RELATIVE CYCLOTOMIC STRUCTURES AND EQUIVARIANT COMPLEX COBORDISM

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Abstract. We describe a structure on a commutative ring (pre)cyclotomic spectrum $\underline{\underline{R}}$ that gives rise to a (pre)cyclotomic structure on topological Hochschild homology (THH) relative to its underlying commutative ring spectrum. This lets us construct TC relative to $\underline{\underline{R}}$, denoted $TC^{\underline{\underline{R}}}$, and we prove some descent results relating $TC^{\underline{\underline{R}}}$ and TC. We explore several examples of this structure on familiar $\mathbb{T}$-equivariant commutative ring spectra including the periodic $\mathbb{T}$-equivariant complex cobordism spectrum $MUP_{\mathbb{T}}$ and a new (connective) equivariant version of the complex cobordism spectrum MU.

This is a preliminary version that depends on the work in progress [Cyc23].

1. Introduction

The remarkable success of trace methods over the past 30 years derives from the relationship between algebraic K-theory and topological cyclic homology (TC). In contrast to algebraic constructions of cyclic homology, TC intrinsically depends on topology, namely the cyclotomic structure on topological Hochschild homology (THH). The cyclotomic structure ultimately depends on working over the sphere spectrum $\mathbb{S}$; this structure does not exist in algebra, working over the integers $\mathbb{Z}$. Moreover, the surprising use of THH and TC in recent seminal work of Bhatt-Morrow-Scholze [3, 4] on p-adic Hodge theory reveals another fundamental role of the cyclotomic structure in modern geometry.

The construction of THH makes sense relative to any commutative ring orthogonal spectrum R, but the cyclotomic structure does not. The search for cyclotomic structures on relative THH goes back to the 1990's. From the perspective of trace methods, relative cyclotomic constructions were supposed to give descent spectral sequences computing the absolute construction for further algebraic K-theory computations. More recently, [4] constructed cyclotomic structures relative to the group ring $\mathbb{S}[t]$ and used them to build Breuil-Kisin modules. New work in Floer homotopy theory shows that spectral Fukaya categories will often come with enrichments over some form of complex cobordism; relative cyclotomic structures here are hoped to correspond to interesting and effective structure on symplectic cohomology.

Previous work of the first and second author and collaborators [1, §7] studies the problem of constructing cyclotomic structures on relative THH. Recall from [7,

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4.1] that a p-precyclotomic structure on a $\mathbb{T}$-equivariant orthogonal spectrum $\underline{X}$ consists of a $\mathbb{T}$-equivariant map

$$\rho^* \Phi^{C_p} \underline{X} \longrightarrow \underline{X}$$

where ρ^* is change of groups along the pth root isomorphism $\rho\colon \mathbb{T} \xrightarrow{\cong} \mathbb{T}/C_p$ and Φ^{C_p} denotes the point-set multiplicative geometric fixed point functor of [15, V.4.3]. The p-precyclotomic spectrum is p-cyclotomic if the induced map from the derived geometric fixed points to $\underline{X}$ is an $\mathscr{F}_p$-equivalence where $\mathscr{F}_p$ is the family of p-groups $C_{p^n} < \mathbb{T}$. (We review these and related structures, including associative and commutative ring p-(pre)cyclotomic structures, in Section 4.) In this paper, we typically denote a p-precyclotomic spectrum with a double underlined symbol, e.g., $\underline{\underline{R}}$, and use the notational shorthand of the single underlined symbol $\underline{R}$ to denote its underlying equivariant spectrum (for the group $\mathbb{T}$ or sometimes C_p) and the non-underlined symbol R for the underlying non-equivariant spectrum.

Cyclotomic structures have been the main focus in the literature on trace methods because (absolute) THH naturally can be endowed with such a structure. However, p-precyclotomic spectra provide a minimal structure sufficient for constructing (p-typical) TC (e.g., see [7, 6.7]). We focus on precyclotomic spectra in this paper because most of the structures we describe on examples of interest are not p-cyclotomic but only p-precyclotomic. (See Section 8 for a discussion on converting p-precyclotomic to p-cyclotomic spectra with equivalent TC.)

Given a commutative ring p-(pre)cyclotomic spectrum $\underline{\underline{R}}$, we then have a canonical counit map $THH(R) \to \underline{R}$ of $\mathbb{T}$-equivariant commutative ring orthogonal spectra. If this map is p-(pre)cyclotomic, then the R-relative topological Hochschild homology $THH^R(-)$ has a natural p-(pre)cyclotomic structure. This condition is not formal (see [1, p. 2146]) and it is not a priori clear when to expect it to hold.

The purpose of this paper is to give a new framework for constructing p-precyclotomic structures on relative THH, which implies the criterion of [1, 7.6] and which we can check in some interesting examples. Our framework depends on a new self map associated to a commutative ring p-precyclotomic $\underline{\underline{R}}$. In Section 4, we construct a self map $\Psi\colon R \to R$ in the homotopy category of non-equivariant commutative ring orthogonal spectra. The map Ψ is the composite of the multiplicative transfer

$$R \longrightarrow \Phi^{C_p} \underline{R}$$

(see also Definition 4.7) and the non-equivariant map underlying the p-precyclotomic structure map

$$\rho^* \Phi^{C_p} \underline{R} \longrightarrow \underline{R}.$$

We refer to Ψ as the p-cyclotomic power operation and prove the following theorem, which we state jointly in the p-cyclotomic and p-precyclotomic cases.

Theorem A. *Let $\underline{\underline{R}}$ be a commutative ring p-(pre)cyclotomic spectrum, let R denote its underlying non-equivariant commutative ring orthogonal spectrum, and assume that $\Psi\colon R \to R$ is the identity in the homotopy category of commutative ring orthogonal spectra. Then $\underline{\underline{R}}$-relative topological Hochschild homology $THH^{\underline{\underline{R}}}(-)$ can be given the structure of a p-(pre)cyclotomic $\underline{\underline{R}}$-module.*

We then get TC relative to $\underline{\underline{R}}$, $TC^{\underline{\underline{R}}}$, by applying to $THH^{\underline{\underline{R}}}$ the classic p-typical topological cyclic homology construction $TC(-;p)$ (see for example [7, 6.3]). For

$\underline{\underline{R}} = \mathbb{S}$ with its canonical cyclotomic structure $TC^{\mathbb{S}}(A)$ is the usual (p-typical) $TC(A)$.

In the case when $\underline{\underline{R}} = THH(A)$ for some commutative ring orthogonal spectrum A, the p-cyclotomic power operation is the Adams operation ψ^p of [1, §10], which is typically not the identity for $A \not\simeq \mathbb{S}$. (See Proposition 4.10.)

We call a commutative ring p-precyclotomic spectrum along with a choice of homotopy from Ψ to the identity a *p-(pre)cyclotomic base*. Not all $\mathbb{T}$-equivariant commutative ring spectra admit such a structure: the standard equivariant structure on complex K-theory cannot be a p-(pre)cyclotomic base (see Example 7.5); however, in Section 7, we show that a number of interesting spectra do. Notably, we show that $\mathbb{S}[t]$, $\mathbb{F}_p$, and MUP (with given p-(pre)cyclotomic structures explained there) all admit the structure of a p-(pre)cyclotomic base. In addition, we construct such a structure on MU for a new precyclotomic structure, which we denote $\underline{\underline{mu}}$. (Note: the underlying $\mathbb{T}$-equivariant spectrum $\underline{mu}$ of $\underline{\underline{mu}}$ is not homotopical or geometric $\mathbb{T}$-equivariant complex cobordism, but is a new equivariant connective Thom spectrum with underlying non-equivariant Thom spectrum MU.)

Theorem B. *The genuine $\mathbb{T}$-equivariant commutative ring orthogonal spectrum $\underline{MUP} = MUP_{\mathbb{T}}$ admits the structure of a p-precyclotomic base. The commutative ring spectrum MU admits the structure of a p-precyclotomic base (with a new $\mathbb{T}$-equivariant structure).*

Finally, we state a descent theorem (proved as Theorem 9.1 in Section 9) relating relative TC to absolute TC for a connective p-precyclotomic base $\underline{\underline{R}}$ and connective R-algebra A. In the statement, we use the commutative R-algebra structure on A to get a commutative $R^{(n+1)}$-algebra structure on A (for all $n \geq -1$) by restriction along the iterated multiplication $R^{(n+1)} \to R$. We argue in Section 9 that the Adams resolution of $\mathbb{S}$ with respect to R, $R^{(\bullet+1)}$, has the canonical structure of a cosimplicial object in the category of p-precyclotomic bases, $\underline{\underline{R}}^{(\bullet+1)}$, and using $TC^{(-)}$ as a functor of the p-precyclotomic base, we get an augmented cosimplicial spectrum

$$TC^{\underline{\underline{R}}^{(\bullet+1)}}(A)$$

(where for $\bullet = -1$, $\underline{\underline{R}}^{(\bullet+1)} = \mathbb{S}$).

Theorem C. *Let $\underline{\underline{R}}$ be a $\mathscr{F}_p$-connective p-precyclotomic base, whose underlying commutative ring p-precyclotomic spectrum is cofibrant. Let A be a connective cofibrant commutative R-algebra. The canonical map*

$$TC(A) \longrightarrow \mathrm{Tot}(TC^{\underline{\underline{R}}^{(\bullet+1)}}(A))$$

is a weak equivalence.

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2. Equivariant preliminaries

In this section, we review the prerequisites in equivariant stable homotopy theory needed for defining and working with precyclotomic and cyclotomic spectra. Although modern treatments of cyclotomic spectra minimize the explicit use of genuine equivariant stable homotopy theory in the setup and require none for end-users, no such simplification can exist for precyclotomic spectra, especially in the nonconnective setting. Our main examples of interest, the cobordism spectra $\underline{\underline{MUP}}$ and $\underline{\underline{mu}}$, have precyclotomic structures that are not cyclotomic structures.

We take as our model for the equivariant stable category the point-set category of $\mathbb{T}$-equivariant orthogonal spectra indexed on the complete universe

$$U = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}(n)^{\infty}$$

where $\mathbb{C}(0)$ denotes the complex numbers with the trivial $\mathbb{T}$-action, $\mathbb{C}(1)$ denotes the complex numbers with its natural $\mathbb{T}$-action (as the unit complex numbers), and in general, for any n, the element $\zeta \in \mathbb{T}$ acts on $\mathbb{C}(n)$ as multiplication by ζ^n. For such a spectrum T, homotopy groups are defined by

$$\pi^H_q T = \operatorname*{colim}_{V<U} \operatorname*{colim}_{n\geq -q} \pi_{n+q}((\Omega^V T(V \oplus \mathbb{R}^n))^H)$$

for $H < \mathbb{T}$ a closed subgroup. We work with the family $\mathscr{F}_{\mathrm{fin}}$ of finite subgroups of $\mathbb{T}$ and the family $\mathscr{F}_p \subset \mathscr{F}_{\mathrm{fin}}$ of p-subgroups of $\mathbb{T}$. For any family $\mathscr{F}$, an $\mathscr{F}$-*equivalence* is a map that induces an isomorphism on π^H_* for $H \in \mathscr{F}$ (but not necessarily for $H \notin \mathscr{F}$). The $\mathscr{F}$-equivalences are the weak equivalences in a model structure called the $\mathscr{F}$-model structure, defined in [15, IV.6.5]. Technically, we want to use the $\mathscr{F}$-model structures in [Cyc23], which are more convenient for the work below. The forgetful functor from $\mathbb{T}$-equivariant associative ring and commutative ring orthogonal spectra to $\mathbb{T}$-equivariant orthogonal spectra creates the weak equivalences and fibrations for the $\mathscr{F}$-model structures on the categories of $\mathbb{T}$-equivariant associative and commutative ring orthogonal spectra.

We will often take a $\mathbb{T}$-equivariant orthogonal spectrum (indexed on U) and look at its underlying non-equivariant spectrum indexed in $\mathbb{R}^\infty$. As indicated in the introduction, we will typically denote an equivariant object with an underlined symbol (e.g., $\underline{R}$) and its underlying non-equivariant object with the non-underlined symbol (R in the case of $\underline{R}$). When we need to explicitly denote the forgetful functor, we use the following notation.

Notation 2.1. Let i denote the forgetful functor from $\mathbb{T}$-equivariant orthogonal spectrum (indexed on U) to non-equivariant spectra (indexed on $\mathbb{R}^\infty$), and its structured variants for categories of commutative ring orthogonal spectra, associative ring orthogonal spectra, and modules over commutative or associative ring orthogonal spectra.

We write Φ for the endofunctor on $\mathbb{T}$-equivariant orthogonal spectra obtained as the composite of the (point-set multiplicative) C_p geometric fixed point functor Φ^{C_p} of [15, V.4.3] followed by change of group functor along the pth root isomorphism $\rho\colon \mathbb{T} \cong \mathbb{T}/C_p$:

Notation 2.2. Let $\Phi\underline{X} := \rho^*(\Phi^{C_p}\underline{X})$.

The change of group functor ρ^* implicitly involves a change of universe functor, using the isomorphism $\rho^* U^{C_p} \cong U$ that comes from the standard isomorphisms

$\rho^*\mathbb{C}(pn) \cong \mathbb{C}(n)$. The functor Φ comes with a lax symmetric monoidal structure: a map

$$\iota\colon \mathbb{S} \longrightarrow \Phi\mathbb{S}$$

and a natural transformation

$$\lambda\colon \Phi\underline{X} \wedge \Phi\underline{Y} \longrightarrow \Phi(\underline{X} \wedge \underline{Y})$$

that is coherently associative, commutative, and unital (in the point-set category of $\mathbb{T}$-equivariant orthogonal spectra). The map ι is an isomorphism and λ is often an isomorphism: the paper [Cyc23] studies this question and identifies a large class of orthogonal spectra with the property that when $\underline{X}$ is in this class, λ is an isomorphism for any $\underline{Y}$. In this paper, we denote the class of spectra with this property as $\mathscr{E}$. In this notation, [Cyc23] proves that the class $\mathscr{E}$ contains the cofibrant $\mathbb{T}$-equivariant orthogonal spectra (and therefore the cofibrant $\mathbb{T}$-equivariant associative ring orthogonal spectra), the cofibrant $\mathbb{T}$-equivariant commutative ring orthogonal spectra, and the cofibrant $\mathbb{T}$-equivariant A-modules for any $\mathbb{T}$-equivariant associative ring orthogonal spectrum whose underlying $\mathbb{T}$-equivariant orthogonal spectrum is in $\mathscr{E}$. (It is expected that λ may not be an isomorphism in general, but no counterexample is currently known to the authors.)

We also have a non-equivariant version of the class $\mathscr{E}$ that consists of the cofibrant objects in the "extended" model structure of [20, 3.2]. When A is an associative ring orthogonal spectrum whose underlying orthogonal spectrum is in the non-equivariant class $\mathscr{E}$, then $THH(A)$ is in the equivariant class $\mathscr{E}$. This happens in particular when A is a cofibrant R-algebra for R a commutative ring orthogonal spectrum that is cofibrant in the standard model structure or the model structure of [Cyc23]. Moreover, when the underlying orthogonal spectrum of A is in the class $\mathscr{E}$, the diagonal map $THH(A) \to \Phi THH(A)$ of [1, 2.19] is an isomorphism [Cyc23], and this endows $THH(A)$ with a cyclotomic structure as in [1, 1.5].

Using the lax symmetric monoidal structure, Φ refines to an endofunctor on various categories of structured spectra, including the category of $\mathbb{T}$-equivariant associative ring orthogonal spectra and the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra. For a $\mathbb{T}$-equivariant associative ring orthogonal spectrum A, Φ refines to a functor from A-modules to ΦA-modules.

The endofunctor Φ on $\mathbb{T}$-equivariant orthogonal spectra has a left derived functor, $\mathbb{L}\Phi$, which can be computed by applying Φ to a cofibrant approximation (in the standard stable model structure, the $\mathscr{F}_{\mathrm{fin}}$-model structure, or the $\mathscr{F}_p$-model structure of [15, III§4–5,IV§6] or its variants in [Cyc23]). This also works to construct the derived functor on the category of $\mathbb{T}$-equivariant associative ring orthogonal spectra. For the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra, the functor Φ does not preserve weak equivalences between objects cofibrant in the standard model structure, but a main result of [Cyc23] is to construct a model structure on $\mathbb{T}$-equivariant commutative ring orthogonal spectra, with the identity functor a left Quillen equivalence with the standard model structure, but having the property that Φ preserves weak equivalences between cofibrant objects. In addition [Cyc23] constructs another model structure (right Quillen equivalent to this one via the identity functor) with the property that Φ preserves cofibrations and acyclic cofibrations. Moreover, [Cyc23] shows that the derived functor of Φ for $\mathbb{T}$-equivariant commutative ring orthogonal spectra agrees with the derived functor of Φ on the underlying $\mathbb{T}$-equivariant orthogonal spectra.

We write $N_e^{C_p}$ for the Hill-Hopkins-Ravenel norm [12, §2.2.3], [1, §2.2] from (non-equivariant) orthogonal spectra to C_p-equivariant orthogonal spectra (indexed on U restricted to C_p). This functor is defined on all orthogonal spectra but when restricted to commutative ring orthogonal spectra it gives the free functor from the category of commutative ring orthogonal spectra to the category of C_p-equivariant commutative ring orthogonal spectra

$$\mathscr{C}om^{C_p}(N_e^{C_p}A, \underline{B}) \cong \mathscr{C}om(A, B)$$

(for any commutative ring orthogonal spectrum A and C_p-equivariant commutative ring orthogonal spectrum $\underline{B}$). As such, a C_p-equivariant commutative ring orthogonal spectrum $\underline{R}$ comes with a canonical map of C_p-equivariant commutative ring orthogonal spectra

$$N_e^{C_p}R \longrightarrow \underline{R} \tag{2.3}$$

given by the counit of the adjunction.

As explained in [1], $THH(-)$ as a functor from associative ring orthogonal spectra to $\mathbb{T}$-equivariant orthogonal spectra is the norm $N_e^{\mathbb{T}}(-)$. On commutative ring orthogonal spectra, THH is the free functor to $\mathbb{T}$-equivariant commutative ring orthogonal spectra. (This refines a 1997 theorem of McClure-Schwänzl-Vogt [17].) As a consequence, for $\underline{R}$ a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum and A a commutative ring orthogonal spectrum, we have an adjunction

$$\mathscr{C}om^{\mathbb{T}}(THH(A), \underline{R}) \cong \mathscr{C}om(A, R)$$

(homeomorphism of mapping spaces) relating maps of commutative ring orthogonal spectra to maps of $\mathbb{T}$-equivariant commutative ring orthogonal spectra. The derived functor of THH represents the derived functor of the free functor, and we have the analogous adjunction in the homotopy category. We summarize the situation in the following proposition.

Proposition 2.4 ([1, 4.3], [17])**.** *Restricted to the category of commutative ring orthogonal spectra, THH is the free functor from commutative ring orthogonal spectra to $\mathbb{T}$-equivariant commutative ring orthogonal spectra,*

$$\mathscr{C}om^{\mathbb{T}}(THH(-), -) \cong \mathscr{C}om(-, i(-)).$$

3. Relative THH

The idea of THH relative to a commutative ring orthogonal spectrum R is to use the smash product of R-modules $\wedge_R$ in place of the smash product of spectra in the cyclic bar construction. In this section, we review and clarify the $\mathbb{T}$-equivariant structures on relative THH, particularly in the case when R has the extra structure of being the underlying non-equivariant commutative ring orthogonal spectrum of a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum $\underline{R}$.

We start with R a commutative ring orthogonal spectrum and A an associative R-algebra. We then have a cyclic bar construction N_R^{cy} constructed using the smash product over R, $\wedge_R$:

$$N_R^{\mathrm{cy}}(A) = |q \mapsto \underbrace{A \wedge_R \cdots \wedge_R A}_{q+1 \text{ factors}}|.$$

This is the geometric realization of a cyclic spectrum, and that structure endows it with a natural $\mathbb{T}$-action. We do point-set change of universe $I_{\mathbb{R}^\infty}^U$ to get a $\mathbb{T}$-equivariant orthogonal spectrum indexed on U.

Definition 3.1. For a commutative ring orthogonal spectrum R, define the point-set functor $THH^{\epsilon R}$ from the category of associative R-algebras to the category of $\mathbb{T}$-equivariant orthogonal spectra as the composite of the cyclic bar construction with respect to $\wedge_R$ and change of universe:

$$THH^{\epsilon R}(A) := I_{\mathbb{R}^\infty}^{U} N_R^{\mathrm{cy}}(A).$$

(The reason for the ϵ in the notation will become clear in Proposition 3.4 and Definition 3.6 below.)

The previous definition does not capture all the structure: $N_R^{\mathrm{cy}}(A)$ has the structure of an R-module, but we have defined $THH^{\epsilon R}(A)$ only as an equivariant spectrum. To describe the equivariant module structure, we use the following notation.

Notation 3.2. For a non-equivariant orthogonal spectrum X, let $\epsilon X = I_{\mathbb{R}^\infty}^{U} X$ denote the $\mathbb{T}$-equivariant orthogonal spectrum (indexed on U) obtained from change of universe by X with the trivial $\mathbb{T}$-action.

We emphasize that in the previous notation, ϵ denotes a point-set functor. It preserves associative ring and commutative ring structures. It also preserves weak equivalences between cofibrant orthogonal spectra and between cofibrant associative ring orthogonal spectra (even for maps of orthogonal spectra that are not ring maps). It preserves ring map weak equivalences between cofibrant commutative ring orthogonal spectra (in the standard model structures or the model structures of [Cyc23]; see [Cyc23]); however, we repeat the warning of [1, 6.1].

Warning 3.3. The functor ϵ preserves weak equivalences between cofibrant objects in the category of commutative ring orthogonal spectra, and so admits a left derived functor $\epsilon^{\mathbb{L}}$ from the homotopy category of commutative ring orthogonal spectra to the homotopy category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra. However, the composite with the derived forgetful functor to the equivariant stable category is not the left derived functor of ϵ from orthogonal spectra to $\mathbb{T}$-equivariant orthogonal spectra.

$$\begin{tikzcd}
\mathrm{Ho}\,\mathscr{C}om \arrow[r, "\epsilon^{\mathbb{L}}"] \arrow[d] \arrow[dr, phantom, "\times"] & \mathrm{Ho}\,\mathscr{C}om^{\mathbb{T}} \arrow[d] \\
\mathrm{Ho}\,\mathscr{S} \arrow[r, "\epsilon^{\mathbb{L}}"'] & \mathrm{Ho}\,\mathscr{S}^{\mathbb{T}}
\end{tikzcd}$$

Specifically, let R be a cofibrant commutative ring orthogonal spectrum and let $X \to R$ be a cofibrant approximation in the category of orthogonal spectra. Then the underlying non-equivariant spectrum of the derived geometric fixed point functor $\mathbb{L}\Phi^{C_p}\epsilon X$ is equivalent to X, whereas by [Cyc23], the underlying non-equivariant spectrum of $\mathbb{L}\Phi^{C_p}\epsilon R$ is equivalent to $R \otimes (* \amalg BC_p)$ (tensor in the category of commutative ring orthogonal spectra).

We lift $THH^{\epsilon R}$ as a functor to $\mathbb{T}$-equivariant ϵR-modules using the following result of [1, 1.8]. In it, the $THH(R)$-module structure on ϵR comes from the map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra $THH(R) \to \epsilon R$ adjoint under Proposition 2.4 to the canonical isomorphism $R \cong i\epsilon R$.

Proposition 3.4 ([1, 1.8])**.** *Let R be a commutative ring orthogonal spectrum and A an associative R-algebra. There is a natural isomorphism of $\mathbb{T}$-equivariant orthogonal spectra*

$$THH^{\epsilon R}(A) \cong THH(A) \wedge_{THH(R)} \epsilon R.$$

Moreover, if R is cofibrant as a commutative ring orthogonal spectrum (in the standard model structure or the model structure of [Cyc23]*), and A is a cofibrant associative R-algebra or a cofibrant commutative R-algebra, then $THH^{\epsilon R}(A)$ represents the derived smash product*

$$THH^{\epsilon R}(A) \simeq THH(A) \wedge^{\mathbb{L}}_{THH(R)} \epsilon R.$$

The left derived functor $(-)\wedge^{\mathbb{L}}_{THH(R)} \epsilon R$ from $\mathbb{T}$-equivariant $THH(R)$-modules to $\mathbb{T}$-equivariant orthogonal spectra factors as the composite of the derived forgetful functor and the derived functor $(-) \wedge^{\mathbb{L}}_{THH(R)} \epsilon R$ from $\mathbb{T}$-equivariant $THH(R)$-modules to $\mathbb{T}$-equivariant ϵR-modules.

$$\begin{tikzcd}
\mathrm{Ho}\,\mathscr{M}od^{\mathbb{T}}_{THH(R)} \arrow[r, "(-)\wedge^{\mathbb{L}}_{THH(R)}\epsilon R"] \arrow[rr, bend right, "(-)\wedge^{\mathbb{L}}_{THH(R)}\epsilon R"'] & \mathrm{Ho}\,\mathscr{M}od^{\mathbb{T}}_{\epsilon R} \arrow[r] & \mathrm{Ho}\,\mathscr{S}^{\mathbb{T}}
\end{tikzcd}$$

The previous result also implies the following homotopical property of THH^R.

Proposition 3.5. *Let $R' \to R$ be a weak equivalence of cofibrant commutative ring orthogonal spectra, let A' be a cofibrant associative R'-algebra and let A be either a cofibrant associative R-algebra or a cofibrant commutative R-algebra. A weak equivalence of R'-algebras $A' \to A$ then induces an $\mathscr{F}_{\mathrm{fin}}$-equivalence of $\mathbb{T}$-equivariant $\epsilon R'$-modules $THH^{\epsilon R'}(A) \to THH^{\epsilon R}(A)$.*

Now let $\underline{R}$ be a $\mathbb{T}$-equivariant commutative ring orthogonal spectra and let $R = i\underline{R}$. We then get a map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra $THH(R) \to \underline{R}$ adjoint under Proposition 2.4 to the identity map $R \to i\underline{R}$. This endows relative THH with a different equivariant structure that incorporates the equivariance on $\underline{R}$.

Definition 3.6. Let $\underline{R}$ be a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum and let $R = i\underline{R}$ be the underlying non-equivariant commutative ring orthogonal spectrum. Let

$$THH(R) \longrightarrow \underline{R}$$

be the map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra adjoint under Proposition 2.4 to the identity map $R \to i\underline{R}$. Define the functor $THH^{\underline{R}}$ from the category of (non-equivariant) R-algebras to the category of $\mathbb{T}$-equivariant $\underline{R}$-modules by

$$THH^{\underline{R}}(A) := THH(A) \wedge_{THH(R)} \underline{R}.$$

The same argument as Proposition 3.4 proves the following results.

Proposition 3.7. *Let $\underline{R}$ be a cofibrant $\mathbb{T}$-equivariant commutative ring orthogonal spectrum (in the standard model structure or the model structure of* [Cyc23]*) and let $R = i\underline{R}$ be the underlying non-equivariant commutative ring orthogonal spectrum. Let A a cofibrant associative R-algebra or a cofibrant commutative R-algebra (in the standard model structures or the model structures of* [Cyc23]*). Then $THH^{\underline{R}}(A)$ represents the derived smash product*

$$THH^{\underline{R}}(A) \simeq THH(A) \wedge^{\mathbb{L}}_{THH(R)} \underline{R}.$$

Proposition 3.8. *Let $\underline{R}' \to \underline{R}$ be an $\mathscr{F}$-equivalence of cofibrant $\mathbb{T}$-equivariant commutative ring orthogonal spectra (in either the standard model structure or the model structure of* [Cyc23]*) for any family of proper subgroups of $\mathbb{T}$. Let A' be a cofibrant associative R'-algebra and let A be either a cofibrant associative R-algebra or a cofibrant commutative R-algebra. A weak equivalence of R'-algebras $A' \to A$ then induces an $\mathscr{F}$-equivalence of $\mathbb{T}$-equivariant $\underline{R}'$-modules $THH^{\underline{R}'}(A) \to THH^{\underline{R}}(A)$.*

When we discuss the derived functor of relative $THH^{\underline{R}}(A)$, we mean the derived functor of both variables $\underline{R}$ and A, which exists by the previous proposition. The following definition makes this precise.

Definition 3.9.

(i) Let $\mathscr{P}air_{\mathscr{A}lg}$ be the category where an object consists of a pair $(\underline{R}, A)$ with $\underline{R}$ a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum and A an associative R-algebra, and a map $(\underline{R}', A') \to (\underline{R}, A)$ consists of a map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra $\underline{R}' \to \underline{R}$ and a map of associative R'-algebras $A' \to A$ (relative to the given map $\underline{R}' \to \underline{R}$).

(ii) Let $\mathscr{P}air_{\mathscr{M}od}$ be the category of pairs $(\underline{R}, M)$ with $\underline{R}$ a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum and M a $\mathbb{T}$-equivariant $\underline{R}$-module with the evident maps (analogous to the definition of $\mathscr{P}air_{\mathscr{A}lg}$).

(iii) For $\mathscr{F}$ a family of subgroups of $\mathbb{T}$, we let the $\mathscr{F}$-equivalences in $\mathscr{P}air_{\mathscr{A}lg}$ be the maps $(\underline{R}', A') \to (\underline{R}, A)$ where $\underline{R}' \to \underline{R}$ is an $\mathscr{F}$-equivalence and $A' \to A$ is a weak equivalence; we let the $\mathscr{F}$-equivalences in $\mathscr{P}air_{\mathscr{M}od}$ be the maps $(\underline{R}', M') \to (\underline{R}, M)$ where $\underline{R}' \to \underline{R}$ and $M' \to M$ are $\mathscr{F}$-equivalences.

Terminology 3.10. By the *derived functor of relative THH*, we mean the total left derived functor of $(\underline{R}, A) \mapsto THH^{\underline{R}}(A)$ as a functor from $\mathscr{P}air_{\mathscr{A}lg}$ to $\mathscr{P}air_{\mathscr{M}od}$ with the $\mathscr{F}_p$-equivalences, and we denote it as $\mathbb{L}THH$.

Since a weak equivalence of $\mathbb{T}$-equivariant commutative ring orthogonal spectra $\underline{R}' \to \underline{R}$ induces an equivalence of $\mathscr{F}_p$-homotopy categories of modules, for a fixed $\mathbb{T}$-equivariant commutative ring orthogonal spectrum $\underline{R}$, $\mathbb{L}THH^{\underline{R}}(-)$ restricts to give a functor from the homotopy category of associative R-algebras to the $\mathscr{F}_p$-homotopy category of $\mathbb{T}$-equivariant $\underline{R}$-modules. When $\underline{R}$ is cofibrant in either the standard model structure on $\mathbb{T}$-equivariant commutative ring orthogonal spectra or the model structure of [Cyc23], this restriction $\mathbb{L}THH^{\underline{R}}(-)$ agrees with the total left derived functor of $THH^{\underline{R}}(-)$ from associative R-algebras to $\mathbb{T}$-equivariant $\underline{R}$-modules (with the $\mathscr{F}_p$-equivalences). To preclude any confusion, we will usually take $\underline{R}$ to be cofibrant in statements involving the derived functor of relative THH.

4. Cyclotomic and precyclotomic structures

In this section, we review the definitions of p-cyclotomic and p-precyclotomic structures on $\mathbb{T}$-equivariant orthogonal spectra. We review the theory of commutative ring objects in these categories and the universal property of THH in this setting. We define the p-cyclotomic power operation Ψ for commutative ring p-precyclotomic spectra and compare it to the Adams operation ψ^p on THH.

Definition 4.1. A p-precyclotomic spectrum $\underline{\underline{X}}$ consists of a $\mathbb{T}$-equivariant orthogonal spectrum $\underline{X}$ and a map of $\mathbb{T}$-equivariant orthogonal spectra $r\colon \Phi\underline{X} \to \underline{X}$. A

p-cyclotomic spectrum is a p-precyclotomic spectrum where the induced map

$$\mathbb{L}\Phi\underline{X} \longrightarrow \Phi\underline{X} \xrightarrow{r} \underline{X}$$

in the $\mathbb{T}$-equivariant stable category is an $\mathscr{F}_p$-equivalence. A map of p-precyclotomic spectra $\underline{\underline{X}} \to \underline{\underline{Y}}$ consists of a map of the underlying $\mathbb{T}$-equivariant orthogonal spectra $f\colon \underline{X} \to \underline{Y}$ that makes the p-precyclotomic structure maps commute.

$$\begin{tikzcd}
\Phi\underline{X} \arrow[r, "r_{\underline{\underline{X}}}"] \arrow[d, "\Phi f"'] & \underline{X} \arrow[d, "f"] \\
\Phi\underline{Y} \arrow[r, "r_{\underline{\underline{Y}}}"'] & \underline{Y}
\end{tikzcd}$$

A map of p-cyclotomic spectra is a map of p-precyclotomic spectra.

The papers [7] and [1] consider the more sophisticated structure of precyclotomic spectra; however, we work exclusively in the p-precyclotomic setting and therefore simplify terminology:

Terminology 4.2. For the purposes of this paper, we write **precyclotomic** for p-precyclotomic and **cyclotomic** for p-cyclotomic. We write (pre)cyclotomic to handle both cases together, with an implicit "respectively".

Because the lax symmetric monoidal structure on Φ is not (known to be) strong, we do not have a symmetric monoidal structure on precyclotomic spectra that refines the usual smash product on $\mathbb{T}$-equivariant orthogonal spectra. As a consequence, we cannot do the usual thing and define associative ring precyclotomic spectra as monoids for the smash product. Instead we follow [Cyc23] to define ring structures as follows. For the following definition we note that for a $\mathbb{T}$-equivariant associative ring orthogonal spectrum $\underline{A}$, $\Phi\underline{A}$ inherits the structure of a $\mathbb{T}$-equivariant associative ring orthogonal spectrum using the lax symmetric monoidal structure maps for Φ: it has unit and product

$$\mathbb{S} \xrightarrow{\iota} \Phi\mathbb{S} \xrightarrow{\Phi\eta} \Phi\underline{A}, \qquad \Phi\underline{A} \wedge \Phi\underline{A} \xrightarrow{\lambda} \Phi(\underline{A} \wedge \underline{A}) \xrightarrow{\Phi\mu} \Phi\underline{A}$$

where η and μ are the unit and product for A.

Definition 4.3 ([Cyc23])**.**

(i) An *associative ring precyclotomic spectrum* consists of a precyclotomic spectrum $\underline{\underline{A}}$ together with a $\mathbb{T}$-equivariant associative ring orthogonal spectrum structure on $\underline{A}$ such that the structure map is a map of $\mathbb{T}$-equivariant associative ring orthogonal spectra. A map of associative ring precyclotomic spectra is a map of precyclotomic spectra that on the underlying $\mathbb{T}$-equivariant orthogonal spectra is a map of $\mathbb{T}$-equivariant associative ring orthogonal spectra.

(ii) A *commutative ring precyclotomic spectrum* is an associative ring precyclotomic spectrum whose underlying $\mathbb{T}$-equivariant associative ring orthogonal spectrum is commutative. The category of commutative ring precyclotomic spectra is a full subcategory of the category of associative ring precyclotomic spectra.

(iii) Commutative and associative ring cyclotomic spectra are commutative and associative ring precyclotomic spectra (respectively) whose underlying precyclotomic spectra are cyclotomic. The categories of associative ring cyclotomic spectra, and commutative ring cyclotomic spectra are full subcategories of the category of associative ring precyclotomic spectra.

(iv) In any of the categories above, a *weak equivalence* is a map that is an $\mathscr{F}_p$-equivalence of the underlying $\mathbb{T}$-equivariant orthogonal spectra.

For an associative or commutative ring precyclotomic spectrum $\underline{\underline{A}}$, we have corresponding notions of (pre)cyclotomic $\underline{\underline{A}}$-modules. When $\underline{M}$ is an $\underline{A}$-module, $\Phi\underline{M}$ obtains a canonical $\Phi\underline{A}$-modules structure; we use this in the following definition.

Definition 4.4. Let $\underline{\underline{A}}$ be an associative ring precyclotomic spectrum. A (pre)-cyclotomic $\underline{\underline{A}}$-module is a (pre)cyclotomic spectrum $\underline{\underline{M}}$, together with the structure of an $\underline{A}$-module on $\underline{M}$ making the following action diagram commute.

$$\begin{tikzcd}
\Phi\underline{A} \wedge \Phi\underline{M} \arrow[r] \arrow[d, "r_{\underline{\underline{A}}} \wedge r_{\underline{\underline{M}}}"'] & \Phi\underline{M} \arrow[d, "r_{\underline{\underline{M}}}"] \\
\underline{A} \wedge \underline{M} \arrow[r] & \underline{M}.
\end{tikzcd}$$

A map of (pre)cyclotomic $\underline{\underline{A}}$-modules is a map of precyclotomic spectra that is also a map of $\underline{A}$-modules.

When the underlying $\mathbb{T}$-equivariant orthogonal spectrum of $\underline{\underline{A}}$ is in the class $\mathscr{E}$ of Section 2 (for example, when $\underline{A}$ is cofibrant as a $\mathbb{T}$-equivariant commutative or associative ring spectrum in any of the model categories we consider), the smash product with $\underline{A}$ monad $\underline{A}\wedge(-)$ on $\mathbb{T}$-equivariant orthogonal spectra lifts to a monad $\underline{\underline{A}} \wedge (-)$ on precyclotomic spectra, and a precyclotomic $\underline{\underline{A}}$-module is precisely an algebra over this monad as usual.

We concentrate on the case of commutative ring precyclotomic spectra and we recall from [Cyc23] a "shortcut" for describing (up to weak equivalence) the mapping space in this category in terms of mapping spaces in $\mathbb{T}$-equivariant commutative ring orthogonal spectra. Given commutative ring precyclotomic spectra $\underline{\underline{A}}$ and $\underline{\underline{B}}$, the space of maps of commutative ring precyclotomic spectra $\mathscr{C}om^{\mathrm{Cyc}}(\underline{\underline{A}}, \underline{\underline{B}})$ can be identified as the equalizer of

$$\mathscr{C}om^{\mathbb{T}}(\underline{A}, \underline{B}) \rightrightarrows \mathscr{C}om^{\mathbb{T}}(\Phi\underline{A}, \underline{B})$$

where one map in the system takes the map $f\colon \underline{A} \to \underline{B}$ to $f \circ r_{\underline{\underline{A}}}$ and the other takes it to $r_{\underline{\underline{B}}} \circ \Phi f$.

To translate this into a homotopical result, we use the model structure on the category of commutative ring precyclotomic spectra of [Cyc23]. The fibrations and weak equivalences in this structure are created by the forgetful functor to precyclotomic spectra, which in turn are created by the forgetful functor to a variant $\mathscr{F}_p$-model structure on $\mathbb{T}$-equivariant orthogonal spectra. For a cofibrant commutative ring precyclotomic spectrum $\underline{\underline{A}}$, the underlying $\mathbb{T}$-equivariant commutative ring spectra $\underline{A}$ and $\Phi\underline{A}$ are cofibrant. When in addition $\underline{\underline{B}}$ is fibrant, the map from the equalizer above to the corresponding homotopy equalizer is a weak equivalence and [Cyc23] gives the following shortcut to computing the derived mapping spaces.

Proposition 4.5 ([Cyc23])**.** *Let $\underline{\underline{A}}$, $\underline{\underline{B}}$ be commutative ring precyclotomic spectra, and assume that for $\underline{\underline{A}}$ the underlying $\mathbb{T}$-equivariant commutative ring orthogonal spectrum is cofibrant and for $\underline{\underline{B}}$ the underlying $\mathbb{T}$-equivariant commutative ring orthogonal spectrum is fibrant in the model structure of* [Cyc23]*. Then the derived* *mapping space* $\mathbb{R}\mathscr{C}om^{\mathrm{Cyc}}(\underline{\underline{A}}, \underline{\underline{B}})$ *of commutative ring precyclotomic spectra maps from* $\underline{\underline{A}}$ *to* $\underline{\underline{B}}$ *is represented by the homotopy equalizer of the maps*

$$r^*_{\underline{\underline{A}}}, r_{\underline{\underline{B}}*} \circ \Phi \colon \mathscr{C}om^{\mathbb{T}}(\underline{A}, \underline{B}) \rightrightarrows \mathscr{C}om^{\mathbb{T}}(\Phi \underline{A}, \underline{B})$$

Specializing to the case when $\underline{\underline{A}} = THH(R)$ for a cofibrant commutative ring orthogonal spectrum R, not only is the underlying $\mathbb{T}$-equivariant commutative ring orthogonal spectrum $\underline{A}$ cofibrant, but the precyclotomic structure map $\Phi\underline{A} \to \underline{A}$ is an isomorphism. Using this isomorphism, the derived mapping space $\mathbb{R}\mathscr{C}om^{\mathrm{Cyc}}(\underline{\underline{A}}, \underline{\underline{B}})$ is represented by the homotopy equalizer of the maps

$$\mathrm{id}, r^{*-1}_{\underline{\underline{A}}} \circ (r_{\underline{\underline{B}}*} \circ \Phi) \colon \mathscr{C}om^{\mathbb{T}}(\underline{A}, \underline{B}) \rightrightarrows \mathscr{C}om^{\mathbb{T}}(\underline{A}, \underline{B}).$$

The universal property of Proposition 2.4 lets us identify the derived mapping space $\mathbb{R}\mathscr{C}om^{\mathrm{Cyc}}(\underline{\underline{A}}, \underline{\underline{B}})$ in this case as a homotopy equalizer

$$\mathrm{id}, \varsigma \colon \mathscr{C}om(R, B) \rightrightarrows \mathscr{C}om(R, B) \tag{4.6}$$

for some map ς, which we now describe. It requires a structure on commutative ring precyclotomic spectra that we call the cyclotomic power operation.

Definition 4.7. Let $\underline{\underline{B}}$ be a commutative ring precyclotomic spectrum. The *cyclotomic power operation* is the composite

$$\Psi \colon B \longrightarrow \Phi^{C_p} N_e^{C_p} B \longrightarrow \Phi^{C_p} \underline{B} \xrightarrow{r_{\underline{\underline{B}}}} B,$$

where the first map is the diagonal, the second map is Φ^{C_p} applied to the map (2.3), and the third map is the precyclotomic structure map.

Proposition 4.8. *The map* ς *in* (4.6) *is the map* Ψ_* *given by post-composition with the cyclotomic power operation on* $\underline{\underline{B}}$.

Proof. Given a map $f \in \mathscr{C}om(R, B)$, consider the diagram in (non-equivariant) commutative ring orthogonal spectra

$$\begin{tikzcd}
R \arrow[r, "\cong"] \arrow[d] \arrow[dd, bend right, "f"'] & \Phi^{C_p}(N_e^{C_p} R) \arrow[d] \arrow[dr, dotted] & \\
THH(R) \arrow[r, "\cong"] \arrow[d, "\tilde{f}"] & \Phi^{C_p}(N_e^{C_p} THH(R)) \arrow[r] \arrow[d, "\Phi^{C_p} N_e^{C_p} \tilde{f}"] & \Phi^{C_p} THH(R) \arrow[d, "\Phi^{C_p} \tilde{f}"] \\
B \arrow[r] & \Phi^{C_p}(N_e^{C_p} B) \arrow[r] & \Phi^{C_p} \underline{B}
\end{tikzcd} \tag{4.9}$$

where $\tilde{f}$ is the underlying non-equivariant map of the map $THH(R) \to \underline{B}$ adjoint to f under Proposition 2.4 and in the bottom two rows, the maps are the first two maps in the definition of Ψ. In the top row the isomorphism is the diagonal. The solid part of the diagram then commutes, and if we fill in the dotted map $R \to \Phi^{C_p} THH(R)$ as the map induced on Φ^{C_p} by the inclusion $N_e^{C_p} R \to THH(R)$, the whole diagram commutes. When we interpret the top isomorphism followed by the dotted map as a (non-equivariant) map $R \to \Phi THH(R)$, the adjoint map

$$THH(R) \longrightarrow \Phi THH(R)$$

under Proposition 2.4 is the map $r_{\underline{\underline{A}}}^{-1}$. Viewing the overall composite map $R \to \Phi^{C_p}B$ as a (non-equivariant) map $R \to \Phi B$, the adjoint map

$$THH(R) \longrightarrow \Phi B$$

is then $r_{\underline{\underline{A}}}^{-1} \circ \Phi\tilde{f}$. □

The diagram in the proof above also identifies the cyclotomic power operation Ψ on $THH(R)$.

Proposition 4.10. *Let R be a cofibrant commutative ring orthogonal spectrum. Then $\Psi\colon THH(R) \to THH(R)$ is the Adams operation given by tensoring R with the p-fold covering map on $\mathbb{T}$ (see, for example,* [1, §10]*).*

Proof. The proof of Proposition 4.8 presents a diagram (4.9) with the composite map $R \to \Phi^{C_p}THH(R)$ in it the map adjoint (under Proposition 2.4) to $r_{THH(R)}^{-1}$, the inverse of the cyclotomic structure map for $THH(R)$. The operation Ψ on $THH(R)$ is the composite of the middle row of that diagram $THH(R) \to \Phi^{C_p}THH(R)$ with the cyclotomic structure map $r_{THH(R)}$. It follows that the composite of the inclusion $R \to THH(R)$ with the operation Ψ is again the inclusion $R \to THH(R)$. If the operation Ψ were equivariant, the adjunction of Proposition 2.4 would identify it as the identity map; however, it is not equivariant for the usual $\mathbb{T}$-action on THH. The proof of the statement is essentially a careful check that it is equivariant when the target is given the $\mathbb{T}$-action pulled back from the p-fold covering map $\mathbb{T} \to \mathbb{T}$.

Working on the point-set level, $\mathbb{T}$-equivariantly, $THH(R)$ is given by the cyclic bar construction followed by change of universe

$$THH(R) = I_{\mathbb{R}^\infty}^{U} N^{\mathrm{cy}}(R) = I_{\mathbb{R}^\infty}^{U}(R \otimes \mathbb{T}).$$

(As functors from commutative ring orthogonal spectra to $\mathbb{T}$-equivariant orthogonal spectra indexed on $\mathbb{R}^\infty$, we can identify the cyclic bar construction as the tensor with $\mathbb{T}$, and we use these descriptions interchangeably.) Working non-equivariantly is in particular working in the universe $\mathbb{R}^\infty$; keeping track of universes, the first map in Ψ, the diagonal map is

$$N^{\mathrm{cy}}(R) \xrightarrow{\cong} \Phi^{C_p}(N_e^{C_p} N^{\mathrm{cy}}(R)).$$

We recall that $N_e^{C_p}$ is a continuous point-set functor from non-equivariant orthogonal spectra indexed on $\mathbb{R}^\infty$ to C_p-equivariant orthogonal spectra indexed on the complete universe U_{C_p} (obtained by restricting the $\mathbb{T}$-action on U to C_p), and Φ^{C_p} is a continuous point-set functor on the same categories in the opposite direction. The map above is natural in maps on $N^{\mathrm{cy}}(R)$ in (non-equivariant) orthogonal spectra and so is a map of $\mathbb{T}$-equivariant orthogonal spectra indexed on $\mathbb{R}^\infty$ (using the $\mathbb{T}$-action on $N^{\mathrm{cy}}(R)$). For the next map in Ψ, written

$$\Phi^{C_p} N_e^{C_p} THH(R) \longrightarrow \Phi^{C_p} THH(R),$$

we are looking at $THH(R)$ on the right as a C_p-equivariant orthogonal spectrum indexed on U_{C_p}. This map is Φ^{C_p} applied to the map

$$N_e^{C_p} N^{\mathrm{cy}}(R) \longrightarrow I_{\mathbb{R}^\infty}^{U_{C_p}} N^{\mathrm{cy}}(R)$$

of C_p-equivariant orthogonal spectra indexed on U_{C_p}. Again by naturality, using the action of $\mathbb{T}$ on $N^{\mathrm{cy}}(R)$ in the category of non-equivariant orthogonal spectra, the displayed map is $\mathbb{T}$-equivariant in the category of C_p-equivariant orthogonal spectra indexed on U_{C_p}. The composite

$$N^{\mathrm{cy}}(R) \xrightarrow{\cong} \Phi^{C_p}(N_e^{C_p} N^{\mathrm{cy}}(R)) \longrightarrow \Phi^{C_p}(I_{\mathbb{R}^\infty}^{U_{C_p}} N^{\mathrm{cy}}(R)) \tag{4.11}$$

is a map of $\mathbb{T}$-equivariant orthogonal spectra indexed on $\mathbb{R}^\infty$ and so is determined by the restriction of its underlying map of non-equivariant spectra along the inclusion $R \to N^{\mathrm{cy}}(R)$, which was analyzed in (4.9). Rewritten to emphasize the universes, it is the map

$$R \xrightarrow{\cong} \Phi^{C_p}(N_e^{C_p} R) \longrightarrow \Phi^{C_p}(I_{\mathbb{R}^\infty}^{U_{C_p}} N^{\mathrm{cy}}(R)) \tag{4.12}$$

induced by the map

$$N_e^{C_p} R = I_{\mathbb{R}^\infty}^{U_{C_p}}(R \otimes C_p) \longrightarrow I_{\mathbb{R}^\infty}^{U_{C_p}}(R \otimes \mathbb{T}) = I_{\mathbb{R}^\infty}^{U_{C_p}} N^{\mathrm{cy}}(R).$$

Because the target of (4.12) is C_p-fixed, we can treat it as a $\mathbb{T}/C_p$-equivariant orthogonal spectrum indexed on $\mathbb{R}^\infty$ and the identification of (4.12) above shows that under the isomorphism $\rho\colon \mathbb{T} \cong \mathbb{T}/C_p$, the adjoint map

$$R \otimes (\mathbb{T}/C_p) \longrightarrow \Phi^{C_p}(I_{\mathbb{R}^\infty}^{U_{C_p}} N^{\mathrm{cy}}(R))$$

becomes the underlying non-equivariant map of $r_{THH(R)}^{-1}$. It follows that the map (4.11) is the composite of the p-fold cover map $\mathbb{T} \to \mathbb{T}$ and $r_{THH(R)}^{-1}$. Thus, composing with the underlying non-equivariant map of $r_{THH(R)}$, Ψ is the map induced by the p-fold cover. □

5. Relative cyclotomic structures

The purpose of this section is to explain our new framework for the existence of cyclotomic structures on relative THH. The work of [1] shows that for a commutative ring (pre)cyclotomic spectrum $\underline{\underline{R}}$ and an R-algebra A, $THH^{\underline{R}}(A)$ obtains a natural (pre)cyclotomic structure precisely when the canonical map of $\mathbb{T}$-equivariant commutative ring spectra $THH(R) \to \underline{R}$ is a (pre)cyclotomic map on the point-set level. In this section, we generalize this to the case when the map $THH(R) \to \underline{R}$ lifts to a map of commutative ring (pre)cyclotomic spectra in the homotopy category. Our framework depends on the characterization in the previous section of derived mapping spaces by allowing us to specify in terms of the cyclotomic power operation when the canonical map $THH(R) \to \underline{R}$ lifts.

Specifically, let $\underline{\underline{R}}$ be a commutative ring precyclotomic spectrum, and assume without loss of generality that $\underline{\underline{R}}$ is cofibrant and fibrant in that category for the model structure of [Cyc23]. Then the underlying non-equivariant commutative ring orthogonal spectrum R is cofibrant and fibrant in the model structure of [Cyc23] (by [Cyc23]). In this case, Proposition 4.8 specializes to the following result.

Proposition 5.1. *Let $\underline{\underline{R}}$ be a cofibrant-fibrant commutative ring precyclotomic spectrum. Then the derived mapping space $\mathbb{R}\mathscr{C}om^{\mathrm{Cyc}}(THH(R), \underline{\underline{R}})$ is modeled by the homotopy equalizer of the self-maps*

$$\mathrm{id}, \Psi_*\colon \mathscr{C}om(R,R) \longrightarrow \mathscr{C}om(R,R).$$

In particular, the canonical map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra $THH(R) \to \underline{R}$ lifts (up to homotopy) to a map of commutative ring precyclotomic spectra from $THH(R) \to \underline{\underline{R}}$ exactly when Ψ is homotopic to the identity. We encapsulate this in the following definition.

Definition 5.2. A *(pre)cyclotomic base* is a commutative ring (pre)cyclotomic spectrum $\underline{\underline{R}}$ together with a choice of homotopy from Ψ to the identity in the category of commutative ring orthogonal spectra.

The following is the main foundational theorem of the paper and gives a more specific formulation of Theorem A of the introduction.

Theorem 5.3. *Let $\underline{\underline{R}}$ be a (pre)cyclotomic base, whose underlying $\mathbb{T}$-equivariant commutative ring orthogonal spectrum is cofibrant in the model structure of* [Cyc23] *(for example, when $\underline{\underline{R}}$ is cofibrant in the model structure of* [Cyc23] *on commutative ring (pre)cyclotomic spectra). Then the derived relative THH functor $\mathbb{L}THH^{\underline{\underline{R}}}$ lifts to a functor from associative R-algebras to (pre)cyclotomic $\underline{\underline{R}}$-modules.*

Proof. Replacing $\underline{\underline{R}}$ by a weakly equivalent commutative ring (pre)cyclotomic spectrum if necessary, we can assume without loss of generality that $\underline{\underline{R}}$ is fibrant in the model structure of [Cyc23]. By Proposition 3.8, for A cofibrant as an associative R-algebra, $\mathbb{L}THH^{\underline{\underline{R}}}(A)$ is represented by the point set construction

$$THH^{\underline{\underline{R}}}(A) = THH(A) \wedge_{THH(R)} \underline{R} \simeq THH(A) \wedge^{\mathbb{L}}_{THH(R)} \underline{R}.$$

By the hypothesis on Ψ and Proposition 4.8, the discussion around (4.6) implies that the diagram in the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra

$$\begin{tikzcd}
\Phi THH(R) \arrow[r] \arrow[d, "r_{THH(R)}"'] & \Phi\underline{R} \arrow[d, "r_{\underline{\underline{R}}}"] \\
THH(R) \arrow[r] & \underline{R}
\end{tikzcd}$$

commutes up to (the given) homotopy. (This specifies an element of the homotopy equalizer in Proposition 4.5 for $\underline{\underline{A}} = THH(R)$ and $\underline{\underline{B}} = \underline{\underline{R}}$.) While $THH(R)$ is cofibrant in the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra, it is generally not cofibrant in the category of commutative ring cyclotomic spectra. Let $\underline{\underline{T}} \to THH(R)$ be a cofibrant approximation in the latter category; the corresponding diagram in $\underline{T}$ and $\underline{R}$ then also commutes up to (the restriction of the given) homotopy, specifying an element of the homotopy equalizer in Proposition 4.5 (for $\underline{\underline{A}} = \underline{\underline{T}}$ and $\underline{\underline{B}} = \underline{\underline{R}}$). Since $\underline{\underline{T}}$ is cofibrant and $\underline{\underline{R}}$ is fibrant, Proposition 4.5 implies that the composite map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra $\underline{T} \to \underline{R}$ is homotopic to a precyclotomic map $g\colon \underline{\underline{T}} \to \underline{\underline{R}}$: the map from the equalizer to the homotopy equalizer in Proposition 4.5 is a weak equivalence. The space of choices of such a g together with a path H in the homotopy equalizer from g to the point specified above is weakly contractible, and we choose an element (g, H). As a component of the path H, we get a homotopy G from the composite map $\underline{\underline{T}} \to THH(R) \to \underline{\underline{R}}$ to g. We write $g^*\underline{R}$ for $\underline{R}$ with the $\mathbb{T}$-equivariant commutative $\underline{T}$-algebra structure from the map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra g, and we use G to give $\underline{R}$ a $\mathbb{T}$-equivariant $\underline{T} \otimes I$-module structure. We then have a zigzag of weak equivalences

$$THH(A) \wedge^{\mathbb{L}}_{THH(R)} \underline{R} \xleftarrow{\simeq} THH(A) \wedge^{\mathbb{L}}_{\underline{T}} \underline{R} \xrightarrow{\simeq} THH(A) \wedge^{\mathbb{L}}_{\underline{T} \otimes I} \underline{R} \xleftarrow{\simeq} THH(A) \wedge^{\mathbb{L}}_{\underline{T}} g^*\underline{R}.$$

Let $\underline{\underline{M}}$ be a cofibrant approximation of $g^*\underline{\underline{R}}$ in the model structure on (pre)cyclotomic T-modules of [Cyc23]. Then $(-) \wedge_{\underline{T}} \underline{M}$ represents the derived functor $(-) \wedge^{\mathbb{L}}_{\underline{T}} g^*\underline{R}$ and $THH(-) \wedge_{\underline{T}} \underline{M}$ is another point-set model for $\mathbb{L}THH^{\underline{\underline{R}}}(-)$.

By construction, T is in the class of objects $\mathscr{E}$ discussed in Section 2 on which Φ and the smash product behave. By [Cyc23], for any $\mathbb{T}$-equivariant $\underline{T}$-module $\underline{N}$, the canonical map

$$\Phi\underline{N} \wedge_{\Phi\underline{T}} \Phi\underline{M} \longrightarrow \Phi(\underline{N} \wedge_{\underline{T}} \underline{M})$$

is an isomorphism. For $\underline{N} = THH(A)$, using the precyclotomic structure maps on $THH(A)$, $\underline{\underline{T}}$, and $\underline{\underline{M}}$, we get a precyclotomic structure map

$$\Phi(THH(A) \wedge_{\underline{T}} \underline{M}) \cong \Phi THH(A) \wedge_{\Phi\underline{T}} \Phi\underline{M} \longrightarrow THH(A) \wedge_{\underline{T}} \underline{M}.$$

For any family $\mathscr{F}$ of proper subgroups of $\mathbb{T}$, the composite map

$$\mathbb{L}\Phi(THH(A) \wedge_{\underline{T}} \underline{M}) \longrightarrow THH(A) \wedge_{\underline{T}} \underline{M}$$

is an $\mathscr{F}$-equivalence whenever the precyclotomic structure map of $\underline{\underline{M}}$ is. □

As a consequence, we can define relative topological cyclic homology in this context. Since we are concentrating on p-(pre)cyclotomic spectra, the relevant version of topological cyclic homology is p-typical TC, denoted as $TC(-;p)$ in [7] (and elsewhere). To avoid ambiguity between TC of a precyclotomic object and TC of a ring spectrum, we use the following notation.

Notation 5.4. We write TC_{cyc} for the composite functor $TC(R(-);p)$ from precyclotomic spectra to spectra, where $TC(-;p)$ is the p-typical TC-construction of [7, 6.3] (for example) and R is a fibrant approximation functor in the category of precyclotomic spectra. We write TC for the functor $TC_{\mathrm{cyc}}(THH(-))$ from associative or commutative ring orthogonal spectra to orthogonal spectra.

The construction of $TC(-;p)$ in the previous paragraph uses the maps between categorical fixed points

$$\mathrm{R}, \mathrm{F}\colon \underline{X}^{C_{p^{n+1}}} \longrightarrow \underline{X}^{C_{p^n}}$$

where F is the inclusion and R is the composite of the canonical map

$$\underline{X}^{C_{p^{n+1}}} \cong (\rho^*(\underline{X}^{C_p}))^{C_{p^n}} \longrightarrow (\Phi\underline{X})^{C_{p^n}} \longrightarrow \underline{X}^{C_{p^n}}$$

where the middle map is the map of [15, 4.4] from the categorical fixed points to the geometric fixed points and the last map is the C_{p^n} fixed points of the precyclotomic structure map $r_{\underline{\underline{X}}}$. We note that the construction only requires a precyclotomic structure and not a cyclotomic structure. For the homotopically correct construction of p-typical TC, we need the homotopically correct categorical fixed points, which we ensure by fibrant approximation of $\underline{X}$ in the category of precyclotomic spectra in the definition of TC_{cyc}.

In the case when the underlying $\mathbb{T}$-equivariant spectrum $\underline{X}$ is p-complete (i.e., all of the categorical fixed point spectra are p-complete as non-equivariant spectra), $TC_{\mathrm{cyc}}(\underline{\underline{X}})$ is just the derived mapping spectrum of maps out of the sphere spectrum in the category of (pre)cyclotomic spectra [7, 6.8]:

$$TC_{\mathrm{cyc}}(\underline{\underline{X}}) \cong \mathbb{R}F^{\mathrm{Cyc}}(\mathbb{S}, \underline{\underline{X}}).$$

Returning to the case of relative THH, we use the following notation for TC of these (pre)cyclotomic spectra.

Notation 5.5. Let $\underline{\underline{R}}$ be a (pre)cyclotomic base. Write $TC^{\underline{\underline{R}}}$ for derived TC_{cyc} of derived $THH^{\underline{\underline{R}}}$ with the (pre)cyclotomic structure of Theorem 5.3.

While this adequately constructs a derived functor $TC^{\underline{\underline{R}}}$ (from the homotopy category of R-algebras to the stable category), the constructions in Section 9 require a point-set model that is functorial in $\underline{\underline{R}}$ as well as A. We describe such a model in Section 10.

The construction in Theorem 5.3 of the precyclotomic structure depended on the choice of homotopy in the precyclotomic base structure on $\underline{\underline{R}}$. We observe that homotopic homotopies construct weakly equivalent precyclotomic structures. In the following proposition, let D denote the 2-disk obtained as the (unreduced) cone on two copies of $[0,1]$ with corresponding endpoints identified.

Proposition 5.6. *Let $\underline{\underline{R}}'$, $\underline{\underline{R}}''$ be precyclotomic bases with the same underlying commutative ring precyclotomic spectrum $\underline{\underline{R}}$. A map of commutative ring orthogonal spectra $R \otimes D \to R$ which restricts on each copy of $[0,1]$ to the homotopy intrinsic to the precyclotomic base structures on $\underline{\underline{R}}'$ and $\underline{\underline{R}}''$ induces a natural isomorphism in the homotopy category of precyclotomic spectra*

$$\mathbb{L}THH^{\underline{\underline{R}}'}(-) \simeq \mathbb{L}THH^{\underline{\underline{R}}''}(-).$$

6. Precyclotomic bases, equivariant factorization homology, and global commutative ring spectra with multiplicative deflations

The purpose of this section is give a conceptual explanation of the motivation behind the definition of a precyclotomic base. The discussion is purely motivational and should not be regarded as rigorously justified. The material here was inspired by a October 2021 talk given by Asaf Horev at MIT. Horev discussed the structure of equivariant factorization homology and its relationship to the cyclotomic structure on THH. Extending these observations from spectra to categories of R-modules and examining the required structure on R leads to the framework of precyclotomic bases, as we explain below. Abstracting the categorical framework using ideas of Bachmann-Hoyois [2, §9] (see also the third author's work in [22, §2]) gives rise to the notion of global commutative ring spectrum with multiplicative deflations (Definition 6.4). This structure arises in nature on the global equivariant Thom spectrum $\underline{MUP}$ and on a new global equivariant structure on MU, motivating the key examples of cyclotomic bases that we discuss in more detail in the next section.

As we progress in this section, we will require ∞-categorical constructions, and statements should be read in ∞-categorical terms. In particular, functors in this section should be read in their homotopical rather than point-set forms.

We begin by recalling the relevant parts of standard structure of genuine equivariant factorization homology. Let G be a finite group and for simplicity, let $\underline{A}$ be a genuine G-equivariant commutative ring orthogonal spectrum with underlying (non-equivariant) commutative ring orthogonal spectrum A. Then equivariant factorization homology associates to each G-manifold M a genuine G-equivariant orthogonal spectrum $\int_M \underline{A}$ with the following features (among others):

(i) When $M = G/H$ is a transitive G-set, there is a canonical equivalence $\int_M \underline{A} \simeq N_H^G \operatorname{res}_H^G \underline{A}$. In particular, $\int_{G/e} \underline{A} \simeq N_e^G A$ depends only on the underlying non-equivariant commutative ring A.

(ii) When M has a free action of G, there is a natural equivalence

$$\Phi^G \int_M \underline{A} \simeq \int_{M/G} A.$$

(In the more general case when $\underline{A}$ is some kind of equivariant disk algebra, representation tubes of the form $G \times_H V$ should replace orbits in (i).)

Two remarks are in order regarding the features above: first, in the special case $M = G/e$, the equivalence in (ii) extends the diagonal identity $\Phi^G N_e^G A \simeq A$, which is the case $M = G$. Second, the observation about restriction in (i) generalizes to (ii): when M is free, $\int_M \underline{A}$ depends only on the non-equivariant commutative ring orthogonal spectrum A. Indeed, features (i) and (ii) are closely related, with (ii) deriving from (i).

The above features relate to the cyclotomic structure on THH in the following way. We take the manifold M to be $\mathbb{T}$ and G to be a finite subgroup C_n of $\mathbb{T}$. The genuine C_n-equivariant orthogonal spectrum $\int_{\mathbb{T}} M$ comes with a compatible $\mathbb{T}$-action extending the inherent C_n-action. As the subgroups C_n vary, they fit together to produce a genuine $\mathbb{T}$-equivariant $\mathscr{F}_{\text{fin}}$-colocal orthogonal spectrum. Since $\mathbb{T}$ is a free C_n-manifold, we also have (non-equivariant) equivalences

$$\Phi^{C_n} \int_{\mathbb{T}} A \simeq \int_{\mathbb{T}/C_n} A,$$

natural in C_n-equivariant self-maps of $\mathbb{T}$, and in particular natural in the $\mathbb{T}/C_n$-action on both sides. Applying the Borel equivariant version of ρ_n^* for $\rho_n \colon \mathbb{T} \cong \mathbb{T}/C_n$ the nth root map, we can view the above map as a Borel equivalence of $\mathbb{T}$-spectra

$$\rho_n^* \Phi^{C_n} \int_{\mathbb{T}} A \simeq \rho_n^* \int_{\mathbb{T}/C_n} A \cong \int_{\mathbb{T}} A.$$

Looking at all the C_n together (assuming the equivalences are appropriately compatible with inclusions of subgroups), we deduce a genuine $\mathbb{T}$-equivariant $\mathscr{F}_{\text{fin}}$-equivalence

$$\Phi \int_{\mathbb{T}} A \simeq \int_{\mathbb{T}} A.$$

Thus, the cyclotomic structure on THH is a consequence of the basic features of equivariant factorization homology.

In order to apply this observation to THH relative to a commutative ring orthogonal spectrum R, we would replace (equivariant) spectra with (equivariant) R-modules for some equivariant structure $\underline{R}$ on R. To start, we need the category of R-modules to admit norm functors ${}_{\underline{R}}N_e^G$ from (non-equivariant) R-modules to G-equivariant $\underline{R}$-modules, and geometric fixed point functors ${}_{\underline{R}}\Phi^G$ from G-equivariant $\underline{R}$-modules to (non-equivariant) R-modules, related by a natural equivalence

$$\mathrm{Id} \xrightarrow{\simeq} {}_{\underline{R}}\Phi^G {}_{\underline{R}}N_e^G,$$

which is symmetric monoidal and appropriately compatible with restrictions to subgroups (as encapsulated below).

Applying the norm N_e^G in spectra to an R-module X naturally yields an $N_e^G R$-module $N_e^G X$. To convert this to a $\underline{R}$-module, a norm multiplication $N_e^G R \to \underline{R}$ suffices, and we can set

$${}_{\underline{R}}N_e^G X := N_e^G X \wedge_{N_e^G R} \underline{R}.$$

If we also assume these norm multiplications satisfy the usual compatibilities of these norm multiplications, by [5, 6.11] the resulting structure essentially amounts to a genuine G-equivariant commutative ring orthogonal spectrum structure on $\underline{R}$ with underlying non-equivariant commutative ring orthogonal spectrum R.

Analogously, applying the geometric fixed point functor Φ^G in spectra to an $\underline{R}$-module $\underline{X}$ naturally yields a $\Phi^G\underline{R}$-module $\Phi^G\underline{X}$. To convert this to a R-module, it suffices to have a map of commutative ring orthogonal spectra $\Phi^G\underline{R} \to R$. That is, we define

$$_{\underline{R}}\Phi^G X := \Phi^G X \wedge_{\Phi^G\underline{R}} R.$$

We say more below about making these fit together along restriction maps, which leads to equivariance considerations, but for now we note that for $G = C_p$, this is the underlying non-equivariant map of a precyclotomic structure map.

Now consider the existence of a diagonal map $X \xrightarrow{\simeq} {}_{\underline{R}}\Phi^G_e\, {}_{\underline{R}}N_e^G X$. By definition, the target functor is the composite

$$_{\underline{R}}\Phi^G\, {}_{\underline{R}}N_e^G X = \Phi^G(N_e^G X \wedge_{N_e^G R} \underline{R}) \wedge_{\Phi^G\underline{R}} R \simeq (\Phi^G N_e^G X) \wedge_{\Phi^G N_e^G R} \Phi^G\underline{R} \wedge_{\Phi^G\underline{R}} R.$$

Using the diagonal equivalence for Φ^G and N_e^G in spectra, this functor is naturally equivalent to the extension of scalars $X \wedge_R R$ for the self-map of R given by the composite

$$\Psi \colon R \xrightarrow{\simeq} \Phi^G N_e^G R \longrightarrow \Phi^G\underline{R} \longrightarrow R.$$

An identification of the composite as the identity in the ∞-category of commutative ring orthogonal spectra then constructs a natural diagonal equivalence

$$X \xrightarrow{\simeq} {}_{\underline{R}}\Phi^G_e\, {}_{\underline{R}}N_e^G X$$

for R-modules. For $G = C_p$, the operation Ψ is precisely the operation in the definition of cyclotomic base, and (for $\underline{R}$ a precyclotomic spectrum) the identification in the ∞-category is essentially a choice of homotopy that gives the structure of a cyclotomic base.

All this was a discussion of norms functors and geometric fixed point functors in R-modules; we now turn to equivariant factorization homology. As in our simplification in the discussion of the features of equivariant factorization homology in spectra, we restrict to the commutative case. In this case, equivariant factorization homology extends to a functor on all G-spaces given by prolongation of norms. To make this work, extending norms to all finite G-sets by smash product, we need norms to be functorial in maps of G-sets. Assuming enough structure on $\underline{R}$ (including at least the structure above, more about which below), we can do this for commutative $\underline{R}$-algebras: for a commutative $\underline{R}$-algebra $\underline{A}$,

$$G/H_1 \amalg \cdots \amalg G/H_n \mapsto ({}_{\underline{R}}N_{H_1}^G \operatorname{res}_{H_1}^G \underline{A}) \wedge_R \cdots \wedge_R ({}_{\underline{R}}N_{H_n}^G \operatorname{res}_{H_n}^G \underline{A})$$

extends to a functor ${}_{\underline{R}}N\underline{A}$ from finite G-sets to $\underline{R}$-modules (or commutative $\underline{R}$-algebras). We then define

$$M \mapsto \int_M^{\underline{R}} \underline{A}$$

to be the functor (in M) from G-spaces to $\underline{R}$-modules (or commutative $\underline{R}$-algebras) given by the left Kan extension of ${}_{\underline{R}}N\underline{A}$ under the inclusion of finite G-sets in all G-spaces. (Because disjoint unions of G-sets go to coproducts of commutative $\underline{R}$-algebras, using commutative $\underline{R}$-algebras as the target and then forgetting to $\underline{R}$-modules gives a naturally equivalent functor.) The diagonal equivalence $A \xrightarrow{\simeq} {}_{\underline{R}}\Phi^G N_e^G A$ extends to a natural diagonal equivalence

$$_{\underline{R}}N\underline{A}(S/G) \xrightarrow{\simeq} {}_{\underline{R}}\Phi^G({}_{\underline{R}}N\underline{A}(S))$$

for free G-sets S and prolongs to a natural diagonal equivalence

$$\int_{M/G}^{\underline{R}} \underline{A} \xrightarrow{\simeq} {}_{\underline{R}}\Phi^G\Big(\int_M^{\underline{R}} \underline{A}\Big)$$

for free G-spaces M, with both $\int_{M/G}^{\underline{R}} \underline{A}$ and $\int_M^{\underline{R}} \underline{A}$ depending only on the underlying non-equivariant commutative R-algebra structure on $\underline{A}$. In other words, this theory has the features (i) and (ii) of equivariant factorization homology discussed above. In the case when $M = \mathbb{T}$ and we let G range over the finite subgroups $C_n < \mathbb{T}$ as above, with some good will and an extension of the structure of $\underline{R}$ to $\mathbb{T}$-equivariant commutative orthogonal spectra, the observations above on $\int_{\mathbb{T}}$ now generalize to $\int_{\mathbb{T}}^{\underline{R}}$: for a commutative R-algebra A, the $\mathscr{F}_{\mathrm{fin}}$-colocal $\mathbb{T}$-equivariant $\underline{R}$-module $\int_{\mathbb{T}}^{\underline{R}} A$ comes with an equivalence

$$\rho^*\left({}_R\Phi^{C_p}\int_{\mathbb{T}}^{\underline{R}} A\right) \simeq \int_{\mathbb{T}}^{\underline{R}} A.$$

We note that this is not necessarily a cyclotomic structure map but does induce a precyclotomic structure map

$$\Phi\int_{\mathbb{T}}^{\underline{R}} A = \rho^*\left(\Phi^{C_p}\int_{\mathbb{T}}^{\underline{R}} A\right) \longrightarrow \rho^*\left({}_R\Phi^{C_p}\int_{\mathbb{T}}^{\underline{R}} A\right) \simeq \int_{\mathbb{T}}^{\underline{R}} A.$$

The discussion above took a direct and streamlined approach to connect the ideas behind equivariant factorization homology in the category of R-modules and precyclotomic bases. One way to fill in some of the missing structure is to use the formulation of global equivariant E_∞ ring spectra in [2, §1.4, 9] and [22, §2]. Let Span(Gpd) denote the ∞-category whose objects are finite groupoids (that is, finite π_1 and finitely many components) X and where morphisms between X and Y are given by the space of spans $X \leftarrow Z \rightarrow Y$, with composition of spans evidenced by homotopy cartesian squares.

Definition 6.1. A (multiplicative) global equivariant context is a functor $\mathscr{C}(-)$: $\mathrm{Span(Gpd)} \to \mathrm{Cat}_\infty$ which sends disjoint unions to products.

Although we will not give details, global equivariant stable homotopy theory fits into this framework: there is a global equivariant context $\underline{\mathrm{Sp}}$ where $\underline{\mathrm{Sp}}(BG)$ is equivalent to the ∞-category of genuine G-equivariant orthogonal spectra. We note that every map in Span(Gpd) is equivalent to a disjoint union of composites of maps of the form $BG \leftarrow BH$, $BK \leftarrow BG$, $BH \to BG$, and $BG \to BK$ for inclusions of subgroups $H \to G$ and quotient maps of quotient groups $G \to K$. These component pieces have classical interpretations in equivariant stable homotopy theory:

Notation 6.2. Given a global equivariant context $\mathscr{C}$ and a short exact sequence of groups $0 \to H \hookrightarrow G \twoheadrightarrow K \to 0$, we denote:

$$\begin{aligned}
{}_{\mathscr{C}}\mathrm{res}^G_H &:= \mathscr{C}(BG \leftarrow BH \to BH) : \mathscr{C}(BG) \longrightarrow \mathscr{C}(BH)\\
{}_{\mathscr{C}}\mathrm{inf}^G_K &:= \mathscr{C}(BK \leftarrow BG \to BG) : \mathscr{C}(BK) \longrightarrow \mathscr{C}(BG)\\
{}_{\mathscr{C}}N^G_H &:= \mathscr{C}(BH \leftarrow BH \to BG) : \mathscr{C}(BH) \longrightarrow \mathscr{C}(BG)\\
{}_{\mathscr{C}}\Phi^H &:= \mathscr{C}(BG \leftarrow BG \to BK) : \mathscr{C}(BG) \longrightarrow \mathscr{C}(BK).
\end{aligned}$$

Proposition 6.3. *There is a global equivariant context* $\underline{\mathrm{Sp}}(-)\colon \mathrm{Span}(\mathrm{Gpd}) \to \mathrm{Cat}_\infty$ *which sends BG to the ∞-category Sp^G of G-spectra and the functors of 6.2 to the corresponding familiar functors in genuine equivariant stable homotopy theory.*

As the full subcategory $\mathrm{Span}(\mathrm{Fin}) \subset \mathrm{Span}(\mathrm{Gpd})$ spanned by the finite sets is the free symmetric monoidal ∞-category on a point, the restriction of any global equivariant context $\mathscr{C}$ to Span(Fin) exhibits $\mathscr{C}(*)$ as a symmetric monoidal ∞-category. More generally, each of the $\mathscr{C}(X)$ are endowed with a natural symmetric monoidal structure, and the functors of 6.2 are naturally symmetric monoidal.

Definition 6.4. Let $\Xi\colon \widetilde{\mathrm{Sp}} \to \mathrm{Span}(\mathrm{Gpd})$ denote the coCartesian fibration corresponding to the multiplicative global equivariant context $\underline{\mathrm{Sp}}$ of Proposition 6.3 and let $\mathfrak{res} \subset \mathrm{Mor}(\mathrm{Span}(\mathrm{Gpd}))$ be the subset of spans of the form

$$X \xleftarrow{f} Y \xrightarrow{=} Y$$

where f is a finite cover of the classifying spaces (up to weak equivalence). Then a global commutative ring spectrum with multiplicative deflations is a section of Ξ which is coCartesian over $\mathfrak{res}$.

Remark 6.5. The notion of a global commutative ring spectrum with mulitplicative deflations is closely related to notions appearing in work of Schwede [19] (where it might be called an ultracommutative monoid with multiplicative deflations), Bachmann-Hoyois [2], and the third author [22].

Let $\underline{R}\colon \mathrm{Span}(\mathrm{Gpd}) \to \widetilde{\mathrm{Sp}}$ be a global commutative ring spectrum with multiplicative deflations. Then we have $\underline{R}(BG) \in \mathrm{Sp}^G$, as Sp^G is the fiber of Ξ over BG. We think of $\underline{R}(BG)$ as the underlying G-equivariant commutative ring spectrum of $\underline{R}$. It admits natural maps (with reference to 6.2):

$$\begin{aligned} \mathrm{res}^G_H\, \underline{R}(BG) &\simeq \underline{R}(BH) \\ \inf^G_K \underline{R}(BK)) &\longrightarrow \underline{R}(BG) \\ N^G_H \underline{R}(BH) &\longrightarrow \underline{R}(BG) \\ \Phi^H\, \underline{R}(BG) &\longrightarrow \underline{R}(BK). \end{aligned}$$

In particular, looking at the last two maps with H and K the trivial group (using the trivial group as both a subgroup and quotient group of G), $\underline{R}$ has the structure maps needed in the factorization homology discussion. The identity

$$(BG \leftarrow BG \to *) \circ (* \leftarrow * \to BG) = (* \leftarrow * \to *)$$

shows that the composition in $\mathrm{Sp} = \mathscr{C}(*)$

$$\Phi^G N^G_e \underline{R}(B) \longrightarrow \Phi^G \underline{R}(BG) \longrightarrow \underline{R}(*)$$

is equivalent to the identity map $\underline{R}(*) = \underline{R}(*)$, which is the condition on these maps that we identified above in the factorization homology discussion and which formed the underlying condition in the definition of precyclotomic base.

We note that treatment of the equivariance of geometric fixed points requires consideration of quotient groups for general G; for finite subgroups of $\mathbb{T}$, the root isomorphisms $\rho_n\colon \mathbb{T} \cong \mathbb{T}/C_n$ provides a way around this. For $G = C_{pn}$, the structure on $\underline{R}$ gives maps in Sp^{C_{pn}/C_p},

$$\Phi^{C_p} \underline{R}(BC_{pn}) \longrightarrow \underline{R}(B(C_{pn}/C_p))$$

which forget under restriction to the map

$$\Phi^{C_p}\underline{R}(BC_p) \longrightarrow \underline{R}(*).$$

Recalling that ρ^* denotes the functor $\mathrm{Sp}^{C_{pn}/C_p} \xrightarrow{\simeq} \mathrm{Sp}^{C_n}$ coming from the pth root isomorphism $\rho\colon C_n \cong C_{pn}/C_p$ and the convention $\Phi = \rho^*\Phi^{C_p}$, the previous maps take the form

$$\Phi\underline{R}(BC_{pn}) \longrightarrow \underline{R}(BC_n)$$

in Sp^{C_n}, and they are compatible under restriction with the system of finite subgroups C_n of $\mathbb{T}$.

When $\underline{R}$ is a global commutative ring spectrum with multiplicative deflations, then the above discussion suggests that there should be a genuine equivariant context $\underline{\mathrm{Mod}}_{\underline{R}}(-)$ with

$$\underline{\mathrm{Mod}}_{\underline{R}}(BG) \simeq \mathrm{Mod}_{\underline{R}_G}(\mathrm{Sp}^G),$$

restriction given by the forgetful functors, the norm functor given by ${}_RN_H^G$, and the geometric fixed point functor given by ${}_R\Phi^H$. Then, one should be able to define equivariant factorization homology relative to $\underline{R}$ as equivariant factorization homology internal to the global equivariant context $\underline{\mathrm{Mod}}_{\underline{R}}(-)$, at least in the simplified form discussed above, i.e., in terms of prolonging a norm functor from finite G-sets to all G-spaces.

In general, one does not expect naturally occuring global ring spectra $\underline{R}$ to admit "multiplicative deflation" maps $\Phi^G\underline{R} \to \underline{R}$. For instance, for global equivariant K-theory, we have that p is invertible in $\Phi^{C_p}\underline{KU}$ but not in KU, so it cannot admit such structure. Nevertheless, in addition to the sphere spectrum $\mathbb{S}$, we have the following example of interest, which we discuss in more detail in the following section.

Example 6.6. The complex cobordism ring spectrum MU and its periodic variant MUP extend to global commutative ring spectra with multiplicative deflations. To see this, let $\mathrm{Vect}_{\mathbb{C}}$ denote the topological category of finite dimensional complex vector spaces and consider the functor

$$\underline{ku}\colon \mathrm{Span}(\mathrm{Gpd}) \longrightarrow \mathrm{Sp}$$

which sends X to the underlying spectrum of the symmetric monoidal topological category $\mathrm{Fun}(X, \mathrm{Vect}_{\mathbb{C}})$ and sends a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ to the functor given by restriction along f followed by left Kan extension along g. There should be a natural global J-homomorphism

$$\underline{ku} \longrightarrow \underline{\mathrm{pic}}(\underline{\mathrm{Sp}})$$

(essentially given by sending $V \in \mathrm{Fun}(X, \mathrm{Vect}_{\mathbb{C}})$ to the invertible equivariant spectrum S^V), and the global Thom spectrum of this would be a global commutative ring spectrum with multiplicative deflations lifting the usual global equivariant structure on $\underline{MUP}$. One can also expect to obtain MU as the underlying commutative ring spectrum of an global commutative ring spectrum with multiplicative deflations using a variant of this procedure, by taking the global Thom spectrum of $\underline{bu} = \tau_{\geq 2}\underline{ku}$ instead of $\underline{ku}$. We note that this is not the usual global equivariant structure on MU for equivariant homotopical bordism nor for geometric equivariant bordism. We denote this global equivariant spectrum as $\underline{mu}$. We explain these examples carefully in Example 7.4.

7. Examples

In this section, we observe that many examples of interest fit into our framework. The original example of a cyclotomic base is the sphere spectrum, where the operation Ψ is the identity map on the point-set level. Work of Bhatt-Morrow-Scholze [4, §11.1] on relative TC suggests that the spherical monoid ring $\mathbb{S}[t] = \Sigma^\infty_+ \mathbb{N}$ should have such a structure. The other examples we discuss were not previously known.

7.1. Example: $\mathbb{S}[G]$ and $\mathbb{S}[\mathbb{N}]$.

Let G be an abelian group. We consider the opposite constant $\mathbb{T}$-Mackey functor on G, which we denote as $\underline{G}^{\mathrm{op}}$: specifically, for a closed subgroup $H < \mathbb{T}$, $\underline{G}^{\mathrm{op}}(\mathbb{T}/H) = G$, and for a pair of closed subgroups $K < H < \mathbb{T}$, the transfer $\underline{G}^{\mathrm{op}}(\mathbb{T}/K) \to \underline{G}^{\mathrm{op}}(\mathbb{T}/H)$ is the identity, and the restriction $\underline{G}^{\mathrm{op}}(\mathbb{T}/H) \to \underline{G}^{\mathrm{op}}(\mathbb{T}/K)$ is multiplication by $\chi(H/K)$, that is, multiplication by the index if finite and zero if not. Associated to $\underline{G}^{\mathrm{op}}$ is the (genuine) $\mathbb{T}$-equivariant Eilenberg-Mac Lane spectrum $H\underline{G}^{\mathrm{op}}$, and for any model, the derived zeroth space $\Omega^\infty H\underline{G}^{\mathrm{op}}$ has a canonical genuine $\mathbb{T}$-equivariant E_∞-space structure, or more concisely, an $E_\infty^{\mathbb{T}}$-space structure. Fixing a model, $\Omega^\infty H\underline{G}^{\mathrm{op}}$, the underlying non-equivariant grouplike E_∞ space is equivalent to G, and we regard $\Omega^\infty H\underline{G}^{\mathrm{op}}$ as a $\mathbb{T}$-equivariant version of G.

We see from the concrete description of $\underline{G}^{\mathrm{op}}$ that the $\mathbb{T}$-Mackey functor $\pi_0(\rho^*((H\underline{G}^{\mathrm{op}})^{C_p}))$ is again $\underline{G}^{\mathrm{op}}$. We then get a weak equivalence

$$\rho^*((H\underline{G}^{\mathrm{op}})^{C_p}) \longrightarrow H\underline{G}^{\mathrm{op}}.$$

Applying the derived zeroth space functor, we get an $E^\infty_{\mathbb{T}}$-space map

$$\rho^*((\Omega^\infty H\underline{G}^{\mathrm{op}})^{C_p}) \longrightarrow \Omega^\infty H\underline{G}^{\mathrm{op}} \tag{7.1.1}$$

that is a weak equivalence.

Let $\underline{R} = \mathbb{S}[\Omega^\infty H\underline{G}^{\mathrm{op}}]$, the equivariant spherical group ring on $\Omega^\infty H\underline{G}^{\mathrm{op}}$: as genuine G-spectrum, $\underline{R}$ is the unreduced suspension spectrum on $\Omega^\infty H\underline{G}^{\mathrm{op}}$,

$$\underline{R} = \Sigma^\infty_+(\Omega^\infty H\underline{G}^{\mathrm{op}}),$$

but the $E_\infty^{\mathbb{T}}$-space structure on $\Omega^\infty H\underline{G}^{\mathrm{op}}$ endows $\underline{R}$ with the structure of an $E_\infty^{\mathbb{T}}$ ring spectrum. Using cofibrant replacement, we can find a weakly equivalent $E_\infty^{\mathbb{T}}$ ring spectrum which is a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum.

Notation 7.1.2. Let $\mathbb{S}_{\mathbb{T}}[G]$ denote a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum model of $\mathbb{S}[\Omega^\infty H\underline{G}^{\mathrm{op}}]$.

The point-set multiplicative geometric fixed point functor satisfies

$$\Phi^{C_p}\underline{R} = \Phi^{C_p}(\Sigma^\infty_+(\Omega^\infty H\underline{G}^{\mathrm{op}})) = \Sigma^\infty_+((\Omega^\infty H\underline{G}^{\mathrm{op}})^{C_p}),$$

and the map (7.1.1) gives a map of $E_\infty^{\mathbb{T}}$ ring spectra

$$r\colon \Phi\underline{R} \longrightarrow \underline{R},$$

which we use as the cyclotomic structure map for $\underline{R}$, giving a cyclotomic structure map for $\mathbb{S}_{\mathbb{T}}[G]$, which is a map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra.

Theorem 7.1.3. $\mathbb{S}_{\mathbb{T}}[G] \simeq \mathbb{S}[\Omega^\infty \underline{G}^{\mathrm{op}}]$ *with the cyclotomic structure above admits the structure of a cyclotomic base.*

Proof. We need to show that the operation

$$\Psi\colon \mathbb{S}_{\mathbb{T}}[G] \longrightarrow \mathbb{S}_{\mathbb{T}}[G],$$

is the identity in the homotopy category of commutative ring orthogonal spectra. Recall that this operation is the composite of the norm diagonal followed by multiplication

$$\mathbb{S}_{\mathbb{T}}[G] \longrightarrow \Phi^{C_p} N_e^{C_p} \mathbb{S}_{\mathbb{T}}[G] \longrightarrow \Phi^{C_p} \mathbb{S}_{\mathbb{T}}[G]$$

and (the non-equivariant map underlying) the cyclotomic structure map $\Phi\mathbb{S}_{\mathbb{T}}[G] \to \mathbb{S}_{\mathbb{T}}[G]$. We can understand the displayed map in terms of the spherical group ring $\mathbb{S}[\Omega^\infty \underline{G}^{\mathrm{op}}]$: it is just the induced map on $\mathbb{S}[-]$ of the transfer

$$\Omega^\infty \underline{G}^{\mathrm{op}} \longrightarrow (\Omega^\infty \underline{G}^{\mathrm{op}})^{C_p},$$

which is just Ω^∞ applied to the transfer

$$H\underline{G}^{\mathrm{op}} \longrightarrow (H\underline{G}^{\mathrm{op}})^{C_p},$$

which by definition of $\underline{G}^{\mathrm{op}}$ is homotopic to the identity, under the identification $(H\underline{G}^{\mathrm{op}})^{C_p} \simeq H\underline{G}^{\mathrm{op}}$ defining the cyclotomic structure map. □

Technically the definition of cyclotomic base requires a choice of point-set homotopy. When the model $S_{\mathbb{T}}[G]$ is cofibrant and fibrant in the model structure on commutative ring cyclotomic spectra of [Cyc23], the underlying (non-equivariant) commutative ring orthogonal spectrum is cofibrant and fibrant in the model structure of [Cyc23], and the argument above constructs a contractible space of homotopies. Any choices produce weakly equivalent cyclotomic bases, as per Proposition 5.6 in Section 5.

We can calculate TC of this example (up to p-completion), using the original method of Bökstedt-Hsiang-Madsen [8]. The exposition of Madsen in [14, 4.4.3] simplifies the argument of [8] and it is clear that it applies to any cyclotomic spectrum $\underline{X}$ where the map from the categorical to geometric fixed points $X^{C_p} \to \Phi^{C_p}\underline{X}$ is $\mathbb{T}/C_p$-equivariantly split. (Nikolaus-Scholze [18, IV.3.4] make an analogous observation in their setting, in terms of the notion of a Frobenius lift.) Madsen's splitting always happens for suspension spectra. In this case, the statement specializes to the following.

Proposition 7.1.4. *For $\mathbb{S}_{\mathbb{T}}[G]$ as above, $TC_{\mathrm{cyc}}(\mathbb{S}_{\mathbb{T}}[G])^\wedge_p$ fits into a homotopy fiber square*

$$\begin{tikzcd}
TC_{\mathrm{cyc}}(\mathbb{S}_{\mathbb{T}}[G])^\wedge_p \arrow[r] \arrow[d] & ((\Sigma\mathbb{S}[G])_{h\mathbb{T}})^\wedge_p \arrow[d, "\mathrm{tr}"] \\
\mathbb{S}[G]^\wedge_p \arrow[r, "{[p]-\mathrm{id}}"'] & \mathbb{S}[G]^\wedge_p .
\end{tikzcd}$$

where $[p]$ denotes the map on $\mathbb{S}[G]$ induced by the multiplication by p-map (the p-power map) on the abelian group G.

The above discussion in terms of groups can in be extended to monoids that inject into their group completion; we treat in detail only the case of the natural numbers $0, 1, 2, 3 \cdots$. Starting with $G = \mathbb{Z}$, we get an $E_\infty^{\mathbb{T}}$-space $\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}}$. Non-equivariantly, the components of $\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}}$ are canonically in one-to-one correspondence with the integers; let $(\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0}$ be the subspace of those components corresponding to the natural numbers $0, 1, 2, \ldots$. The $\mathbb{T}$-action and $E_\infty^{\mathbb{T}}$ structure on $\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}}$ restrict to $(\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0}$. The spherical monoid ring $\mathbb{S}[(\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0}]$

then obtains a $E_\infty^{\mathbb{T}}$ ring structure. This is the appropriate equivariant version of the E_∞ ring spectrum $\mathbb{S}[t]$ for the cyclotomic structure studied by Bhatt-Morrow-Scholze in [4, 11.1].

Notation 7.1.5. Write $\mathbb{S}_{\mathbb{T}}[t]$ for a commutative ring orthogonal spectrum model for the $E_\infty^{\mathbb{T}}$ ring spectrum $\mathbb{S}[(\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0}]$.

The map (7.1.1) restricts to a weak equivalence of $E_\infty^{\mathbb{T}}$ spaces

$$\rho^*(((\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0})^{C_p}) \longrightarrow (\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0}.$$

We use this to give $\mathbb{S}_{\mathbb{T}}[t]$ the structure of a commutative ring cyclotomic spectrum.

Remark 7.1.6. It is easy to see that the cyclotomic structure above is the (classic) cyclotomic structure on $\mathbb{S}[t]$ described by Bhatt-Morrow-Sholze in [4, 11.5]: the composite

$$\mathbb{S}[t] := \mathbb{S}[\mathbb{N}] \simeq \mathbb{S}_{\mathbb{T}}[t] \xleftarrow{\simeq} \Phi(\mathbb{S}_{\mathbb{T}}[t]) \longrightarrow \rho^*((\mathbb{S}_{\mathbb{T}}[t])^{tC_p}) \simeq \rho^*((\mathbb{S}[\mathbb{N}])^{tC_p})$$

is induced by the multiplication by p map $\mathbb{N} \to \mathbb{N}$. (In the display "$\simeq$" denotes $\mathbb{T}$-equivariant Borel equivalence and $\mathbb{S}[\mathbb{N}]$ has the trivial $\mathbb{T}$-action.)

Bhatt-Morrow-Sholze [4, §11.1] study cyclotomic structures relative to $\mathbb{S}[t]$; our theory also gives such structures:

Theorem 7.1.7. $\mathbb{S}_{\mathbb{T}}[t] \simeq \mathbb{S}[(\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0}]$ *with the cyclotomic structure above admits the structure of a cyclotomic base, i.e.,* $\Psi\colon \mathbb{S}_{\mathbb{T}}[t] \to \mathbb{S}_{\mathbb{T}}[t]$ *is the identity in the homotopy category of* $E_\infty^{\mathbb{T}}$ *ring spectra.*

The proof follows from the corresponding proof above for $\underline{\mathbb{Z}}^{\mathrm{op}}$, noting that the transfer

$$\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}} \longrightarrow (\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})^{C_p}$$

restricts to

$$(\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0} \longrightarrow ((\Omega^\infty H\underline{\mathbb{Z}}^{\mathrm{op}})_{\geq 0})^{C_p}.$$

7.2. **Example: The multiplicative Borel precyclotomic structure.**

Given a (non-equivariant) cofibrant commutative ring orthogonal spectrum R, we can apply the functor ϵ of Notation 3.2 to get a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum ϵR; we then tensor in the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra with the $\mathbb{T}$-space $E\mathbb{T}$ to form a new $\mathbb{T}$-equivariant commutative ring orthogonal spectrum $\epsilon R \otimes E\mathbb{T}$. We have the following identification of its geometric fixed points.

Proposition 7.2.1. *Let R be a cofibrant non-equivariant commutative ring orthogonal spectrum. There exists a natural isomorphism of $\mathbb{T}$-equivariant commutative ring orthogonal spectra*

$$\Phi(\epsilon R \otimes E\mathbb{T}) \cong \epsilon R \otimes \rho^*(E\mathbb{T}/C_p).$$

Proof. Working with the commutative ring orthogonal spectrum $R \otimes X$ for a space X, the diagonal map $N_e^{\mathbb{T}/C_p} \to \Phi^{C_p} N_e^{\mathbb{T}}$ of [1, 4.6] induces a $\mathbb{T}$-equivariant map

$$\delta\colon \epsilon R \otimes \rho^*(X \times \mathbb{T}/C_p) \longrightarrow \Phi(\epsilon R \otimes (X \times \mathbb{T})),$$

natural in the commutative ring orthogonal spectrum R and the space X, which is an isomorphism when R is cofibrant and X is a cell complex. Moreover, since R is cofibrant as a commutative ring orthogonal spectrum, ϵR is cofibrant as a

$\mathbb{T}$-equivariant commutative ring orthogonal spectrum, and the above generalized diagonal map is an isomorphism. Moreover, by the universal property of $N_e^{\mathbb{T}/C_p}$ (the $\mathbb{T}/C_p$ analogue of Proposition 2.4), any equivariant map

$$\epsilon R \otimes \rho^*(X \times \mathbb{T}/C_p) \longrightarrow \Phi(\epsilon R \otimes (X \times \mathbb{T}))$$

is uniquely characterized by the composite map of non-equivariant commutative ring orthogonal spectra

$$R \otimes X \longrightarrow i(\epsilon R \otimes \rho^*(X \times \mathbb{T}/C_p)) \longrightarrow i(\Phi(\epsilon R \otimes (X \times \mathbb{T}))).$$

Now consider the diagram

$$\begin{tikzcd}
\epsilon R \otimes \rho^*(\mathbb{T} \times \mathbb{T}/C_p) \arrow[r, "\delta"] \arrow[d] & \Phi(\epsilon R \otimes (\mathbb{T} \times \mathbb{T})) \arrow[d] \\
\epsilon R \otimes \rho^*(\mathbb{T}/C_p) \arrow[r, "\delta"'] & \Phi(\epsilon R \otimes \mathbb{T})
\end{tikzcd}$$

where the vertical maps are induced by the action map $\mathbb{T} \times \mathbb{T}/C_p \to \mathbb{T}/C_p$ and $\mathbb{T} \times \mathbb{T} \to \mathbb{T}$. Looking at the underlying non-equivariant diagram and precomposing with the non-equivariant map

$$R \otimes \mathbb{T} \longrightarrow i(\epsilon R \otimes \rho^*(\mathbb{T} \times \mathbb{T}/C_p))$$

both composite maps

$$R \otimes \mathbb{T} \longrightarrow i(\Phi(\epsilon R \otimes \mathbb{T})) \cong R \otimes \mathbb{T}$$

are the identity map, so the diagram commutes. Applying these observations to $X = \mathbb{T}$ defined in terms of the bar construction

$$E\mathbb{T} = |B_\bullet(*, \mathbb{T}, \mathbb{T})| = |\mathbb{T} \times \cdots \times \mathbb{T} \times \mathbb{T}|,$$

we see that δ then induces a natural isomorphism of simplicial $\mathbb{T}/C_p$-equivariant commutative ring orthogonal spectra

$$\epsilon R \otimes \rho^* B_\bullet(*, \mathbb{T}, \mathbb{T})/C_p \longrightarrow \Phi(\epsilon R \otimes B_\bullet(*, \mathbb{T}, \mathbb{T})).$$

Taking geometric realization, we get the natural isomorphism of the statement. □

We observe that the $\mathbb{T}$-space $\rho^*(E\mathbb{T}/C_p)$ is a free $\mathbb{T}$-CW complex non-equivariantly homotopy equivalent to BC_p; we choose and fix a $\mathbb{T}$-equivariant homotopy equivalence $\rho^*(E\mathbb{T}/C_p) \to BC_p \times E\mathbb{T}$.

Definition 7.2.2. The *multiplicative Borel precyclotomic structure* on a cofibrant (non-equivariant) commutative ring orthogonal spectrum R consists of the genuine $\mathbb{T}$-equivariant commutative ring orthogonal spectrum $\epsilon R \otimes E\mathbb{T}$, with the precyclotomic structure map

$$\Phi(\epsilon R \otimes E\mathbb{T}) \cong \epsilon R \otimes \rho^*(E\mathbb{T}/C_p) \longrightarrow \epsilon R \otimes (BC_p \times E\mathbb{T}) \longrightarrow \epsilon R \otimes (E\mathbb{T})$$

induced by the isomorphism of the proposition above, the homotopy equivalence chosen above, and the collapse map $BC_p \to *$.

Proposition 7.2.3. *For R a cofibrant (non-equivariant) commutative ring orthogonal spectrum, the multiplicative Borel precyclotomic spectrum $\underline{R} = \epsilon R \otimes E\mathbb{T}$ admits the natural structure of a precyclotomic base.*

Proof. It is clear from the construction of the isomorphism of Proposition 7.2.1 that the map

$$\underline{R} \longrightarrow \Phi^{C_p} N_e^{C_p} \underline{R} \longrightarrow \Phi^{C_p} \underline{R}$$

is the map $\epsilon R \otimes E\mathbb{T} \to \epsilon R \otimes (E\mathbb{T}/C_p)$ induced by the quotient map $E\mathbb{T} \to E\mathbb{T}/C_p$. The composite of this and the precyclotomic structure map is then the self-map of $\epsilon R \otimes E\mathbb{T}$ induced by the map

$$E\mathbb{T} \longrightarrow E\mathbb{T}/C_p \longrightarrow BC_p \times E\mathbb{T} \longrightarrow E\mathbb{T},$$

which is evidently (non-equivariantly) homotopic to the identity. □

We have a variant of this cyclotomic base structure using ϵR without first tensoring with $E\mathbb{T}$: the multiplicative tom Dieck splitting of [Cyc23] identifies $\Phi\epsilon R$ (up to weak equivalence) as $R \wedge (R \otimes BC_p)$ with the multiplicative transfer $R \to \Phi^{C_p} R$ the map

$$R \cong R \otimes * \longrightarrow R \otimes BC_p \longrightarrow R \wedge (R \otimes BC_p)$$

induced by the inclusion of the base point in BC_p. If we give ϵR the precyclotomic structure

$$\Phi\epsilon R \simeq R \wedge (R \otimes BC_p) \longrightarrow R$$

which is the identity on the R factor and the map

$$R \otimes BC_p \longrightarrow R \otimes * \cong R$$

on the other factor, then $\Psi \colon R \to R$ is homotopic to the identity.

As a particular example, we get two $\mathbb{T}$-equivariant versions of $H\mathbb{Z}$ with precyclotomic base structures. This does not contradict Hesselholt's observation [1, 7.1] on the non-existence of a cyclotomic structure on relative $THH^{H\mathbb{Z}}$, because the structures induced on $THH^{H\mathbb{Z}}$ are precyclotomic and not cyclotomic.

7.3. Example: $H\underline{\mathbb{F}}_p$.

Let $H\underline{\mathbb{F}}_p$ be a cofibrant model for the $\mathbb{T}$-equivariant commutative ring spectrum specified by the Eilenberg-Mac Lane spectrum on the constant Mackey functor on $\mathbb{F}_p$ (with constant restriction maps and zero transfer maps). In this case the geometric fixed points $\Phi^{C_p} H\underline{\mathbb{F}}_p$ is the connective cover of the Tate fixed point spectrum $H\underline{\mathbb{F}}_p^{tC_p}$. In particular, it is a connective $\mathbb{T}/C_p$-equivariant commutative ring orthogonal spectrum with $\underline{\pi}_0 \cong \underline{\mathbb{F}}_p$ as a Green functor. It follows that there exists a unique map in the homotopy category of $\mathbb{T}$-equivariant commutative orthogonal spectra $\Phi H\underline{\mathbb{F}}_p \to H\underline{\mathbb{F}}_p$ and a unique homotopy type in commutative ring precyclotomic spectra with this map as the structure map. The cyclotomic power map Ψ is a map of commutative ring orthogonal spectra $H\mathbb{F}_p \to H\mathbb{F}_p$ and is therefore homotopic to the identity, with a contractible space of choices for the homotopy.

In this example, it is easy to compute relative TC:

Proposition 7.3.1. *There exists a weak equivalence of commutative ring spectra*

$$TC^{H\underline{\mathbb{F}}_p}(H\underline{\mathbb{F}}_p) \simeq H\mathbb{F}_p^{S^1} \simeq H\mathbb{F}_p \wedge DS^1.$$

Proof. We can compute TC (as a commutative ring spectrum) as the homotopy equalizer of the maps R and F on

$$TF^{H\underline{\mathbb{F}}_p}(H\underline{\mathbb{F}}_p) := \operatorname{holim}_n H\underline{\mathbb{F}}_p^{C_{p^n}}$$

where the limit is taken along the inclusion of fixed point maps F (see for example [14, §2.5] for this fact and the definition of R and F). The maps F are maps of commutative ring spectra $H\mathbb{F}_p \to H\mathbb{F}_p$ and so are homotopic to the identity (as maps of commutative ring spectra) in an essentially unique way. This gives us a weak equivalence of commutative ring spectra $TF^{H\underline{\mathbb{F}}_p}(H\underline{\mathbb{F}}_p) \simeq H\mathbb{F}_p$. The self-maps R and F of $TF^{H\underline{\mathbb{F}}_p}(H\underline{\mathbb{F}}_p)$ are maps of commutative ring spectra and so also homotopic to the identity in an essentially unique way. $\square$

Although the underlying non-equivariant spectra of both $THH^{H\underline{\mathbb{F}}_p}(HA)$ and $THH^{\epsilon H\mathbb{F}_p}(HA)$ are equivalent to $HH^{\mathbb{F}_p}(A)$ (for any $\mathbb{F}_p$-algebra A), we note that these are quite different equivariant spectra. For example, non-equivariantly

$$\mathbb{L}\Phi(THH^{\underline{\mathbb{F}}_p}(H\mathbb{F}_p)) \simeq \Phi(\underline{\mathbb{F}}_p) \simeq \tau_{\geq 0}H\underline{\mathbb{F}}_p^{tC_p},$$

which has as its homotopy groups a single copy of $\mathbb{F}_p$ in each non-negative degree, whereas

$$\mathbb{L}\Phi(THH^{\epsilon H\mathbb{F}_p}(H\mathbb{F}_p)) \simeq H\mathbb{F}_p \wedge (H\mathbb{F}_p \otimes BC_p)$$

(see Warning 3.3), which has as its homotopy groups a free module over the dual Steenrod algebra. For more on $\epsilon_* H\mathbb{F}_p$ (and a precyclotomic base structure on it), see the example on multiplicative Borel precyclotomic structures above.

7.4. **Example: connective homotopical and periodic equivariant complex cobordism.**

Given an equivariant map of E_∞^G-spaces $\alpha\colon X \to \Omega^\infty \underline{KU}$, we get a G-equivariant E_∞^G ring Thom spectrum $M(\alpha)$ [13, X.3], which is usually denoted MX by abuse of notation. $M(\mathrm{id})$ is the equivariant Thom spectrum $\underline{MUP}$, periodic equivariant complex cobordism. The inclusion of $\underline{BU}$ in $\Omega^\infty \underline{KU}$ as the zero component (non-equivariantly: on fixed points it includes as the components of virtual dimension zero in the representation ring) produces the equivariant Thom spectrum $\underline{MU}$, homotopical equivariant complex cobordism. In this section, we introduce a new G-equivariant E_∞^G ring Thom spectrum $\underline{mu}$ which we call connective homotopical equivariant complex cobordism, which is the Thom spectrum of a map of E_∞^G-spaces $\Omega^\infty \underline{bu} \to \Omega^\infty \underline{KU}$, where on each fixed point subspace $\Omega^\infty \underline{bu}$ includes as the zero component. We emphasize that despite similar notation $\underline{mu}$ is more related to equivariant homotopical cobordism than equivariant geometric cobordism. (The Thom spectrum model of the geometric theory for equivariant unoriented cobordism (for groups satisfying Wasserman's condition) is now generally denoted mO [19], but in older literature was denoted mo [16, §XV].)

The equivariant space $\Omega^\infty \underline{bu}$ and map $\Omega^\infty \underline{bu} \to \Omega^\infty \underline{KU}$, without further structure, are easily constructed by Elmendorf's theorem: working with the orbit diagram for $\Omega^\infty \underline{KU}$, we take the subdiagram that at the orbit G/H consists of the zero component. Non-equivariantly, the zero component is a product of copies of BU indexed on the irreducible representations of H. The fact that these subspaces should be closed under transfers suggests that there should exist an E_∞^G-model. To construct this, we note that as an E_∞^G-space

$$\Omega^\infty \underline{KU} \simeq \Omega^\infty \tau_{\geq 0} \underline{KU},$$

where $\tau_{\geq 0}\underline{KU}$ is the genuine G-equivariant spectrum characterized (up to weak equivalence) by $\pi_q^H = 0$ for $q < 0$ and having the extra structure of a map to $\underline{KU}$ which is an isomorphism of ($\mathbb{Z}$-graded) homotopy groups in non-negative degrees. There is an essentially unique map of genuine G-equivariant spectra from $\tau_{\geq 0}\underline{KU}$

to $H\underline{RU}$ that is the identity on $\underline{\pi}_0$, where $\underline{RU}$ is the complex representation ring Mackey functor. We then get a map of E_∞^G-spaces,

$$\Omega^\infty \underline{KU} \longrightarrow \Omega^\infty H\underline{RU}$$

and we obtain an E_∞^G-space model of $\Omega^\infty \underline{bu}$ as the homotopy fiber.

Notation 7.4.1. Let $\underline{mu}$ be the E_∞^G Thom spectrum associated to the E_∞^G-space map $\Omega^\infty \underline{bu} \to \Omega^\infty \underline{KU}$.

The geometric fixed point functor commutes with the Thom spectrum functor in the following sense. For clarity, we add a subscript to denote the group of equivariance, as in $\Omega^\infty \underline{KU}_G$ (for the E_∞^G-space $\Omega^\infty \underline{KU}$). For a subgroup $H < G$, let WH denote its Weyl group. We have a map of E_∞^{WH} spaces

$$\phi^H \colon (\Omega^\infty \underline{KU}_G)^H \longrightarrow \Omega^\infty \underline{KU}_{WH} \tag{7.4.2}$$

induced by replacing an H-equivariant complex vector space with its H-fixed point subspace. Starting with a G-equivariant map

$$\alpha \colon X \longrightarrow \Omega^\infty \underline{KU}$$

defining a genuine G-equivariant Thom spectrum MX, taking H-fixed points, we get a map of WH-spaces

$$\alpha^H \colon X^H \longrightarrow (\Omega^\infty \underline{KU})^H \simeq BU_{WH} \times R(H).$$

Let

$$\phi^H \alpha^H \colon X^H \longrightarrow \Omega^\infty \underline{KU}_{WH}$$

be the composite of α^H with ϕ^H. Then the geometric fixed points $\Phi^H MX$ is weakly equivalent to the genuine WH-equivariant Thom spectrum $M(X^H)$ for the map $\phi^H\alpha^H$. When α is a E_∞^G map, α^H is an E_∞^{WH} map, $M(X^H) = M(\phi^H\alpha^H)$ is an E_∞^{WH} ring spectrum, and we get a weak equivalence of E_∞^{WH} ring spectra $(MX)^{\Phi H} \simeq M(X^H)$.

We now discuss precyclotomic structures and set $G = \mathbb{T}$. In the case of $\underline{MUP}$, the existence of such a structure was originally observed by Brun [10, 8.3]. In our notation, the map (7.4.2) for $H = C_p$ induces a map of $E_\infty^{\mathbb{T}/C}$ ring spectra

$$\Phi^{C_p}(\underline{MUP}_{\mathbb{T}}) \longrightarrow \underline{MUP}_{\mathbb{T}/C_p}.$$

Applying the functor ρ^*, we then get a map of $E_\infty^{\mathbb{T}}$ ring spectra

$$r \colon \Phi\underline{MUP} \longrightarrow \underline{MUP},$$

which we take as the precyclotomic structure map. We note that this map is not a weak equivalence as the domain and codomain are not even abstractly weakly equivalent; see [21, 4.10].

In the case of $\underline{mu}$, the diagram of genuine $\mathbb{T}/C_p$-equivariant spectra

$$\begin{tikzcd}
(\tau_{\geq 0} KU_{\mathbb{T}})^{C_p} \arrow[r] \arrow[d] & H\underline{RU}^{C_p} \arrow[d] \\
\tau_{\geq 0} KU_{\mathbb{T}/C_p} \arrow[r] & H\underline{RU}
\end{tikzcd}$$

commutes up to homotopy where the lefthand vertical map is the spectrum-level analogue of (7.4.2) and the righthand vertical map is induced by the map of

Mackey functors that at level $(\mathbb{T}/C_p)/(K/C_p)$ is the unique (abelian group) homomorphism $R(K) \to R(K/C_p)$ that sends a C_p-trivial irreducible K-representation to the corresponding K/C_p-representation and sends C_p-non-trivial irreducible K-representations to 0. The map r therefore lifts up to homotopy to a map of $E_\infty^{\mathbb{T}}$ ring spectra

$$r\colon \Phi\underline{mu} \longrightarrow \underline{mu},$$

which we use for the precyclotomic structure. Again this map is not a weak equivalence: non-equivariantly, the left-hand side is equivalent to $MU^{(p)} \simeq MU \wedge (BU_+)^{(p-1)}$ while the right-hand side is MU.

We observe that the trick above does not work to define a precyclotomic structure on $\underline{MU}$. Non-equivariantly, MU is the Thom spectrum of $BU \times \{0\} \to BU \times \mathbb{Z}$ and $\Phi^{C_p}MU$ is the Thom spectrum of

$$BU \times I(R(C_p)) \longrightarrow BU \times R(C_p) \longrightarrow BU \times \mathbb{Z}.$$

The map $V \mapsto V^H$ does not send the augmentation ideal $I(R(C_p))$ to $0 \in \mathbb{Z}$, so the map $r\colon \Phi MUP \to MUP$ does not restrict to a map $\Phi MU \to MU$.

Theorem 7.4.3. *Commutative ring precyclotomic spectrum models for $\underline{\underline{MUP}}$ and $\underline{\underline{mu}}$ admit the structure of a cyclotomic base.*

Proof. Let $\underline{R}$ be a commutative ring precyclotomic spectrum models for $\underline{\underline{MUP}}$ or $\underline{\underline{mu}}$ and let $X = \Omega^\infty \underline{KU}$ or $\Omega^\infty \underline{bu}$, respectively. The first part of operation Ψ,

$$R \longrightarrow \Phi^{C_p} N_e^{C_p} R \longrightarrow \Phi^{C_p} \underline{R}$$

is the map of Thom spectra induced by the transfer $X \to X^H$ [Cyc23], and the second part is r, which is the map of Thom spectra induced by the map $X^H \to X$ described above. Both maps are Ω^∞ of spectrum-level maps, and the composite on the spectrum level is the identity in the stable category. □

7.5. **Non-example: $\underline{KU}$.**

Although the geometric construction of the precyclotomic structure map on $\underline{MUP}$ suggests that the equivariant complex K-theory spectrum $\underline{KU}$ might provide another example of a precyclotomic base, in fact, it cannot even be a precyclotomic spectrum. By [11, 3.1], the integer p is a unit in $\pi_0(\Phi^{C_p}\underline{KU})$, and so there are no maps of ring spectra $\Phi\underline{KU} \to \underline{KU}$.

8. Precyclotomic spectra and Nikolaus-Scholze $TC(-;p)$

Nikolaus-Scholze [18] describes a shortcut for calculating TC_{cyc} of connective cyclotomic spectra. The purpose of this section is to convert a connective precyclotomic spectrum to a cyclotomic spectrum with equivalent TC_{cyc}, which can then be calculated by the Nikolaus-Scholze formula. This amounts to describing concretely the restriction to connective objects of the right adjoint from the homotopy category of precyclotomic spectra to the homotopy category of cyclotomic spectra.

The idea for the functor is to take the homotopy inverse limit of iterates of Φ. This is complicated by the fact that to preserve weak equivalences, we need both the left derived functor of Φ and the right derived functor of inverse limit; implicitly this involves both cofibrant and fibrant approximation. We use L and R for cofibrant and fibrant approximation functors, respectively, in the category of precyclotomic spectra. In our main construction that follows, the fibrant approximation functor R is only applied to cofibrant objects, and in the case of a commutative ring precyclotomic spectrum, we have a variant construction using cofibrant and fibrant approximation in the category of commutative ring precyclotomic spectra (q.v. [Cyc23]). The construction is a homotopy inverse limit in the category of precyclotomic spectra: for fibrant objects this may be constructed as the spacewise homotopy inverse limit (either using the Bousfield-Kan cobar construction or the mapping microscope construction) in orthogonal spectra, given the evident precyclotomic structure map.

Construction 8.1. For a precyclotomic spectrum $\underline{\underline{X}}$ with structure map r, we note that $\Phi\underline{X}$ has the canonical structure of a precyclotomic spectrum (that we denote $\Phi\underline{\underline{X}}$) with structure map Φr, and that the map $r\colon \Phi\underline{\underline{X}} \to \underline{\underline{X}}$ is a map of precyclotomic spectra. Writing Φ^n for the nth iterate of Φ, we get an inverse system of precyclotomic spectra $\Phi^n\underline{\underline{X}}$. Let $\Phi^\infty\underline{\underline{X}}$ be the homotopy inverse limit of the inverse system of precyclotomic spectra $R\Phi^n L\underline{\underline{X}}$.

By construction, Φ^∞ preserves weak equivalences of precyclotomic spectra. Moreover, $\Phi^\infty\underline{\underline{X}}$ comes with a point-set map of precyclotomic spectra $\Phi^\infty\underline{\underline{X}} \to RL\underline{\underline{X}}$ and a map in the homotopy category $\Phi^\infty\underline{\underline{X}} \to \underline{\underline{X}}$, induced by the identity $R\Phi^0 L\underline{\underline{X}} = RL\underline{\underline{X}}$. These maps are weak equivalences of precyclotomic spectra when X is cyclotomic, and in particular $\Phi^\infty\underline{\underline{X}}$ is itself cyclotomic in this case. We also have the following observation:

Proposition 8.2. *Let $\underline{\underline{X}}$ be a precyclotomic spectrum that is $\mathscr{F}_p$-connective (i.e., $\underline{\underline{X}}^{C_{p^n}}$ is connective for all $n \geq 0$). Then $\Phi^\infty\underline{\underline{X}}$ is a cyclotomic spectrum.*

Proof. Let $\underline{\underline{X}}_n = R\Phi^n L\underline{\underline{X}}$. It suffices to show that the natural map

$$\Phi L(\operatorname{holim}_n R\underline{\underline{X}}_n) \longrightarrow \operatorname{holim}_n(R\Phi L R\underline{\underline{X}}_n)$$

is an $\mathscr{F}_p$-equivalence. Looking at the norm cofiber sequence inside and outside the holim, and taking $C_{p^{k-1}}$-fixed points, we get a map of cofiber sequences of non-equivariant spectra

$$\begin{array}{ccccc}
(R((\operatorname{holim}_n R\underline{\underline{X}}_n) \wedge E\mathbb{T}))^{C_{p^k}} & \longrightarrow & (R\operatorname{holim}_n R\underline{\underline{X}}_n)^{C_{p^k}} & \longrightarrow & (R^2\Phi L\operatorname{holim}_n R\underline{\underline{X}}_n)^{C_{p^{k-1}}} \\
\downarrow & & \downarrow & & \downarrow \\
R\operatorname{holim}_n(R(\underline{\underline{X}}_n \wedge E\mathbb{T}))^{C_{p^k}} & \longrightarrow & R\operatorname{holim}_n(R\underline{\underline{X}}_n)^{C_{p^k}} & \longrightarrow & R\operatorname{holim}_n(R\Phi L R\underline{\underline{X}}_n)^{C_{p^{k-1}}},
\end{array}$$

and it suffices to observe that the first two vertical maps are weak equivalences. The second vertical map is weak equivalence without hypotheses on $\underline{\underline{X}}_n$ and the first is weak equivalence when the underlying non-equivariant spectra of $\underline{\underline{X}}_n$ are connective. (To see this, let $E\mathbb{T}_q$ denote the $\mathbb{T}$-equivariant q-skeleton of $E\mathbb{T}$. Because $E\mathbb{T}_q$ is a finite $\mathbb{T}$-spectrum, smashing with it commutes up to weak equivalent with homotopy limits. Since both $((\operatorname{holim}_n \underline{\underline{X}}_n) \wedge E\mathbb{T}/E\mathbb{T}_q)^{C_{p^k}}$ and $\operatorname{holim}_n(\underline{\underline{X}}_n \wedge E\mathbb{T}_q)^{C_{p^k}}$ are at least q-connected, the first map is a weak equivalence.) $\square$

Proposition 8.3. *Let $\underline{T}$ be a cyclotomic spectrum and $\underline{\underline{X}}$ a precyclotomic spectrum. Then the map $\Phi^\infty\underline{\underline{X}} \to \underline{\underline{X}}$ in the homotopy category of precyclotomic spectra induces a weak equivalence*

$$\mathbb{R}F_{\mathrm{cyc}}(\underline{T}, \Phi^\infty\underline{\underline{X}}) \longrightarrow RF_{\mathrm{cyc}}(\underline{T}, \underline{\underline{X}})$$

where $\mathbb{R}F_{\mathrm{cyc}}$ denotes the derived mapping spectrum of precyclotomic maps.

Proof. We can assume without loss of generality that $\underline{\underline{T}}$ is cofibrant; then since $\underline{\underline{T}}$ is cyclotomic, $\Phi\underline{\underline{T}}$ is also cyclotomic, and $r\colon \Phi\underline{\underline{T}} \to \underline{\underline{T}}$ is a weak equivalence. Because Φ, viewed as an endofunctor on precyclotomic spectra, is spectrally enriched, it follows that the map $r\colon \mathbb{L}\Phi\underline{\underline{X}} \to \underline{\underline{X}}$ induces an isomorphism in the stable category

$$\mathbb{R}F_{\mathrm{cyc}}(\underline{\underline{T}}, \mathbb{L}\Phi\underline{\underline{X}}) \longrightarrow \mathbb{R}F_{\mathrm{cyc}}(\underline{\underline{T}}, \underline{\underline{X}}),$$

with the composite

$$\mathbb{R}F_{\mathrm{cyc}}(\underline{\underline{T}}, \underline{\underline{X}}) \xrightarrow{\Phi} \mathbb{R}F_{\mathrm{cyc}}(\Phi\underline{\underline{T}}, \mathbb{L}\Phi\underline{\underline{X}}) \xrightarrow{(r^{-1})^*} \mathbb{R}F_{\mathrm{cyc}}(\underline{\underline{T}}, \mathbb{L}\Phi\underline{\underline{X}})$$

giving the inverse isomorphism. (This assertion is the tautological observation that for any precyclotomic map $f\colon \underline{\underline{T}} \to \underline{\underline{X}}$, the diagram

$$\begin{tikzcd}
\Phi\underline{\underline{T}} \arrow[r, "\Phi f"] \arrow[d, "r"'] & \Phi\underline{\underline{X}} \arrow[d, "r"] \\
\underline{\underline{T}} \arrow[r, "f"'] & \underline{\underline{X}}
\end{tikzcd}$$

commutes, combined with the definition of $r_{\Phi\underline{\underline{X}}}$ as $\Phi r_{\underline{\underline{X}}}$.) By induction, the map $\mathbb{L}\Phi^{n+1}\underline{\underline{X}} \to \mathbb{L}\Phi^{n}\underline{\underline{X}}$ induces a weak equivalence

$$\mathbb{R}F_{\mathrm{cyc}}(T, \mathbb{L}\Phi^{n+1}\underline{\underline{X}}) \longrightarrow \mathbb{R}F_{\mathrm{cyc}}(\underline{\underline{T}}, \mathbb{L}\Phi^{n}\underline{\underline{X}})$$

for all $n \geq 1$. Since $\Phi^{\infty}\underline{\underline{X}}$ is the homotopy limit in precyclotomic spectra, the map displayed in the statement is also a weak equivalence. $\square$

By the usual corepresentability result for TC_{cyc} [7, 6.8], the previous proposition implies in particular that $\Phi^{\infty}\underline{\underline{X}} \to \underline{\underline{X}}$ induces a weak equivalence on p-completed TC_{cyc}. In fact, the sharper corepresentability result [7, 6.7] implies the following sharper result. (We remind the reader that TC_{cyc} here means the composite of the functor denoted $TC(-;p)$ in [7] with fibrant approximation.)

Proposition 8.4. *The natural map $\Phi^{\infty}\underline{\underline{X}} \to \underline{\underline{X}}$ in the homotopy category of precyclotomic spectra induces a weak equivalence on TC_{cyc}.*

9. Descent for $TC^{\underline{\underline{R}}}$

In this section we study descent for $TC^{\underline{\underline{R}}}$. If we assume that the underlying commutative ring precyclotomic spectrum of $\underline{\underline{R}}$ is cofibrant, then any smash power $\underline{\underline{R}}^{(n)}$ of $\underline{\underline{R}}$ obtains the canonical structure of a precyclotomic base. Moreover, the maps $\underline{\underline{R}}^{(m)} \to \underline{\underline{R}}^{(n)}$ induced by (iterated) the inclusions of the unit and (iterated) multiplication in the various factors are maps of cyclotomic bases. Then the usual "Adams resolution" cosimplicial spectrum $\underline{\underline{R}}^{\bullet+1}$ is a cosimplicial object in the category of cyclotomic bases and we can combine it with TC in various ways.

We have in mind the case when $\underline{\underline{R}} = \underline{\underline{H\mathbb{F}_p}}$ (of Example 7.3) or $\underline{\underline{R}} = \underline{\overline{mu}}$ (of Example 7.4) and the two theorems stated below apply in particular to these cases. In the first theorem we need to assume that $\underline{R}$ is $\mathscr{F}_p$-connective, which means that for every n, the fixed points $\underline{R}^{C_{p^n}}$ is connective (as a non-equivariant spectrum); by standard arguments, this is equivalent to the assumption that $\Phi^n\underline{R}$ is connective (as a non-equivariant spectrum) for every n. In the second theorem, in addition to $\mathscr{F}_p$-connectivity of $\underline{R}$, we also need a finite type hypothesis, and a condition on π_0

called "solid" by Bousfield-Kan [9, I.4.5] but (in light of emerging terminology in the field) we call "core": we say that $\pi_0 R$ is *core* when the multiplication map

$$\pi_0 R \otimes_{\mathbb{Z}} \pi_0 R \longrightarrow \pi_0 R$$

is an isomorphism. More generally, say that $\pi_0 R$ is *p-adically core* when this isomorphism becomes an isomorphism after applying the zeroth left derived functor of p-completion. We need $\pi_0 R$ to be p-adically core and also $\pi_0(\Phi^n \underline{R})$ to be p-adically core for all n.

In the first descent theorem, we look at a commutative R-algebra A. The multiplication $R^{(n)} \to R$ makes A a commutative $R^{(n)}$-algebra for all n. Moreover, the maps of cyclotomic bases $\underline{\underline{R}}^{(m)} \to \underline{\underline{R}}^{(n)}$ induce maps on TC,

$$TC^{\underline{\underline{R}}^{(m)}}(A) \longrightarrow TC^{\underline{\underline{R}}^{(n)}}(A).$$

We then have a cosimplicial object

$$T^\bullet = TC^{\underline{\underline{R}}^{(\bullet+1)}}(A)$$

with cofaces induced by inclusions of units and codegeneracies induced by multiplication in the usual way (see the next section for this functoriality of TC). We prove the following theorem below.

Theorem 9.1. *Let $\underline{\underline{R}}$ be a $\mathscr{F}_p$-connective precyclotomic base, whose underlying commutative ring precyclotomic spectrum is cofibrant. Let A be a connective cofibrant commutative R-algebra. The canonical map*

$$TC(A) \longrightarrow \mathrm{Tot}(TC^{\underline{\underline{R}}^{(\bullet+1)}}(A))$$

is a weak equivalence.

For the second descent theorem, we start with an arbitrary associative ring orthogonal spectrum A. We can then form THH relative to $\underline{\underline{R}}^{(n)}$ of the $R^{(n)}$-algebra $R^{(n)} \wedge A$:

$$THH^{\underline{\underline{R}}^{(n)}}(R^{(n)} \wedge A)$$

and the TC relative to $\underline{\underline{R}}^{(n)}$:

$$TC^{\underline{\underline{R}}^{(n)}}(R^{(n)} \wedge A).$$

The iterated unit and multiplication maps $\underline{\underline{R}}^{(m)} \to \underline{\underline{R}}^{(n)}$ induce maps

$$TC^{\underline{\underline{R}}^{(m)}}(R^{(m)} \wedge A) \longrightarrow TC^{\underline{\underline{R}}^{(n)}}(R^{(n)} \wedge A)$$

and in particular, we have the augmented cosimplicial object

$$T^\bullet = TC^{\underline{\underline{R}}^{(\bullet+1)}}(R^{(\bullet+1)} \wedge A).$$

In different notation, this construction generalizes to the case when $\underline{\underline{R}}$ is just a commutative ring precyclotomic spectrum and not necessarily a precyclotomic base: we have an isomorphism of cosimplicial objects

$$TC^{\underline{\underline{R}}^{(\bullet+1)}}(R^{(\bullet+1)} \wedge A) := TC_{\mathrm{cyc}}(THH^{\underline{\underline{R}}^{(\bullet+1)}}(R^{(\bullet+1)} \wedge A)) \cong TC_{\mathrm{cyc}}(\underline{\underline{R}}^{(\bullet+1)} \wedge THH(A)).$$

We prove the following theorem.

Theorem 9.2. *Let $\underline{\underline{R}}$ be an $\mathscr{F}_p$-connective cofibrant commutative ring precyclotomic spectrum such that p is not unit in $\pi_0 R$. We assume that*

(i) $\pi_0(\Phi^n \underline{R})$ *is p-adically core for all $n \geq 0$ and*

(ii) *the underlying non-equivariant spectrum of $\Phi^n \underline{R}$ is finite p-type for all n.*

Let A be a connective associative ring orthogonal spectrum that is cofibrant as an associative ring orthogonal spectrum or as a commutative ring orthogonal spectrum. Then the canonical map

$$TC(A) \longrightarrow \mathrm{Tot}(TC_{\mathrm{cyc}}(R^{(\bullet+1)} \wedge THH(A)))$$

is a p-equivalence. If $\underline{\underline{R}}$ is an $\mathscr{F}_p$-connected precyclotomic base, then the canonical map

$$TC(A) \longrightarrow \mathrm{Tot}(TC^{\underline{\underline{R}}^{(\bullet+1)}}(R^{(\bullet+1)} \wedge A))$$

is a p-equivalence.

Proof of Theorems 9.1 and 9.2. We note that the TC_{cyc} construction is a homotopy limit of C_{p^n}-fixed points and C_{p^n}-fixed points commute with Tot; thus, it suffices to show that the maps

$$THH(A) \longrightarrow \mathrm{Tot}(THH^{\underline{\underline{R}}^{(\bullet+1)}}(A))$$
$$THH(A) \longrightarrow \mathrm{Tot}(R^{(\bullet+1)} \wedge THH(A))$$

induce a weak equivalence or p-equivalence on C_{p^n} fixed point spectra for all $n \geq 0$ in the respective cases for Theorems 9.1 and 9.2. We argue by induction on n. To write a uniform argument, we let $T^\bullet$ denote either of the cosimplicial objects and prove "π-local equivalences" where $\pi = \mathbb{Z}$ in the case of Theorem 9.1 and $\pi = \mathbb{Z}/p$ in the case of Theorem 9.2, so that a π-local equivalence means weak equivalence in the former case and p-equivalence in the latter case.

For Theorem 9.2, the base case $n = 0$ follows from standard results on convergence of the Adams spectral sequence. For Theorem 9.1, it is useful to note that

$$\begin{aligned} THH^{\underline{\underline{R}}^{(q+1)}}(A) &\cong THH^{\underline{\underline{R}}}(A) \wedge_{THH(A)} \cdots \wedge_{THH(A)} THH^{\underline{\underline{R}}}(A) \\ &= THH^{\underline{\underline{R}}}(A)^{\wedge_{THH(A)}(q+1)} \end{aligned}$$

and the cosimplicial object is the Adams resolution in the category of $THH(A)$-modules of $THH(A)$ by the $THH(A)$-algebra $THH^{\underline{\underline{R}}}(A)$. The hypothesis that R and A are connective (and commutative) implies that the map $THH(A) \to THH^{\underline{\underline{R}}}(A)$ is a 1-equivalence and from here the base case $n = 0$ follows from standard arguments: the normalized E_1-term for the homotopy group spectral sequence is $2q$-connected in cosimplicial degree q, which implies that the qth fiber of the cosimplicial filtration on Tot is q-connected. This implies that smashing over $THH(A)$ with $THH^{\underline{\underline{R}}}(A)$ commutes with Tot, and the map of cosimplicial objects

$$THH^{\underline{\underline{R}}}(A) \cong THH^{\underline{\underline{R}}}(A) \wedge_{THH(A)} THH(A) \longrightarrow THH^{\underline{\underline{R}}}(A)^{\wedge_{THH(A)}(1+\bullet+1)}$$

(for the constant cosimplicial object $THH^{\underline{\underline{R}}}(A)$) is a cosimplicial homotopy equivalence.

The base case shows that the map is a non-equivariant π-local equivalence and implies that the induced map of homotopy orbits

$$THH(A)_{hC_{p^n}} \xrightarrow{\simeq_\pi} \mathrm{Tot}(T^\bullet)_{hC_{p^n}} \tag{9.3}$$

is a (non-equivariant) π-local equivalence in the respective cases. Moreover, connectivity of the fibers in the cosimplicial filtration implies that homotopy orbits commute with Tot,

$$\mathrm{Tot}\big(T^\bullet\big)_{hC_{p^n}} \xrightarrow{\simeq} \mathrm{Tot}\big(T^\bullet_{hC_{p^n}}\big).$$

Consider the norm cofiber sequence

$$(T^\bullet)_{hC_{p^n}} \longrightarrow (T^\bullet)^{C_{p^n}} \longrightarrow (\Phi T_\bullet)^{C_{p^{n-1}}} \longrightarrow \Sigma \cdots .$$

Commuting fixed points and homotopy orbits with Tot, up to weak equivalence, we get a cofiber sequence

$$\mathrm{Tot}(T^\bullet)_{hC_{p^n}} \longrightarrow \mathrm{Tot}(T^\bullet)^{C_{p^n}} \longrightarrow \mathrm{Tot}(\Phi(T^\bullet))^{C_{p^{n-1}}} \longrightarrow \Sigma \cdots$$

and compatible maps from the cofiber sequence

$$THH(A)_{hC_{p^n}} \longrightarrow THH(A)^{C_{p^n}} \longrightarrow (\Phi THH(A))^{C_{p^{n-1}}} \longrightarrow \Sigma \cdots$$

By (9.3) the map in the first position is a π-local equivalence, and by induction the map in the third position is a π-local equivalence: we can identify it up to weak equivalence as the induced map on $C_{p^{n-1}}$ fixed points of the corresponding descent problem with $\underline{\underline{R}}$ replaced by $\Phi\underline{\underline{R}}$. It follows that the map in the middle position is a π-local equivalence. This completes the induction. □

10. Functoriality of relative TC in the cyclotomic base

This section is devoted to studying the functoriality of $TC^{\underline{\underline{R}}}(A)$ in $\underline{\underline{R}}$ as well as A.

Let $\underline{\underline{R}}$ be a precyclotomic base. Then, as observed in the proof of Theorem 5.3, the precyclotomic base structure gives a homotopy that makes the diagram

$$\begin{tikzcd} \Phi THH(R) \arrow[r] \arrow[d, "r_{THH(R)}"'] & \Phi\underline{R} \arrow[d, "r_{\underline{R}}"] \\ THH(R) \arrow[r] & \underline{R} \end{tikzcd}$$

commute in the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra. We write

$$f_0 \colon \Phi THH(R) \longrightarrow \Phi\underline{R} \longrightarrow \underline{R}$$
$$f_1 \colon \Phi THH(R) \longrightarrow THH(R) \longrightarrow \underline{R}$$
$$F \colon \Phi THH(R) \otimes I \longrightarrow \underline{R}$$

for the left-then-down composite, the down-then-right composite, and the homotopy, respectively. We write f_i^*R and F^*R for the $\mathbb{T}$-equivariant commutative $THH(R)$- and $(THH(R) \otimes I)$-algebra structures on R induced by f_i and F. We use the notation

$$c \colon THH(R) \longrightarrow \underline{R}$$

for the canonical map of Definition 3.6.

For the functoriality used in Theorem 9.1, even when A is cofibrant as an R-algebra, we cannot expect A to be cofibrant as an $R^{(n)}$-algebra for all n, and so we need to use a point-set model for the derived functor of $THH^{\underline{R}}(A)$ that requires minimal cofibrancy on A.

First we need to ensure that $THH(A)$ has the right $\mathbb{T}$-equivariant ($\mathscr{F}_{\mathrm{fin}}$-colocal) homotopy type. For this, it suffices to assume that A is flat for the smash product in orthogonal spectra (meaning that $A \wedge (-)$ preserves all weak equivalences of orthogonal spectra); moreover, when A is flat for the smash product in orthogonal spectra, $THH(A)$ is flat for the smash product in $\mathbb{T}$-equivariant orthogonal spectra. To get the cyclotomic structure map on $THH(A)$, the construction of [1, 4.2] requires an additional hypothesis; for this, it suffices that the underlying orthogonal spectrum of A be in the (non-equivariant) class $\mathscr{E}$ discussed in Section 2. To ensure both that A is flat for the smash product in orthogonal spectra and that the underlying orthogonal spectrum of A is in the class $\mathscr{E}$, it is enough that A be a cofibrant R-algebra for some commutative ring orthogonal spectrum R that is cofibrant in the standard model structure or the model structure of [Cyc23].

Next we need a point-set model for the derived smash product $THH(A)\wedge_{THH(R)} \underline{R}$. Let $\underline{M}$ be a $\mathbb{T}$-equivariant orthogonal spectrum, $\underline{R}$ be a cofibrant $\mathbb{T}$-equivariant commutative ring orthogonal spectrum in the model structure of [Cyc23]. If $\underline{R}'$ is a cofibrant commutative $\underline{R}$-algebra, the $\underline{M} \wedge_{\underline{R}} \underline{R}'$ represents the derived smash product. The structure of a commutative $\underline{R}$-algebra $\underline{R}'$ is just a map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra $g\colon \underline{R} \to \underline{R}'$ and $\underline{R}'$ is a cofibrant commutative $\underline{R}$-algebra exactly when this map is a cofibration. In the case when $\underline{R}'$ is not necessarily cofibrant as a commutative $\underline{R}$-algebra but is cofibrant as a $\mathbb{T}$-equivariant commutative ring orthogonal spectra, we can use the mapping cylinder construction to produce a homotopy equivalent cofibrant commutative $\underline{R}$-algebra. Let

$$Ig = (\underline{R} \otimes I) \cup_{(\underline{R}\otimes\{1\})} \underline{R}'$$

with the pushout done in the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra (i.e., "$\cup$"$= \wedge$), gluing along the given map g:

$$\underline{R} \otimes \{1\} \cong \underline{R} \xrightarrow{g} \underline{R}'.$$

Returning to $THH^{\underline{R}}(A)$, when $\underline{R}$ is a cofibrant $\mathbb{T}$-equivariant commutative ring orthogonal spectrum, $Ic \to \underline{R}$ is a cofibrant approximation in the category of commutative $THH(R)$-algebras, and when in addition the underlying orthogonal spectrum of the R-algebra A is flat for the smash product in orthogonal spectra, $THH(A) \wedge_{THH(R)} Ic$ then represents the derived functor $THH^{\underline{R}}(A)$.

Assume the underlying orthogonal spectra of R and A are in the (non-equivariant) class $\mathscr{E}$ of Section 2 so that $THH(R)$ and $THH(A)$ have cyclotomic structures and consider the following zigzag of maps.

$$\begin{tikzcd}
\Phi(THH(A) \wedge_{THH(R)} Ic) & & THH(A) \wedge_{THH(R)} Ic \\
\Phi THH(A) \wedge_{\Phi THH(R)} I(\Phi c) \arrow[u] \arrow[d, dotted] & \Phi THH(A) \wedge_{\Phi THH(R)\otimes I} IF & \\
\Phi THH(A) \wedge_{\Phi THH(R)} If_0 \arrow[ur] & & \Phi THH(A) \wedge_{\Phi THH(R)} If_1 \arrow[ul] \arrow[uu, dashed]
\end{tikzcd}$$

When $\underline{R}$ is cofibrant as a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum in the model structure of [Cyc23]), $THH(R)$, $\Phi THH(R)$, and ΦR are also cofibrant. By the discussion above, when in addition A is flat for the smash product in orthogonal spectra, then the solid arrows are weak equivalences and the dashed arrow is an $\mathscr{F}_{\mathrm{fin}}$-equivalence. When in addition $\underline{\underline{R}}$ is cyclotomic, the dotted arrow is an $\mathscr{F}_p$-equivalence. This zigzag motivates the following definition.

Definition 10.1. We define the category of zigzag-cyclotomic spectra as follows. An object consists of $\mathbb{T}$-equivariant orthogonal spectra $\underline{X}, \underline{X}_1, \underline{X}_2, \underline{X}_3$ and $\mathbb{T}$-equivariant maps

$$\Phi\underline{X} \xleftarrow{r_3} \underline{X}_3 \xrightarrow{r_2} \underline{X}_2 \xleftarrow{r_1} \underline{X}_1 \xrightarrow{r_0} \underline{X}.$$

A map of zigzag-cyclotomic spectra $\underline{\underline{f}}\colon \underline{\underline{X}} \to \underline{\underline{Y}}$ consists of maps of $\mathbb{T}$-equivariant orthogonal spectra f, f_1, f_2, f_3 as pictured, making the following diagram commute.

$$\begin{tikzcd}
\Phi\underline{X} \arrow[d, "\Phi f"'] & \underline{X}_3 \arrow[l, "r_3"'] \arrow[r, "r_2"] \arrow[d, "f_3"] & \underline{X}_2 \arrow[d, "f_2"] & \underline{X}_1 \arrow[l, "r_1"'] \arrow[r, "r_0"] \arrow[d, "f_1"] & \underline{X} \arrow[d, "f"] \\
\Phi\underline{Y} & \underline{Y}_3 \arrow[l, "r_3"] \arrow[r, "r_2"'] & \underline{Y}_2 & \underline{Y}_1 \arrow[l, "r_1"] \arrow[r, "r_0"'] & \underline{Y}
\end{tikzcd}$$

For $\mathscr{F}$ any family of subgroups of $\mathbb{T}$, an $\mathscr{F}$-equivalence is a map where each of f, f_1, f_2, f_3 is an $\mathscr{F}$-equivalence.

In order to do a TC construction, we need an Ω-spectrum replacement functor in this category. For this is suffices to have an Ω-spectrum replacement functor R (or more precisely, functor R and natural transformation $\eta\colon \mathrm{Id} \to R$) in the category of $\mathbb{T}$-equivariant orthogonal spectra that comes with a natural transformation $\theta\colon \Phi R \to R\Phi$ that makes the following diagram commute.

$$\begin{tikzcd}
& \Phi\underline{X} \arrow[dl, "\Phi\eta_{\underline{X}}"'] \arrow[dr, "\eta_{\Phi\underline{X}}"] & \\
\Phi R\underline{X} \arrow[rr, "\theta_{\underline{X}}"'] & & R\Phi\underline{X}
\end{tikzcd}$$

We call such a structure a Φ-compatible Ω-spectrum replacement functor. Theorem 4.7 of [6] and its proof assert the existence of such functors (and construct two). We now choose and fix a Φ-compatible Ω-spectrum replacement functor R.

We construct a functor TC_z from zigzag-cyclotomic spectra to orthogonal spectra as follows.

Construction 10.2. For a zigzag-cyclotomic spectrum $\underline{\underline{X}}$, define $TR_z(\underline{\underline{X}})$ to be the homotopy limit (constructed via the Bousfield-Kan cobar construction or mapping microscope) of the following diagram.

$$\begin{tikzcd}
& (R\underline{X}_3)^{C_{p^n}} \arrow[d, "(Rr_3)^{C_{p^n}}"'] \arrow[dr, "(Rr_2)^{C_{p^n}}"] & (R\underline{X}_1)^{C_{p^n}} \arrow[d, "(Rr_1)^{C_{p^n}}"'] \arrow[dr, "(Rr_0)^{C_{p^n}}"] & & \cdots \arrow[d, "(Rr_3)^{C_{p^{n-1}}}"'] \\
\cdots \arrow[r] & (R\Phi\underline{X})^{C_{p^n}} & (R\underline{X}_2)^{C_{p^n}} & (R\underline{X})^{C_{p^n}} \arrow[r] & (R\Phi\underline{X})^{C_{p^{n-1}}}
\end{tikzcd}$$

where the unlabeled map $(R\underline{X})^{C_{p^n}} \to (R\Phi\underline{X})^{C_{p^{n-1}}}$ (for each $n \geq 1$) is the composite

$$(R\underline{X})^{C_{p^n}} \cong (\rho^*(R\underline{X})^{C_p})^{C_{p^{n-1}}} \longrightarrow (\Phi R\underline{X})^{C_{p^{n-1}}} \longrightarrow (R\Phi\underline{X})^{C_{p^{n-1}}}$$

induced by the canonical map from the fixed point to the geometric fixed points and the Φ-compatibility structure of R. Naturality of the inclusion of fixed points map F implies that the diagrams

$$\begin{tikzcd}
(R\Phi\underline{X})^{C_{p^n}} \arrow[d, "\mathrm{F}"'] & (R\underline{X}_3)^{C_{p^n}} \arrow[l, "(Rr_3)^{C_{p^n}}"'] \arrow[r, "(Rr_2)^{C_{p^n}}"] \arrow[d, "\mathrm{F}"'] & (R\underline{X}_2)^{C_{p^n}} \arrow[d, "\mathrm{F}"'] & (R\underline{X}_1)^{C_{p^n}} \arrow[l, "(Rr_1)^{C_{p^n}}"'] \arrow[r, "(Rr_0)^{C_{p^n}}"] \arrow[d, "\mathrm{F}"'] & (R\underline{X})^{C_{p^n}} \arrow[d, "\mathrm{F}"'] \\
(R\Phi\underline{X})^{C_{p^{n-1}}} & (R\underline{X}_3)^{C_{p^{n-1}}} \arrow[l, "(Rr_3)^{C_{p^{n-1}}}"] \arrow[r, "(Rr_2)^{C_{p^{n-1}}}"'] & (R\underline{X}_2)^{C_{p^{n-1}}} & (R\underline{X}_1)^{C_{p^{n-1}}} \arrow[l, "(Rr_1)^{C_{p^{n-1}}}"] \arrow[r, "(Rr_0)^{C_{p^{n-1}}}"'] & (R\underline{X})^{C_{p^{n-1}}}
\end{tikzcd}$$

and

$$\begin{tikzcd}
(R\underline{X})^{C_{p^{n+1}}} \arrow[r] \arrow[d, "\mathrm{F}"'] & (R\Phi\underline{X})^{C_{p^n}} \arrow[d, "\mathrm{F}"'] \\
(R\underline{X})^{C_{p^n}} \arrow[r] & (R\Phi\underline{X})^{C_{p^{n-1}}}
\end{tikzcd}$$

commute and so F induces a self-map F of $TR_z(\underline{\underline{X}})$. Define $TC_z(\underline{\underline{X}})$ to be the homotopy equalizer of F and the identity on $TR_z(\underline{\underline{X}})$.

The following is clear from construction.

Proposition 10.3. *If $\underline{\underline{X}} \to \underline{\underline{Y}}$ is an $\mathscr{F}_p$-equivalence of zigzag cyclotomic spectra, then the induced maps on TR_z and TC_z are weak equivalences.*

To compare TC_{cyc} and TC_z, we use the following functor from precyclotomic spectra to zigzag cyclotomic spectra.

Definition 10.4. Given a precyclotomic spectrum $\underline{\underline{X}}$, let $z\underline{\underline{X}}$ be the zigzag cyclotomic spectrum

$$\Phi\underline{X} \xleftarrow{=} \Phi\underline{X} \xrightarrow{r_{\underline{\underline{X}}}} \underline{X} \xleftarrow{=} \underline{X} \xrightarrow{=} \underline{X}.$$

Theorem 10.5. *There is a zigzag of natural weak equivalences connecting $TC_{\mathrm{cyc}}(\underline{\underline{X}})$ and $TC_z(z\underline{\underline{X}})$.*

Proof. Let $TR_{\mathrm{cyc}}(\underline{\underline{X}})$ be the homotopy limit of $(R\underline{X})^{C_{p^n}}$ under the R maps

$$\begin{aligned} \mathrm{R}\colon (R\underline{X})^{C_{p^n}} \cong (\rho^*(R\underline{X})^{C_p})^{C_{p^{n-1}}} &\longrightarrow (\Phi R\underline{X})^{C_{p^{n-1}}} \\ &\xrightarrow{\theta^{C_{p^{n-1}}}} (R\Phi\underline{X})^{C_{p^{n-1}}} \xrightarrow{Rr_{\underline{\underline{X}}}^{C_{p^{n-1}}}} (R\underline{X})^{C_{p^{n-1}}}. \end{aligned}$$

Then $TC_{\mathrm{cyc}}(\underline{\underline{X}})$ is naturally weakly equivalent to the homotopy equalizer of the self-maps induced by F and R on $TR_{\mathrm{cyc}}(\underline{\underline{X}})$, and since the self-map of $TR_{\mathrm{cyc}}(\underline{\underline{X}})$ induced by R is naturally homotopic to the identity, TC_{cyc} is naturally weakly equivalent to the homotopy equalizer of the identity and the self-map induced by F on $TR_{\mathrm{cyc}}(\underline{\underline{X}})$. We defined $TC_z(z\underline{\underline{X}})$ as the homotopy equalizer of the identity and the self-map induced by F on $TR_z(z\underline{\underline{X}})$, and so it suffices to construct a natural weak equivalence $TR_{\mathrm{cyc}}(\underline{\underline{X}}) \to TR_z(z\underline{\underline{X}})$ that is compatible up to natural homotopy with F.

For the diagram defining TR_z, consider the subdiagram that consists of $(R\underline{X})^{C_{p^n}}$ and everything to the right of it as displayed above

$$\begin{tikzcd}
(R\underline{X}_1)^{C_{p^n}} \arrow[dr] & & (R\underline{X}_3)^{C_{p^{n-1}}} \arrow[d] \arrow[dr] & (R\underline{X}_1)^{C_{p^{n-1}}} \arrow[d] \arrow[dr] & \\
& (R\underline{X})^{C_{p^n}} \arrow[r] & (R\Phi\underline{X})^{C_{p^{n-1}}} & (R\underline{X}_2)^{C_{p^{n-1}}} & \cdots
\end{tikzcd}$$

The homotopy limit of this diagram can be constructed as an iterated homotopy pullback and the point-set limit is an iterated point-set pullback. In the particular case of $z\underline{\underline{X}}$, all the vertical non-diagonal maps are the identity, and so the point-set limit maps to $(R\underline{X})^{C_{p^n}}$ by an isomorphism, and this maps to the homotopy limit by a homotopy equivalence. Under these isomorphisms (for n varying), the maps induced on the limits by inclusion of subdiagrams are the iterated maps

$$\mathrm{R}^{n-m}\colon (R\underline{X})^{C_{p^n}} \longrightarrow (R\underline{X})^{C_{p^m}}.$$

The map from the sequential homotopy limit of point-set inverse limits to the homotopy inverse limit of the whole diagram then induces a weak equivalence $TR_{\mathrm{cyc}}(\underline{\underline{X}})$ to $TR_z(z\underline{\underline{X}})$ that is compatible with the F self-maps (without needing a homotopy). □

We now return to the problem of constructing $TC^{\underline{\underline{R}}}(A)$ as a point-set functor. We require a functor of $\underline{\underline{R}}$ and A in the following sense.

Definition 10.6. Let $\mathscr{P}air_{\mathrm{pb}}$ denote the category where

- the objects are ordered pairs $(\underline{\underline{R}}, A)$ with $\underline{\underline{R}}$ a precyclotomic base and A an associative R-algebra, where $\underline{\underline{R}}$ is cofibrant in the category of $\mathbb{T}$-equivariant commutative ring orthogonal spectra (for the standard model structure or the model structure of [Cyc23]) and the underlying orthogonal spectrum of A is in the (non-equivariant) class $\mathscr{E}$ of Section 2 and is flat for the smash product in orthogonal spectra.
- the set of maps in $\mathscr{P}air_{\mathrm{pb}}$ from $(\underline{\underline{R}}, A) \to (\underline{\underline{R}}', A')$ consists of the set of ordered pairs of maps $\underline{\underline{f}}\colon \underline{\underline{R}} \to \underline{\underline{R}}'$ and $g\colon A \to A'$ where $\underline{\underline{f}}$ is a map of precyclotomic bases and g is a map of R-algebras for the R-algebra structure f^*A' on A' induced by the map $f\colon R \to R'$ underlying $\underline{\underline{f}}$.

We say that a map $(\underline{\underline{f}}, g)$ in $\mathscr{P}air_{\mathrm{pb}}$ is a weak equivalence when the map $\underline{\underline{f}}$ is a weak equivalence of precyclotomic spectra (i.e., $\underline{f}$ is an $\mathscr{F}_p$-equivalence of the underlying $\mathbb{T}$-equivariant orthogonal spectra) and g is a weak equivalence.

As motivated by the discussion above, we have the following functor from $\mathscr{P}air_{\mathrm{pb}}$ to zigzag cyclotomic spectra.

Construction 10.7. Let $(\underline{\underline{R}}, A)$ be an object in $\mathscr{P}air_{\mathrm{pb}}$. Define the zigzag cyclotomic object $THH_z^{\underline{\underline{R}}}(A)$ by

$$\begin{aligned}
\underline{X}_3 &:= \Phi THH(A) \wedge_{\Phi THH(R)} I(\Phi c)\\
\underline{X}_2 &:= \Phi THH(A) \wedge_{\Phi THH(R)\otimes I} IF\\
\underline{X}_1 &:= \Phi THH(A) \wedge_{\Phi THH(R)} If_1\\
\underline{X} &:= THH(A) \wedge_{THH(R)} Ic
\end{aligned}$$

where the maps are as described at the start of the section. We note that this is functorial for $(\underline{\underline{R}}, A)$ in $\mathscr{P}air_{\mathrm{pb}}$ and sends weak equivalences to $\mathscr{F}_p$-equivalences.

If we fix a cyclotomic base $\underline{\underline{R}}$ whose underlying $\mathbb{T}$-equivariant commutative ring orthogonal spectrum is cofibrant (in either the standard model structure or the model structure of [Cyc23]), then we get a functor from cofibrant R-algebras to $\mathscr{P}air_{\mathrm{pb}}$ (sending A to $(\underline{\underline{R}}, A)$). The essence of the following theorem is that the construction $TC_z(THH_z^{\underline{\underline{R}}}(A))$ gives a point-set model of $TC^{\underline{\underline{R}}}(A)$ functorial in both $\underline{\underline{R}}$ and A.

Theorem 10.8. *For $\underline{\underline{R}}$ a fixed cyclotomic base whose underlying $\mathbb{T}$-equivariant commutative ring orthogonal spectrum is cofibrant in the model structure of* [Cyc23]*), then the composite functor $TC_z(THH_z^{\underline{\underline{R}}}(-))$ viewed as a functor from cofibrant R-algebras to the stable category is naturally isomorphic to the functor $TC^{\underline{\underline{R}}}(-)$.*

The proof of the theorem fills the remainder of this section.

As per the statement $\underline{\underline{R}}$ is fixed, and as in Section 5 we fix a cofibrant approximation $q\colon \underline{\underline{T}} \to THH(R)$ in the category of commutative ring precyclotomic spectra, a map of commutative ring precyclotomic spectra $g\colon \underline{\underline{T}} \to \underline{\underline{R}}$ and a path H from the image of g in the homotopy equalizer

$$r^*_{\underline{\underline{T}}}, r_{\underline{\underline{R}}*} \circ \Phi\colon \mathscr{C}om^{\mathbb{T}}(\underline{T}, \underline{R}) \rightrightarrows \mathscr{C}om^{\mathbb{T}}(\Phi\underline{T}, \underline{R})$$

to the element specified by $c \circ q$ and $F \circ q$. Such a path is adjoint to a map of $\mathbb{T}$-equivariant commutative ring orthogonal spectra

$$\Phi\underline{T} \otimes I^2 \longrightarrow \mathbb{R}$$

which by abuse of notation, we will also denote by H. The four faces of the square I^2 are the following homotopies $\Phi\underline{T} \otimes I \to \underline{R}$:

$$\begin{aligned}
&[0,1] \times \{0\} : F \circ q\\
&[0,1] \times \{1\} : K = \text{constant homotopy } g \circ r_{\underline{\underline{T}}} = r_{\underline{\underline{R}}} \circ \Phi g\\
&\{0\} \times [0,1] : r_{\underline{\underline{R}}} \circ \Phi G\\
&\{1\} \times [0,1] : G \circ r_{\underline{\underline{T}}}
\end{aligned}$$

where G is the homotopy from g to $c \circ q$, as in the notation in the proof of Theorem 5.3. Since $\underline{R}$ is cofibrant as a $\mathbb{T}$-equivariant commutative ring orthogonal spectrum, the derived functor $THH(A) \wedge^{\mathbb{L}}_{\underline{T}} g^*\underline{R}$ in that proof is represented by $THH(A) \wedge_{\underline{T}} I(g)$ (when A is a cofibrant R-algebra as in the hypothesis of Theorem 10.8), where the action of $\underline{T}$ on $THH(A)$ is always via $q\colon \underline{T} \to THH(R)$.

To prove Theorem 10.8, it suffices to construct a natural zigzag of $\mathscr{F}_p$-equivalences of zigzag cyclotomic spectra relating $\underline{\underline{X}} = THH_z^{\underline{\underline{R}}}(-)$ and $\underline{\underline{Z}} = z(THH(-) \wedge_{\underline{T}} I(g))$. Let $\underline{\underline{Y}}$ be the functor from cofibrant $\overline{R}$-algebras to zigzag precyclotomic spectra defined by

$$\begin{aligned}
\underline{Y}_3 &= \Phi THH(-) \wedge_{\Phi\underline{T}} I(\Phi g)\\
\underline{Y}_2 &= \Phi THH(-) \wedge_{\Phi\underline{T}\otimes I} IK\\
\underline{Y}_1 &= \Phi THH(-) \wedge_{\Phi\underline{T}} I(r_{\underline{\underline{R}}} \circ \Phi g)\\
\underline{Y} &= THH(-) \wedge_{\underline{T}} Ig
\end{aligned}$$

with maps

- $\Phi\underline{Y} \leftarrow \underline{Y}_3$ induced by the lax symmetric monoidal structure map for Φ
- $\underline{Y}_3 \to \underline{Y}_2$ the composite

$$\Phi THH(-) \wedge_{\Phi\underline{T}} I(\Phi g) \longrightarrow \Phi THH(-) \wedge_{\Phi\underline{T}} I(r_{\underline{\underline{R}}} \circ \Phi g) \longrightarrow \Phi THH(-) \wedge_{\Phi\underline{T}\otimes I} IK$$

where the second map is induced by the inclusion of $\underline{T} \otimes \{0\}$ in $\underline{T} \otimes I$ and the first map is induced by the map $I(\Phi g) \to I(r_{\underline{\underline{R}}} \circ \Phi g)$ (induced by $r_{\underline{\underline{R}}} \colon \Phi \underline{R} \to \underline{R}$);

- $\underline{Y}_2 \leftarrow \underline{Y}_1$ induced by the inclusion of $\underline{T} \otimes \{1\}$ in $\underline{T} \otimes I$; and
- $\underline{Y}_1 \to \underline{Y}$ induced by the precyclotomic structure maps $r_{THH(-)}$ and $r_{\underline{\underline{T}}}$ (using the precyclotomic equation $r_{\underline{\underline{R}}} \circ \Phi g = g \circ r_{\underline{\underline{T}}}$).

We then have a natural map of zigzag cyclotomic spectra $\underline{\underline{Y}} \to \underline{\underline{Z}} = z(THH(-) \wedge_{\underline{T}} I(g))$ with component maps

$$\underline{Y}_3 = \Phi THH(-) \wedge_{\Phi \underline{T}} I(\Phi g) \longrightarrow \Phi(THH(-) \wedge_{\underline{T}} Ig) = \underline{Z}_3$$

induced by the lax symmetric monoidal structure map of Φ;

$$\underline{Y}_2 = \Phi THH(-) \wedge_{\Phi \underline{T} \otimes I} IK \longrightarrow THH(-) \wedge_{\underline{T}} Ig = \underline{Z}_2$$

induced by $r_{THH(-)}$, $r_{\underline{\underline{T}}}$, and the collapse map $\underline{T} \otimes I \to \underline{T}$;

$$\underline{Y}_1 = \Phi THH(-) \wedge_{\Phi \underline{T}} I(r_{\underline{\underline{R}}} \circ \Phi g) \longrightarrow THH(-) \wedge_{\underline{T}} Ig = \underline{Z}_1$$

the map induced by $r_{THH(-)}$ and $r_{\underline{\underline{T}}}$; and

$$\underline{Y} = THH(-) \wedge_{\underline{T}} Ig = \underline{Z}$$

the identity. Because $THH(A)$ and $\underline{\underline{T}}$ are cyclotomic, each of these maps is an $\mathscr{F}_p$-equivalence.

We construct the natural map of zigzag cyclotomic spectra $\underline{\underline{Y}} \to \underline{\underline{X}} = THH^{\underline{\underline{R}}}_{z}(-)$ as follows. The map

$$\underline{Y}_3 = \Phi THH(-) \wedge_{\Phi \underline{T}} I(\Phi g) \longrightarrow \Phi THH(-) \wedge_{\Phi THH(R)} I(\Phi c) = \underline{X}_3$$

is induced by the map $q \colon \underline{T} \to THH(R)$ and a map of commutative $\underline{T}$-algebras $I(\Phi g) \to (\Phi q)^* I(\Phi c)$ we now define. Using the multiplication by 2 isomorphism $[0,1] \cong [0,2]$, $I(\Phi g)$ is isomorphic to

$$(\Phi \underline{T} \otimes [0,1]) \cup_{\Phi \underline{T} \otimes \{1\}} (\Phi \underline{T} \otimes [1,2]) \cup_{\Phi \underline{T} \otimes \{2\}} (\Phi g)^* \underline{R}$$

(where the pushout is done in $\mathbb{T}$-equivariant commutative ring orthogonal spectra, i.e.,"$\cup$"$= \wedge$). The $\Phi \underline{T} \otimes [0,1]$ piece maps by $\Phi q \otimes [0,1]$, and the $\Phi \underline{T} \otimes [1,2]$ piece maps using the subtract 1 isomorphism $[1,2] \to [0,1]$ and the homotopy ΦG (which starts at $\Phi c \circ \Phi q$ and ends at Φg). The map

$$\underline{Y}_2 = \Phi THH(-) \wedge_{\Phi \underline{T} \otimes I} IK \longrightarrow \Phi THH(-) \wedge_{\Phi THH(R) \otimes I} IF = \underline{X}_2$$

is induced by the map of commutative $(\underline{\underline{T}} \otimes I)$-algebras

$$IK \cong (\Phi \underline{T} \otimes I \otimes [0,1]) \cup_{\Phi \underline{T} \otimes \{1\}} (\Phi \underline{T} \otimes I \otimes [1,2]) \cup_{\Phi \underline{T} \otimes \{2\}} K^* \underline{R} \longrightarrow (\Phi q \otimes I)^* IF$$

that sends $\Phi \underline{T} \otimes I \otimes [0,1]$ by $\Phi q \otimes I \otimes [0,1]$ and maps $\Phi \underline{T} \otimes I \otimes [0,1]$ using H (which can be viewed as a homotopy from $F \circ q$ to $K \colon \Phi \underline{T} \otimes I \to \underline{R}$). The maps

$$\underline{Y}_1 = \Phi THH(-) \wedge_{\Phi \underline{T}} I(r_{\underline{\underline{R}}} \circ \Phi g) \longrightarrow \Phi THH(-) \wedge_{\Phi THH(R)} I(f_1) = \underline{X}_1$$
$$\underline{Y} = \Phi THH(-) \wedge_{\underline{T}} Ig \longrightarrow \Phi THH(-) \wedge_{THH(R)} Ic = \underline{X}_1$$

are defined in the same way as the maps above but with $\underline{Y}_1 \to \underline{X}_1$ using $\Phi q \otimes I$ and $G \circ r_{\underline{\underline{T}}}$ and $\underline{Y} \to \underline{X}$ using $q \otimes I$ and G. A straightforward check of diagrams (using the precyclotomic equations $q \circ r_{\underline{\underline{T}}} = r_{THH(R)} \circ \Phi q$ and $g \circ r_{\underline{\underline{T}}} = r_{\underline{\underline{R}}} \circ \Phi g$) shows that these maps construct a map of zigzag cyclotomic spectra. Because the map $\underline{T} \to THH(R)$ is by definition an $\mathscr{F}_p$-equivalence, the resulting map $\underline{\underline{Y}} \to \underline{\underline{X}}$ is a $\mathscr{F}_p$-equivalence of zigzag cyclotomic spectra.

Work in Progress

[Cyc23] Andrew J. Blumberg and Michael A. Mandell. The homotopy theory of cyclotomic spectra: 10 years later. In preparation.

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