If $M^{n+k}$ is a closed oriented manifold, we have a cup product $H^{k}(M) \otimes H^{n+k}(M) \rightarrow Z$. This is symmetric, bilinear, and multilinear, and it has a signature $	ext{sgn}(L)$, which is the difference between the number of positive eigenvalues and the number of negative eigenvalues.

If $M = \partial W$ (W oriented) then $\text{sgn}(M) = 0$, so it is a cobordism invariant.

For $M^{4k+2}$, $H^{2k+1} \otimes H^{2k+1} \rightarrow Z$ is nonvanishing skew symmetric bilinear and therefore has a canonical form. So there is no
such invariant

Homotopy theory began with the Hopf map

\[ S^3 \rightarrow S^2 = \mathbb{C} \cup \{ \infty \} \]

\[ (2_1, 2_2) \rightarrow (2_1/2_2, 2_2 \neq 0) \]

\[ 2_1^2 + 2_2^2 = 0 \]

\[ S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \} \]

\[ S^3_+ = \{ (z_1, z_2) : |z_1| \geq |z_2| \} \]

\[ S^3_- = \{ \} \]
If \( h \) extends to \( D^4 \) then

\[ h^{-1}(O_+) = S_+ , \ h^{-1}(O_-) = S_- \]

linking \# 1

If \( h \) extends this linking \# would be 0.

Pontryagin (193)

\[ \mathbb{S}^{n+k} \overset{f}{\longrightarrow} \mathbb{S}^k \]

For a regular value \( x \in \mathbb{S}^k \), \( f^{-1}(x) = \mathbb{N}^n \)

with a framing i.e. an embedding

\[ \mathbb{N}^n \times \mathbb{R}^k \overset{g_{n+k}}{\longrightarrow} \mathbb{S}^{n+k} \]

\[ \mathbb{T}^{n+k} \mathbb{S}^k = S^{2m} \text{ for } k \gg 0 \]
Portyaga in used this method to show $\pi_1 = 2/2$ and $\pi_2 = 0$ via an interesting mistake, assuming a certain map is linear rather than quadratic.

Let $V$ be an $F_2$-vector space with quad. form $q$, i.e. $q(x+y) = q(x) + q(y) = (x, y)$ bilinear.

"democratic invariant"

$q: V \rightarrow \mathbb{F}_2$, $q(x)$ is the vote of $x$ majority rules. The radical is

$$\{ y \in V : (x, y) = 0 \ \forall x \in \mathbb{F}_2 \}$$
Lemma. The election is not in the provided.
if no radical element votes negatively.
\[ d(e) = \begin{cases} 
\pm 1, & \text{for a tree} \\
0, & \text{otherwise}
\end{cases} \]

Kervaire, 1960 Let \( M^{10} \) be 4-connected. Define
\[ \varphi : H^5 M \to \mathbb{F}_2 \]
and showed
\[ d(e) = 1 \] (trivial Arf invariant) if \( M \in C^0 \).

Construct a PL \( M \) smooth outside a single pit with \( d(e) = -1 \). Hence \( M \) is not the hky. Light of a smooth mfd.
Let $M$ be 4-connected and smooth outside $x \in M$. For every $x \in H^5(M)$ we can find an embedding $S^5 \to M$ mapping $x$ (Hard Whitney embedding theorem).

Let $\nu(x) = \begin{cases} 0 & \text{if normal bundle of } S^5 \to M \text{ is trivial} \\ 1 & \text{if not} \end{cases}$

Any such bundle is stably trivial, so $V = \nu(S^5)$ (tangent bundle).

On trivial

This $\nu$ is quadratic. Consider $S^5 \times S^5$.
Kervaire–Milnor paper “Groups of homotopy spheres”

Milnor introduced plumbing surgery.

Kervaire’s \( M^{10} \) is a plumbing construction.

Let \( E = \text{tangent bundle of } S^5 \)

\[
\begin{align*}
E_1 \xrightarrow{f_1} S^5 \times D^5 & \cong D^5 \times D^5 \\
E_2 \xrightarrow{f_2} S^5 \times D^5 & \cong D^5 \\
\end{align*}
\]

Place \( E_1 \) and \( E_2 \) along common \( D^5 \times D^5 \) and get \( M^{10} \) with \( 2(M^{10}) = \Sigma^9 \) homeo.

Hence \( \Sigma^9 \) is exotic \( S^9 \).
Kervaire–Milnor (KM). Let $W^n$ be a framed $m$-fold with $\partial W = N$, meaning

$$W^n \times D^k \hookrightarrow D^{n+k}$$

$$\partial W \times D^k \hookrightarrow 2D^{n+k}$$

Use framed surgery to simplify homotopy type of $W$. Suppose $W$ is $(l-1)$-connected and $S^l \subset W$ framed embedding.

$$D^{l+1} \hookrightarrow D^{l+1} \subset D^{n+k}$$

$S^l$ has a $k$-frame $U (S^l \subset D^{n+k})$ if this $k$-frame extends to $D^{l+1} \subset D^{n+k}$,
then we can use this embedding to change $W$ to $W'$ via framed surgery. There are no difficulties below the middle dimension.

KM showed that $QW = N \left[ \frac{m}{2} \right]$ - connected

$W \left[ \frac{m}{2} \right]$ - connected

can go from $N \times IR^k \subset IR^{2d+k}$ to $N' (d-1)$-conn.

The obstruction to making it $(2d-1)$ connected is

$\text{sgn}(W)$ if $2W \neq 0$, $l = 2g$, $n \neq 0$,

$\text{Ker}(w) = d(u)$, $u$ defined by normal bundle
normal bundle of embedded sphere on by obstruction to extending the k-frame over a disk.

How can we determine this obstruction without doing surgery first?

Question: When are their framed maps N
(n = 2q + 1) in $\mathbb{R}^{2n+k}$ with $H_1(M) \neq 0$?

$s^1 \times s^1$, $s^3 \times s^3$, $s^7 \times s^7$ YES

dim 10 NO

For many years it was thought answer was no unless $n = 2 \cdot 6 \cdot 14$
1963 E.Y. Brown showed $K_I$ could be defined for thin mfs, for $n = 8k + 2$.

1965 Brown and Peterson showed it vanishes for $k = 0$.

1968 Browder gave different definition.

For $q$ odd, let $M^2$ be framed in some mfs $W = q + k$.

Thom's work implied:

a) Given $x \in H_2(M; \mathbb{Z}_2)$, $\exists$ mfs $N^3 \subset M^2$.
such that \( x \in [W^3] \rightarrow x \)

\( T \) \( y \cdot x = \partial y \) for \( y \in H^3(W, M) \)

\( \exists U \subset W \times [0, 1] \) with \( \partial U = N \)

Over \( N \) in \( M \subset W \), given \( k \)-frame over \( M \)

hence over \( N \subset W \)

Obstruction to extending over \( U \) is \( \tau^W_1 \big| V \)

\( \tau^W_1 = \text{relative characteristic class} \)

(Kervaire)

\( = W^W \) class
Win roughly \( K(2/2, g) \to W \to BD \)

\[ \downarrow \]
\[ K(2/2, g+1) \]

This leads to reduction of framing problem to easier homotopy theory??