2}}$ | $g_{1p^k}$ and | 0 |
|            | $h_{0p^{k+1}}$ (if it exists) |            |   |

| $n = 2k$ | $p^k$ | 0 | $p^l$ |
| $g_{p^{k+1}}$ | $a_{1p^k}$ | multiple of $a_1$ |
| $h_{0p^{k+1}}$ | $a_{2p^k}$ | multiple of $a_2$ |

<p>| $n = 2k$ | $g_{p^{k+2}}$ | $h_{0p^{k+1}}$ and | multiple of $a_2$ |
| $k = -2$ (p) | $a_{2p^k}$ |            |   |</p>
<table>
<thead>
<tr>
<th>( n )</th>
<th>relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 2k-1 )</td>
<td>( p \beta p^k = \phi_0 p^k = p \gamma_1 p^k = 0 )</td>
</tr>
<tr>
<td>( n = 2k-1 )</td>
<td>( (k+1) \alpha_1 \beta p^k = 0 )</td>
</tr>
<tr>
<td>( k \equiv -1 \ (p) )</td>
<td>( \alpha_1 \beta p^{k+1} \equiv 0 \mod a_2 \beta p^k )</td>
</tr>
<tr>
<td>( n = 2k-1 )</td>
<td>( \phi_0 \beta p^{k+1} \equiv 0 \mod a_2 \beta p^k )</td>
</tr>
<tr>
<td>( k \equiv -2 \ (p) )</td>
<td>( \alpha_1 \beta p^{k+1} = 0 )</td>
</tr>
<tr>
<td>( n = 2k )</td>
<td>( \alpha_1 p^k = p \beta p^{k+1} )</td>
</tr>
<tr>
<td>( n = 2k )</td>
<td>( (k+2) \alpha_1 \beta p^{k+1} = 0 )</td>
</tr>
<tr>
<td>( k \equiv -2 \ (p) )</td>
<td>( p \beta p^{k+2} \equiv -\phi_0 p^{k+1} \mod a_2 p^k )</td>
</tr>
</tbody>
</table>
TABLE 1.3

Operations on $\tau_n$ for $p = 2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>operations</th>
<th>indeterminacy</th>
<th>$\tau_{2*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \leq 0$ (4)</td>
<td>$p^n$</td>
<td>$2p^n$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$p^{n+1}$</td>
<td>$p^n$</td>
<td>$n$</td>
</tr>
<tr>
<td></td>
<td>$p^{n+3}$</td>
<td>$2p^{n+3}, vp^n$</td>
<td>multiple of $v$</td>
</tr>
<tr>
<td></td>
<td>$h_1p^{n+2}$</td>
<td>$2h_1p^{n+2}, vp^n$</td>
<td>multiple of $v$</td>
</tr>
<tr>
<td>$n = 1$ (4)</td>
<td>$p^n$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_1p^{n+1}$</td>
<td>$n^2p_n$</td>
<td>0 or $n^2$</td>
</tr>
<tr>
<td></td>
<td>$p^{n+2}$</td>
<td>$2p^{n+2}$</td>
<td>0 or $n^2$</td>
</tr>
<tr>
<td></td>
<td>$h_1p^{n+5}$</td>
<td>$2h_1p^{n+5}, v^2p_n$</td>
<td>0 or $v^2$</td>
</tr>
<tr>
<td></td>
<td>$h_2p^{n+4}$</td>
<td>$2h_2^2p^{n+4}, v^2p_n$</td>
<td>0 or $v^2$</td>
</tr>
<tr>
<td></td>
<td>$h_3p^{n+3}$</td>
<td>$2h_3^2p^{n+3}, v^2p_n$</td>
<td>0 or $v^2$</td>
</tr>
<tr>
<td>$n = 1$ (8)</td>
<td>$p^{n+6}$</td>
<td>$2p^{n+6}$</td>
<td>0 or $v^2$</td>
</tr>
<tr>
<td>$n = 2$ (4)</td>
<td>$p^n$</td>
<td>$2p^n$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$p^{n+1}$</td>
<td>$p^n$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$h_1p^{n+4}$</td>
<td>$2h_1p^{n+4}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_2p^{n+3}$</td>
<td>$2h_2^2p^{n+3}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_3p^{n+2}$</td>
<td>$2h_3^3p^{n+2}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_2p^{n+3}$</td>
<td>$v^2p_n$</td>
<td>0 or $v^2$</td>
</tr>
<tr>
<td>$n = 2$ (8)</td>
<td>$p^{n+5}$</td>
<td>$2p^{n+5}$</td>
<td>0</td>
</tr>
<tr>
<td>$n = 3$ (4)</td>
<td>$p^n$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_1p^{n+1}$</td>
<td>0</td>
<td>0 or $n^2$</td>
</tr>
<tr>
<td></td>
<td>$h_1p^{n+3}$</td>
<td>$2h_1p^{n+3}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_2p^{n+2}$</td>
<td>$n^2h_1p^{n+1}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$h_2p^{n+2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 3$ (8)</td>
<td>$p^{n+4}$</td>
<td>$2p^{n+4}$</td>
<td>0</td>
</tr>
</tbody>
</table>
TABLE 1.4

Relations among operations for $p = 2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Conditions</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 0$ (4)</td>
<td>2$p^{n+1} = 0$</td>
<td>$2h_1p^{n+2} = n_2p^{n+1}$</td>
</tr>
<tr>
<td>$n = 0$ (8)</td>
<td>$2p^{n+3} = h_1p^{n+2}$</td>
<td>$\eta p^{n+3} = 0$</td>
</tr>
<tr>
<td></td>
<td>$2v p^{n+3} = v h_1p^{n+2} = 0$</td>
<td></td>
</tr>
<tr>
<td>$n = 4$ (8)</td>
<td>$2p^{n+3} = h_1p^{n+2} + v p^n$</td>
<td>$\eta p^{n+3} = v p^{n+1}$</td>
</tr>
<tr>
<td></td>
<td>$v h_1p^{n+2} = v_2p^n$</td>
<td></td>
</tr>
<tr>
<td>$n = 1$ (4)</td>
<td>$2p^n = 0$</td>
<td>$2h_1p^{n+5} = h_1^2p^{n+4}$</td>
</tr>
<tr>
<td></td>
<td>$2h_1p^{n+1} = n_2p^n$</td>
<td>$2h_1^2p^{n+4} = h_1^3p^{n+3}$</td>
</tr>
<tr>
<td></td>
<td>$2p^{n+2} = h_1p^{n+1}$</td>
<td>$2h_1^3p^{n+3} = 0$</td>
</tr>
<tr>
<td></td>
<td>$\eta h_1p^{n+1} = 0$</td>
<td></td>
</tr>
<tr>
<td>$n = 1$ (8)</td>
<td>$\eta p^{n+2} = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2v p^{n+2} = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2p^{n+6} = h_1p^{n+5}$</td>
<td></td>
</tr>
<tr>
<td>$n = 5$ (8)</td>
<td>$\eta p^{n+2} = v p^n$</td>
<td>$v p^{n+2} = 0$</td>
</tr>
<tr>
<td>$n = 2$ (4)</td>
<td>$2p^{n+1} = \eta p^n$</td>
<td>$2h_1p^{n+4} = h_1^2p^{n+3}$</td>
</tr>
<tr>
<td></td>
<td>$\eta p^{n+1} = 0$</td>
<td>$2h_1^2p^{n+3} = h_1^3p^{n+2}$</td>
</tr>
<tr>
<td></td>
<td>$4v p^n = 0$</td>
<td>$2h_1^3p^{n+2} = 0$</td>
</tr>
<tr>
<td>$n = 2$ (8)</td>
<td>$2p^{n+5} = h_1p^{n+4}$</td>
<td>$\eta p^{n+5} = h_2p^{n+3}$</td>
</tr>
<tr>
<td>$n = 6$ (8)</td>
<td>$v p^{n+1} = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2h_2p^{n+3} = v^2p^n$</td>
<td>$v h_1p^{n+4} \equiv 0 \mod v^2p^n$</td>
</tr>
</tbody>
</table>
\[
\begin{align*}
& n = 3 \ (4) \quad 2p^n = 0 \quad 2h_1 p^{n+3} = h_1^2 p^{n+2} \\
& \quad \eta p^n = 0 \quad 2h_1^2 p^{n+2} = \eta^2 h_1 p^{n+1} \\
& \quad 2h_1 p^{n+1} = 0 \quad 2h_2 p^{n+2} = 0
\end{align*}
\]

\[
\begin{align*}
& n = 3 \ (8) \quad 2p^{n+4} = h_1 p^{n+3} \\
& \quad \eta p^{n+4} = h_2 p^{n+2} \\
& \quad \eta h_2 p^{n+2} = \nu^2 p^n
\end{align*}
\]

\[
\begin{align*}
& n = 7 \ (8) \quad \nu p^n = 0 \\
& \quad \eta h_1 p^{n+3} = 0 \\
& \quad \eta h_2 p^{n+2} = 0
\end{align*}
\]
Cartan Formulas

For later computations we need the Cartan formulas for the first operation above the $p^{th}$ power.

Proposition I.10. Let $p = 2$, $x \in \pi_n Y$, $y \in \pi_m Y$. Assume $n+m$ is even. Then

$$p^{n+m+1}(xy) = \begin{cases} 
  p^{n+1}(x)y^2 + x^2 p^{m+1}(y) + c_{n,m} x^2 y^2 & n \equiv m \equiv 0 \ (2) \\
  S_{n,m}(x,y) & n \equiv 3 \ (4) \text{ or } m \equiv 3 \ (4) \\
  S_{n,m}(x,y) + c_{n,m} x^2 y^2 & n \equiv m \equiv 1 \ (4)
\end{cases}$$

where $S_{n,m} : \pi_n \times \pi_m \to \pi_{n+m+1}$ is an operation such that

$$E^0(S_{n,m}) = p^n p^{m+1} + p^{n+1} p^m$$

and

$$2S_{n,m}(x,y) = \begin{cases} 
  n x^2 y^2 & n \equiv m \equiv 1 \ (4) \\
  0 & n \equiv 3 \ (4) \text{ or } m \equiv 3 \ (4)
\end{cases},$$

and where $c_{n,m}$ is an integer depending only on $n$ and $m$.

Proposition I.11. Let $p > 2$, $x \in \pi_n Y$ and $y \in \pi_m Y$. Then

(i) if $n = 2j$ and $m = 2k$,

$$\beta_{p^{j+k+1}}(xy) = \beta_{p^{j+1}}(x) y^p + x^p \beta_{p^{k+1}}(y) + d_{n,m} x^p y^p$$

where $d_{n,m}$ is an integer depending only on $n$ and $m$.

(ii) if $n = 2j$ and $m = 2k-1$,

$$\beta_{p^{j+k}}(xy) = x^p \beta_{p^k}(y)$$

(iii) if $n = 2j-1$ and $m = 2k-1$,

$$\beta_{p^{j+k}}(xy) = S_{j,k}(x,y)$$

where $S_{j,k} : \pi_{2j-1} Y \times \pi_{2k-1} Y \to \pi_{2(j+k)-1} Y$ is an operation such that

$$E^0(S_{j,k}) = \beta_{p^j} \cdot p^k + p^j \cdot \beta_{p^k} \text{ and } pS_{j,k}(x,y) = 0.$$
Computations

Our final results contain extensions to all \(H\) ring spectra of classical results about \(\pi_*S\) due to Toda, Barratt, Mahowald, Gray and Milgram, as well as some low dimensional calculations at the prime 2.

Let \(\equiv\) denote equality up to multiplication by a unit.

Proposition 1.12. If \(p = 2\) then \(P^1(2) = \eta\).

Proposition 1.13. If \(p > 2\) then \(\alpha P^1(p) \equiv \alpha\) and \(\alpha P^{p-1}(\alpha) = \beta_1\).

Combined with the Cartan formulas 1.10 and 1.11, these yield the following results.

Proposition 1.14. Let \(x \in \pi_*Y\) and \(n = 2j\). If \(p = 2\) then \(P^{n+1}(2x) = nx^2\). If \(p > 2\) then \(\alpha P^{j+1}(px) \equiv \alpha_j x^p\) and \(\alpha P^{p-1}(\alpha_j x) = \beta_1 x^p\). The indeterminacy of each is 0.

Corollary 1.15. Let \(x \in \pi_*Y\). If \(p = 2\), \(n \not\equiv 1 \pmod{4}\) and \(2x = 0\), then \(\eta x^2 = 0\). If \(p > 2\) and \(px = 0\) then \(\beta_1 x^p = 0\). In particular, \(\beta_1 \chi_1 = 0\).

In the next proposition, the statement "\(aP^j(x) = y \pmod{A}\)" means that \(A\) is the indeterminacy of \(aP^j\) when applied to \(x\). If the indeterminacy is not mentioned, it is 0.

Proposition 1.16. The following hold in \(\pi_*S\) localized at 2.

1. \(P^1(\eta) = \eta^2\)
2. \(P^3(\nu) = \nu^2, h_1^1P^4(\nu) = \eta \nu \text{ or } \nu, h_1^2P^5(\nu) = 0\).
3. \(P^3, h_1^1P^4, h_1^2P^6, h_2^1P^5\) annihilate \(2\nu\) and \(4\nu\).
4. \(P^6, P^7, h_1^2P^9, \text{ and } h_1^3P^8\) annihilate \(\nu^2\).
5. \(P^7(\sigma) = \sigma^2, h_1^1P^8(\sigma) = \eta \sigma \text{ or } \sigma^* + \eta \sigma,\)
   \[h_1^1P^{10}(\sigma) = \nu^* \pmod{<2\nu^* + \eta\nu>},\]
   \[h_2^1P^9(\sigma) = 2\nu^* \pmod{<4\nu^* + \eta\nu>}.\]
6. \(P^7(2\sigma) = 0, h_1^1P^8(2\sigma) = 0, h_2^1P^9(2\sigma) = 0,\)
   \[h_1^1P^{10}(2\sigma) = 2\nu^* \pmod{<4\nu^* + \eta\nu>},\]
   \[h_2^1P^9(2\sigma) = 4\nu^* \pmod{<\eta\nu>}.\]
(vii) $p^7$, $h_1^8$, $h_1^2p^9$ and $h_0^2p^9$ annihilate $4q$,
$$h_1^{10}(4q) = 4q^* \mod \langle \eta \bar{\nu} \rangle.$$

Remarks: These are listed by the result to which they refer.

(1.4): The indeterminacy of the Toda bracket $\langle \tilde{a}, c_i, \xi D_{p}^{i-1}(x) \rangle$ in Proposition 1.4 is $\xi D_{p}^{i-1}(x) \circ \eta D_{p}^{i-1}S + (\eta_{np+1}) \circ \tilde{a}$, while the indeterminacy of $a^*(x)$ is only $\xi D_{p}^{i-1}(x) \circ \eta D_{p}^{i-1}S$. This reflects the fact that $a^*(x)$ uses the canonical null homotopy $D_{p}^{i-1}(x)$ of $D_{p}^{i-1}(x) \circ c_i$, whereas the Toda bracket allows any null homotopy of $\xi D_{p}^{i-1}(x) \circ c_i$.

(1.8): Since $\pi_{pq-2}$ is the first nonzero homotopy group of $S$ in a dimension congruent to $-2 \mod q$, we get
$$a_j \beta P^{k+1}x = 0$$
for $j < p-1$ satisfying the hypotheses of (1.8).

(1.9): (i) In the range of dimensions listed, the operations and relations given in Tables 1.1 through 1.4 generate all the operations and relations over $\pi_* S$. For examples, when $n \equiv 0 \pmod{4}$ and $p = 2$:

(a) $\eta P^2$ and $\eta^2 P^2$ are nonzero operations because the relations listed do not force them to be 0;

(b) the relation $4h_1 P^{n+2} = 0$ follows from the listed relation
$$2h_1 P^{n+2} = \eta^2 P^{n+1},$$
and is therefore omitted;

(c) the redundant operation $h_1 P^{n+2}$ is included because the relation
$$2P^{n+3} = h_1 P^{n+2}$$
which makes it redundant reflects a universally hidden extension:

\[
\begin{array}{c}
| 2n+3 | 2n+2 |
| \hline
| P^{n+2} | \hline
| \hline
| h_0 P^{n+3} = 0 \text{ in } E_\infty \text{ and } 2P^{n+3}x \text{ is detected by } h_1 P^{n+2}x.|
\end{array}
\]
(ii) The operations of degree \(n+3\) for \(n \equiv 0\) \((4)\) and \(p = 2\) are particularly interesting. If \(n \equiv 0\) \((8)\) then by \([59]\) \(\pi_{2n+3}D_2S^n = Z_2 \oplus Z_9\). It is generated by \(v^p\) and \(p^{n+3}\) with relations

\[
2p^{n+3} = h_1 p^{n+2}
\]
and
\[
4p^{n+3} = 2h_1 p^{n+2} = n^2 p^{n+1}.
\]

If \(n \equiv 4\) \((8)\) then \([59]\) gives \(\pi_{2n+3}D_2S^n = Z_4 \oplus Z_{16}\) and it is generated by \(h_1 p^{n+2}\) (of order 4) and \(p^{n+3}\) (of order 16) with relations

\[
2h_1 p^{n+2} = n^2 p^{n+1}
\]
\[
2p^{n+3} = h_1 p^{n+2} + v^p
\]
\[
4p^{n+3} = n^2 p^{n+1} + 2v^p
\]
\[
8p^{n+3} = 4v^p
\]

(iii) Entries in the \(T_p\) column such as "0 or \(n^2\)" indicate that we have not calculated \(\tau_{p,\pi}\). Such entries simply list the elements of \(\pi_{\#}S\) in the relevant dimension. Even this limited information is useful in Proposition 1.16.

(1.10) and (1.11): Let \(\psi: A \times A \otimes A\) be the diagonal of the Steenrod algebra \((\psi(p^n) = \sum p^k \otimes p^{n-k})\). If

\[
E^0(a) = \sum a_i A_i, \quad a_i \in E_2(S, S), \quad A_i \in A
\]
then

\[
E^0(\delta_*(a)) = \sum a_i \psi(A_i).
\]

This defines \(\delta_*(a)\) and, hence, the formula for \(\alpha_#(xy)\), modulo higher filtration in \(E_\infty(S, D)\).

(1.15): This proof that \(P_{\pi_1}B_1 = 0\) differs from Toda's in that Toda views the product in \(\pi_{\#}S\) as composition and studies \(D_p(S \cup_p e^{n+1})\) while we view it as the smash product and study \(D_pS^{n} \wedge D_pS^{n}\). Toda shows that

\[
D_p(S^{n} \cup_p e^{n+1}) \supset S^{n} \cup_{\alpha_1} e^{np+q}
\]
and

\[
D_p(S^{n} \cup_{\alpha_1} e^{n+q}) \supset S^{n} \cup_{\beta_1} e^{n^p+1}.
\]

Thus, if \(px = 0\) or \(\alpha_1 x = 0\) then \(\alpha_1 x^p = 0\) or \(\beta_1 x^p = 0\), respectively. The proof given in 1.15 uses the values of the operations on \(p\) and \(\alpha_1\), rather than the structure of \(D_p\) of their cofibers.
Segal [49] saw that the Cartan formula for homotopy operations should provide a proof that \( a_1 \beta_1^p = 0 \), but his explicit formulas were incorrect.

There is still another proof that \( a_1 \beta_1^p = 0 \) which uses virtually none of the machinery of homotopy operations, but does require that we have calculated enough of \( \pi_8 S \) to know that the \( p^2q-3 \) stem is either 0 or \( \mathbb{Z}_p \). Given this, the relation

\[
-a_1 \beta_1^p = p\beta P^2 - P(\beta_1)
\]

from Table 1.2 implies that \( \beta_1^p = 0 \).

**Remark 1.17:** This is a quick survey of results on homotopy operations which are not included here. Toda [106] shows that the extended powers propagate several relations. For example, if \( \langle a_1, p, x \rangle = 0 \) then \( \beta_s x^p = 0 \mod a_1 \) for \( 1 < s < p \). As corollaries he shows that \( \beta_2 \beta_1^p = 0 \) and the \( \beta_s \) are nilpotent, foreshadowing Nishida's proof, a few years later, that all positive dimensional elements of \( \pi_8 S \) are nilpotent.

Gray [36] obtained results similar to 1.15 using homotopy operations which are associativity or commutativity obstructions for ring spectra.

Oka and Toda [92] have extensive information on the cell structure of \( D_p(S^n \cup_p e^{n+1}) \) which they use, in particular, to show that \( \gamma_1 \neq 0 \).

Milgram [80] also uses extended powers \( D_2(S^n \cup_2 e^{n+1}) \) to define homotopy operations which can be iterated to yield infinite families of elements in \( \pi_8 S \), presumably related to the elements detected by K-theory.

Cooley, in his thesis [30], uses extended powers to compute some Toda brackets and to derive 1.7 as well as the relation \( \epsilon x^2 = 0 \) if \( x \in \pi_n, n = 2, 3, 7 \) (8), which is not in 1.7.

Milgram [79 and 81] computes the Coker J part of the operations on \( \pi_8 S \) and \( \pi_9 S \) using Steenrod operations in \( E_2(S, S) \).

§2. Extended powers of spheres

In this section we collect the results on extended powers of spheres which are needed to prove the results of §1. They will also be essential to our results on differentials in the next chapter. First, we recall the values of the \( K \) and J groups of lens spaces. Then, we identify the spectra \( D_p S_n, n \) cyclic, as the suspension spectra of stunted lens spaces and determine when they are stably reducible or coreducible. Also, we show that, after localizing at \( p \), \( D_p S^n \) is a wedge summand of \( D_p S^n \), which gives a simple cell structure to \( D_p S^n \).
Throughout this section, let \( p \) be a prime, let \( \pi \subset \Sigma_p \) be the \( p \)-Sylow subgroup generated by the \( p \)-cycle \( (1 \, 2 \cdots p) \), and let \( W^k \) be the \( k \)-skeleton of a contractible \( \pi \) or \( \Sigma_p \) free CW complex \( W \). (Definitions 2.1 and 2.7 provide the \( \pi \) free CW complexes which we shall use most frequently.)

The results for \( p = 2 \) are analogous to the results for odd primes, but are sufficiently simpler that we state them separately. We begin with odd primes.

Definition 2.1. Let \( p > 2 \) and let \( \rho = \exp(2\pi i/p) \). Let \( \pi \) act on the unit sphere \( S^{2k+1} \subset \mathbb{C}^{k+1} \) by letting a generator of \( \pi \) send \((z_0, \ldots, z_k)\) to \((\rho z_i)\). Let
\[
\mathbb{L}^{2k+1} = S^{2k+1}/\pi,
\]
\[
\mathbb{L}^{2k} = \{[z_0, \ldots, z_k] \in \mathbb{L}^{2k+1} \mid z_k^p \text{ is real and } > 0\},
\]
and
\[
\mathbb{L}^{n+k} = \mathbb{L}^{n+k}/\mathbb{L}^{n-1},
\]
where \([z_0, \ldots, z_k]\) denotes the equivalence class of \((z_0, \ldots, z_k)\) and \( \mathbb{L}^{2k-1} \) is embedded in \( \mathbb{L}^{2k} \) by setting \( z_k = 0 \). We call \( \mathbb{L}^{n+k} \) a stunted lens space.

Each representation of \( \pi \) on \( \mathbb{C}^{k+1} \) without trivial subrepresentations yields a free \( \pi \) action on \( S^{2k+1} \) and a corresponding lens space \( \mathbb{L}^{2k+1}/\pi \). Since they are all stably equivalent we have simply chosen our favorite. Note, however, that the others reappear briefly in the proof of Proposition 2.4.

It is easy to see that \( L^n - L^{n-1} \) is an open \( n \) cell. Thus \( L^{n+k} \) has one cell in each dimension between \( n \) and \( n+k \) inclusive. Note that \( L^n = L^n \) and \( L^0 = (L^n)^+ \), the union of \( L^n \) and a disjoint basepoint.

Since \( L^0 = S^0/\pi \) is a \( K(\pi,1) \), \( H^*(L^0; \mathbb{Z}_p) = E(x) \otimes P(gx) \), with \(|x| = 1\), and the Steenrod operations are specified by
\[
P^i(x^e(gx)^j) = (j^i)x^e(gx)^{j+i(p-1)}.
\]
The isomorphisms
\[
H^{i+n+k}_{L^n} \rightarrow H^{i+n+k}_{L^{n+k}} \leftarrow H^{i+n+k}_{L^{n+k}}
\]
for \( n < i < n+k \) then determine \( H^{n+k}_{L^{n+k}} \) as an \( A_{p} \) module.

Definition 2.2. Let \( p > 2 \) and let \( \pi \) act on \( \mathbb{C} \) by multiplication by \( \rho \). Let \( \xi \in KU(L^{2k+1}) \) be the bundle
\[
S^{2k+1} \times_{\pi} \mathbb{C} \rightarrow S^{2k+1} \times_{\pi} \{0\} = \mathbb{L}^{2k+1},
\]
let \( \zeta_1 = r(\xi^1) \in KO(L^{2k+1}) \) where \( r: KU \to KO \) forgets complex structure, let
\( \zeta = J(\zeta_1) \in J(\hat{L}^{2k+1}) \), and let \( \sigma = \zeta - 1 \in \tilde{KU}(\hat{L}^{2k+1}) \). Let \( \xi, \zeta_1, \zeta \) and \( \sigma \) also denote the restrictions of these elements to \( \hat{L}^{2k} \).

We collect some results from [47], [48] and [58] in the following theorem.

**Theorem 2.3.** Let \( L^{2k} \subset L^{2k+1} \) be the inclusion and let \( \langle x \rangle \) denote the cyclic group generated by \( x \).

(i) \( i: KU(L^{2k+1}) \to KU(L^{2k}) \) is an isomorphism and

\[
\tilde{KU}(L^{2k}) = \langle \sigma \rangle \oplus \langle \sigma^2 \rangle \oplus \cdots \oplus \langle \sigma^{p-1} \rangle
\]

(ii) \( r: KU(L^{2k+1}) \to KU(L^{2k}) \) is an epimorphism,

\[
\tilde{KU}(L^{2k+1}) = \tilde{KU}(L^{2k}) \oplus \tilde{KU}(S^{2k+1}),
\]

and \( i^* \) is projection onto the first summand under this isomorphism.

(iii) \( J(L^{2k}) = \langle Jr(\sigma) \rangle = \langle \zeta - 2 \rangle \) and has order \( p^{[k/(p-1)]} \),

\[
J(L^{2k+1}) = J(L^{2k}) \oplus J(S^{2k+1})
\]

and \( i^* \) is projection onto the first summand under this isomorphism.

Also, \( J(\zeta_1) = \zeta \) for \( i = 1,2,\ldots,p-1 \).

**Proof.** This is all in [47], [48] and [58] except \( J(\zeta_1) = \zeta \), which follows from the Adams conjecture:

\[
J(\zeta_1) = Jr^1 \zeta = Jr^i \zeta = Jr^i = \zeta. \]


The extended powers \( D^kS^n \) are suspension spectra of Thom spaces of complex bundles over \( L^k = W^k/\pi \). Thus Theorem 2.3 ensures us that the following theorem (proved in [81]) identifies all such spectra. Note that its proof does not require \( p \) to be a prime.

**Theorem 2.4.** If \( s \geq 0 \), the Thom complex of \( r + s \zeta \) over \( L^k \) satisfies

\[
\hat{L}^\infty T(r + s \zeta) = \hat{L}^\infty \hat{L}^{2s+k}. \]

**Proof.** The contribution of the trivial \( r \) dimensional fibration is obvious and may be ignored. We will actually prove a much more precise result. If \( a \) is an \( n \)-dimensional representation of \( \pi \), we let \( H^n(a) \) and \( S^{n-1}(a) \) denote \( H^n \) and \( S^{n-1} \) with \( \pi \) action given by \( a \). If the action is free on \( S^{n-1} \) we obtain a closed manifold \( L(a) = S^{n-1}(a)/\pi \). If \( a \) and \( b \) are two such representations of dimension \( n \) and \( k \) respectively, let \( a|L(b) \) be the bundle
We claim that there is a homeomorphism

\[ T(\alpha|L(\gamma)) \cong L(\gamma \oplus \alpha)/L(\alpha), \]

where \( L(\alpha) \) is embedded in \( L(\gamma \oplus \alpha) \) as the last \( n \) coordinates. This will imply Theorem 2.4 for odd \( k \) (since \( L(\gamma) \) is odd dimensional, \( p \) being odd). The even case will follow by removing the top cell on each side, since the homeomorphism will be cellular if we give the Thom complex \( T(\alpha|L(\gamma)) \) the natural cell structure compatible with that of \( L(\gamma) \).

To establish the claim, let \( f: S^{-1}(\gamma) \times \mathbb{R}^n(\alpha) \rightarrow S^{-1}(\gamma \oplus \alpha) \) be induced by the natural inclusion \( S^{-1}(\gamma) \times \mathbb{R}^n(\alpha) \rightarrow S^{-1}(\gamma \oplus \alpha) \) followed by the radial retraction \( \mathbb{R}^{n+k} \rightarrow S^{n+k-1} \). It is easy to check that \( f \) is one-to-one and maps onto everything except the copy of \( L(\alpha) \) embedded as the last \( n \) coordinates. Just as easily, one sees that \( f \) sends the zero section of \( \alpha|L(\gamma) \) to the embedding of \( L(\gamma) \) as the first \( k \) coordinates. It follows that \( \alpha|L(\gamma) \) is the normal bundle of this embedding \( L(\gamma) \hookrightarrow L(\gamma \oplus \alpha) \) and that its Thom complex is \( L(\gamma \oplus \alpha)/L(\alpha) \). //

The fact that \( \gamma \in J(L^k) \) has finite order enables us to define stunted lens spectra in positive and negative dimensions.

Definition 2.9. Let \( \psi(k) = \lfloor k/2(p-1) \rfloor \). If \( n \) is any integer, \( \epsilon = 0 \) or \( 1 \), and \( k \geq \epsilon \), let

\[ \gamma_{2n+k} = \gamma_{2(n-r)+2r+k}, \]

for \( r \equiv n (p^\epsilon(k)) \) such that \( r > 0 \).

The following result shows that the spectrum \( \gamma_{2n+k} \) is well-defined up to equivalence in \( \mathbb{R}^k \). Recall that an \( n \)-dimensional complex \( X \) is reducible if \( X/X^{n-1} = S^n \) and the projection \( X \rightarrow S^n \) has a right inverse. Dually, an \( (n-1) \)-connected complex \( X \) is coreducible if \( X^n = S^n \) and the inclusion \( S^n \hookrightarrow X \) has a left inverse. Let \( W = S^\infty \), let \( q: W \rightarrow L^\infty \) be the quotient map and let \( W^k = q^{-1}(L^k) \). Then we may define \( D^k_X = W^k \times X(p) \).

Theorem 2.6. Let \( S^n \) be the \( p \)-local \( n \)-sphere spectrum. Then

(i) \( D^k_S = \gamma_{2n+k} \).

(ii) \( \gamma_{2n+k} \) is coreducible iff \( n \equiv 0 (p^\epsilon(k)) \), while \( \gamma_{2n+1} \) is coreducible iff \( k = 1 \).
(iii) If $\varepsilon = 0$ or 1, $k \geq \varepsilon$ and $n \equiv r \pmod{(p^\psi(k))}$ then

$$\nu^{2n+k} \equiv \Sigma^{2(n-r)} \nu^{2r+k} \pmod{2n+\varepsilon}.$$ 

(iv) $\nu_b$ and $\nu_{-a-1}$ are $(-1)$ dual spectra.

(v) If $\varepsilon = 0$ or 1 and $k > \varepsilon$ then $\nu^{2n+k}$ is reducible iff either $k = \varepsilon$ or $k$ is odd and $2n+k+1 \equiv 0 \pmod{(p^\psi(k))}$.

**Proof.** If $n \geq 0$ then $D^k_{S^n} = W^k \times \nu^*_S = \Sigma \nu_T(n\gamma_k)$, where $\gamma_k$ is the restriction to $L^k$ of the bundle over $L^n = B_n$ induced by the regular representation of $w$. Since $\gamma_k = 1 + \zeta_1 + \cdots + \zeta_m$, $J(n\gamma_k) = n + nm_\zeta$ (where $2m = p-1$). By Theorem 2.4,

$$\Sigma \nu_T(n\gamma_k) = \nu^{n+nm_\zeta(p-1)+k}.$$ 

If $n < 0$ then, by [Equiv, VI.5.3 and 5.4] $W^k \times \nu^*_S = \Sigma \nu_T(n\gamma_k)$ for sufficiently large $N$, and since $J(n\gamma_k) = n + nm_\zeta$, we find that $W^k \times \nu^*_S = \Sigma \nu_T(n+n + nm_\zeta) = \nu^{n+nm_\zeta(p-1)+k}$ by Definition 2.5 and Theorems 2.4 and 2.3.(iii). This proves (i).

By Theorem 2.4, $\nu^{2n+k} \equiv \nu^{2n+1} \pmod{2n+k}$. By [15], $\nu = \nu_T(n\zeta)$ is coreducible if and only if $\tilde{J}(n\zeta) = 0$, so the first half of (ii) follows by Theorem 2.3.(iii). For the second part of (ii) we need only note that the Bockstein is nonzero on $H^{2n+1}$ if $k > 1$.

To prove (iii), note that $\tilde{J}(n\zeta) = \tilde{J}(r\zeta)$ if $n \equiv r \pmod{(p^\psi(k))}$ by Theorem 2.3.(iii).

To prove (iv), first consider $\nu^{2n+k}$ with $k$ odd. By Theorem 2.4,

$$\nu^{2n+k} \equiv \nu^{2n+1} \pmod{2n+k}.$$ 

Since $k$ is odd, $L$ is a closed manifold. By considering the fibration $S^1 + L^k \rightarrow CL^{[k/2]}$, we see that the tangent bundle of $L$ is $([k/2] + 1)\zeta - 1$. Atiyah’s duality theorem [15, Theorem 3.3] implies that the $(-1)$ dual of $\nu^{2n+k}$ is $\nu^{2n+1} \equiv (n+([k/2]+1)\zeta) = \nu^{2n+1} \equiv \nu^{2n+1} \pmod{2n+k-1}$. To prove (iv) for the other three possible combinations of odd or even top and bottom cells, we use the duality between inclusion of the bottom cell of a complex and projection onto the top cell of its dual.
Finally, (v) follows from (ii) and (iv) by the duality between reductions and coreductions. //

Now we present the analogs of 2.1 through 2.6 for $D_P S^n$ instead of $D S^n$. Since the transfer splits $D_P S^n$ off as a wedge summand of $D S^n$, we can use this as a short-out to the results we need. Let $X(p)$ denote the $p$-localization of a spectrum or space $X$. The following result is proved in [7].

**Proposition 2.7.** There is a CW spectrum $L$ with one cell in each nonnegative dimension congruent to 0 or -1 modulo $2(p-1)$, such that $L = (E^m B^+ P)_p$.

**Definition 2.8.** Let $L^k$ be the $k$-skeleton of $L$ and let $L^k_n = L^k_n / L^k_{n-1}$ if $n > 0$. If $n < 0$, $c = 0$ or $1$, and $k \geq c$, let $L^k_{2n+c} = L^2(n-r)_{2r+k}$ for $r \geq 0$.

Note that $n$ and $k$ are not uniquely determined by $L^k_n$ as they are by $L^k_n$. For example, $L^2_1 = L^2_2 = \cdots = L^q_{q-1}$, where $q = 2(p-1)$, since $L$ has no cells in dimensions $1, 2, \ldots, q-2$.

**Theorem 2.9.** Let $S^n$ be the $p$-local $n$-sphere spectrum and let $q = 2(p-1)$. Then

(i) $D_P S^j = \Sigma^j L^\omega_{jq}$ and $D_P S^j-1 = \Sigma^j-1_{jq-1}$. The maps $D_P S^n \to D_P S^n$ and $L^k_n \to L^k_n$ induced by the inclusion $\nu \subset \Sigma_P$ are projections onto wedge summands.

(ii) $L^j_{iq+k}$ is coreducible iff $j \equiv 0 (p^+ k)$, while $L^j_{jq-1}$ is coreducible iff $k = -1$.

(iii) If $c = 0$ or $1$ and $i \equiv j (p^+ k+2c)$ then

$L^j_{jq+k} = \Sigma^{(j-1)c}_{iq+k} L^j_{iq+c}$.

(iv) If $c$ and $\delta$ are 0 or 1 then $L^j_{jq-\delta}$ is $(-1)$ dual to $L^{-}\delta_{jq+c}$.

(iv) If $c = 0$ or $1$ then $L^j_{jq+c}$ has a reducible $jq+c$ cell iff either $k = c = 0$ or $k = iq-1$ and $i+j = 0 (p^+ c-1)$.

Note: Part (i) shows that bottom dimensions of the form $jq-c$, $c = 0$ or $1$, are more natural in this context than $jq+c$. This accounts for the exponent $p^+ (k+2c)$ in (iii), where $p^+ k$ might be expected.
Proof. By the remark preceding the theorem, the first statement in (i) can be abbreviated to $D_pS^n = L_{n(p-1)}^n$. The transfer $(\pi^*\mathbb{B}_p(p)) + \pi^*\mathbb{B}_p$ splits off $L^\infty$ and $L^\infty_{n(p-1)}$ as wedge summands of $L^\infty$ and $L^\infty_{n(p-1)}$ respectively. Similarly, the transfer splits off $D_pS^n$ as a wedge summand of $D_nS^n$. The maps

$$D_pS^n \xrightarrow{t_1} D_nS^n = L^\infty_{n(p-1)} \xrightarrow{i_1} L^\infty_{n(p-1)}$$

and

$$L^\infty_{n(p-1)} \xrightarrow{t_2} L^\infty_{n(p-1)} = D_pS^n \xrightarrow{i_2} D_pS^n,$$

where $t_1$ and $t_2$ are transfers, and $i_1$ and $i_2$ are induced by the inclusion $\pi \subset \mathbb{E}_p$ are inverse equivalences because their composites induce isomorphisms in mod p homology. This proves (i). Now (ii)-(v) follows from 2.6 and (i). //

The preceding theorem does not assert that $W^k \times_{\mathbb{E}_p} S^{n(p)} = L^\infty_{n(p-1)+k}$ where $W^k$ is the k-skeleton of a contractible free $\mathbb{E}_p$ space, because this is not true. In general, $W^k \times_{\mathbb{E}_p} S^{n(p)}$ will have homology in dimension $np+k$ which goes to 0 in $D_pS^n$ and in $L^\infty_{n(p-1)+k}$. Since we are only interested in homology which is nonzero in $D_pS^n$, $L^\infty_{n(p-1)+k}$ is more useful to us than is $W^k \times_{\mathbb{E}_p} S^{n(p)}$.

Therefore we will let $D_p^kS^n = L^\infty_{n(p-1)+k}$, rather than $W^k \times_{\mathbb{E}_p} S^{n(p)}$.

The preceding theorem also shows that we may ignore the distinction between $L^\infty_{n(p-1)+k}$ without harm. We used $L^\infty_{n(p-1)+k}$ and $D_pS^n$ as a stepping stone to information about $D_pS^n$ because $J$ theory only gives information about coreducibility of Thom complexes, and we need Atiyah's S-duality theorem to convert this to information about reducibility. The S-duality theorem of Atiyah only applies to Thom complexes of bundles over manifolds so cannot be used on bundles over the skeleta of $B_p$, or over the even skeleta of $B_\pi$. Conveniently, the odd skeleta of $B_\pi$ are manifolds (if we use a lens space for $B_\pi$). To obtain analogous information about $D_rS^n$ for nonprime $r$, a similar technique works. First, we split $D_pS^n$ off of $D_rS^n$ using the transfer, where $r \subset \mathbb{E}_r$ is a $p$-Sylow subgroup. Then the structure of $r$ (a Cartesian product of iterated wreath products of $r$) suggests manifolds mapping to $B_r$ which we can use just as the odd skeleta of $B_\pi$ are used here.

We now turn to the analogs of 2.1 through 2.6 for $p = 2$. 
Definition 2.10. Let \( n \geq 0 \), let \( \pi = \varepsilon_2 \) act antipodally on \( S^n \) and let
\[
\mathbb{P}^n = S^n/\pi
\]
and
\[
\mathbb{P}^{n+k}_n = \mathbb{P}^{n+k}/\mathbb{P}^{n-1}_n.
\]
We call \( \mathbb{P}^{n+k}_n \) a stunted projective space. Let \( \xi \) in \( KO(\mathbb{P}^n) \) be the canonical real \( n \)-line bundle and let \( \lambda = \xi - 1 \in KO(\mathbb{P}^n) \).

Remarks. (1) If \( p = 2 \) we will agree to let \( \mathbb{P}^n \) and \( \mathbb{P}^n \) mean \( \mathbb{P}^n \) and let \( \mathbb{P}^{n+k}_n \) and \( \mathbb{P}^{n+k}_n \) mean \( \mathbb{P}^{n+k} \) so that uniform statements of results for all primes can be given. The \( \mathbb{P}^n \) and \( \mathbb{P}^{n+k}_n \) notation will still appear frequently because many of the results are not the same for even and odd primes.

(2) It is easy to see that \( \mathbb{P}^n - \mathbb{P}^{n-1} \) is an open \( n \)-cell so that \( \mathbb{P}^{n+k}_n \) has one cell in each dimension between \( n \) and \( n+k \) inclusive. Since \( \mathbb{P}^n = S^n/\mathbb{Z}_2 \) is a \( K(\mathbb{Z}_2,1) \), \( H^*(\mathbb{P}^n;\mathbb{Z}_2) = P(x) \) with \( |x| = 1 \) and
\[
S^j x^j = (x^j_{j=0}) x^{i+j}.
\]
The isomorphisms
\[
H^i \mathbb{P}^{n+k}_n \cong H^i \mathbb{P}^{n+k}_n \oplus H^i \mathbb{P}^n
\]
for \( n \leq i \leq n+k \) thus determine \( H^* \mathbb{P}^{n+k}_n \) as an \( \mathcal{A}_2 \) module.

Theorem 2.11. Let \( \phi(n) \) be the number of integers \( j \) congruent to 0, 1, 2, or 4 mod 8 such that \( 0 < j \leq n \). Then \( KO(\mathbb{P}^n) = \langle \lambda \rangle \) and has order \( 2^{\phi(n)} \). Furthermore,
\[
J:KO(\mathbb{P}^n) \to J(\mathbb{P}^n)
\]
is an isomorphism.

Proof. \( KO(\mathbb{P}^n) \) is computed in [1]. The computations there and the Adams conjecture imply the last statement. //

Theorem 2.12. If \( s \geq 0 \) the Thom complex of \( r+s \xi \) over \( \mathbb{P}^n \) satisfies
\[
\Sigma^r \mathbb{T}(r + s \xi) = \Sigma^r \mathbb{P}^{n+s}.
\]

Proof. The proof of Proposition 2.4 can easily be adapted to prove this as well.

As for odd primes, we can now define stunted projective spectra starting and ending in any positive or negative dimensions.
Definition 2.13. For \( k > 0 \) and any \( n \) let
\[
\bar{p}_n^{n+k} = \sum_{r=n}^{r=n+k} \bar{p}_r
\]
for any \( r \equiv n \pmod{2\phi(k)} \), \( r \geq 0 \).

The following result shows that \( \bar{p}_n^{n+k} \) is well defined up to equivalence in \( h\mathcal{A} \). Let \( S^k \) have the antipodal action of \( \pi \). We define \( D^k_\pi X = S^k\kappa_\pi X^{(2)} \).

Theorem 2.14. Let \( S^N \) be the 2-local \( n \)-sphere spectrum. Then

(i) \( D^k_\pi S^n = \Sigma^n \bar{p}_n^{n+k} \)
(ii) \( \bar{p}_n^{n+k} \) is coreducible if and only if \( n \equiv 0 \pmod{2\phi(k)} \)
(iii) If \( n \equiv m \pmod{2\phi(k)} \) then \( \bar{p}_n^{n+k} = \Sigma^n \bar{p}_m^{n+k} \)
(iv) \( \bar{p}_a^b \) and \( \bar{p}_{a-1}^b \) are \((-1)\) dual spectra
(v) \( \bar{p}_n^{n+k} \) is reducible if and only if \( n+k+1 \equiv 0 \pmod{2\phi(k)} \).

Proof (i) follows for \( n \geq 0 \) from Theorem 2.12 once we observe that the regular representation \( \gamma_k \) is \( 1 + \xi \). For \( n < 0 \) we have
\[
D^k_\pi S^n = D^k_\pi (\Sigma^n S) = \Sigma^{-N} E T(N + n\gamma_k)
\]
by VI.5.3 and VI.5.4 of [Equiv], for sufficiently large \( N \). Hence \( D^k_\pi S^n = \Sigma^n \bar{p}_n^{n+k} \) for \( n < 0 \) also, again by 2.12.

Parts (ii) through (v) follow exactly as in 2.6. In (iv) we use the fact that \( P^N \) is a closed manifold with tangent bundle \( (n+1)\xi - 1 \). //

The last results in this section identify the top dimensional component of any attaching map of \( D_pS^n \) by combining Theorems 2.6 and 2.14 with Milnor's result on Thom complexes of sphere bundles over suspensions. First we must define the maps under consideration. As in §1, \( q = 2(p-l) \) and \( \epsilon = 0 \) or 1 (\( q = 1 \) and \( \epsilon = 0 \) if \( p = 2 \)).

Definition 2.15. Define a function \( v_p \) by
\[
v_p(n) = \max\{v\mid h_{n-v}^n \text{ is reducible}\}.
\]
Let \( v = v_p(n) \) and define \( a_p(n) : \pi_{v-1}S \) to be \( \Sigma^{v-n} \) of the composite
\[
S^{n-1} \longrightarrow \Sigma^{v-n} \longrightarrow S^{n-v}
\]
in which the first map is a lift of the attaching map of the \( n \) cell and the second is projection onto the top cell of \( I^{n-v} \).
The indeterminacy in the definition of \( \alpha_p(n) \) is the kernel of the homomorphism induced on \( \pi_{n-1} \) by the inclusion of the bottom cell of \( L_{n-v} \).

We will often omit the subscript \( p \) for typographical simplicity. The notations \( v \) and \( a \) are intended to be mnemonic: \( v \) stands for "vector field number" and \( a \) stands for "attaching map." Actually, \( v \) is not quite the vector field number as defined by Adams [1]; \( v_2(n) \) is \( p(n-1) \) in Adams' notation. The function \( v_p \) tells us how far we can compress each of the attaching maps of \( L^n \). The attaching map of the \( n \) cell factors through \( L_{n-v} \) if and only if \( L_{n-v+1} \) is reducible. Thus, it factors through \( L_{n-v} \) but not through \( L_{n-v-1} \), where \( v = v_p(n) \). By the definition of \( v_p(n) \), \( \alpha_p(n) \) is nonzero. We obtain a good hold on \( v_p \) and \( \alpha_p \) from the following two lemmas. Let \( \varepsilon_p(j) \) be the exponent of \( p \) in the prime factorization of \( j \).

**Proposition 2.16.** If \( p > 2 \) then, with \( q = 2(p-1) \),

\[
v_p(jq - \varepsilon) = \begin{cases} 1 & \varepsilon = 0 \\ q(1 + \varepsilon_p(j)) & \varepsilon = 1 \end{cases}
\]

If \( p = 2 \) then \( v_2(j) = 2a + 2b \), where \( \varepsilon_2(j+1) = 4a + b \) and \( 0 \leq b \leq 3 \).

**Proposition 2.17.** If \( v_p(n) = 1 \) then \( \alpha_p(n) \) is the map of degree \( p \). If \( v_p(n) > 1 \) then \( \alpha_p(n) \oplus 1 \) generates \( \text{Im} J \oplus \mathbb{Z}(p) \) in dimension \( v_p(n) - 1 \).

**Proof of 2.16.** Theorem 2.14.(v) shows that \( v_2(j) \) is the maximum \( s \) such that \( \varepsilon_2(j+1) = \psi(s-1) \). The formula for \( v_2(j) \) follows easily from this. Theorem 2.9.(v) shows that if \( p > 2 \) then \( v_p(jq) = 1 \) while \( v_p(jq-1) \) is the maximum \( s \) such that \( \varepsilon_p(jq) = \psi(s-1) \). The formula for \( v_p(jq-\varepsilon) \) follows immediately. //

**Proof of 2.17.** Let \( n = jq - \varepsilon, v = v_p(n) \) and \( a = \alpha_p(n) \). We wish to construct a map of cofiber sequences

\[
\begin{array}{c}
S^{n-1} \to L_{n-v}^{n-1} \to L_{n-v} \to S^n \\
S^{n-1} \to a S^{n-v} \to Ca \to S^n
\end{array}
\]

where \( Ca = S^{n-v} \cup e^N \), \( b \) is the inclusion of the bottom cell, and \( a \oplus 1 \) generates \( \text{Im} J \oplus \mathbb{Z}(p) \). By S-duality and Theorems 2.9.(iv) and 2.14.(iv), it is equivalent to construct a map of cofiber sequences

\[
\begin{array}{c}
S^n \leftarrow L_{n-v} \leftarrow L_{n-v}^{n-1} \to S^{n-1} \\
S^n \leftarrow \text{Ca} S^{n-1} \leftarrow \text{Ca} \to S^{n-1}
\end{array}
\]
in which \( b^* \) is the collapse onto the top cell and \( a \) is as before. The lemma is trivial when \( v = 1 \) so we may assume \( v > 1 \) and hence, that \( n \) is odd. Let \( \gamma \) be the bundle \(-(n+1)\xi \) if \( p = 2 \) and \(-J(p-1)\xi \) if \( p > 2 \) over \( L^V \). Then \( L^V_{n-1} = T(\gamma) \). By the definition of \( v \), \( \gamma \) is trivial over \( L^V_{n-1} \) but not over \( L^V \). This implies \( \gamma = \pi^* \nu \) where \( \pi : L^V \to L^V_{n-1} = S^V \) is the collapsing map and \( 0 \neq \nu \in KO(S^V) \). By [85], \( T(\nu) \) has attaching map \( J(\nu) \). Thus, the inclusions of the fiber \( S^{n-1} \) into \( T(\nu) \) and \( T(\nu) \) induce a map (*) of cofiber sequences with \( a = J(\nu) \). Since \( v \) is greater than 1, it is even when \( p > 2 \) by 3.2. Thus, 2.3.(iii) and 2.9.(i) when \( p > 2 \), and 2.11 when \( p = 2 \), imply that the kernel of \( J(L^V) \to J(L^V_{n-1}) \) is \( \mathbb{Z}_p \). Hence \( J(\gamma) \) generates it, being nonzero. Since \( \pi^* (a) = J(\gamma) \), \( a \in J(S^V) \) must generate \( J(S^V) \otimes \mathbb{Z}_p \).

In the notation of 1.6, Propositions 2.16 and 2.17 are summarized by the equations

\[
\begin{align*}
a_2(j) & = \beta_{\epsilon_2}(j+1) \\
a_p(jq) & = p \\
\text{and} \quad a_p(jq-1) & = a_1e_p(j)
\end{align*}
\]

where \( \cong \) denotes equality up to multiplication by a unit of \( \mathbb{Z}_p \).

§3. Proofs for section 1 and other calculations

This section primarily consists of proofs of results of §1 with the additional necessary results (3.1-3.4) interspersed. Note, however, that the spectral sequence charts in Figures 3.1 to 3.9 can be very useful in conjunction with Theorem 1.10 since they show where in the Adams spectral sequence the elements detecting the results of homotopy operations must lie.

Proof of 1.1. \( \beta^*(x) = x^p \) by IV.7.3.(iii). Clearly, \( E^0(i) = e_0 \otimes 1^p_n = e_0 \), so the second statement is immediate from the definition:

\[
\beta^p_j^i = (-1)^n_\epsilon n e_{j-n}^i (p-1).
\]

Proof of 1.2. Recall from III §1 that the homology operations are defined by

\[
Q^j_\epsilon x = \xi^*(e_{j-n} \otimes x^2) \quad \text{if } p = 2,
\]

and

\[
\beta Q^j_\epsilon x = \xi^*((-1)^n_\epsilon n e_{j-n}^i (p-1) \otimes x^p) \quad \text{if } p > 2.
\]

To prove 1.2 we simply calculate. If \( p = 2 \) and \( E^0(a) = p^i \) then
The proof is essentially the same when \( p > 2 \).

Proof of 1.3. This is just the naturality of the spectral sequence \( E_r(S, \mathcal{G}) \).

Proof of 1.4. Consider the following commutative diagram, in which the row is the cofiber sequence of \( c_1 \) and \( a' \) is a lift of \( a \) to \( D^i P \).

Clearly \( a^*(x) = \xi D_p(x) a = \xi D^i_p(x) a' \) and this lies in the Toda bracket \( \langle a, c_1, \xi D^i_p(x) \rangle \). If \( a \) and \( \beta \) both lift to \( D^i P \) and project to \( \tilde{a} \) on \( S^{np+i} \), then \( a - \beta \) lifts to \( D^{i-1} P \) so that \( a^*(x) - \beta^*(x) \) is in \( \xi D^{i-1}_p(x) \circ \pi D^{i-1} P \).

Conversely, if \( \gamma \) lifts \( D^{i-1} P \) then \( a + \gamma \) also lifts to \( D^i P \) and projects to \( a' \) on \( S^{np+i} \).

Proof of 1.5. By definition, \( \beta \cdot D_1 \) is defined on \( \pi_1 \) if and only if \( e_1 \) is a permanent cycle in \( E_r(S, D_1) \). Thus (i) and (ii) are equivalent. Let \( D_1 \) be \( D \) truncated at the \( np+i \) cell. The map of spectral sequences \( E_r(S, D_1) \rightarrow E_r(S, S^{np+i}) \) induced by the projection \( D^i S \rightarrow S^{np+i} \) sends \( e_1 \) to the identity map of \( S^{np+i} \). If \( D^i S \) is reducible then there is a map back which splits \( E_r(S, S^{np+i}) \) off \( E_r(S, D_1) \), forcing \( e_1 \) to be a permanent cycle. Conversely, if \( e_1 \) is a permanent cycle then any map detecting it will be a reduction. Thus (ii) and (iii) are equivalent. Finally, (iii) and (iv) are equivalent by Theorems 2.6.(v), 2.9.(v) and 2.14.(v).
Proof of 1.7. To show $\beta_j \circ x^2 = 0$, where $\beta_j \in \pi_{v-1}S$, we need only show that $p_{n+1}^{n+v}$ is reducible and $p_n^{n+v}$ is not, since this implies that the $n+v$ cell is attached only to the $n$ cell of $p_n^{n+v}$, and Proposition 2.17 implies that the attaching map is a generator of $\text{Im} J$ in $\pi_{v-1}S$. If $j = 4a + b$ then $v = 8a + 2b$, so 2.14.(v) implies that $n$ must satisfy

$$n + 8a + 2b \equiv -1 \ (2j)$$

and

$$n + 8a + 2b \not\equiv -1 \ (2j+1).$$

To show $\beta_j \circ p_{n+1}^{n+1}x$ is a multiple of $x^2$, we must show that $p_{n+1}^{n+v+1}$ is not reducible, but $p_{n+2}^{n+v+1}$ is reducible, for then the top cell will be attached to the cells carrying $x^2$ and $p_{n+1}^{n+1}x$. The rest of the proof is the same as in the first case. //

Proof of 1.8. To show that $\alpha_j \circ \beta^{k} x = 0$, for $x \in \pi_n Y$ and $n = 2k-1$, is trivial when $j = 0$. Simply note that $L_{kq-1}^{kq}$ is a mod $p$ Moore spectrum. When $j > 0$ we must show $L_{kq}^{(k+j)q-1}$ is reducible, while $L_{kq-1}^{(k+j)q-1}$ is not. By 2.9.(v) we need $k+j \equiv 0 \ (p^{j-1})$ but $k+j \not\equiv 0 \ (p^j)$.

When $n = 2k$, the relation $\alpha_j \circ \beta^{k+1} x = a \circ x^p$ for some $a$ is also trivial when $j = 0$. We need only note that $L_{kq+q}^{kq}$ is a mod $p$ Moore spectrum. For $j > 0$, we must show that $L_{kq+q-1}^{(k+j)q+q-1}$ is reducible, but $L_{kq+q-1}^{(k+j)q+q-1}$ is not. By 2.9.(v) we must have $k+j+1 \equiv 0 \ (p^{j-1})$ but $k+j+1 \not\equiv 0 \ (p^j)$. //

When $n = 2k$, if we try to show $\alpha_j \circ x^p = 0$ by this technique we find we must assume $k+j \equiv 0 \ (p^{j-1})$ and $k+j \not\equiv 0 \ (p^{j-1})$, so that no information is available.

Before we compute the first few homotopy groups of $D_p S^n$ (and hence the first few homotopy operations), we describe the attaching maps of the first few cells. Exact definitions of the maps used in the following proposition can be found in the proof.

Proposition 3.1. Let $p = 2$.

(1) If $n \equiv 1 \ (4)$ then $p_n^{n+3} = S_n \cup e_n^{n+1} \cup S_n^{n+2} \cup e_n^{n+3}$

(2) If $n \equiv 2 \ (4)$ then $p_n^{n+3} = S_n \vee S_n^{n+1} \cup e_n^{n+2} \cup e_n^{n+3}$

(3) If $n \equiv 3 \ (4)$ then $p_n^{n+3} = S_n \cup e_n^{n+1} \cup e_n^{n+2} \cup S_n^{n+3}$

(4) If $n \equiv 0 \ (4)$ then $p_n^{n+3} = S_n \vee S_n^{n+1} \cup e_n^{n+2} \vee S_n^{n+3}$.
Proof. Much of the structure of $p_n^{n+3}$ is determined by $Sq^1$ and $Sq^2$ in $H^n p_n^{n+3}$. We will assume this information and fill in the rest. Suppose $n \equiv 0 \pmod{4}$. Then 2.14 implies $p_n^{n+3}$ is both reducible and coreducible, so only the middle two cells are attached. When $n \equiv 1 \pmod{4}$, collapsing the bottom cell of the previous case yields $p_n^{n+2} \simeq S^n \cup e^1 \cup S_{n+2}$. Computing $Sq^1$ and $Sq^2$ shows $e^{n+3}$ is attached to $S_{n+2}$ by a map of degree 2, and is attached to the Moore spectrum by a map which projects to $n$ on $S_{n+1}$. This projection induces an epimorphism

$$Z_4 = \pi_n^{n+2}(S^n \cup e^{n+1}) \longrightarrow \pi_{n+2} S_{n+1} = Z_2.$$ 

Therefore, the attaching map is a generator $n$ of $\pi_{n+2}(S^n \cup e^{n+1})$.

When $n \equiv 2 \pmod{4}$, we start with $p_n^{n+2} = S^n \cup e^{n+1} \cup e^{n+2}$. The long exact homotopy sequence of $S^n \cup S_{n+1} \cup p_n^{n+2}$ shows that the inclusion $S_{n+1} + p_n^{n+2}$ induces an isomorphism on $\pi_{n+2}$. Since $Sq^2$ is nonzero on $H^{n+1} p_n^{n+3}$, the $n+3$ cell is attached by the map

$$S^{n+2} \longrightarrow S^{n+1} \longrightarrow p_n^{n+2},$$

which we also call $n$.

Finally, when $n \equiv 3 \pmod{4}$, we start with $p_n^{n+2} = S^n \cup e^{n+1} \cup e^{n+2}$. The map $p_n^{n+2} \simeq S^n \cup e^{n+2}$ which collapses the bottom cell, induces on $\pi_{n+2}$ a monomorphism

$$\pi_{n+2} p_n^{n+2} = Z_2 \oplus Z \longrightarrow Z_2 \oplus Z = \pi_{n+2} S_{n+1} \oplus \pi_{n+2} S_{n+2}$$

which sends $(a,b)$ to $(a,2b)$. Computing $Sq^1$ and $Sq^2$ shows that the attaching map of the $n+3$ cell is $(0,1) \in \pi_{n+2} p_n^{n+2}$, which projects to the map of degree 2 on $S_{n+2}$. We simply call this map $2$. \]

Proposition 3.2. Let $p > 2$.

1. $L_{jq+q}^j = S_{jq}^{jq} \cup e^{jq+q-1} \bigcup_{a_1} e^{jq+q} - (j+2)a_1$.

2. $L_{jq}^{jq+q-l} = S_{jq-l}^{jq} \bigcup_{a_1} e^{jq+q-1} - (j+1)a_1$.

Proof. Recall that the first three nonzero homotopy groups of $S$ localized at $p$ are $\pi_0 = Z$, $\pi_{-1} = Z_p$ generated by $a_1$, and $\pi_{-2} = Z_p$ generated by $a_2$. Thus $L_{jq}^{jq+q-l} = S_{jq}^{jq} \cup e^{jq+q-1}$ is the only possibility. Computing $s$ and $p^1$ in $H_{L_{jq}^{jq+q}}$ shows that $L_{jq}^{jq+q} = S_{jq}^{jq} \cup e^{jq+q-1} - (j+1)a_1$. Finally, the long exact homotopy sequence of $S_{jq}^{jq} \cup e^{jq+q-1} + L_{jq}^{jq+q}$ shows that the inclusion of $S_{jq}^{jq+q}$ induces an
isomorphism of $\pi_{jq+2q-2}$. Thus the attaching map of the $jq+2q-1$ cell factors through $S^{jq+q-1}$ and is determined to be $-(j+2)a_1$ by computing $p^1$.

Collapsing the bottom cell and redefining $j$ we find that $L_{jq-1}^{jq+q-1} = S^{jq-1} \cup_p e^{jq} \bigcup_{(j+1)a_1} e^{jq+q-1}$. The long exact homotopy sequence of $S^{jq-1} \to L_{jq-1}^{jq+q-1}$ shows that the attaching map of the $jq+q$ cell is determined by its projections onto $S^{jq}$ and $S^{jq+q-1}$. Computing $P^1$ and $\beta$ shows these to be $-Ja_1$ and $p$ respectively. 

Diagrams of the cohomology with $Sq^1$ and $Sq^2$ or $\beta$ and $P^1$ indicated are convenient mnemonic devices. For $p = 2$ we have

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\end{align*}

For $p > 2$, we have

\begin{align*}
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\end{align*}

We can also think of these diagrams as indicating cells by dots and attaching maps by lines, and this is how we have labelled the diagrams for $p > 2$.

The spectral sequence $E_\ast(S, \mathcal{P})$ will enable us to glean a maximal amount of information from Propositions 3.1 and 3.2. We begin with $p = 2$. Recall, from [66], the initial segment of the $HZ_2$ Adams spectral sequence for $\pi_\ast S$. 
Vertical lines represent multiplication by $h_0$, detecting the map of degree 2, and diagonals represent multiplication by $h_1$, detecting $n$. We shall only use the first 8 stems ($t-s \leq 8$). Let $p$ be the sequence

$$p_{n+8} \leftarrow p_{n+7} \leftarrow \ldots \leftarrow p_{n+1} \leftarrow p_{n}.$$

(Omitting the $L^n$ from $D_{2}^{S} = L^{n}p_{n+1}$ means a class in $E_{*}(S, \mathcal{D})$ will have stem degree equal to the amount by which the corresponding homotopy operation raises degrees.)

Proposition V.7.5 says that $E_{2}(S, \mathcal{D})$ is free over $E_{2}(S, S)$ on generators in each degree from $n$ to $n+k$. Write $x(i)$ for the element of $E_{2}(S, \mathcal{D})$ which is $x \in E_{2}(S, S)$ in the $i$ summand, if $i \geq n$. Let $x(i)$ mean 0 if $i < n$.

**Theorem 3.3.** In $E_{t}(S, \mathcal{D})$, for $t-s \leq 6$,

$$d_{2}x(i) = h_{0}x(i-1) \quad \text{if } i \equiv 0 \quad (2),$$

$$d_{3}x(i) = h_{1}x(i-2) \quad \text{if } i \equiv 1 \quad (4),$$

and

$$d_{2}x(i) = h_{2}x(i-4) \quad \text{if } i \equiv 0,1,2,3 \quad (8).$$

In the same range, $E_{n}(S, \mathcal{D})$ is given by Figures 3.1 through 3.4.

**Note:** Dotted vertical lines indicate "hidden extensions". That is, they represent multiplications by 2 which cause an increase of more than 1 in filtration. Similarly, dotted diagonals indicate the effect of multiplication by $n$ when this causes an increase in filtration of more than 1. See the proof of 1.9 for their derivation.
Figure 3.1

*) hit by $d_5(l(n+7))$ iff $n \equiv 4 \pmod{8}$

**) 2 times $l(n+1)$ is $h_1(n+2)$ if $n \equiv 0 \pmod{8}$

and it is $"h_1(n+2) + h_2(n)"$ if $n \equiv 4 \pmod{8}$

***) if $n \equiv 4 \pmod{8}$

Figure 3.2

*) differential iff $n \equiv 5 \pmod{8}$

**) if $n \equiv 5 \pmod{8}$
Proof of 3.3: The differentials listed correspond to attaching maps which can be detected by $\text{Sq}^1$, $\text{Sq}^2$ and $\text{Sq}^4$, and they hold in the spectral sequences for $\mathcal{S}'$, $\mathcal{S}''$ and $\mathcal{S}'''$ below.

\begin{align*}
\mathcal{S}' & \quad S^{i-1} \text{ or } e^1 \quad \text{ or } S^{i-1} \quad \ldots \\
\mathcal{S}'' & \quad S^{i-2} \text{ or } e^1 \quad S^{i-2} \quad S^{i-2} \quad \ldots \\
\mathcal{S}''' & \quad S^{i-4} \text{ or } e^1 \quad S^{i-4} \quad S^{i-4} \quad \ldots
\end{align*}
The differential \( d_2x(i) = h_0x(i-1) \) if \( i \equiv 0 \pmod{2} \) is immediate, since \( l(i) \notin E_2 \) and by dimensional considerations \( d_2l(i) = h_0(i-1) \) is the only possible \( d_2 \) on \( l(i) \). The module structure over \( E_2(S,S) \) now gives \( d_2x(i) = h_0x(i-1) \).

The \( d_2 \) differential is slightly more complicated. There are two cases. If \( i \equiv 1 \pmod{2} \) then the \( i \) cell is not attached to the \( i-1 \) cell, but is attached to the \( i-2 \) cell by \( l_1 \); \( d_2l_1(i) = l_1(i-2) \) follows as for \( d_2 \), and this implies \( d_2x(i) = l_1x(i-2) \). If \( i \equiv 0 \pmod{2} \) then \( l(i) \notin E_2 \) since \( d_2l(i) = h_0(i-1) \). However, the map of spectral sequences induced by \( \xi + \beta'' \)

\[
\begin{array}{ccccccccc}
S^{i-2} & S^{i-1} & \cup & \eta + e^1 & S^{i-2} & S^{i-1} & S^{i-2} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
S^{i-2} & \cup & \eta + e^1 & S^{i-2} & S^{i-2} & S^{i-1}
\end{array}
\]

shows that elements of \( E_3(S,S') \) must satisfy \( d_3x(i) = l_1x(i-2) + k \) where \( k \) is the kernel of \( E_3(S,S') + E_3(S,S'' \eta) \), that is, \( k \) must have the form \( y(i-1) \). By inspection \( k \) must be 0 in the dimensions considered. Now, by truncating \( \beta \) at the \( i \) cell, then collapsing the \( i-3 \) skeleton we can compare \( E_3(S,\beta) \) to \( E_3(S,S') \). Again we have \( d_3x(i) = l_1x(i-2) + k \), where \( k \) is now a sum of elements coming from the \( i-3 \) cell or below. The first possibility is when \( n \equiv 0 \pmod{4} \). We must decide between \( d_3h_1(n+4) = l_1^2(n+2) \) and \( d_3h_1(n+4) = l_1^2(n+2) + h_2(n+1) \). Let \( p^n, p^{n+1}, h_1p^{n+2} \), and \( p^{n+2} \) denote elements detected by \( l(n), l(n+1), h_1(n+2) \), and \( l(n+3) \), respectively. Comparing with Mahowald's calculations [59], we find that \( 2 \circ p^{n+2} = h_1p^{n+2} \) or \( h_1p^{n+2} + v \circ p^n \), depending on \( n \) mod 8. Composing with \( \eta \) yields \( \eta \circ h_1p^{n+2} = 0 \). But if \( d_3h_1(n+4) \) were \( h_1^2(n+2) + h_2(n+1) \) we would have \( \eta \circ h_1p^{n+2} = \eta \circ p^{n+1} \). Therefore we must have \( d_3h_1(n+4) = h_1^2(n+2) \). The same argument, with minor variations, finishes all the \( d_3 \) differentials.

Finally, the \( d_5 \) differentials follow by similar comparisons with \( E_5(S, \beta'' \eta) \). In all but one case, there is nothing in filtrations less than or equal to the filtration of \( h_2x(i-4) \) so the comparison with \( E_5(S, \beta'' \eta) \) is sufficient. The one remaining case is when \( n \equiv 1 \pmod{4} \). Here \( h_1^3(n+3) \) lies between \( h_2(n+4) \) and \( h_2^2(n) \). Since the \( n+4 \) cell is not attached to the \( n+3 \) cell, the \( d_5h_2(n+4) = h_2^2(n) \) is right here also.

There are no further possible differentials by inspection. The hidden extensions here are all evident from Mahowald's computation in [59] of the Adams spectral sequence of \( P_n^\beta \). //

Note. The spectral sequence \( E_r(S, \beta) \) has far more hidden extensions than \( E_r(S,P_n^\beta) \) since the cells are spread apart in \( E_r(S, \beta) \) whereas they all occur in the same filtration in \( E_r(S,P_n^\beta) \). By IV.7.6, the same hidden extensions occur among the elements generated by the \( \beta^x \) for a fixed \( x \).
Proof of 1.9 when $p = 2$: A permanent cycle $x(i)$ corresponds to an operation $x^p$. Thus, Table 1.3 is simply a list of the elements of $E_\infty(S, \mathfrak{S})$, omitting most of those which are multiples by elements of $\pi_* \mathfrak{S}$ of other elements of $E_\infty(S, \mathfrak{S})$. The indeterminacy of an operation consists of those elements in the same stem and higher filtration, so it too can be read off Figures 3.1 through 3.4. With the exception of $\tau_* (p^n)$ and $\tau_* (p^{n+1})$, the values of $\tau_*$ listed are the only elements of $\pi_* \mathfrak{S}$ in the relevant dimension. Since $\pi_{2n} D_2^{2n} = \mathbb{Z}_2$ when $n$ is odd, $\tau_* (p^n) = 0$ in this case.

When $n$ is even, $i : S^{2n} + D_2 S^n$ induces an isomorphism of $\pi_{2n}$. By II.1.10, the composite $i : D_2 S^n + D_2 S^n$ is multiplication by 2 on $H_2 \simeq \pi_{2n}$. Thus $\tau_* (p^n) = 2$.

To calculate $\tau_* (p^{n+1})$, first suppose $n \geq 2$ (4). By Theorem 3.3, $\pi_{2n+2} D_2 S^n = 0$. Therefore, $\eta (p^{n+1}) = 0$ and hence $\eta (p^{n+1}) = 0$. This forces $\tau_* (p^{n+1})$ to be 0, not $\pi_*$. When $n \leq 0$ (4), Theorem 3.3 gives $\pi_{2n+1} D_2 S^n = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with generators $p^{n+1}$ and $\eta (p^n)$. By II.2.8, $\tau_* (p^{n+1})$ is not zero and hence must be $\eta$.

Determining the relations in Table 1.4 amounts to determining the $\pi_* S$ module structure of $\pi_* D_2 S^n$. The indeterminacy of the operations in Table 1.3 induces a similar indeterminacy in the relations of Table 1.4. The relations are to be interpreted as asserting equality modulo the sum of the indeterminacies of the two sides. Thus, in order to prove that they hold, we need only show that they hold for some choice of representatives. The $E_\infty$ terms in Theorem 3.3 force the following thirteen relations:

\[
\begin{align*}
2p^n &= 0 \quad \text{n} \equiv 1, 3 \quad (4) \\
\eta h_1 p^{n+1} &= 0 \quad \text{n} \equiv 1 \quad (4) \\
2v p^{n+2} &= 0 \quad \text{n} \equiv 1 \quad (8) \\
v p^{n+2} &= 0 \quad \text{n} \equiv 5 \quad (8) \\
4v p^n &= 0 \quad \text{n} \equiv 2 \quad (4) \\
\eta p^{n+1} &= 0 \quad \text{n} \equiv 2 \quad (4) \\
v p^{n+1} &= 0 \quad \text{n} \equiv 6 \quad (8) \\
\eta p^n &= 0 \\
2h_1 p^{n+1} &= 0 \quad \text{n} \equiv 3 \quad (4) \\
2h_2 p^{n+2} &= 0 \\
v p^n &= 0 \\
\eta h_1 p^{n+3} &= 0 \quad \text{n} \equiv 7 \quad (8) \\
h_2 p^{n+2} &= 0
\end{align*}
\]

Another eighteen relations follow by considering the attaching maps given in Proposition 3.1, the spectral sequences in Theorem 3.3 and the reducibility and coreducibility given in Theorem 2.14. These are
For example, when \( n = 0 \) (8), the attaching map of the \( n+4 \) cell gives \( 2p_{n+3} = h_1p_{n+2} \). Then \( 2vp_{n+3} = v_h_1p_{n+2} \) must be either 0 or \( v^2p_n \) by the \( E_\infty \) term in Figure 3.1. But \( p_{n+7} \) is coreducible, so \( v^2p_n \) is impossible. Similarly, when \( n = 4 \) (8), the attaching map of the \( n+4 \) cell gives \( 2p_{n+3} = h_1p_{n+2} + v_p_n \). (Note that, since \( p_{n+3} \) is coreducible, \( v_p_n \) need not be considered a part of the indeterminacy of \( 2p_{n+3} \) or \( h_1p_{n+2} \).) Thus \( 2vp_{n+3} = v_h_1p_{n+2} + v^2p_n \). But \( vp_{n+3} \) is either 0 or \( v^2p_n \) by the \( E_\infty \) term in Figure 3.1. Thus \( 2vp_{n+3} = 0 \) and hence \( v_h_1p_{n+2} = -v^2p_n = v^2p_n \).

Four more relations come from the fact that \( s_{n+2}(S^n \cup e^{n+1}) \equiv Z_4 \), so that the composite of 2 and a map which projects to \( n \) on \( S^{n+1} \), lifts to \( n^2 \) on \( S^n \). These are

\[
\begin{align*}
2h_1p_{n+1} &= n^2p_n \\
2h_1p_{n+5} &= h_1^2p_{n+4} \\
2h_1p_{n+4} &= h_1^2p_{n+3} \\
2h_1p_{n+3} &= h_1^2p_{n+2} \\
\end{align*}
\]
The relations

\[ 2h_1^{2n+4} = h_1^{2n+3}, \quad n = 1 \ (4) \]
\[ 2h_1^{3n+3} = 0 \]
\[ 2h_1^{2n+3} = h_1^{3n+2}, \quad n = 2 \ (4) \]
\[ 2h_1^{2n+2} = h_1^{3n+1}, \quad n = 3 \ (4) \]
\[ \pi h_2^{n+2} = \sqrt{2} h_n^{n-2}, \quad n = 3 \ (8) \]

are the only possibilities consistent with Mahowald's calculations [59] (note that these are not hidden extensions in his spectral sequence).

Finally, the relation \( 2h_2^{2n+3} = \sqrt{2} h_n^{n-2} \) when \( n = 6 \ (8) \) follows by comparison with the spectral sequence for the cofiber of the inclusion \( \pi^{n+2} \to \pi^{n+4} \). In the cofiber, \( 2\pi^{n+3} = \sqrt{2} \pi^n \) is obvious from the attaching maps. //

Now consider the odd primary case. Recall, from [55], that, in degrees less than \( pq-2 \), the \( \mathbb{H}_p \) Adams spectral sequence has elements

\[ a_0 \in E_2^{i, j} \quad \text{detecting } p^i, \quad i = 0, 1, 2, \ldots, \]
\[ h_0 \in E_2^{1, q} \quad \text{detecting } a_1 \epsilon \pi_{q-1}, \]

and
\[ g_{i-1} \in E_2^{i, iq+i-1} \quad \text{detecting } a_i \epsilon \pi_{iq-1}, \quad \text{for } 2 \leq i \leq p. \]

Let \( \mathcal{B} \) be the sequence

\[ \ldots \rightarrow I_{n(p-1)+ps} \rightarrow I_{n(p-1)+ps-1} \rightarrow I_{n(p-1)+1} \rightarrow I_{n(p-1)} \]

Since \( I_{n(p-1)} \) has cells only in dimensions \( n(p-1) \) and greater which are congruent to 0 or \(-1\) mod \( q \), \( E_2(S, \mathcal{B}) \) is free over \( E_2(S, S) \) on generators in those degrees.

Write \( x(j, \varepsilon) \) for the element of \( E_2(S, \mathcal{B}) \) which is \( x \in E_2(S, S) \) in the \( jq-\varepsilon \) summand, if \( jq-\varepsilon \geq n(p-1) \). We agree to let \( x(j, \varepsilon) = 0 \) if \( jq-\varepsilon < n(p-1) \).

**Theorem 3.4.** In \( E_2(S, \mathcal{B}) \), \( d_2(x(j, 0)) = a_0 x(j, 1) \) and

\[ d_{2p-1}(x(j, 1)) = -j h_0 x(j-1, 1). \]

In low dimensions \( E_{2p}(S, \mathcal{B}) \) is given by Figures 3.5 through 3.9.

**Notes:** (1) The dotted arrows to the left represent possible \( d_{2p} \) differentials which we have not computed. This is why the theorem only claims to give \( E_{2p}(S, \mathcal{B}) \). The indicated \( d_{2p} \) is the only possible remaining differential in the range listed. This is true for dimensional reasons except when \( n = 2k-1 \) and \( k = -2 \ (p) \). Here the
possibility that \( d_{4p-2}(l(k+2,1)) \) is nonzero is excluded by the fact that \( L_{kq-1}^{(k+2)q-1} \) is reducible when \( k \equiv -2 \) (p) by Theorem 2.9.(v).

(2) Dashed vertical lines represent hidden extensions. Precisely, if \( x \) and \( y \) are detected by \( \bar{x} \) and \( \bar{y} \), the notation

\[
\bar{y} \quad \bar{x}
\]

means that \( px \equiv jy \mod \) modulo higher filtrations. Of course, if \( j \) is 0 this means the extension is trivial. We replace \( j \) by a question mark if we have not settled the extension.

Figure 3.5 \( n = 2k, k \equiv -2 \) (p)

Figure 3.6 \( n = 2k, k \not\equiv -2 \) (p)
Figure 3.7  \( n = 2k-1, \; k = -1 \) (p)

Figure 3.8  \( n = 2k-1, \; k = -2 \) (p)

Figure 3.9  \( n = 2k-1, \; k \neq -1 \) or -2 (p)
Proof of 3.4. The differentials follow from the attaching maps in Proposition 3.2 just as 3.3 follows from 3.1. Applying them gives the values of $E_{2p}(S_5, S)$ listed in Figures 3.5 through 3.9. The indicated hidden extensions all come from the attaching maps of the even cells of $L_n^{m(p-1)}$. //

Proof of 1.9 when $p > 2$: A permanent cycle $x(j,s)$ corresponds to a homotopy operation $x^g S_1$. Thus Table 1.1 is a list of those elements in Figures 3.5 through 3.9 which must be permanent cycles by Theorem 3.4. The indeterminacy is obtained from Figures 3.4 through 3.9 as for $p = 2$. The values of $\tau_p$ listed are the only elements of $\pi_p S$ in the relevant dimensions, except for $\tau_p(p^k) = p!$, which follows from II.1.10.

The relations in Table 1.2 are all determined by the attaching maps from Proposition 3.2. //

Proof of 1.10. By IV. 7.3(v), to determine $p^{n+m+1}(xy)$ we must calculate the image of $p^{n+m+1} \varepsilon_2(n+m+1)D_2S^{n+m}$ under $\delta_2 : \pi_2 D_2S^{n+m} + \pi_2(D_2S^n \wedge D_2S^m)$. We need only consider

$$p^{n+m+2} \varepsilon_n \wedge p^{n+2} \wedge p^{m+2}$$

for dimensional reasons. If $B_{n,m}$ is the skeletal filtration of $P_n^{m+2}$, then $E_2(S, B_{n,m})$ is generated over $E_2(S, S)$ by elements $l(j,k)$ with $n < j < n+2$ and $m < k < m+2$ corresponding to the cells of $P_n^{m+2}$ and $P_m^{n+2}$ in an obvious fashion. The attaching maps of $P_n^{m+2}$ and $P_m^{n+2}$ determine the differentials in low dimensions from which we get $E_2(S, B_{n,m})$. The extension questions in $\pi_2(n+m+1)$ are also determined by $P_n^{m+2}$ and $P_m^{n+2}$ when $n \equiv m \equiv 0 \ (2)$. When $n \equiv m \equiv 1 \ (2)$ we need the fact that the top cell of the smash product of two mod 2 Moore spaces is attached to the bottom cell by $\eta$, to settle the extension question. We conclude that if $n \equiv m \equiv 0 \ (2)$ then $\pi_2(n+m+1)$ is generated by $p^{n+1}p^m$, $p^np^{m+1}$, and $\eta p^np^m$ with relations

$$2p^{n+1}p^m = \begin{cases} 0 & \text{if } n \equiv 0 \ (4) \\ \eta p^np^m & \text{if } n \equiv 2 \ (4) \end{cases}$$

and

$$2p^np^{m+1} = \begin{cases} 0 & \text{if } m \equiv 0 \ (4) \\ \eta p^np^m & \text{if } m \equiv 2 \ (4) \end{cases} .$$

If $n \equiv m \equiv 1 \ (2)$ then $\pi_2(n+m+1)$ is generated by an element we call $S_{n,m}$ which is detected by $l(n+1,m) + l(n,m+1)$ with the relation

$$2S_{n,m} = \begin{cases} 0 & \text{if } n \equiv 3 \text{ or } m \equiv 3 \ (4) \\ \eta p^np^m & \text{if } n \equiv m \equiv 1 \ (4) \end{cases} .$$
From the image of $S_{n,m}$ in $E^*(S, n,m)$ we can see that

$$E^0(S_{n,m}) = p^{n+1}p^m + p^{n+m+1}.$$ 

Finally $\delta_*(p^{n+m+1})$ is determined modulo the kernel of the Hurewicz homomorphism by commutativity of the following diagram, in which the isomorphisms are Thom isomorphisms

$$
\begin{array}{ccc}
D_2^{n+m} & \xrightarrow{\delta_*} & D_2^n \wedge D_2^m \\
\downarrow h & & \downarrow h \\
H_2^{n+m} & \xrightarrow{\Delta_*} & H_2^{n} \wedge H_2^m \\
\|l & & \|l \\
H_2 BZ_2 & \xrightarrow{\Delta_*} & H_2(BZ_2 \times BZ_2)
\end{array}
$$

Since $p^{n+m}$ generates the kernel of the Hurewicz homomorphism we are done. //

**Proof of 1.11.** The commutative diagram above shows that the Hurewicz homomorphism must map the Cartan formula for a homotopy operation into the Cartan formula for its Hurewicz image. Case (i), $n = 2j$ and $m = 2k$, follows by an argument formally identical to, but easier than, the proof of 1.10 when $n = m = 0$ (2). Case (ii) is immediate from the homology Cartan formula because in this case we're in the Hurewicz dimension. Case (iii) follows just as in the proof of 1.10 when $n = m = 3$ (4). //

**Proof of 1.12.** In $E_2(S,S)$, $Sq^1(h_0) = h_1$ by [3]. Therefore, $p^1(2) = n$. //

**Proof of 1.13.** By definition $\beta^1(p)$ is a unit times the composite

$$S^{2p-3} \xrightarrow{\beta^1} D_p S \xrightarrow{D(p)} D_p S \xrightarrow{\xi} S,$$

where $\beta^1$ is the inclusion of the $2p-3$ cell. By II.1.8, $D_p(p) \equiv 1 \mod p$, and by II.2.8, $\tau_p \circ \beta^1 \neq 0$. Since $\xi_1 p = 1$, the composite and hence $\beta^1(p)$ are nonzero. The fact that $\beta^p\bmo(1) = 1$ follows from the fact that in the Adams spectral sequence, $\beta^p\bmo(h_0) = b^1_1$ using the notation of [66]. The latter can be computed directly from the definition of $\beta^p\bmo$ using the definitions

$$h_0 = [\xi_1], \quad b^1_1 = \sum_{i+j = p-1} (1,j)[\xi^p_1 | \xi^p_1]$$

in the bar construction. Alternatively, we may refer to Liulevicius' computation [55, pp. 26, 30] using [66, II-6.6] to translate it into our notation. //
Proof of 1.14. This is now immediate:
\[ p^{n+1}(2x) = p^1(2)x^2 + 4p^{n+1}(x) + 4c_{0n}nx^2 \]
\[ = nx^2 \]
since \( 2p^{n+1}(x) \) is either 0 or \( nx^2 \) by 1.10. Similarly,
\[ sp^{j+1}(px) = sp^1(p)x^p + p^0sp^{j+1}(x) + d_{0n}a_1p^px^p \]
\[ = sp^1(p)x^p + \sum p^1a_1p^j(x) \]
\[ = a_1x^p \]
since \( p^0j+1(x) = j_{a_1p^j}(x) \). Finally \( sp^{j+1}(a_1x) = x^p sp^{j+1}(a_1x) = x^p a_1 \). The indeterminacy is always zero because where it is not automatically zero it is \( 4nx^2 \) or \( p^0a_1x^p \). //

Proof of 1.15. If \( p = 2 \) then \( nx^2 = 0 \) by Theorem 1.10 when \( n \equiv 3 \) (4) (even if \( 2x \neq 0 \)) while \( 0 = p^{n+1}(2x) = nx^2 \) by Proposition 1.14 when \( n \equiv 0 \) (2). If \( p > 2 \) then \( x^p = 0 \) if \( n \) is odd, while if \( n = 2j \), Proposition 1.14 implies that \( 0 = sp^{j+1}(px) = a_1x^p \) and \( 0 = sp^{j+p-1}(a_1x) = \beta_1x^p \). When \( x = \beta_1 \) the second of these formulas is \( a_1\beta_1^p = 0 \). //

Proof of 1.16. Several of the computations follow from \( P^n(x) = x^2 \) if \( x \equiv_n 0 \), others from \( \pi_4 = \pi_5 = \pi_12 = \pi_13 = 0 \). Similarly, several indeterminacies are zero from Theorem 1.10 or because they lie in filtrations which are 0. We will prove the remainder of the results.

Since \( P^4(h_2) = h_3 \), \( h_1P^4(o) \) is detected by \( h_1h_3 \) so is either no or \( \bar{v} \). By 1.10,
\[ h_1^2P^5(o) = 2h_1^2P^6(o) = 0 \] since \( 2\pi_{10} = 0 \). Similarly, \( h_1^2P^4(2v) = 0 \) by calculating Steenrod operations in Ext. Since \( \tau_{2k}(h_1^6) = 0 \), we get \( h_1^2P^6(2v) = 2h_1^2P^6(o) = 0 \), and since \( \tau_{2k}(h_2^5) = 0 \), we get \( h_2^2P^5(2v) = 2h_2^2P^5(o) = 0 \). By 1.10,
\[ h_1^2P^5(2v) = 2h_1^2P^5(o) = 0 \] also. The operations on \( 4v \) can all be calculated from the additivity rule \( a^*(4v) = 2a^*(2v) + \tau_{2k}(a)(2v) = 2a^*(2v) \).

Since \( 2\pi_{17} = 0 \), the relations \( h_1^2P^9(o^2) = 2h_1^2P^{10}(o^2) \) and \( h_1^2P^9(o^2) = 2h_1^2P^9(o^2) \) force these elements to be 0 mod 0.

Since \( P^8(h_3) = h_4 \), \( h_1P^8(o) \) is detected by \( h_1h_4 \) so must be \( \eta^* \) or \( \eta^* + \eta^p \). Since \( 2h_1^2P^9 = \eta^2h_1P^8 \) and \( \eta^2h_1P^8(o) \) is detected by \( h_1^2h_4 = h_0h_2h_4 \), it follows that \( h_1^2P^9(o) \) is detected by \( h_0h_2h_4 \). Since \( 2h_1^1P^{10} = h_1^2P^9 \) it follows that \( h_1P^{10}(o) \) is
detected by $h_2h_4$. Thus $h_1P_{10}(o) = \nu^*$ or $\nu^* + \eta u$ modulo $<2\nu^*>$, which is its indeterminacy, and similarly for $h_1P_9(o)$.

Since $P^7(2o) = 4\sigma^2 = 0$, we have

$$h_1P_8(2o) = 2h_1P_8(o) + \begin{cases} 0 \\ \eta \sigma^2 = 0 + 0 = 0 \end{cases}$$

The remaining operations are additive except for

$$h_1P_8(4o) = 2h_1P_8(2o) + \begin{cases} 0 \\ \eta \sigma^2 = 0 + 0 = 0 \end{cases}$$